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HAUSDORFF CONTENT AND RATIONAL APPROXIMATION IN FRACTIONAL LIPSCHITZ NORMS

BY

ANTHONY G. O'FARRELL[1]

ABSTRACT. For $0 < \alpha < 1$, we characterise those compact sets X in the plane with the property that each function in the class $\mathrm{lip}(\alpha, X)$ that is analytic at all interior points of X is the limit in $\mathrm{Lip}(\alpha, X)$ norm of a sequence of rational functions. The characterisation is in terms of Hausdorff content.

1. If E is a closed subset of the complex plane $\mathbf{C}$, and f is a bounded complex-valued function on E we define the *modulus of continuity* ω_f by setting

$$\omega_f(r) = \sup\{|f(x) - f(y)|: x, y \in E, |x - y| \leqslant r\}$$

whenever $r \geqslant 0$. Thus ω_f is a nondecreasing function, $\omega(0) = 0$, and f is uniformly continuous on E if and only if ω_f is continuous at zero. For $0 < \alpha < 1$ we define

$$\|f\|_{\alpha,E} = \sup\{r^{-\alpha}\omega_f(r): r > 0\},$$

$$\mathrm{Lip}(\alpha, E) = \{f: \|f\|_{\alpha,E} < \infty\},$$

$$\mathrm{lip}(\alpha, E) = \{f \in \mathrm{Lip}(\alpha, E): r^{-\alpha}\omega_f(r) \to 0 \text{ as } r\downarrow 0\}.$$

When given the norm

$$\|f\|'_{\alpha,E} = \|f\|_{\alpha,E} + \|f\|_{u,E}$$

(where $\|f\|_{u,E}$ is the sup norm), $\mathrm{Lip}(\alpha, E)$ becomes a Banach algebra, and $\mathrm{lip}(\alpha, E)$ is a closed point-separating subalgebra [9]. This paper concerns the question of approximation in $\mathrm{Lip}(\alpha, X)$, for compact sets X, by rational functions with poles off X.

Before stating the main result, we must define the *Hausdorff contents* M^β and M^β_*. A *measure function* is a nonnegative increasing function defined on

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$\mathbf{R}^+ = \{t \in \mathbf{R}: t \geqslant 0\}$. If h is a measure function and $F \subset \mathbf{C}$, then the Hausdorff content $M_h(F)$ is the infimum of all sums

$$\sum_{S \in \mathfrak{S}} h(\operatorname{diam} S),$$

where $\mathfrak{S}$ runs over all countable coverings of F by closed (or open) balls. In case $h(r) = r^\beta$ for some $\beta > 0$, we write $M_h = M^\beta$. The set function M_*^β is defined by setting

$$M_*^\beta(F) = \sup\{M_h(F): h \text{ is a measure function}, \\ h(r) \leqslant r^\beta, r^{-\beta}h(r) \to 0 \text{ as } r \downarrow 0\}.$$

THEOREM. *Let X be a compact subset of* $\mathbf{C}$, *and let* $0 < \alpha < 1$. *In order that every function in* $\operatorname{lip}(\alpha, X)$ *which is analytic on the interior of X be the limit in* $\operatorname{Lip}(\alpha, X)$ *norm of a sequence of rational functions, it is necessary and sufficient that there exist a constant* $\mu > 0$ *such that*

$$M^{1+\alpha}(D \setminus X) \geqslant \mu\, M_*^{1+\alpha}(D \setminus \operatorname{int} X)$$

whenever D is an open disc.

It is worth noting that the condition for approximation is purely metric, in contrast to the conditions which have been obtained for uniform approximation [12].

The necessity of the condition is proved in §§2–8. We introduce capacities in §2 and show that if two spaces have the same closure then the corresponding capacities coincide. In §§3–7 we apply a generalisation of Melnikov's Theorem [10] in order to relate the capacities corresponding to rational functions and lip α analytic functions to the contents $M^{1+\alpha}$ and $M_*^{1+\alpha}$. The proof of sufficiency in §§10–15 is modelled on the Vitushkin approximation scheme [12], [6], [8] as modified by Davie [3]. We make heavy use of the metric character of the capacities. We give some applications in §§16–23.

Throughout the paper, α is fixed, $0 < \alpha < 1$; $\mathbf{Z}$ denotes the set of integers, and $\mathbf{Z}^+ = \mathbf{Z} \cap \mathbf{R}^+$; Σ is the Riemann sphere; $\mathfrak{D}$ is the space of complex-valued C^∞ functions with compact support. If f is continuous on $\mathbf{C}$ and $\varphi \in \mathfrak{D}$ we define

$$T_\varphi f(z) = \frac{1}{\pi} \int \frac{f(z) - f(\zeta)}{z - \zeta} \frac{\partial \varphi}{\partial \bar{\zeta}}\, dm(\zeta),$$

where m denotes Lebesgue measure on the plane. For an exposition of the properties of this "T_φ-operator", see [6]. A set B of continuous functions on $\mathbf{C}$ is said to be *T-invariant* if $T_\varphi f \in B$ whenever $f \in B$ and $\varphi \in \mathfrak{D}$. The operator T_φ is bounded with respect to the $\operatorname{Lip}(\alpha, \mathbf{C})$ norm, for each $\varphi \in \mathfrak{D}$.

In fact

$$\|T_\varphi f\|_{\alpha,\mathbf{C}} \leqslant K\eta_f(d)\{\|\varphi\|_u + d\|\nabla\varphi\|_u\},$$

where K is a constant depending only on α,

$$d = \operatorname{diam}\operatorname{spt}\varphi, \qquad \eta_f(d) = \sup\{s^{-\alpha}\omega_f(s) : 0 < s \leqslant d\}.$$

The symbol X always stands for a compact subset of $\mathbf{C}$, $\mathfrak{R}(X)$ is the subspace of $\operatorname{Lip}(\alpha, \mathbf{C})$ consisting of those functions which agree on some neighbourhood of X with a rational function, and $\tilde{\mathfrak{R}}(X)$ is the space of functions in $\operatorname{Lip}(\alpha, \mathbf{C})$ which are analytic on a neighbourhood of X. If B is any subspace of $\operatorname{Lip}(\alpha, X)$, then the closure of B with respect to the norm $\|\cdot\|'_{\alpha,X}$ is denoted $[B]_{\alpha,X}$, or just $[B]_\alpha$. If B contains the constants, then this coincides with the closure with respect to the norm $\|\cdot\|_{\alpha,X}$. *For any* X,

$$[\mathfrak{R}(X)]_{\alpha,X} = [\tilde{\mathfrak{R}}(X)]_{\alpha,X}.$$

This assertion is the α version of Runge's Theorem, and the classical proof of Runge's Theorem is easily modified to prove it.

As a technical convenience, we assume that the diameter of X does not exceed $\frac{1}{4}$.

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2. We follow established custom in denoting the algebra of all continuous complex-valued functions on X by $C(X)$ and denoting the subalgebra of functions analytic on $\operatorname{int}(X)$ by $A(X)$. We further define

$$A^\alpha(X) = \operatorname{Lip}(\alpha, X) \cap A(X), \qquad A_\alpha(X) = \operatorname{lip}(\alpha, X) \cap A(X),$$

so that A^α and A_α are closed subalgebras of Lip α. In view of the extension theorem [11, Chapter VI], a subspace $V \subset A^\alpha(X)$ may be regarded as a subspace of $\operatorname{Lip}(\alpha, \mathbf{C})$ (we may identify V with the set of functions in $\operatorname{Lip}(\alpha, \mathbf{C})$ whose restrictions to X lie in V), so T-invariance makes sense for such subspaces. To each T-invariant subspace V of $A^\alpha(X)$ we associate a **capacity** $\gamma(V, \circ)$, a nonnegative increasing function defined on the family $\{D\}$ of open discs: we say a function $f \in V$ is *D-admissible* if f is analytic off a compact subset of D, $f(\infty) = 0$, and $\|f\|_{\alpha,\mathbf{C}} \leqslant 1$; we set

$$\gamma(V, D) = \sup\{|f'(\infty)| : f \in V, f \text{ is } D\text{-admissible}\}.$$

Lemma. *Let V and W be T-invariant subspaces of $A^\alpha(X)$. Suppose V and W have the same closure in* $\operatorname{Lip}(\alpha, X)$ *norm. Then $\gamma(V, D) = \gamma(W, D)$ for every open disc D.*

Proof. It suffices to show that

$$\gamma(V, D) = \gamma([V]_\alpha, D).$$

It is clear that

$$\gamma(V, D) \leqslant \gamma([V]_\alpha, D).$$

To prove the opposite inequality, let D be a fixed open disc and let $\varepsilon > 0$ be given. Choose $f \in [V]_\alpha$ such that f is D-admissible and

$$|f'(\infty)| > \gamma([V]_\alpha, D) - \varepsilon.$$

Choose a sequence $\{f_n\}_1^\infty$ of elements of V such that $\|f_n - f\|_{\alpha,X} \to 0$. For each n the extension theorem ensures the existence of a function

$$g_n \in \operatorname{Lip}(\alpha, \mathbf{C})$$

such that $g_n = f_n - f$ on X and $\|g_n\|_{\alpha,\mathbf{C}} \leqslant 4\|f_n - f\|_{\alpha,X}$. Let $h_n = f + g_n$. Then $h_n \in V$ and $\|h_n - f\|_{\alpha,\mathbf{C}} \to 0$ as $n \to +\infty$. Choose $\varphi \in \mathcal{D}$ such that $\operatorname{spt} \varphi \subset D$ and $\varphi \equiv 1$ on a neighbourhood of the set of singularities of f. Then $T_\varphi f = f$, $T_\varphi h_n \in V$, and

$$\begin{aligned}\|T_\varphi h_n - f\|_{\alpha,\mathbf{C}} &= \|T_\varphi (h_n - f)\|_{\alpha,\mathbf{C}} \\ &\leqslant K\|h_n - f\|_{\alpha,D}\{\|\varphi\|_u + \operatorname{diam} D \|\nabla \varphi\|_u\},\end{aligned}$$

by §1. Thus $\|T_\varphi h_n - f\|_{\alpha,\mathbf{C}} \to 0$, and hence $(T_\varphi h_n)'(\infty) \to f'(\infty)$, so that

$$\gamma(V, D) \geqslant \gamma([V]_\alpha, D) - \varepsilon.$$

Since this holds for each $\varepsilon > 0$, we conclude that $(*)$ holds.

We do not know whether or not the converse to this lemma is true in general.

3. In order to apply Lemma 2 to rational approximation we have to describe the capacities $\gamma(V, \cdot)$ in the cases $V = \mathfrak{R}(X)$ and $V = A_\alpha(X)$. **Melnikov's Theorem** provides the key. It relates certain capacities to the Hausdorff contents M_h. Before stating it we define a special class of "modulus of continuity functions".

Consider a concave increasing function $\omega(r)$, defined for $r \geqslant 0$ and constant for $r \geqslant 1$, with $\omega(0) = 0$, and such that

(1) $\omega'(r)$ exists for $r > 0$;

(2) there exists a constant $L_1 > 0$ such that $\omega(r) \leqslant L_1 r\omega'(r)$ for $0 < r < \frac{1}{2}$;

(3) there exists a constant $L_2 > 1$ such that $r\omega'(r) \leqslant (L_2 - 1)\omega(r)/L_2$ for $0 < r < \frac{1}{2}$.

Such a ω we call a *modulated function*. To each modulated function is associated a measure function h, defined by $h(r) = r\omega(r)$, and a capacity $\tau(\omega, \cdot)$ defined on arbitrary bounded sets $E \subset \mathbf{C}$ by

$$\tau(\omega, E) = \sup\{|f'(\infty)| : f \text{ is analytic on a neighbourhood of } \Sigma \setminus E, f(\infty) = 0, \omega_f \leqslant \omega\}.$$

Here ω_f refers to the modulus of continuity of f as a function on **C**.

MELNIKOV'S THEOREM. *Let ω be a modulated function. Then there is a constant $K(\omega)$ such that*

$$K^{-1}M_h(E) \leqslant \tau(\omega, E) \leqslant KM_h(E)$$

whenever E is compact or E is open and bounded. $K(\omega)$ may be taken to be $K_0(L_1 + L_2)$, where K_0 is a certain universal constant.

Actually, this is a slight extension of Melnikov's result. He proved it in case $\omega(r) = r^\beta$ for some β, $0 < \beta < 1$, and in that case $K(\omega)$ may be taken to be $K_0\beta^{-1}(1-\beta)^{-1}$. His proof [10] carries over with trivial changes. We omit the details.

An example of a modulated function other than the various r^β, $0 < \beta < 1$, is obtained by fixing $0 < \delta < 1$ and setting

$$\omega(r) = \begin{cases} r^\delta\{\delta^{-1} - \log 2r\}, & 0 < r < \frac{1}{2}, \\ \delta^{-1}2^{-\delta}, & \frac{1}{2} \leqslant r < \infty. \end{cases}$$

4. LEMMA. *Let $\omega(r)$ be a nonnegative function such that $\omega(r) \leqslant r^\alpha$ and $r^{-\alpha}\omega(r) \to 0$. Let $\varepsilon > 0$ and $\beta > \alpha$ be given. Then there exists a modulated function $\omega_1(r)$ with the following properties*:

(1) $(1-\varepsilon)\omega(r) \leqslant \omega_1(r) \leqslant r^\alpha$ *for* $0 \leqslant r \leqslant \frac{1}{2}$,

(2) $\alpha\omega_1(r) \leqslant r\omega_1'(r) \leqslant \beta\omega_1(r)$ *for* $0 \leqslant r \leqslant \frac{1}{2}$,

(3) $r^{-\alpha}\omega_1(r) \to 0$ *as* $r\downarrow 0$.

PROOF. In proving this, we may suppose that $\beta < \alpha(1-\varepsilon)^{-1}$. Choose a monotonically-decreasing sequence of piecewise smooth functions ψ_j such that

(4) $\beta(1-\varepsilon)\omega(r)/\alpha \leqslant \psi_j(r) \leqslant r^\alpha$,

(5) $\alpha\psi_j(r) \leqslant r\psi_j'(r) \leqslant \beta\psi_j(r)$,

(6) $\psi_j(r) < r^\alpha/j$ in a neighbourhood of the origin.

Such ψ_j's may be constructed as follows: Choose $\delta_j > \alpha$, put

$$\varphi_j(r) = \max\{r^\alpha/j, r^{\delta_j}\}, \quad \text{and}$$

$$\psi_j(r) = \min\left\{\alpha\int_0^r \frac{\varphi_j(s)}{s}\,ds, \psi_{j-1}(r)\right\}.$$

If δ_j is sufficiently close to α, properties (4), (5) and (6) are satisfied, as is seen by a routine calculation.

Set $\varphi(r) = \lim \psi_j(r)$. It follows easily that

$$\omega_1(r) = \alpha\int_0^r \frac{\varphi(s)}{s}\,ds$$

satisfies properties (1), (2), and (3). Verification is again routine. This completes the proof.

Fix $\beta = (1+\alpha)/2$. For each $f \in \operatorname{lip}(\alpha, \mathbf{C})$ with $\|f\|_\alpha \leqslant 1$, and each $\varepsilon > 0$, choose a modulated function $\omega_1(r)$ such that

$(1-\varepsilon)\omega_f(r) \leqslant \omega_1(r) \leqslant r^\alpha$,

$\alpha\omega_1(r) \leqslant r\omega_1'(r) \leqslant \beta\omega_1(r)$,

$r^{-\alpha}\omega_1(r) \to 0$ as $r \downarrow 0$.

Let $\mathfrak{F}_\alpha$ denote the family of all functions ω_1 obtained in this way. Clearly, we may apply Melnikov's Theorem to all $\omega_1 \in \mathfrak{F}_\alpha$ at once, using the same constant K.

5. Corollary. *Let $X \subset \mathbf{C}$ be compact, $V = \tilde{\mathfrak{R}}(X)$. Then for all open discs D*

$$K^{-1}\gamma(V, D) \leqslant M^{1+\alpha}(D \setminus X) \leqslant K\gamma(V, D),$$

where K depends only on α.

Proof. Choose a sequence of open sets $\{U_n\} \downarrow X$ such that each set $\operatorname{bdy}(U_n)$ is a finite union of smooth curves. Then

$$M^{1+\alpha}(D \setminus X) = \lim_{n\uparrow\infty} M^{1+\alpha}(D \setminus U_n).$$

Next, for $n = 1, 2, 3, \ldots$, we have

$$A^\alpha(X_n) \subset V \subset \bigcup_{m=1}^{\infty} A^\alpha(X_m),$$

where $X_n = \operatorname{clos}(U_n)$. Hence for each open disc D,

$$\gamma(A^\alpha(X_n), D) \leqslant \gamma(V, D) \leqslant \lim_{m\uparrow\infty} \gamma(A^\alpha(X_m), D).$$

Applying Melnikov's Theorem with $\omega(r) = r^\alpha$ and $E = D \setminus X_n$ (so that $\tau(\omega, E) = \gamma(A^\alpha(X_n), D)$), we obtain

$$K^{-1}\gamma(A^\alpha(X_n), D) \leqslant M^{1+\alpha}(D \setminus X_n) \leqslant K\gamma(A^\alpha(X_n), D),$$

for $n = 1, 2, 3, \ldots$, where K depends only on α. Taking limits we get the desired result.

6. In the definition of $M_*^{1+\alpha}$ it suffices to consider those h of the form $r\omega(r)$ for $\omega \in \mathfrak{F}_\alpha$.

7. Corollary. *Let $W = A_\alpha(X)$. Then for all open discs D,*

$$K^{-1}\gamma(W, D) \leqslant M_*^{1+\alpha}(D \setminus \operatorname{int} X) \leqslant K\gamma(W, D),$$

where K depends only on α.

Proof. Let $f \in W$ be D-admissible, and let $\varepsilon > 0$ be given. Then there exists $\omega \in \mathfrak{F}_\alpha$ such that $(1-\varepsilon)\omega_f \leqslant \omega$. Thus

$$(1-\varepsilon)|f'(\infty)| \leqslant \tau(w, D \setminus \operatorname{int} X).$$

If $h(r) = r\omega(r)$, then Melnikov's Theorem yields

$$\tau(\omega, D \setminus \operatorname{int} X) \leqslant K(\omega) M_h(D \setminus \operatorname{int} X).$$

Thus

$$(1-\varepsilon)\gamma(W, D) \leqslant K M_*^{1+\alpha}(D \setminus \operatorname{int} X),$$

where $K = \sup\{K_0(L_1 + L_2): \omega \in \mathfrak{F}_\alpha\}$ depends only on α. This proves the first inequality.

For the second, fix $\omega \in \mathfrak{F}_\alpha$, and let $h(r) = r\omega(r)$. Let $f \in C(\Sigma)$ be analytic off $(D \setminus \operatorname{int} X)$, with $\omega_f \leqslant \omega$, $f(\infty) = 0$. Then $f \in W$ and f is D-admissible. Hence $|f'(\infty)| \leqslant \gamma(W, D)$. Thus $\tau(\omega, D \setminus \operatorname{int} X) \leqslant \gamma(W, D)$. By Melnikov's Theorem

$$K(\omega)^{-1} M_h(D \setminus \operatorname{int} X) \leqslant \gamma(W, D).$$

Since this holds for every $\omega \in \mathfrak{F}_\alpha$, we conclude that

$$K^{-1} M_*^{1+\alpha}(D \setminus \operatorname{int} X) \leqslant \gamma(W, D),$$

with K as above.

8. Combining the results of §§1, 2, 5, and 7, we deduce the necessity of the condition of the theorem. In fact, if $[\mathfrak{R}]_\alpha = A_\alpha(X)$, then

$$M^{1+\alpha}(D \setminus X) \geqslant K M_*^{1+\alpha}(D \setminus \operatorname{int} X),$$

for every open disc D, where $K > 0$ is a constant which depends only on α.

9. Remark. One might wonder whether it is always possible, given a modulated function ω, to find functions $f \in A(X)$ such that $\omega_f \leqslant \omega$ but $\omega(r)^{-1}\omega_f(r) \nrightarrow 0$ as $r \to 0$. Putting it another way, if $\omega_1(r)\omega_2(r)^{-1} \to 0$ as $r \to 0$, are there any functions f in $A(X)$ such that $\omega_f \leqslant \omega_2$ but $\omega_f \neq o(\omega_1)$? The answer is yes. This follows from some results of Dolženko [4].

10. the first step towards proving the sufficiency of the approximation condition is a lemma which gives an estimate for the uniform norm in terms of the Lip α norm.

Lemma. *Suppose $E \subset \mathbf{C}$ is bounded, f is analytic on $\Sigma \setminus E$, $f(\infty) = 0$, and $f \in \operatorname{Lip}(\alpha, \mathbf{C})$. Then*

$$\|f\|_{u,\mathbf{C}} \leqslant 2^{1+\alpha}(\operatorname{diam} E)^\alpha \|f\|_{\alpha,\mathbf{C}}.$$

Proof. There is a circle C of radius diam E which encloses E. Since $f(\infty) = 0$, then $\int_C f\, d\vartheta = 0$. Hence, if $f = u + iv$, then $\int_C u\, d\vartheta = \int_C v\, d\vartheta = 0$. Thus u and v each have a zero on C. Thus for x inside S,

$$|u(x)| \leqslant (2 \operatorname{diam} E)^\alpha \|f\|_\alpha, \qquad |v(x)| \leqslant (2 \operatorname{diam} E)^\alpha \|f\|_\alpha,$$

hence

$$|f(x)| \leqslant 2^{1+\alpha}(\operatorname{diam} E)^\alpha \|f\|_\alpha,$$

and the result follows by the maximum principle.

The above estimate is somewhat crude, in that it depends only on the diameter of E. A more refined version is obtain in §14.

11. Now fix X compact in $\mathbf{C}$ and abbreviate $\mathfrak{R} = \mathfrak{R}(X)$, $A = A_\alpha(X)$, $\gamma(D) = \gamma(\mathfrak{R}, D)$, $\gamma_A(D) = \gamma(A, D)$. Let $c(D)$ denote the centre of the disc D, and let τD denote the disc with centre $c(D)$ and radius equal to τ times the radius of D. For any function f which is analytic on a neighbourhood of ∞ we may write

$$f(z) = a_0 + \frac{a_1}{z - c(D)} + \frac{a_2}{(z - c(D))^2} + \dots$$

for large z. Here $a_0 = f(\infty)$, $a_1 = f'(\infty)$, and we define $\beta(f, D) = a_2$. If $a_0 = a_1 = 0$, then $\beta(f, D)$ does not depend on D.

Lemma. *Let D be an open disc of radius r, and let $f \in \mathfrak{R}$ be D-admissible. Then*

$$|\beta(f, D)| \leqslant Kr\gamma(D),$$

where K is a constant depending only on α. For $f \in A$ the same inequality holds, but with γ replaced by γ_A.

Proof. Let $f \in \mathfrak{R}$ be D-admissible. Then f is analytic off D, $f(\infty) = 0$, and $\|f\|_\infty \leqslant 1$. We define the function $g \in \mathfrak{R}$ by setting

$$g(z) = (z - c(D))f(z) - f'(\infty).$$

Then $g(\infty) = 0$, $g'(\infty) = \beta(f, D)$, and we claim that $\|g\|_\alpha \leqslant K_8 r$, where K_8 depends only on α.

In proving this claim we may assume $c(D) = 0$. Let $z, w \in \mathbf{C}$, $z \neq w$. We consider four cases, which together cover all the possiblilites.

Case 1. $z, w \in 3D$. Then

$$\begin{aligned}\frac{|zf(a) - wf(w)|}{|z-w|^\alpha} &\leqslant \frac{|z|\,|f(z) - f(w)| + |z-w|\,|f(w)|}{|z-w|^\alpha} \\ &\leqslant 3r\,\|f\|_\alpha + (6r)^{1-\alpha}\|f\|_u \\ &\leqslant K_1 r\,\|f\|_\alpha \quad \text{by §10} \\ &\leqslant K_1 r.\end{aligned}$$

Case 2. $z, w \in \mathbf{C} \setminus 2D$, $|z-w| \geqslant r$. Then

$$\frac{|zf(z) - wf(w)|}{|z - w|^{\alpha}} \leqslant \frac{|zf(z)|}{r^{\alpha}} + \frac{|wf(w)|}{r^{\alpha}}$$

$$\leqslant 2r^{1-\alpha}\left(|f(z)| + |f(w)|\right) \leqslant K_1 r^{1-\alpha}\left\{ \frac{r\|f\|_u}{|z|} + \frac{r\|f\|_u}{|w|} \right\}$$

$$\leqslant K_2 r^{1-\alpha}\|f\|_u \leqslant K_3 r\|f\|_{\alpha} \leqslant K_3 r.$$

In the third inequality we used the *uniform norm decay estimate* [6, p. 201], and in the fifth we again applied §10.

Case 3. $z, w \in \mathbf{C} \setminus 2D$, $|z - w| < r$. Then

$$\frac{|zf(z) - wf(w)|}{|z - w|^{\alpha}}$$

$$= \frac{1}{|z - w|^{\alpha}} \left| \frac{1}{2\pi i} \int_{|\zeta| = r} \zeta f(\zeta) \left\{ \frac{1}{\zeta - z} - \frac{1}{\zeta - w} \right\} d\zeta \right|$$

$$\leqslant \frac{K_4 r \|f\|_u}{|z - w|^{\alpha}} \int_{|\zeta| = r} \frac{|z - w|}{|\zeta - z|\,|\zeta - w|} |d\zeta|$$

$$\leqslant K_5 r^{1+\alpha}\|f\|_{\alpha}|z - w|^{1-\alpha} r^{-1} < K_5 r.$$

Case 4. $z \in 2D$, $w \notin 3D$. Then

$$\frac{|zf(z) - wf(w)|}{|z - w|^{\alpha}} \leqslant \frac{|zf(z)|}{r^{\alpha}} + \frac{|wf(w)|}{r^{\alpha}}$$

$$\leqslant 2r^{1-\alpha}\|f\|_u + \frac{|w|}{r^{\alpha}} \cdot \frac{r\|f\|_u}{|w|} \leqslant K_6 r^{1-\alpha}\|f\|_u \leqslant K_7 r.$$

Hence the claim is true, so that $(K_8 r)^{-1} g$ is D-admissible. Thus

$$|\beta(f, D)| = |g'(\infty)| \leqslant K_8 r \gamma(D).$$

The assertion about A is proved similarly.

12. Decay Lemma (Garnett). *Let D be a disc of radius r, and let $z \in \mathbf{C}$, with $d = \operatorname{dist}(z, D) \geqslant r$. Then*

(1) $$|f(z)| \leqslant K\gamma(D)\|f\|_{\alpha}/d$$

and

(2) $$|f'(z)| \leqslant K\gamma(D)\|f\|_{\alpha}/d^2$$

whenever $f \in \mathfrak{R}$. There is a similar estimate for $f \in A$, with γ replaced by γ_A.

Proof. (1) $D \setminus X$ may be covered by a finite collection $\{S_j\}$ of open squares with sides parallel to the axes, such that

$$\sum (\text{side } S_j)^{1+\alpha} \leqslant 4M^{1+\alpha}(D \setminus X)/\pi,$$

and no square is contained in the union of the rest. Arrange the squares in an order of nondecreasing side-lengths, and form $H_1 = S_1$, $H_2 = S_2 \setminus S_1$, $H_3 = S_3 \setminus S_1 \setminus S_2$, and so on. For each i, let $\Gamma_j = \text{bdy } H_j$, and choose $\zeta_j \in \text{int } H_1$. Observe that the length of Γ_i is at most 4(side S_i). Then

$$\begin{aligned} |f(z)| &= \left| \frac{1}{2\pi i} \sum_j \int_{\Gamma_j} \frac{f(\zeta)}{\zeta - z}\, d\zeta \right| \\ &\leqslant \frac{1}{2\pi} \sum_j \left| \int_{\Gamma_j} \frac{f(\zeta) - f(\zeta_j)}{\zeta - z}\, d\zeta \right| \\ &\leqslant K_1 \sum_j \frac{(\text{side } S_j)^{1+\alpha}}{d} \|f\|_\alpha \leqslant \frac{K_2 M^{1+\alpha}(D \setminus X)\|f\|_\alpha}{d} \\ &\leqslant \frac{K_3 \gamma(D)\|f\|_\alpha}{d}, \quad \text{by Corollary 5.} \end{aligned}$$

The estimate for $f'(z)$ is obtained in a similar way.

To prove the corresponding estimate for $f \in A$, first choose a modulated function ω such that

$$\tfrac{1}{2}\omega_f(r) \leqslant \|f\|_\alpha \omega(r), \qquad 0 \leqslant r \leqslant \tfrac{1}{2},$$

$$\omega(r) \leqslant r^\alpha, \qquad 0 \leqslant r \leqslant \tfrac{1}{2},$$

$$r^{-\alpha}\omega(r) \to 0 \qquad \text{as } r \downarrow 0.$$

Set $h(r) = r\omega(r)$. An argument like that above shows that

$$|f(z)| \leqslant K_4 M_h(D \setminus \text{int } X)\|f\|_\alpha / d,$$

and so

$$|f(z)| \leqslant \frac{K_4 M_*^{1+\alpha}(D \setminus \text{int } X)\|f\|_\alpha}{d} \leqslant \frac{K_5 \gamma_A(D)\|f\|_\alpha}{d}, \quad \text{by §7.}$$

13. Lemma. *Let D be an open disc, $s^{1+\alpha} = M^{1+\alpha}(D \setminus X)$, and let $\{B_j\}$ be a family of discs of radius s, each of which is contained in D, such that no point belongs to more than p of the B_j. Then there is a constant K, depending only on α, such that*

(1) $$\sum_j M^{1+\alpha}(B_j \setminus X) \leqslant KpM^{1+\alpha}(D \setminus X),$$

and also

$$\left\| \sum_j f_j \right\|_\alpha \leqslant Kp \tag{2}$$

whenever $f_j \in \mathfrak{R}$ *is* B_j*-admissible*, $j = 1, 2, \ldots$.

PROOF. Fix $\varepsilon > 0$, and choose a covering $\{D_n\}$ of $D \setminus X$ by discs with radii $\{r_n\}$ such that each r_n is no greater than s, and

$$\sum_n r_n^{1+\alpha} < M^{1+\alpha}(D \setminus X) + \varepsilon.$$

Then the D_n cover each $B_j \setminus X$, and no D_n meets more than $K_1 p$ of the B_j. Thus

$$\sum_j M^{1+\alpha}(B_j \setminus X) \leqslant K_1 p \sum r_n^{1+\alpha} \leqslant K_1 p \{ M^{1+\alpha}(D \setminus X) + \varepsilon \}.$$

This proves (1).

Now let $f_j \in \mathfrak{R}$ be B_j-admissible, $j = 1, 2, \ldots$. Fix $x, y \in \mathbf{C}$ and consider

$$|f_j(x) - f_j(y)| / |x - y|^\alpha.$$

We divide the integers j into classes F_m, corresponding to $m = 0, 1, 2, 3, \ldots$, as follows. We say $j \in F_m$ if m is the greatest integer not exceeding

$$s^{-1} \min\{\operatorname{dist}(x, B_j), \operatorname{dist}(y, B_j)\}.$$

Observe that the number of elements in F_m does not exceed $K_2 pm$.

For $m = 0$ or 1 and $j \in F_m$ we use the crude estimate

$$|f_j(x) - f_j(y)| / |x - y|^\alpha \leqslant \|f_j\|_\alpha \leqslant 1.$$

For $m > 1$, $j \in F_m$ we consider two cases.

Case 1. $|x - y| > s$. Then

$$\begin{aligned} \frac{|f_j(x) - f_j(y)|}{|x - y|^\alpha} &\leqslant \frac{|f_j(x)| + |f_j(y)|}{s^\alpha} \\ &\leqslant \frac{K_3 \gamma(B_j) \|f_j\|_\alpha}{(ms) s^\alpha} \quad \text{by §12} \\ &\leqslant \frac{K_3 \gamma(B_j)}{m s^{1+\alpha}}. \end{aligned}$$

Case 2. $|x - y| \leqslant s$. Since $j \in F_m$ there is an arc Γ joining x to y such that the length of Γ does not exceed $6|x - y|$, and $\operatorname{dist}(\Gamma, B_j) \geqslant ms$. Thus

$$\begin{aligned} \frac{|f_j(x) - f_j(y)|}{|x - y|^\alpha} &= \frac{|\int_\Gamma f'(z)\, dz|}{|x - y|^\alpha} \\ &\leqslant \frac{K_4 |x - y|^{1-\alpha} \gamma(B_j) \|f_j\|_\alpha}{(ms)^2} \leqslant \frac{K_4 \gamma(B_j)}{m^2 s^{1+\alpha}}. \end{aligned}$$

Thus in either case

$$\frac{|f_j(x) - f_j(y)|}{|x-y|^\alpha} \leq \frac{K_5 M^{1+\alpha}(B_j \setminus X)}{s^{1+\alpha}}.$$

Let $f = \Sigma_j f_j$. Then, abbreviating $M^{1+\alpha}(B_j \setminus X) = M_j$, we have

$$\begin{aligned}\frac{|f(x) - f(y)|}{|x-y|^\alpha} &\leq \sum_j \frac{|f_j(x) - f_j(y)|}{|x-y|^\alpha}\\ &\leq K_6 p + \sum_j \frac{K_5 M_j}{s^{1+\alpha}}\\ &\leq K_6 p + K_5 K_1 p \quad \text{by (1)}\\ &= K_7 p.\end{aligned}$$

14. This lemma allows us to improve the estimate for $\|f\|_u$ of §10.

COROLLARY. *Let D be an open disc and let $f \in \mathfrak{R}(X)$ be D-admissible. Then*

$$\|f\|_u \leq K\gamma(D)^{\alpha/(1+\alpha)}.$$

PROOF. In proving this we may assume that X contains a neighbourhood of $3D \setminus D$, and we do.

Cover the set of singularities of f by discs $\frac{1}{2}B_j \subset D$ of side

$$s = M^{1+\alpha}(D \setminus X)^{1/(1+\alpha)}$$

in such a way that no point belongs to more than 100 of the B_j. Choose functions $\varphi_j \in \mathfrak{D}$ such that $0 < \varphi_j \leq 1$, spt $\varphi_j \subset B_j$, $\|\nabla\varphi_j\|_u \leq 4/s$, and $\Sigma\varphi_j \equiv 1$ on $\bigcup \frac{1}{2}B_j$, which is a neighbourhood of the set of singularities of f (cf. [3]). Let $f_j = T_{\varphi_j} f$. Then $f = \Sigma f_j$, $f_j \in \tilde{\mathfrak{R}}$, f_j is analytic off B_j, and $f_j(\infty) = 0$. Also $\|f_j\|_\alpha \leq K_1$ by the T_φ estimate, so that $K_1^{-1} f_j$ is B_j-admissible.

Fix $z \in \mathbf{C}$, and divide the indices j up into classes again: say $j \in G_m$ if m is the greatest integer not exceeding s^{-1} dist(z, B_j). For $m > 1$ and $j \in G_m$ we have

$$|f_j(z)| \leq K_2\gamma(B_j)/ms$$

by the Decay Lemma, §12. Thus

$$|f(z)| \leqslant \sum_j |f_j(z)| \leqslant K_3\|f\|_u + \sum_{m=2}^{\infty} \sum_{j \in G_m} |f_j(z)|$$

$$\leqslant K_4\left\{s^\alpha + \sum_{m=2}^{\infty} \sum_{j \in G_m} \frac{\gamma(B_j)}{ms}\right\} = K_4 s^\alpha\left\{1 + \sum_{m=2}^{\infty} \sum_{j \in G_m} \frac{\gamma(B_j)}{ms^{1+\alpha}}\right\}$$

$$\leqslant K_5 s^\alpha\left\{1 + \left[\sum_j \frac{\gamma(B_j)}{s^{1+\alpha}}\right]^{1/2}\right\} \quad \text{(cf. [6, p. 201, 2.6])}$$

$$\leqslant K_6 s^\alpha, \quad \text{by §13 and §15.}$$

Thus $\|f\|_u \leqslant K_6 s^\alpha \leqslant K_7 \gamma(D)^{\alpha/(1+\alpha)}$.

15. We are now in a position to prove the sufficiency of the condition for approximation. In fact, we will prove a slightly stronger statement.

Suppose there exist constants $\mu > 0$, $\tau > 1$ *such that for each point* $x \in$ bdy X *and each disc* D *centered at* x,

$$M^{1+\alpha}(\tau D \setminus X) \geqslant \mu M_*^{1+\alpha}(D \setminus \operatorname{int} X).$$

Then $[\mathfrak{R}]_\alpha = A_\alpha(X)$.

Throughout the proof $K_1, K_2, K_3, \ldots$ stand for constants which may depend on α, μ, τ and $\|f\|_\alpha$, but not on any other variables.

Suppose μ and τ exist as in the statement. Then for each open disc D of radius r centered at a point of $\mathbf{C} \setminus \operatorname{int} X$ we have

$$M^{1+\alpha}(\tau D \setminus X) \geqslant 4^{-1}\mu M_*^{1+\alpha}(D \setminus \operatorname{int} X),$$

hence $\gamma(\tau D) \geqslant K_1 \gamma_A(D)$ for each such disc D.

Fix $f \in A$. We shall prove that f may be approximated in Lip(α, X) norm by elements of $\tilde{\mathfrak{R}}$. First, we extend f to $\mathbf{C}$ so that the extension (also denoted by f) lies in lip$(\alpha, \mathbf{C})$ and is analytic off some disc. Fix $\delta > 0$. Let $\{D_n\}_1^\infty$ be a covering of $\mathbf{C} \setminus \operatorname{int} X$ by open discs of radius δ centered at points of $\mathbf{C} \setminus \operatorname{int} X$ and such that no disc D_n meets more than 100 others. Let $\{\varphi_n\}_1^\infty \subset \mathfrak{D}$ be a sequence of functions such that $0 \leqslant \varphi_n \leqslant 1$, spt $\varphi_n \subset 2D_n$, $\|\nabla \varphi_n\|_u \leqslant 4\delta^{-1}$, and $\Sigma_1^\infty \varphi_n \equiv 1$ on $\bigcup_1^\infty D_n$. Let $f_n = T_{\varphi_n} f$. Then $f_n \in A$, $f_n \equiv 0$ except for a finite number of indices n, and $f = \Sigma_1^\infty f_n$. Let $\eta(r) = r^{-\alpha}\omega_f(r)$, so that $\eta(r) \to 0$ as $r \downarrow 0$. For each n, f_n is holomorphic off $2D_n$, $f_n(\infty) = 0$, and $\|f_n\|_\alpha \leqslant K_2 \eta(\delta)$.

Now fix n and, following Davie [3], let

$$r = \frac{1}{2\tau} \cdot \min\{\delta, M^{1+\alpha}(3D \setminus X)^{1/(1+\alpha)}\}.$$

Cover the (closed) set of singularities of f_n (a subset of $2D_n \setminus \operatorname{int} X$) by centered discs $B_j \subset 2D_n$ of radius r, in such a way that no point belongs to more than 25 of the B_j. Select a collection $\{\psi_j\} \subset \mathfrak{D}$ of functions such that $0 \leqslant \psi_j \leqslant 1$, spt $\psi_j \subset 2B_j$, $\|\nabla\psi_j\|_u \leqslant 4/r$, and $\Sigma\psi_j \equiv 1$ on $\bigcup B_j$, which is a neighbourhood of the set of singularities of f_n. Let $f_j^* = T_{\psi_j} f_n$. Then $f_j^* \in A$, f_j^* is analytic off $2B_j$, $f_j^*(\infty) = 0$, $\|f_j^*\|_\alpha \leqslant K_4\eta(\delta)$, and $f_n = \Sigma_j f_j^*$. From the definition of γ_A we deduce that

$$|f_j^{*\prime}(\infty)| \leqslant K_4\eta(\delta)\gamma_A(2B_j) \leqslant K_5\eta(\delta)\gamma(2\tau B_j), \quad \text{by hypothesis.}$$

Thus there exist functions $g_j^* \in \mathfrak{R}$ such that g_j^* is analytic off $2\tau B_j$, $g_j^*(\infty) = 0$, $\|g_j^*\|_\alpha \leqslant K_5\eta(\delta)$, and $g_j^{*\prime}(\infty) = f_j^{*\prime}(\infty)$. Let $g_n = \Sigma g_j^*$. Then $g_n \in \mathfrak{R}$, g_n is analytic off $3D_n$, $g_n(\infty) = 0$, and $g_n'(\infty) = f_n'(\infty)$. Also, by Lemma 13 (2), $\|g_n\|_\alpha \leqslant K_6\eta(\delta)$.

We have

$$\beta(f_n - g_n, D_n) = \sum_j \beta(f_j^* - g_j^*, D_n) = \sum_j \beta(f_j^* - g_j^*, G_j),$$

since $f_j^* - g_j^*$ vanishes to second order at ∞. Hence by Lemma 11 and Lemma 13 (1),

$$|\beta(f_n - g_n, D_n)| \leqslant \sum_j K_7 r\gamma(B_j)\eta(\delta) \leqslant K_8\gamma(2D_n)^{(2+\alpha)/(1+\alpha)}\eta(\delta).$$

We may choose a function $h_n \in \mathfrak{R}$, analytic off $2D_n$ and vanishing at ∞, with $\|h_n\|_\alpha \leqslant 2$ and $h_n'(\infty) = \gamma(2D_n)$. Forming

$$k_n = g_n + \beta(f_n - g_n, D_n)(h_n/\gamma)^2 \in \mathfrak{R}$$

(where we have abbreviated $\gamma = \gamma(2D_n)$), we deduce that

$$\begin{aligned}\|k_n\|_\alpha &\leqslant \|g_n\|_\alpha + |\beta(f_n - g_n, D_n)|\gamma^{-2}\|h_n^2\|_\alpha \\ &\leqslant K_6\eta(\delta) + K_8\gamma^{-\alpha/(1+\alpha)}\eta(\delta)\|h_n\|_u \leqslant K_9\eta(\delta)\end{aligned}$$

by Corollary 14. Also k_n is analytic off $2D_n$, $k_n(\infty) = 0$, $k_n'(\infty) = g_n'(\infty) = f_n'(\infty)$, and $\beta(k_n, D_n) = \beta(g_n, D_n) + \beta(f_n - g_n, D_n) = \beta(f_n, D_n)$.

Let $q_n = f_n - k_n$. Then $f = \Sigma k_n + \Sigma q_n$. The first sum belongs to $\tilde{\mathfrak{R}}$. We will show that the second sum tends to zero in Lip(α, **C**) norm as $\delta\downarrow 0$, so that $f \in [\tilde{\mathfrak{R}}]_{\alpha,\mathbf{C}}$.

Clearly $\|q_n\|_\alpha \leqslant K_{10}\eta(\delta)$, so that by Lemma 10, $\|q_n\|_u \leqslant K_{11}\delta^\alpha\eta(\delta)$. Fix two distinct points $x, y \in \mathbf{C}$. In order to estimate

$$|x - y|^{-\alpha}\left\{\sum q_n(x) - \sum q_n(y)\right\}$$

we divide the indices n into classes F_m, in the same way as in the proof of Lemma 13, with $s = 2\delta$. Thus $n \in F_m$ if ns is the greatest integral multiple of s not exceeding

$$\min\{\operatorname{dist}(x, 2D_n), \operatorname{dist}(y, 2D_n)\}.$$

The number of indices in F_m does not exceed $K_{11}(m + 1)$.

The function q_n has a triple zero at ∞, so that $\delta^{-3}(z-c_n)^3q_n(z)$, the function, is analytic on $\Sigma \setminus 2D_n$ (here $c_n = c(D_n)$). For $z \in \text{bdy}(2D_n)$,

$$\left|\delta^{-3}(z-c_n)^3 q_n(z)\right| \leqslant 8\|q_n\|_u \leqslant K_{12}\delta^\alpha\eta(\delta),$$

hence by the maximum principle,

$$(*) \qquad |q_n(z)| \leqslant K_{13}\delta^{3+\alpha}\eta(\delta)d^{-3}$$

whenever $d = \text{dist}(z, 2D_n) > s$.

If $k(z)$ is a bounded function, is analytic off a disc D of radius r, and vanishes at ∞, and $0 < R = \text{dist}(z, D)$, then the *uniform norm derivative decay estimate* [12, p. 201] states that

$$|k'(z)| \leqslant 4r\|k\|_{u,\text{bdy}D}/R^2.$$

If $d = \text{dist}(z, D_n) > 4s$, take $D = \frac{1}{2}dD_n$, so that $\|q_n\|_{u,\text{bdy}D} \leqslant K_{14}\delta^{3+\alpha}\eta(\delta)d^{-3}$ by $(*)$, and conclude that

$$(**) \qquad |q_n'(z)| \leqslant K_{15}\delta^{3+\alpha}\eta(\delta)d^{-4}.$$

If n belongs to one of the first six F_m we use the crude estimate

$$|q_n(x) - q_n(y)|/|x-y|^\alpha \leqslant \|q_n\|_\alpha \leqslant K_6\eta(\delta).$$

If $6 \leqslant m \in \mathbf{Z}$ and $n \in F_m$, we consider two cases.

Case 1. $|x-y| \leqslant s$. We have

$$ms \leqslant \min\{\text{dist}(x, 2D_n), \text{dist}(y, 2D_n)\},$$

so there is a curve Γ joining x to y, the length of which does not exceed $\pi|x-y|$, with the property that $\text{dist}(\Gamma, 2D_n) \geqslant ms$. Thus by $(**)$,

$$\frac{|q_n(x) - q_n(y)|}{|x-y|^\alpha} = \frac{1}{|x-y|^\alpha}\left|\int_\Gamma h_n'(z)\,dz\right|$$
$$\leqslant \pi K_{15}|x-y|^{1-\alpha}s^{3+\alpha}\eta(\delta)(ms)^{-4} \leqslant K_{16}\eta(\delta)m^{-4}.$$

Case 2. $|x-y| > s$. Then by $(*)$,

$$\frac{|q_n(x) - q_n(y)|}{|x-y|^\alpha} < \frac{|q_n(x)| + |q_n(y)|}{s^\alpha}$$
$$\leqslant 2K_{13}s^{3+\alpha}\eta(\delta)(ms)^{-3} = K_{17}\eta(\delta)m^{-3}.$$

Thus in either case

$$\frac{|\Sigma_n q_n(x) - \Sigma q_n(y)|}{|x-y|^{\alpha}} \leqslant \sum_n \frac{|q_n(x) - q_n(y)|}{|x-y|^{\alpha}}$$
$$\leqslant \sum_{m=0}^{5} \sum_{n \in F_m} K_6 \eta(\delta) + \sum_{m=6}^{\infty} \sum_{n \in F_m} K_{18} \eta(\delta) m^{-3}$$
$$\leqslant \left\{ \sum_{m=0}^{5} K_6 K_{11}(m+1) + \sum_{m=6}^{+\infty} K_{18} K_{11}(m+1) m^{-3} \right\} \eta(\delta)$$
$$= K_{19} \eta(\delta).$$

Since $\eta(\delta) \to 0$ as $\delta \downarrow 0$, this proves that $\|\Sigma q_n\|_{\alpha} \to 0$ as $\delta \downarrow 0$, so we are done.

16. As a special case we obtain a characterisation of those compact sets X on which *all* $f \in \text{lip}(\alpha, X)$ may be approximated in $\text{Lip}(\alpha, X)$ norm by rational functions.

COROLLARY. *A necessary and sufficient condition that*

$$[\mathfrak{R}]_{\alpha} = \text{lip}(\alpha, X)$$

is that there exist $\mu > 0$ *such that* $M^{1+\alpha}(D \setminus X) \geqslant \mu r^{1+\alpha}$ *for every open disc* D *of radius* r $(0 < \alpha < 1)$.

This follows from our theorem because $M_*^{1+\alpha}(D) = (2r)^{1+\alpha}$.

17. COROLLARY. *If* X *has zero area and* $0 < \alpha < 1$, *then*

$$[\mathfrak{R}]_{\alpha} = \text{lip}(\alpha, X).$$

PROOF. Let D be any disc of radius r. Then, denoting Lebesgue measure on the plane by m, we have $m(D \setminus X) = m(D) = \pi r^2$. Let $\{B_j\}$ be a covering of $D \setminus X$ by discs with radii $\{r_j\}$, $r_j \leqslant r$. Then

$$\sum r_j^{1+\alpha} \geqslant \frac{\Sigma r_j^2}{r^{1-\alpha}} \geqslant \frac{m(D \setminus X)}{\pi r^{1-\alpha}} = r^{1+\alpha},$$

hence $M^{1+\alpha}(D \setminus X) \geqslant r^{1+\alpha}$. Thus the condition of Corollary 16 is satisfied, with $\mu = 1$.

J. Garnett has shown the author how to give a direct constructive proof of this fact. There is also an entirely different proof, based on duality.

18. COROLLARY. *If* $0 < \alpha < 1$ *and* $M_*^{1+\alpha}(\text{bdy } X) = 0$, *then*

$$[\mathfrak{R}]_{\alpha} = \text{lip}(\alpha, X) \cap A(X).$$

PROOF. If E_1 and E_2 are two subsets of $\mathbf{C}$, then

$$M_*^{1+\alpha}(E_1 \cup E_2) \leqslant M_*^{1+\alpha}(E_1) + M_*^{1+\alpha}(E_2).$$

This is an immediate consequence of the definition of M_*^{β} and the subadditivity of M_h. It follows that

$$M_*^{1+\alpha}(D \setminus \operatorname{int} X) \leqslant M_*^{1+\alpha}(\operatorname{bdy} X) + M_*^{1+\alpha}(D \setminus X)$$
$$\leqslant M_*^{1+\alpha}(\operatorname{bdy} X) + M^{1+\alpha}(D \setminus X),$$

hence if $M_*^{1+\alpha}(\operatorname{bdy} X) = 0$, then the condition of our theorem is satisfied, with $\mu = 1$.

The condition $M_*^{1+\alpha}(E) = 0$ is equivalent to $\mathfrak{H}^{1+\alpha}(E) < \infty$, where $\mathfrak{H}^{1+\alpha}$ is $(1 + \alpha)$-dimensional *Hausdorff measure* [5, (2.10)].

19. Before giving some examples, we need a definition. Let $B(x, r)$ denote the disc $\{z \in \mathbf{C}: |z - x| \leqslant r\}$. If $E \subset \mathbf{C}$ and $\beta > 0$, then the *β-dimensional upper density* of E at the point $x \in \mathbf{C}$ is defined as

$$\limsup_{r \downarrow 0} \frac{M^\beta(E \cap B(x, r))}{r^\beta};$$

the *lower density* is the corresponding lim inf, and in case these two coincide, we refer to the *density*.

20. Example. We construct a compact set $X \subset \mathbf{C}$ such that X is the closure of its interior, and $[\mathfrak{R}]_u = A(X)$, but $[\mathfrak{R}]_\alpha \neq A_\alpha(X)$.

Fix β, $\alpha < \beta < 1$. We begin with a closed square P, and inside P an arc Γ having positive $(1 + \beta)$-dimensional lower density at each of its points [7]. We then remove from P a sequence of thin wavy open strips S_1, S_2, $S_3, \ldots$, so that the S_j "accumulate" only on Γ and accumulate at every point of Γ, and so that $\bigcup_j S_j$ has zero $(1 + \alpha)$-dimensional density at each point of Γ. Then we set $X = P \setminus (\bigcup_j S_j)$. For any small disc D of radius r about any point of Γ, $M_*^{1+\alpha}(D \setminus \operatorname{int} X)$ will be bounded below by some constant times $r^{1+\alpha}$, whereas $M^{1+\alpha}(D \setminus X)$ will be $o(r^{1+\alpha})$. So the condition of the theorem cannot hold for any $\mu > 0$. Thus $[\mathfrak{R}]_\alpha \neq A_\alpha(X)$. Since the diameters of the components of $\mathbf{C} \setminus X$ are bounded away from zero, it follows that $[\mathfrak{R}]_u = A(X)$ (cf. [6, p. 219 (8.3)]).

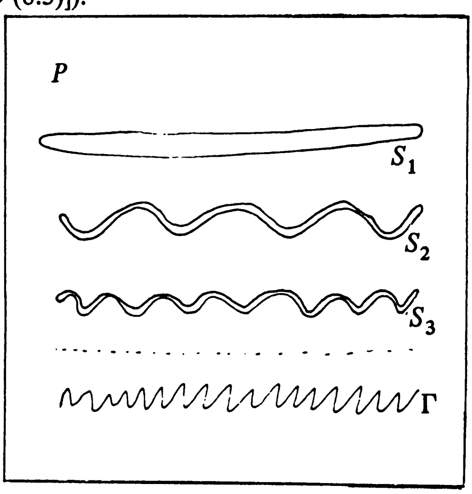

Figure 1

21. EXAMPLE. We construct a set X with empty interior such that the analytic polynomials $\mathcal{P}$ are uniformly dense in $C(X)$, but $[\mathcal{R}]_\alpha \neq \operatorname{lip}(\alpha, X)$.

Choose a sequence of positive numbers l_n such that $\Sigma_1^\infty l_n^\alpha < 1$. Then $\Sigma_1^\infty l_n < 1$ and we may form a Cantor set C of positive length on $[0, 1]$ by deleting successively (open) intervals of length l_n. Let λ denote Lebesgue measure on the line.

LEMMA. $[0, 1] \setminus C$ *has zero* α*-dimensional density at* λ *almost all points of* C.

PROOF. Let (a_n, b_n) be the interval of length l_n in $[0, 1] \setminus C$. Then by Fubini's Theorem,

$$\int_0^1 \sum_{n=1}^\infty \frac{l_n^\alpha}{|z - a_n|^\alpha} d\lambda(z) = \sum_{n=1}^\infty l_n^\alpha \int_0^1 \frac{d\lambda(z)}{|z - a_n|^\alpha}$$
$$\leqslant 2^\alpha (1-\alpha)^{-1} \sum_{n=1}^\infty l_n^\alpha < \infty,$$

so that

$$\sum_{n=1}^\infty \frac{l_n^\alpha}{|z - a_n|^\alpha} < \infty$$

for λ almost all $z \in [0, 1]$. Similarly,

$$\sum_1^\infty \frac{l_n^\alpha}{|z - b_n|^\alpha} < \infty$$

for λ almost all $z \in [0, 1]$. For $z \in C$ the upper α density of $[0, 1] \setminus C$ at z is

$$\limsup_{r \downarrow 0} \frac{M^\alpha([z-r, z+r] \setminus C)}{r^\alpha} \leqslant \limsup_{r \downarrow 0} \frac{\Sigma' l_n^\alpha}{r^\alpha}$$

(where the sum is taken over those n for which $[a_n, b_n]$ meets $[z - r, z + r]$).

$$\leqslant \limsup_{r \downarrow 0} \sum{}' \left\{ \frac{l_n^\alpha}{|z - a_n|^\alpha} + \frac{l_n^\alpha}{|z - b_n|^\alpha} \right\}$$

$$\leqslant \limsup_{r \downarrow 0} \sum_{N_r}^\infty \left\{ \frac{l_n^\alpha}{|z - a_n|^\alpha} + \frac{l_n^\alpha}{|z - b_n|^\alpha} \right\}$$

(where N_r is the first index in Σ')

$$= 0$$

for λ almost all $z \in C$. This proves the lemma.

Now set $X = C \times [0, 1]$. Then $[\mathcal{P}]_u = C(X)$ by Mergelyan's Theorem [6], since X does not separate the plane. But clearly $\mathbf{C} \setminus X$ has zero $(1 + \alpha)$-density at $\mathcal{L}^2$ almost all points of X, so $[\mathcal{R}]_\alpha \neq \operatorname{lip}(\alpha, X)$ by Corollary 16.

22. EXAMPLE. The term *Swiss Cheese* is traditionally applied to any compact set X obtained by removing from the closed unit disc an infinite sequence $\{D_n\}$ of disjoint open discs, with radii $\{r_n\}$ and centres $\{a_n\}$, such that $\Sigma r_n < 1$ and $\bigcup_n D_n$ is dense in the unit disc. For any such X, $[\mathfrak{R}]_u \neq C(X)$ [1], [6], and hence *a fortiori* $[\mathfrak{R}]_\alpha \neq \operatorname{lip}(\alpha, X)$, for $0 < \alpha < 1$.

Fix $0 < \alpha < 1$. A larger class of cheeses is obtained by relaxing the condition on the radii of the excised discs to $\Sigma r_n^{1+\alpha} < \infty$. We call such a cheese an "α-cheese". If X is an α-cheese, then $[\mathfrak{R}]_\alpha \neq \operatorname{lip}(\alpha, X)$. To see this, note that by Fubini's Theorem,

$$\int_X \sum_1^\infty \frac{r_n^{1+\alpha}}{|z - a_n|^{1+\alpha}} \, dm(z) = \sum_1^\infty r_n^{1+\alpha} \int \frac{dm(z)}{|z - a_n|^{1+\alpha}}$$

$$\leqslant \sum_1^\infty r_n^{1+\alpha} 2\pi (1 - \alpha)^{-1} < \infty.$$

Hence

$$\sum_1^\infty \frac{r_n^{1+\alpha}}{|z - a_n|^{1+\alpha}} < \infty \quad \text{a.e. } dm.$$

For m almost all such z, it follows that

$$M^{1+\alpha}(B(z, r) \setminus X)/r^{1+\alpha} \to 0$$

as $r \downarrow 0$. Precisely speaking, the limit is zero for any z for which the series converges, unless z happens to belong to bdy D_n for some n. This is seen by essentially the same argument as that of the last section.

Thus the necessary condition for rational approximation is violated, and so $[\mathfrak{R}]_\alpha \neq \operatorname{lip}(\alpha, X)$.

23. We close with some remarks about polynomial approximation. Let $\mathfrak{P}$ denote the space of analytic polynomials. It is not hard to see that $[\mathfrak{R}]_{\alpha,X} = [\mathfrak{P}]_{\alpha,X}$ if and only if $\mathbf{C} \setminus X$ is connected. Thus $[\mathfrak{P}]_{\alpha,X} = A_\alpha(X)$ if and only if $\mathbf{C} \setminus X$ is connected and there exists a constant $\mu > 0$ such that

$$M^{1+\alpha}(D \setminus X) \geqslant \mu M_*^{1+\alpha}(D \setminus \operatorname{int} X)$$

whenever D is an open disc. Also $[\mathfrak{P}]_{\alpha,X} = \operatorname{lip}(\alpha, X)$ if and only if $\mathbf{C} \setminus X$ is connected and there exists a constant $\mu > 0$ such that

$$M^{1+\alpha}(D \setminus X) \geqslant \mu r^{1+\alpha}$$

whenever D is an open disc and the radius of D is r.

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Department of Mathematics, University of California, Los Angeles, California 90024

Current address: Department of Mathematics, Maynooth College, County Kildare, Ireland