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THE EQUIVALENCE OF COMPLETE REDUCTIONS

BY

R. HINDLEY

ABSTRACT. This paper is about two properties of the $\lambda\beta$-calculus and combinatory reduction, namely (E): all complete reductions ρ and σ of the residuals of a set of redexes in a term X have the same end; and (E^+): ρ and σ leave the same residuals of any other redex in X. Property (E) is deduced from abstract assumptions which do not imply (E^+). Also (E^+) is proved for the usual extensions of combinatory and $\lambda\beta$-reduction, and a weak but natural form of (E^+) is proved for $\lambda\beta\eta$-reduction.

1. Introduction. This paper is about Curry and Feys' Property (E), which says in the λ-calculus ([1, p. 113]) that

(E) *If a term X contains a finite set R of redexes, then there exists a complete reduction relative to R, and all such reductions end at the same term.*

Curry and Feys used (E) as the key to their proof of the Church-Rosser theorem for $\lambda\beta$-reduction, and Church and Rosser before them did the same ([1, pp. 113–115], [2, pp. 20–25]). Church and Rosser originally proved (E) by induction on X, but M. H. Newman pointed out that in order to make the induction work one must prove the following slightly stronger property, which will here be called (E^+):

(E^+) *Property* (E) *holds and all complete reductions relative to R produce the same residuals of each redex in X.*

(For comment see [1, p. 149].) Church and Rosser did not mention (E^+) in their proof of (E) though the arrangement of their proof made the 'preservation of residuals' property fairly obvious. However, this property is rather tedious to check in detail; Curry and Feys showed this in [1, pp. 119–121], preparatory to proving (E^+) and (E) on pp. 123–130.

Property (E) is now no longer needed to prove the Church-Rosser theorem for λ-conversion (see Martin-Löf [5, §2.4]), but I feel that (E) and (E^+) are still interesting properties in their own right, likely to be useful whenever one needs to replace reductions by others.

This paper will show that (E) can be proved using assumptions that do not imply (E^+). It will also prove (E^+) for $\lambda\beta$- and combinatory weak reduction

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with extra 'arithmetical' operators added, and prove a restricted but natural and useful form of (E^+) for $\lambda\beta\eta$-reduction.

I would like to thank the referee for some improvements in the exposition of this paper.

2. **Notation and basic assumptions.** In the first few sections we shall be working in an abstract setting, with arbitrary points instead of λ-terms, and lines from point to point instead of $\lambda\beta$-contractions. The notation of [3, pp. 545–548] will be used, except that the lines called *cells* there (one-step reductions) will be called *contractions* here. The main features of the notation are as follows (for the details, see [3]).

To each ordered pair ξ, η of contractions which are coinitial (i.e. both starting at the same point), there is assumed to exist a set ξ/η of contractions, *the residuals of ξ with respect to η*, all starting at the end of η.

A *reduction* ρ is a finite series $\xi_1 + \cdots + \xi_n$ of contractions, each one starting at the end of the one before. Residuals ξ/ρ of a contraction ξ, and residuals α/ρ of a set α of coinitial contractions, are defined in the natural way (see [3, p. 546]). The *sum*, $\rho + \sigma$, of two reductions is the reduction obtained by putting the contractions of σ in order after the end of ρ; of course it is defined only when the start of σ is the end of ρ. To each point X, the *null reduction at X*, called 0, starts and ends at X and has no contractions.

No assumption is made that the end of a contraction or reduction must be different from its start, nor that coinitial sets are finite.

Given a set α of coinitial contractions, a *development* of α is a reduction $\rho = \xi_1 + \cdots + \xi_n$ such that ξ_1 is in α and each ξ_{i+1} is a residual of α with respect to $\xi_1 + \cdots + \xi_i$. A *complete development* (CD) of α is a development ρ such that α/ρ is empty. For any binary relation $\prec$ between contractions, a *minimal complete development* (MCD) of α with respect to this relation is a CD $\xi_1 + \cdots + \xi_n$ such that ξ_1 is minimal in α (with respect to $\prec$), and each ξ_{i+1} is minimal in $\alpha/(\xi_1 + \cdots + \xi_i)$.

For two reductions ρ and σ, we shall say that ρ *is weakly equivalent to* σ ($\rho \simeq \sigma$) iff ρ has the same start and end as σ. We shall say that ρ *is strongly equivalent to* σ ($\rho \cong \sigma$) iff $\rho \simeq \sigma$ and $\zeta/\rho = \zeta/\sigma$ for all contractions ζ coinitial with both ρ and σ. In this notation, properties (E) and (E^+) say

(E) *Every finite set α of coinitial contractions has a complete development, and all CDs of α are weakly equivalent.*

(E^+) *Property* (E) *holds and all CDs of a finite coinitial set are strongly equivalent.*

In later sections, it will be shown that the assumptions (A1)–(A8) of [3, p. 548] imply (E) but not (E^+). This makes these assumptions interesting relative to (E), so they will now be listed for future reference. They assume that there is a relation $\prec$ between coinitial contractions (corresponding to one redex being a proper part of another), such that

(A1) $\xi \not\prec \eta \Rightarrow \eta < \xi$;

(A2) $\xi \prec \eta$ *and* $\eta \prec \zeta \Rightarrow \xi \prec \zeta$;

(A3) $\xi \not\prec \eta \Rightarrow \xi/\eta$ *has at most one member*;

(A4) $\xi/\xi = \varnothing$;

(A5) $\eta_1 \not\prec \xi$ *and* $\eta_1 \not\prec \eta_2 \Rightarrow \eta_1/\xi \not\prec \eta_2/\xi$;

(A6) *if* $\eta_i \prec \xi$ *for* $i = 1, \ldots, n$, *then there is a minimal* η_i *whose residuals are also minimal among* $\eta_1/\xi \cup \cdots \cup \eta_n/\xi$;

(A7) *to each coinitial pair* ξ, η *correspond MCDs* $\xi//\eta$, $\eta//\xi$ *of the sets* ξ/η, η/ξ *respectively, such that*

$$\xi + \eta//\xi \simeq \eta//\eta;$$

(A8) *if* (A7) *is true and* ζ *is coinitial with* ξ *and* η, *then*

$$\zeta/(\xi + \eta//\xi) = \zeta/(\eta + \xi//\eta)$$

in the following cases:

(i) $\zeta \not\prec \xi$ *and* $\zeta \not\prec \eta$,

(ii) $\eta \prec \xi$ *and* $\zeta \prec \xi$ *and* $\zeta \not\prec \eta$ *and* $\zeta/\xi \not\prec \eta/\xi$.

Most of these assumptions are fairly natural (see [4, p. 20]), but (A6) and (A8)(ii) have no excuse for their existence except that they imply the Church-Rosser theorem and they are satisfied by several of the known systems including $\lambda\beta$-reduction ([4, §§7–8]). (They are not satisfied by $\lambda\beta\eta$-reduction, though.) (A8) is slightly weaker than the corresponding assumption in the abstract work of Newman, Curry and Schroer, which is

(D^+) *For all coinitial* ξ *and* η, $\xi + \eta//\xi \cong \eta + \xi//\eta$.

This (D^+) is the *strong form of* D of Curry and Feys [1, p. 119]; the weak form of D is just (A7). Just as this weak D is a special case of (E), so the strong D is a special case of (E^+).

LEMMA 1. *Assuming only* (A1)–(A4); *every finite set of coinitial contractions has a complete development, in fact an MCD.*

LEMMA 2. *Assuming all except* (A6) *and* (A8)(ii), *for any finite set* α *of coinitial contractions, all MCDs of* α *are weakly equivalent.*

PROOF. See [3, Lemma 8].

Finally, all the λ-calculus notation in this paper will be taken from [1, Chapters 3 and 4]. In particular, the result of substituting N for x in M, and changing bound variables to avoid clashes, will be called $[N/x]M$. However, identity will be denoted here by '$=$', not '$\equiv$', and reduction will be denoted by '$>$' (one step) and '$\gg$' (several steps, possibly none).

3. (A1)–(A8) imply (E). Let α be any finite set of coinitial contractions. We know that α has at least one MCD by Lemma 1, and that all MCDs of α are weakly equivalent by Lemma 2; so to prove that all CDs are weakly equivalent, we need only show that an arbitrary CD π is weakly equivalent to an arbitrary MCD, ρ.

In fact we shall prove the following slightly more general property (Figure 1):

(1) *If α is finite and ρ is an MCD of α and π is any development of α, then $\pi + \rho' \simeq \rho$, for some MCD ρ' of a subset of α/π.*

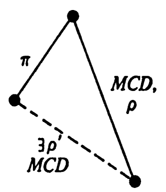

FIGURE 1

This will imply what we want, because if π is complete, then α/π will be empty, so the ρ' given by (1) will be 0 and hence $\pi \simeq \rho$.

PROOF OF (1). The proof uses induction on the number of contractions in π. For the basis ($\pi = 0$), choose $\rho' = \rho$. For the induction-step, suppose that $\pi = \xi + \pi'$, where ξ is a member of α (Figure 2).

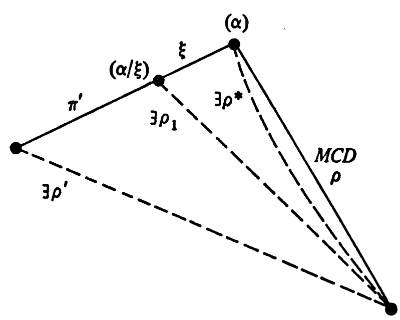

FIGURE 2

Let ρ^* be a ξ-MCD of α, as defined in [3, pp. 551–552]. Then by Lemma 2, ρ^* ends at the same point as ρ. Apply [3, Lemma 9(ii)] to ξ and ρ^*, giving an MCD ρ_1 of α/ξ and an MCD σ of α/ρ^* such that

$$\xi + \rho_1 \simeq \rho^* + \sigma.$$

But ρ^* is complete, so $\sigma = 0$, which gives us

$$\xi + \rho_1 \simeq \rho^* \simeq \rho.$$

(See Figure 2.) Now apply the induction hypothesis to π', ρ_1 and α/ξ to get the result.

(It might happen that α/ξ is infinite. In this case let α' be the finite subset of α/ξ whose residuals actually occur in ρ_1. Then ρ_1 is an MCD of α', and also of any other subset of α/ξ which contains α'. Let α'' be the finite subset of α/ξ whose residuals actually occur in π'. Then $\alpha' \cup \alpha''$ is finite, ρ_1 is an MCD of $\alpha' \cup \alpha''$, and π' is a development of $\alpha' \cup \alpha''$, so we can apply the induction-hypothesis to $\alpha' \cup \alpha''$, π' and ρ_1.)

REMARK. We have just seen that to deduce (E) one does not need the full (D^+), but only the more restricted (A8). This raises the possibility that if all the details were filled in the original Church-Rosser proof of (E), it might turn out that only some, not all, cases of (D^+) were required. I have not checked this.

Another proof of (E) which does not use (D^+) has recently been given in [10, Chapter II, Corollary 1.12].

4. **(A1)–(A8) do not imply (E^+).** This section consists of an example satisfying (A1)–(A8) and therefore (E), but not (E^+) or even its special case (D^+). And furthermore, no redefinition of residuals and the $\prec$-relation will give (E^+) or (D^+) to the system. (One can fairly easily construct slightly simpler examples where (E^+) fails but the failure can be cancelled by merely redefining residuals without adding any new lines to the system; these are not so interesting.)

The example consists of 23 contractions arranged along the edges of a cube, as shown in Figure 3.

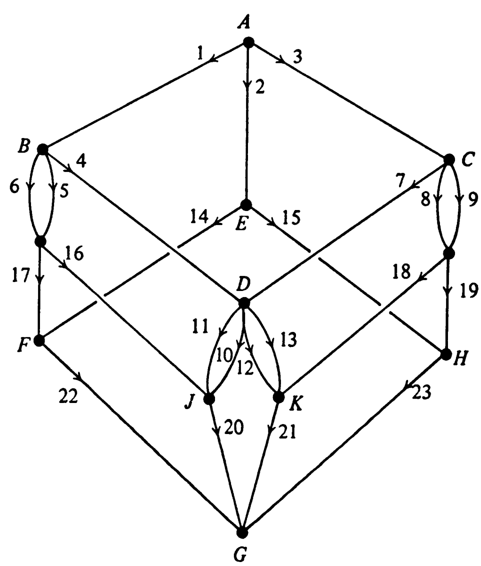

FIGURE 3

DEFINITION OF $\prec$-RELATION. $2 \prec 3, 2 \prec 1$.

DEFINITION OF RESIDUALS. To satisfy (A4), define $\xi/\xi = \varnothing$ for all ξ. Then define

$1/2 = \{14\}$,	$2/1$	$= \{5, 6\}$		(forced on us; no other definition will give (A7)),
$3/2 = \{15\}$,	$2/3$	$= \{8, 9\}$		(forced, in order to get (A7)),
$1/3 = \{7\}$,	$3/1$	$= \{4\}$		(forced, in order to get (A7)),
$5/6 = \{17\}$,	$6/5$	$= \{17\}$		(forced, to give (A7) to 1, 2),
$4/5 = \{16\}$,	$5/4$	$= \{10\}$		(see comment later),
$4/6 = \{16\}$,	$6/4$	$= \{11\}$		(see later),
$7/8 = \{18\}$,	$8/7$	$= \{12\}$		(see later),
$7/9 = \{18\}$,	$9/7$	$= \{13\}$		(see later),
$8/9 = \{19\}$,	$9/8$	$= \{19\}$		(forced, to give (A7) to 2,3)
$10/11 = 11/10$	$= \{20\}$			(to get (A8), 10/11 cannot be $\varnothing$; see below),
$12/13 = 13/12$	$= \{21\}$			(to get (A8), 12/13 cannot be $\varnothing$; see below),
$10/12 = 10/13$	$= 11/12$	$= 11/13$	$= \{21\}$	
$12/10 = 13/10$	$= 12/11$	$= 13/11$	$= \{20\}$	(forced, to get (A7)),
$16/17 = \{22\}$,	$17/16$	$= \{20\}$		(forced),
$18/19 = \{23\}$,	$19/18$	$= \{21\}$		(forced),
$14/15 = \{23\}$,	$15/14$	$= \{22\}$		(forced).

PROOF OF (A1)–(A8). Properties (A1)–(A4) and (A7) are immediate. We get (A5) and (A6) because $\xi \not\prec \eta$ holds for all contractions except 1, 2 and 3. We shall now prove (A8), considering each vertex in turn.

CONTRACTIONS AT D. For all ξ, η, ζ, there are no residuals of ζ with respect to $\xi + \eta//\xi$ or $\eta + \xi//\eta$.

CONTRACTIONS AT B.

$$4/(5 + 6//5) = (4/5)/(6/5) = 16/17 = \{22\}.$$

$$4/(6 + 5//6) = 16/17 = 4/(5 + 6//5).$$

$$5/(6 + 4//6) = (5/6)/(4/6) = 17/16 = \{20\}.$$

$$5/(4 + 6//4) = (5/4)/(6/4) = 10/11 = \{20\}.$$

(To get $\{20\}$ here we need $5/4 \neq 6/4$ and $10/11 \neq \varnothing$; this forces the above definition of residuals, except for the trivial alternative $5/4 = \{11\}$ and $6/4 = \{10\}$. ($5/4 = \{10, 11\}$ would destroy (A7).))

$$6/(5 + 4//5) = \{20\} = 6/(4 + 5//4) \quad \text{similarly.}$$

CONTRACTIONS AT C. Similar to B.

CONTRACTIONS AT A. Contraction 2 does not satisfy the conditions for ζ in (A8), so we need only check the residuals of 1 and 3.

$$1/(2 + 3//2) = 14/15 = \{23\}.$$

$$1/(3 + 2//3) = 7/(8 + 19) \quad \text{or} \quad 7/(9 + 19) = \{23\}.$$

$$3/(2 + 1//2) = \{22\} = 3/(1 + 2//1) \quad \text{similarly.}$$

This completes the proof of (A1)–(A8). And we have seen incidentally that any change in the residuals, except interchanging 5/4 and 6/4, or 8/7 and 9/7, will destroy one of these properties.

PROOF THAT (E^+) FAILS. Consider the residuals of 2:

$$2/(3+1//3) = (2/3)/(1/3) = \{8, 9\}/7 = \{12, 13\};$$

$$2/(1+3//1) = \{5, 6\}/4 = \{10, 11\}.$$

So (E^+) fails. Furthermore, we cannot change the definition of residuals to give (E^+), even if we abandon (A1)–(A8). To see this, check through the residuals of 2 used in the equations above. In order to have (A7), which is contained in (E^+), 1/3 must be $\{7\}$ and 2/3 must be $\{8, 9\}$, or $\{8\}$ or $\{9\}$ (if we defined $8/8 = \{19\}$ or $9/9 = \{19\}$). Then (2/3)/(1/3) must be a subset of $\{12, 13\}$. Similarly (2/1)/(3/1) must be a subset of $\{10, 11\}$, so (E^+) must fail.

REMARK. If the above example seems a bit too abstract, a system of terms and replacement-rules satisfying (A1)–(A8) but not (E^+) can be constructed as follows.

The *terms* are built up by one operation (T_1T_2) from 20 atomic constants 'a', 'b', . . . , 't'. The contraction-rules are

1.	$((ab)c)$	$>$	$((de)f)$,	13.	$(((jk)l)m)$	$>$	(qm),
2.	a	$>$	g,	14.	$((gb)c)$	$>$	(rf),
3.	(ab)	$>$	(hi),	15.	(gb)	$>$	s,
4.	$((de)f)$	$>$	$(((jk)l)m)$,	16.	$((ne)f)$	$>$	$(((pk)l)m)$,
5.	(de)	$>$	(ne),	17.	(ne)	$>$	r,
6.	d	$>$	n,	18.	$((oi)c)$	$>$	(qm),
7.	$((hi)c)$	$>$	$(((jk)l)m)$,	19.	(oi)	$>$	s,
8.	h	$>$	o,	20.	$(((pk)l)m)$	$>$	t,
9.	(hi)	$>$	(oi),	21.	(qm)	$>$	t,
10.	j	$>$	p,	22.	(rf)	$>$	t,
11.	(jk)	$>$	(pk),	23.	(sc)	$>$	t.
12.	$((jk)l)$	$>$	q,				

The reductions of the term $((ab)c)$ can be seen to be isomorphic to Figure 3, with the contractions numbered as above. (No contraction-rule applies twice.)

5. **(A1)–(A8) and (D^+) imply (E^+).** In the last two sections we saw that (A1)–(A8) imply (E) but not (E^+). On the other hand, if (A8) is replaced by the stronger (D^+), we do get (E^+), as follows.

First of all, given (D^+), checking the proof of [3] Lemma 8 shows that all MCDs of a finite coinitial set of contractions become strongly equivalent. And Lemma 9(ii) on [3, p. 552], which asserts a weak equivalence, will, given

(D^+), assert strong equivalence. Then the proof of (1) in §3 above will give

$$\pi + \rho' \cong \rho$$

instead of weak equivalence, and (E^+) is a special case of this result just as (E) was a special case of (1).

6. (E^+) holds for modified and unmodified $\lambda\beta$- and combinatory reductions. In this section (E^+) will be proved for combinatory and $\lambda\beta$-reduction and for these reductions with extra operators adjoined, for example the recursion operator **R**, Curry's iterator **Z**, and the pairing operator (Curry et al. [8, pp. 216–229]).

It will be assumed here that *terms* are the usual combinatory or λ-terms (possibly with type-restrictions), together with a (perhaps empty or infinite) sequence of atomic constants $a_1, a_2, \ldots$ (not necessarily all distinct). *Reduction* will be assumed to be defined by the usual $\lambda\beta$- or combinatory (weak) axiom-schemes, together with one extra axiom-scheme for each a_i, of form

$$\text{(a)} \qquad a_i M_{i1} \cdots M_{in_i} > M_i^*,$$

where the m's are terms built up from the atoms (including perhaps some a's), and certain 'meta-variables'. If $v_1, \ldots, v_{k_i}$ are all the meta-variables in (a), then the result of substituting for these v's any terms $U_1, \ldots, U_{k_i}$ (not containing meta-variables but possibly containing variables) is called an *axiom*:

$$\text{(2)} \quad [U_1/v_1] \cdots [U_{k_i}/v_{k_i}](aM_{i1} \cdots M_{in_i}) > [U_1/v_1] \cdots [U_{k_i}/v_{k_i}]M_i^*.$$

A *redex* (a_i-, β-, or combinatory) is the left-hand side of an axiom, and the right-hand side is the redex's *contractum*. (See [4, pp. 6ff.] for any further details.)

If a redex occurs several times in a term X, we shall need to distinguish between the different occurrences. This will be done by means of *position indices*, which are finite (perhaps empty) sequences of integers, as follows (from [4, p. 5, Definitions 2 and 3], slightly modified following Rosen [7]):

(i) *X occurs in X at position $\varnothing$ (the empty sequence)*;

(ii) *if (UV) occurs in X at position $(n_1, \ldots, n_t)$, then U occurs in X at position $(n_1, \ldots, n_t, 1)$ and V occurs in X at position $(n_1, \ldots, n_t, 2)$*;

(iii) *if $(\lambda x \cdot U)$ occurs in X at position $(n_1, \ldots, n_t)$, then U occurs in X at position $(n_1, \ldots, n_t, 3)$.*

The reason that positions are best defined in this way, and not, for example, as the distance from U to the left-hand end of X, is that with the above definition, contracting a redex which does not overlap U, or lies entirely inside U, will not change U's position. The position of U in X shows the branch of the construction-tree of X that U is on, and how far up that branch U is.

An *occurrence* of U in X will be defined to be a triple (U, p, X) such that U occurs at position p in X. Occurrences in X may also be called *components of* X. Two components of X are *disjoint* iff they are on different branches of the construction-tree of X, i.e. iff neither of their positions is an initial segment of the other. A component (U, p, X) *contains* a component (V, q, X) iff p is an initial segment of q (cf. [4, p. 5]). In future when components are discussed, their positions will be omitted whenever possible.

LEMMA 0 (cf. [4, p. 5, Lemma 13]). *Let P, Q occur at positions p, q respectively in X, and let X^* be the result of replacing Q at position q by another term Q^*:*

(i) *if (P, p, X) is disjoint from (Q, q, X), then P will occur in X^* at position p;*

(ii) *if (P, p, X) contains (Q, q, X), then q is p followed by a sequence which we may call $q - p$, and Q occurs in P at position $q - p$, and the term at position p in X^* is the result P^* of replacing Q in P.*

Continuing now with the notation for redexes: a *contraction* will be the act of replacing one occurrence of a redex in a term by its contractum. Contractions can therefore be identified with occurrences of redexes. And from now on, the word '*redex*' will be used almost always to mean 'occurrence of a redex'; for example, 'all redexes in X' will mean 'all occurrences of all redexes in X', and 'let ξ be a redex in X' will mean 'let ξ be an occurrence of a redex in X'.

ASSUMPTIONS. The following assumptions will be made throughout the present section. They are taken from [4, pp. 9 and 11], and they can easily be seen to be satisfied by the recursion, iterator and pairing operators of Curry et al. [8, pp. 216–229], also by the generalized δ-operators of Curry and Feys [1, p. 93].

(D1) *If the terms are restricted by type-restrictions or any other restrictions; replacements by axioms* (a) *do not change types or violate the restrictions.*

(D2) *A redex cannot be an instance of the left side of two schemes* (a).

(D3) *Each meta-variable in M_i^* on the right of* (a) *also occurs on the left of* (a).

(D4) *No meta-variable occurs twice on the left side of* (a).

(D5) *If an a_k-redex P is a proper part of an a_i-redex Q, then P is in one of the components U_j that were substituted for meta-variables in forming Q from its axiom-scheme.*

(D6) *No meta-variable occurs in function position*([1]) *on the left side of* (a), *and* (λ-*case*): *no $\lambda\beta$-redex occurs in the left side of* (a); (*combinatory case*): *no*

([1]) Function position is the position of the υ in a component (υZ); that is, any position whose last number is 1.

combinatory redex occurs in the left side of (a) *and no a_i is an atomic combinator.*

(D7) (λ-*case only*): *no scheme* (a) *contains free variables other than meta-variables; and if* (a) *contains a part with form $\lambda x \cdot Y$, then Y contains no meta-variables, and the result of making any change of bound variables in* (a) *is also an axiom-scheme.*

MOTIVATION OF (D1)–(D7). Condition (D1) is needed only in systems whose terms are restricted by type-restrictions, to make reduction definable.

Conditions (D2) and (D3) together are equivalent to saying that each a_i-redex has a unique contractum. And (D3) says that contraction introduces no new concepts.

(D5) says that if an a_j-redex P is a proper part of an a_i-redex Q, then P must be in one of the U's (cf. (2) above). Suppose P is in U_j; then by (D4) there is only one substituted occurrence of U_j in Q, so the result Q' of contracting P will still be a redex from the same scheme as Q.

(D6) is just a 'structural' way of saying that (D5) still holds when P is a $\lambda\beta$- (or combinatory) redex.

(D7) says that the M's in (a) are just applicative combinations of meta-variables, atomic constants, and λ-terms without free variables, and that (a) are invariant with respect to change of bound variables. This keeps substitution properties tidy in the λ-calculus(2).

Finally, (D1)–(D7) were shown in [4, Theorems 5 and 7] to imply (A1)–(A8) and hence the Church-Rosser theorem. Essentially equivalent conditions were also proposed by B. K. Rosen in his abstract generalization of the original Rosser proof of the theorem for combinators (see [7]).

DEFINITION OF RESIDUALS. The purpose of residuals is to get (A7), and this makes the required definition fairly obvious. But for ease of reference I shall set it out here in detail. Let ξ and η be occurrences of redexes in a term X.

Case 1. $\xi = \eta$. Define $\xi/\eta = \varnothing$.

Case 2. *ξ is disjoint from η.* Define ξ/η to be the redex which is at the same position as ξ when η is contracted.

(2) Here is an interesting example, due to G. Mitschke, showing the need for (D7) in the λ-calculus. Add to the $\lambda\beta$-calculus a constant D with axioms $DMN > \lambda uv \cdot u$ if $M \neq N$, $DMN > \lambda uv \cdot v$, where M and N are arbitrary β-normal forms, possibly containing free variables. (Such a D would be a discriminator for β-normal forms.) These axioms can be regarded as special cases of (a) if we take $a_1 = a_2 = a_3 = \cdots = D$ and we view each axiom as an axiom-scheme containing no meta-variables. But then (D7) fails, because M and N may contain free variables. And the Church-Rosser property also fails. For example the term $(\lambda x \cdot Dxx)W$, where $W = (\lambda y \cdot yy)(\lambda y \cdot yy)$, reduces to two distinct terms $\lambda uv \cdot v$, $\lambda x \cdot DWW$ which cannot be reduced to a common term. This failure is caused by the substitution of W (with no normal form) for x (a normal form). Furthermore, in this system all terms can be proved equal. In contrast, if we add the above D to combinatory reduction instead of λ-reduction the result will then have the Church-Rosser property (because (D1)–(D6) are all satisfied).

Case 3. *ξ properly contains η*. Define ξ/η to be the component which is at the same position as ξ when η is contracted. This will be the result of replacing η in ξ; it will be a redex by (D4)–(D6) (see motivation above).

Case 4. *ξ is a proper part of η*. If $\eta = (\lambda x \cdot U)V$, then ξ must be in U or in V. If ξ is in U, define the residual of ξ as usual to be the component of $[V/x]U$ with the same position as ξ had in U. (This will be $[V/x]\xi$ if no free variable of V is bound in U.) If ξ is in V, then in each substituted V in $[V/x]U$ there will be a corresponding occurrence of ξ; these are taken for the residuals of ξ.

If η is a combinatory redex $SU_1U_2U_3$ or KU_1U_2, or an (a)-redex

$$[U_1/v_1] \cdots [U_k/v_k](aM_1, \ldots, M_n),$$

then ξ must be in a U_j, and the contractum of η will have a corresponding ξ in each substituted occurrence of U_j; these are taken for the residuals of ξ.

Proof of (E^+). By [4, §§7, 8], (D1)–(D7) imply (A1)–(A8). So in order to prove (E^+), by §5 we need only prove (D^+). This will now be done.

Proof of (D^+). Let T be any term, and ξ, η, ζ be any three contractions (redexes) in T. We must show that

$$\zeta/(\xi + \eta//\xi) = \zeta/(\eta + \xi//\eta). \tag{3}$$

We already know by (A7) that $\xi + \eta//\xi$ has the same end-term, say W, as $\eta + \xi//\eta$; and two distinct components of W cannot occupy the same position, so we only need prove that the residuals $\zeta/(\xi + \eta//\xi)$ occur at the same positions as the residuals $\zeta/(\eta + \xi//\eta)$ in W'.

If ζ is not in ξ and not in η, then contracting ξ, η in any order will leave one single residual of ζ, with the same position in W as ζ originally had in T. This gives (3).

If ζ is in one of ξ, η (say ξ), and η is disjoint from ξ, then the residuals of ζ will be unaffected by when η is contracted, so (3) is easy.

From now on, suppose that one of ξ, η is in the other (say η in ξ), and that ζ is in the larger one (ζ in ξ). Suppose also that ξ, η, ζ are distinct, otherwise (3) would be trivial. There are two possibilites for ξ:

$$\xi = (\lambda x \cdot M)N \quad \text{or} \quad \xi = [U_1/v_1] \cdots [U_k/v_k](aM_1 \cdots M_n),$$

and η and ζ will be in M, N, or $U_1, \ldots, U_k$(3). By change of bound variables, we may assume that no variable is both free and bound in ξ.

Case 1. *ζ is not in η*. Part of this case, the case that no residual ζ/ξ is in any residual η/ξ, has already been covered in the proof of (A8)(ii) in [4], but for future reference I shall give the full proof again here.

(3) The case that η is a combinatory redex will be included in the case that η is an a_i-redex. The combinatory axiom-schemes can be regarded as special axiom-schemes of the type (a), namely $Sv_1v_2v_3 > v_1v_3(v_2v_3)$, $Kv_1v_2 > v_1$, where v_1, v_2, v_3 are meta-variables. When we do this, (D1)–(D6) will remain true, as can easily be checked.

Subcase 1a. *ξ is $(\lambda x \cdot M)N$ with ζ in M.* Contracting ξ first will produce $[N/x]M$, containing a residual $[N/x]\zeta$. And the residuals of η cannot contain this $[N/x]\zeta$; because if η is in M, it does not contain ζ, and if η is in N, its residuals are in the substituted N's in $[N/x]M$. Hence, reducing the η-residuals will not change the position of the ζ-residual.

On the other hand, contracting η first will change ξ to a term $(\lambda x \cdot M')N$ or $(\lambda x \cdot M)N'$, and since η does not contain ζ the position of ζ will be unchanged. Then contracting the whole term to $[N/x]M'$ or $[N'/x]M$ will leave a residual of ζ at the same position as ζ originally had relative to M (cf. Lemma 0).

Subcase 1b. *ξ is $(\lambda x \cdot M)N$ with ζ in N.* By the condition of Case 1, if η is in N, then η does not contain ζ; hence the positions of the residuals of ζ will be independent of whether η is contracted, and will depend only on the original position of ζ in N, and the positions of the free x's in M.

Now suppose η is in M. As above, the positions of the residuals of ζ will depend on the positions of the free x's in M. If none of these free x's are in η, then the positions of the ζ-residuals will be independent of whether η is contracted.

Now suppose η contains some of the free x's in M. Then contracting η first will give

$$\xi' = (\lambda x \cdot M')N,$$

with M' containing the contractum C_η of η. Let $x_1, \ldots, x_k$ be the free occurrences of x (if any) in C_η. Then contracting ξ' will produce

$$[N/x]M', \tag{4}$$

with an N substituted for each of $x_1, \ldots, x_k$, and a residual of ζ inside each N.

On the other hand, if ξ was contracted before η we would get the term $[N/x]M$, and the residual of η would have the form

$$[N/x]\eta = (\lambda y \cdot [N/x]P)[N/x]Q \quad \text{or} \quad (b[N/x]P_1 \cdots [N/x]P_m), \tag{5}$$

since η must have the form $(\lambda y \cdot P)Q$ $(y \neq x)$, or $(bP_1 \cdots P_m)$. Then contracting this residual of η would produce either

$$[[N/x]Q/y][N/x]P = [N/x][Q/y]P,$$

or

$$[N/x]C_\eta \qquad [4, \text{p. } 12, \text{Lemma } 17]. \tag{6}$$

The positions of the substituted N's, and hence the positions of the residuals of ζ, would be the same as in (4) above.

Subcase 1c. *ξ is an a_i-redex.* Let η be in some U_h and ζ be in U_j. If $h \neq j$, then the residuals η/ξ are disjoint from the residuals ζ/ξ, and so the

ζ-residuals will be independent of when η is contracted. If $h = j$, then since ζ is not in η, no residual η/ξ will contain a residual ζ/ξ, so again the residuals of ζ will have the same positions independently of when η is contracted.

Case 2. *ζ is in η*. As above, ξ is a redex with one of the following forms:

$$\xi = (\lambda x \cdot M)N \quad \text{or} \quad aM_1 \cdots M_n,$$

and η is in one of $M, N, M_1, \ldots, M_n$. Now η must have form

$$\eta = (\lambda y \cdot P)Q \quad \text{or} \quad bP_1 \cdots P_m,$$

and ζ must be in one of $P, Q, P_1, \ldots, P_m$.

Subcase 2a. *η is in M*. Contracting η first will change M to a term M', and ξ to

$$\xi' = (\lambda x \cdot M')N,$$

and ζ will have certain residuals in the contractum C_η of η, in M'. Then contracting ξ' will produce

$$[N/x]M', \tag{7}$$

and the substitution will not change the positions of the ζ-residuals relative to M'.

On the other hand, contracting ξ first will produce

$$[N/x]M \tag{8}$$

containing $[N/x]\eta$, and contracting $[N/x]\eta$ will change (8) to a term which by (A7) must be $[N/x]M'$. The residual of ζ in (8) will be $[N/x]\zeta$ in $[N/x]\eta$.

Claim. The residuals of $[N/x]\zeta$ after contracting $[N/x]\eta$ will have the same positions as the residuals of ζ after contracting η (and hence the same positions as the ζ-residuals in (7)).

PROOF OF CLAIM. If η is $(\lambda y \cdot P)Q$ with ζ in Q, then the residuals of ζ are in the substituted Q's in $[Q/y]P$. These Q's occur at the positions of the free occurrences of y, say $y_1, \ldots, y_k$, in P. In $[N/x]\eta$ these y's will occur at the same positions in $[N/x]P$ (since $x \neq y$). Hence the residuals of $[N/x]\zeta$ in $[[N/x]Q/y][N/x]P$ will have the same positions as the ζ-residuals in $[Q/y]P$.

If $\eta = (\lambda y \cdot P)Q$ with ζ in P, then the residual of ζ in $[Q/y]P$ will have the same position as ζ has in P. On the other hand, in $[N/x]P$ the position of $[N/x]\zeta$ will be the same as the position of ζ in P, and this position will not be changed by substituting $[N/x]Q$ for y.

If $\eta = bP_1 \cdots P_m$ with ζ in a P_j, then η must have been obtained from the left-hand side of an axiom-scheme

$$bL_1 \cdots L_m > L^* \tag{9}$$

by substituting some terms $W_1, \ldots, W_q$ for the meta-variables $v_1, \ldots, v_q$ in (9). And ζ must be in some W_j, by (D5) and (D6). The contractum C_η of η will be

$$C_\eta = [W_1/v_1] \cdots [W_q/v_q]L^*,$$

and the residuals of ζ will be in the substituted occurrences of W_j. On the other hand, $[N/x]\eta$ will be the result of substituting $[N/x]W_1, \ldots, [N/x]W_q$ for $v_1, \ldots, v_q$ in the left side of (9), and its contractum will be

$$[[N/x]W_1/v_1] \cdots [[N/x]W_q/v_q]L^*.$$

In this term, the residuals of $[N/x]\zeta$ will have the same positions as the residuals of ζ in C_η, since the position of $[N/x]\zeta$ in $[N/x]W_j$ is the same as that of ζ in W_j.

This proves the above claim, and completes Subcase 2a.

Subcase 2b. *η is in N.* We have $\xi = (\lambda x \cdot M)N$; contracting η first will change N to a term N' and ξ to $(\lambda x \cdot M)N'$, with some residuals of ζ in the contractum C_η of η in N'. Then contracting $(\lambda x \cdot M)N'$ will produce $[N'/x]M$, with the residuals of ζ being a set ζ/η in the C_η in each substitute N'.

On the other hand, contracting ξ first will produce $[N/x]M$, and contracting the η in each substituted N will change this to $[N'/x]M$ with residuals of ζ in each C_η, the same as above.

Subcase 2c. *η is in a U_j.* We have

$$\xi = [U_1/v_1][U_2/v_2] \cdots [U_k/v_k](aM_1 \cdots M_n)$$

for some terms $U_1, \ldots, U_k$, meta-variables $v_1, \ldots, v_k$, and some axiom-scheme $aM_1 \cdots M_n > M^*$. One of $U_1, \ldots, U_k$ must contain η; say it is U_1. Contracting η first will change U_1 to a term U_1'; and then contracting ξ/η will produce

$$[U_1'/v_1][U_2/v_2] \cdots [U_k/v_k]M^*, \tag{10}$$

with residuals of ζ in each substituted U_1', in positions determined by the positions of ζ/η relative to the contractum of η in U_1'.

On the other hand, contracting ξ first will give

$$[U_1/v_1][U_2/v_2] \cdots [U_k/v_k]M^*,$$

and contracting the residual of η in each substituted U_1 will give (10) again, with the same residuals of ζ.

This completes the proof of (D^+), and hence of (E^+).

REMARK 1. Is there a slicker proof of (D^+) than the above tedious case-checking? Here is one possible candidate.

Let T contain redexes ξ, η, ζ. The first step is to prove (A7); then as above, we shall know that $\xi + \eta//\xi$ has the same end-term, say W, as $\eta + \xi//\eta$, and to prove

$$\zeta/(\xi + \eta//\xi) = \zeta/(\eta + \xi//\eta) \tag{3}$$

it will be enough to show that the two sets of residuals have the same positions in W.

Now assume ξ, η, ζ are distinct (otherwise the result will be trivial).

Let T^- be the result of replacing ζ in T by a new variable z not already in T. Let ξ^- and η^- be the 'residuals' of ξ and η in T^-. (If ξ or η is inside ζ, define ξ^- or η^- to be the null reduction, 0.)

Let W^- be the end-term of the reduction

$$\xi^- + (\eta^- // \xi^-).$$

By (A7), W^- will also be the end-term of the reduction

$$\eta^- + (\xi^- // \eta^-).$$

The next step is to prove that *the residuals* $\zeta/(\xi + \eta//\xi)$ *in* W *have exactly the same positions as the occurrences of* z *in* W^-. This result, and the corresponding one for $\eta + \xi//\eta$, together imply (3).

Unfortunately, when the details are filled in this proof is no shorter than the original one, and it is less straightforward. But the result in italics gives a nice 'reason' for why (D^+) is true.

REMARK 2. The above deduction of (E^+) from (D1)–(D7) generalizes [1, Chapter 4], where (E^+) was proved for β-reductions with extra 'δ-axioms' of the form

$$aM_1 \cdots M_n > M^* \qquad (n \geqslant 0),$$

where $M_1, \ldots, M_n$ contained no redexes, free variables or meta-variables, and for no $k < n$ was $aM_1 \cdots M_k$ the left-hand side of another axiom.

The proof in [1] proceeded by first showing that these $\beta\delta$-reductions satisfied (D^+) and certain abstract conditions (H_0)–(H_7); see p. 124 of [1]. Then (E^+) was deduced in the abstract from (D^+), (H_0)–(H_7), and an implicit assumption that all sets of residuals were finite. The corresponding abstract deduction here, §5, is slightly more general than [1] because Curry's (H_2) and (H_5) were slightly more restrictive than (A1)–(A8); as mentioned rather imprecisely at the end of [3], the H's (together with (D^+) and the finiteness of residual-sets) imply the A's but the A's with (D^+) and finiteness do not quite imply the H's([4]).

The present section could have used the H's instead of the A's, because the assumptions of the section can be shown to imply the H's when the 'R **a** S' of [1] is read as 'R is a proper part of S' in the case that S is an a_i-redex.

REMARK 3. If we were proving (E^+) for $\beta - a_i$-reduction directly, without assuming any abstract lemmas, the quickest way would be to first prove (D^+) and then use the original method Church and Rosser used for the pure λ-calculus [2, pp. 20–25].

([4]) The statement in [3, §4] of these facts wrongly omitted all mention of (D^+) and finiteness. It also claimed that any system satisfying the H's could be made to satisfy the A's by defining $\prec$ to be Curry's relation **f**; the '**f**' should have been '**a**'.

7. The lemma of parallel moves. If a system has property (E^+), then it has the Church-Rosser property; but it also has the slightly stronger property that the constructions used in the usual proofs of the Church-Rosser theorem preserve strong equivalence. (All these proofs use complete developments of sets of residuals, implicitly if not explicitly; even in Martin-Löf's proof in [5, §2.4.3] the 'one-step reductions' are actually MCDs.)

The following special case of this property has been used by P. Welch in his analysis of 'inside-out' reductions in [6, Chapter 6], also by J. H. Morris in his doctoral thesis [9].

PARALLEL-MOVES LEMMA. *If a reduction $\rho = \xi_1 + \cdots + \xi_n$ and a contraction η are coinitial, and we define a reduction ρ^* by*

$$\rho^* = \sigma_1 + \cdots + \sigma_n,$$

where σ_1 is any CD of ξ_1/η, and σ_{i+1} is any CD of the residuals of ξ_{i+1} with respect to any CD of $\eta/(\xi_1 + \cdots + \xi_i)$, then ($E^+$) *implies that $\eta + \rho^* \cong \rho +$ any CD of η/ρ*(4a).

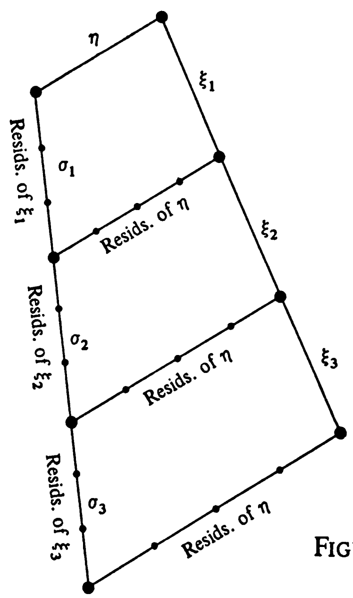

FIGURE 4

(4a) The lemma as stated above is true for any system satisfying (E^+), including of course $\lambda\beta$-reduction.

But an important fact, which was realized too late to include in the text above, is that the lemma's conclusion also extends with only slight modification to $\beta\eta$-reduction, which only satisfies a weak form of (E^+). To be precise, suppose reduction is defined as in §9 by the $\beta\eta$-axioms and a (possibly empty) set of schemes (a) satisfying (D1)–(D9). Let a contraction η and a reduction $\rho = \xi_1 + \cdots + \xi_n$ be coinitial. Then for each i from 1 to $n - 1$ there is a set α_{i+1} which is well separated (see §9) and coinitial with ξ_{i+1}, and such that if we define $\sigma_1, \ldots, \sigma_n$ by setting σ_1 = any CD of ξ_1/η and σ_{i+1} = any CD of (ξ_{i+1}/(any CD of α_{i+1})), then

$$(\eta + \sigma_1 + \cdots + \sigma_n) \cong (\rho + \text{any CD of } \eta/\rho).$$

The set α_{i+1} takes the place of $\eta/(\xi_1 + \cdots + \xi_i)$ in the lemma in the main text. In fact, if η is a β-redex, these two sets will be identical. But this need not be so if η is an η-redex.

Proofs of these facts will be included, it is hoped, in a future paper.

PROOF (from [1, p. 114]). Induction on n. For the induction-step, apply (E^+) to the two reductions

$$\xi_{n+1} + \text{CD of } \eta/(\xi_1 + \cdots + \xi_{n+1}), \quad (\text{CD of } \eta/(\xi_1 + \cdots + \xi_n)) + \sigma_{n+1},$$

which are both CDs of the set $\{\xi_{n+1}\} \cup \eta/(\xi_1 + \cdots + \xi_n)$.

8. **Pairs of arbitrary β-reductions.** In §4 there was a pair of reductions that were weakly but not strongly equivalent (namely $1 + 3//1$ and $3 + 1//3$). In contrast, we have seen that in the $\lambda\beta$-calculus no pairs of complete developments have this property.

But there exist pairs of less restricted $\lambda\beta$-reductions that are weakly but not strongly equivalent. For example, let $\rho = 0$ and σ be the reduction

$$(\lambda x \cdot xx)(\lambda x \cdot xx) > (\lambda x \cdot xx)(\lambda x \cdot xx).$$

Then ρ and σ have the same start and end, but if ζ is the redex $(\lambda \cdot xx)(\lambda x \cdot xx)$, then $\zeta/\rho = \{\zeta\}$, $\zeta/\sigma = \varnothing$. (I do not know if there is an example which does not rely on the destruction of residuals by a reduction whose end is the same as its start(5).)

9. **$\lambda\beta\eta$-reduction.** It is well known that (D^+), (E) and (E^+) all fail for $\lambda\beta\eta$-reduction [1, p. 119]. But this failure is not the whole story. In fact, $\beta\eta$-reduction satisfies a restricted form of (E^+) which will be proved in the present section, and this property will actually be useful in a later paper.

For good measure, the result will be stated and proved for $\beta\eta$-reduction extended by extra axiom-schemes (a). It will be assumed that

(D1)–(D7) *all hold.*

(D8). *In $a_i M_{i1} \cdots M_{in_i}$ the last M is not a single meta-variable or variable.*

(D9). *No M_{ij} contains an η-redex.*

The last two assumptions ensure that if we have an a_i-redex

$$\xi = [U_1/v_1] \cdots [U_k/v_k](a_i M_{i1} \cdots M_{in_i}),$$

and an η-redex $\eta = \lambda x \cdot Mx$, then ξ can only be in η if ξ is in M, and η can only be in ξ if η is in a substituted U_j. These assumptions were needed in [4, p. 15] to deduce the Church-Rosser theorem.

RESIDUALS. These are defined in a natural way. Namely, for β- and a_i-redexes we use the definition in §6, and for $\beta\eta$-redexes we use [1, pp. 116–119]; the definition for η- and a_i-redexes is as follows. Let ξ be an

(5) An example which does not rely on a 'circular' reduction has recently been pointed out to me. It is $I(Ix) > Ix$, where I is $\lambda y \cdot y$. The term $I(Ix)$ can be reduced to Ix in two different ways; either by contracting the redex $I(Ix)$, or by contracting the smaller redex Ix. The first reduction will leave no residual of $I(Ix)$ but one of Ix, whereas the second will leave none of Ix but one of $I(Ix)$.

This example was originally proposed by J.-J. Lévy for a slightly different purpose. It seems likely that further examples of arbitrary complexity exist.

a_i-redex as above, and η be an η-redex as above.

If ξ and η are disjoint, the definition of residuals is obvious. If ξ is in η, then ξ must be in M. Contracting $\lambda x \cdot Mx$ to M will leave ξ unchanged relative to M. Take this ξ as ξ/η. On the other hand, contracting ξ in $\lambda x \cdot Mx$ will change it to a term $\lambda x \cdot M'x$. Take this as η/ξ.

If η is in ξ, then η must be in a U_j. Contracting ξ will produce

$$[U_1/v_1] \cdots [U_k/v_k]M_i^*,$$

and we take η/ξ to consist of the occurrences of η in the substituted occurrences of U_j. On the other hand, contracting η will change U_j to a term U_j', and since v_j only occurs once (by (D4)), this will change ξ to a redex ξ'. Take ξ' as ξ/η.

DEFINITION. Two redexes in a λ-term X are said to be *too close together* iff one is a part of the other, and they have form either $(\lambda x \cdot Mx)N$, $\lambda x \cdot Mx$, or $(\lambda x \cdot (\lambda y \cdot M)x)$, $(\lambda y \cdot M)x$. A set of redexes in X is called *well separated* iff no two of its members are too close together.

Note that if ξ, η are too close together, then contracting one leaves the other without any residuals. Also, the end-result of contracting ξ is the same as contracting η.

LEMMA 3. *If α is a well separated set of $\beta - \eta - a_i$-redexes in a term X, and ρ is any reduction starting at X; then the residuals α/ρ form a well separated set.*

PROOF. It is enough to show that if a term X contains three redexes ξ, η, ζ, with ξ, η well separated, then the residuals ξ/ζ, η/ζ will form a (perhaps empty) well separated set.

First of all, if $\zeta = \xi$ or $\zeta = \eta$, then ξ/ζ or η/ζ will be empty. The same will be true if ζ is too close to ξ or η. In both these cases, the residuals of the other redex will consist of at most a set of disjoint redexes. Hence $\xi/\zeta \cup \eta/\zeta$ will be a well separated set.

From now on, assume that ξ, η, ζ are distinct and well separated.

Case 1. *Neither of ξ, η is in ζ.* Then ξ, η have single residuals in the same relative positions as ξ, η themselves. Hence the residuals are well separated.

Case 2. *One of ξ, η is in ζ and the other is disjoint from ζ.* In this case all the residuals will be disjoint, and hence well separated.

Case 3. *Both of ξ, η are in ζ.* If ζ is $\lambda x \cdot Mx$, then ξ and η must be in M, and when ζ is contracted their residuals will have the same relative positions, so they will be well separated because ξ and η are.

If ζ is $(\lambda x \cdot M)N$, then ξ and η must be in M or N. If they are both in M, then their residuals in $[N/x]M$ will have the same relative positions, and hence will be well separated. If they are both in N, their residuals will consist of disjoint well separated pairs. Now suppose one (say ξ) is in M and the other is in N. Let $\xi' = [N/x]\xi$ be the residual of ξ in $[N/x]M$. (We are ignoring changes of bound variables here.) If a residual η' of η was too close to ξ', we would have to have either

$$\xi' = (\lambda y \cdot Uy)V, \qquad \eta' = \lambda y \cdot Uy$$

or

$$\xi' = (\lambda y \cdot (\lambda z \cdot U)y), \qquad \eta' = (\lambda z \cdot U)y.$$

But these are both impossible, because they could only happen if $N = \eta'$ and ξ had form (xW) or $(\lambda y \cdot x)$, and neither of these is a redex.

If ζ is an a_i-redex, then ζ must have form

$$[U_1/v_1] \cdots [U_k/v_k](a_i M_{i1} \cdots M_{in_i}),$$

and ξ, η must be in substituted U's, say U_j and U_h (possibly with $h = j$). Their residuals will be corresponding occurrences in substituted U's, and since all such U's are disjoint, separation is obvious.

Case 4. *One of* ξ, η (*say* ξ) *is in* ζ *and the other contains* ζ. Since ξ, η, ζ are well separated, contracting ζ will produce exactly one residual η' of η, and the contractum of ζ will be a proper part of η'. Also, the residuals of ξ will be in the contractum of ζ. Suppose some residual ξ' of ξ is too close to η'. Then we must have one of two cases:

(i) $\eta' = (\lambda y \cdot Uy)V$, $\xi' = \lambda y \cdot Uy$;

(ii) $\eta' = \lambda y \cdot (\lambda z \cdot U)y$, $\xi' = (\lambda z \cdot U)y$.

And (i) can only happen if η is a β-redex and ξ is an η-redex, and (ii) can only happen if η is an η-redex and ξ is a β-redex.

Subcase 4a. η *is* $(\lambda y \cdot P)Q$ *with* ζ *in* Q. Let Q' be the result of contracting ζ in Q. Then $\eta' = (\lambda y \cdot P)Q'$. But all the residuals of ξ will be in the contractum of ζ which is in Q', so $(\lambda y \cdot P)$ cannot be a residual of ξ. Hence (i) above cannot happen in this case.

Subcase 4b. η *is* $(\lambda y \cdot P)Q$ *with* ζ *in* P. Let P' be the result of contracting ζ in P. Then we have

$$\eta' = (\lambda y \cdot P')Q,$$

and the residuals of ξ must all be inside P', so none of them can be $\lambda y \cdot P'$. Hence (i) is impossible.

Subcase 4c. η *is an* η*-redex* $\lambda y \cdot Py$, *with* ζ *in* P. Let P' be the result of contracting ζ in P. Then we have $\eta' = \lambda y \cdot P'y$, and all the residuals of ξ must be inside P'. So no ξ-residual can be $P'y$, and hence (ii) is impossible. This proves the lemma.

LEMMA 4. *If* ξ, η, *are any* $\beta - \eta - a_i$*-redexes in a term* T, *then* $\eta + \xi//\eta \simeq \xi + \eta//\xi$, *and if the two are well separated, then for any* ζ *in* T,

$$\zeta/(\eta + \xi//\eta) = \zeta/(\xi + \eta//\xi).$$

PROOF. We shall use the proof of (D^+) for β-reduction in §6, inserting more cases as necessary. First note that the weak equivalence of $\eta + \xi//\eta$ and $\xi + \eta//\xi$ follows from the definition of residuals, even when ξ, η are too close together.

One of the key properties used in §6 was that contracting a redex inside (and distinct from) ζ leaves exactly one residual of ζ, with unchanged position.

If ζ is well separated from ξ and η, then by (D8) and (D9), this property also holds here. The other key property was that substituting into a β- or a_i-redex produces another redex of the same kind. This property also holds for η-redexes. Now let T be any term, and ξ, η, ζ be any three redexes in T, with ξ well separated from η. We must show that

$$\zeta/(\xi + \eta//\xi) = \zeta/(\eta + \xi//\eta).$$

As in §6, it will be enough to show that the positions of the ζ-residuals are the same after both reductions. If ξ, η, ζ are not all distinct, the result is easy, so we shall assume they are all distinct.

Suppose first that ξ, η, ζ are all well separated.

If ζ is not in ξ and not in η, or ζ is in one of ξ, η and ξ is disjoint from η, then the argument of §6 still applies, thanks to separatedness. From now on, assume that both ζ and η are in ξ.

Case 1. *ζ is not in η*. If ξ is a β- or a_i-redex, the proof is the same as in Subcases 1a–c in §6. Note that in those proofs the exact structure of η and ζ played no part, except for the two key properties mentioned above. Even in (5), where we seemed to use the fact that η was a $\beta - a_i$-redex, we did not actually use any particular properties of the contractum of η, so the η-redex case will be the same as the a_i-case.

Subcase 1d. *ξ is an η-redex $\lambda x \cdot Mx$*. In this case, by separatedness η and ζ must both be in M. Contracting η first will change ξ to a term $\lambda x \cdot M'x$, with one residual of ζ, unchanged in position. Then contracting $\lambda x \cdot M'x$ will produce M', with the ζ-residual at the same position in M' as ζ originally had relative to M.

On the other hand, contracting ξ first will produce M, and then contracting η will change this to M', with the position of the ζ-residual unchanged.

Case 2. *ζ is in η*. If ξ is an a_i-redex, we use the proof of §6, Subcase 2c.

If ξ is an η-redex $\lambda x \cdot Mx$, then ζ and η are both in M, and the positions of the residuals of ζ will be determined entirely by the structure of η and the position of ζ in η, and hence will be independent of whether η is contracted first or last.

Now suppose ξ is a β-redex, $\xi = (\lambda x \cdot M)N$. If η is in N, argue as in §6, Subcase 2b. If η is in M and is a $\beta - a_i$-redex, argue as in Subcase 2a.

Finally, suppose that η is in M and is an η-redex:

$$\eta = \lambda y \cdot Py.$$

By separatedness, ζ must be in P. Contracting η to P will change ξ to a term $\xi' = (\lambda x \cdot M')N$, and the residual of ζ will be the ζ in P in M'. Then contracting ξ' will produce

$$[N/x]M', \tag{11}$$

containing $[N/x]P$ with $[N/x]\zeta$ at the same position as ζ originally had in P.

On the other hand, contracting ξ first will produce

$$[N/x]M \tag{12}$$

containing the residual of η, namely

$$[N/x]\eta = \lambda y \cdot ([N/x]P)y,$$

and $[N/x]P$ will contain $[N/x]\zeta$ at the same position as ζ had in P. Then contracting the η-residual to $[N/x]P$ will change (12) to (11), and leave the position of the ζ-residual unchanged, relative to $[N/x]P$.

This completes the proof for the case that ζ, ξ, η are all well separated.

Now suppose ζ is too close to both ξ and η. Then ζ/ξ and ζ/η are both empty, and so

$$\zeta/\,(\xi + \eta//\xi) = \varnothing = \zeta/\,(\eta + \xi//\eta).$$

Finally, suppose ζ is too close to one of ξ, η (say ξ), but not to the other (and that ξ is well separated from η). Then $\zeta/\xi = \varnothing$, so

$$\zeta/\,(\xi + \eta//\xi) = \varnothing.$$

We must show that $\zeta/(\eta + \xi//\eta) = \varnothing$; it will be enough to show that every residual ζ/η is too close to a residual ξ/η, so that when the residuals ξ/η are contracted, all the residuals of ζ will disappear.

Now the pair ζ, ξ has one of the two forms

$$(\lambda x \cdot Mx)N, \qquad (\lambda x \cdot Mx);$$

$$\lambda x \cdot (\lambda y \cdot P)x, \qquad (\lambda y \cdot P)x.$$

If η is in M, N or P, or η is disjoint from ζ and ξ, then contracting η leaves ζ, ξ too close together; hence as above,

$$\zeta/\,(\eta + \xi//\eta) = \varnothing.$$

The only other possibility is that η contains both of ζ, ξ. The possible forms of η are

$$\lambda z \cdot Wz, \qquad (\lambda z U)V, \qquad [U_1/v_1] \cdots [U_k/v_k](aM_1 \cdots M_n).$$

Since η is well separated from both ξ and ζ, the pair ξ, ζ must be in one of W, U, V, $U_1, \ldots, U_k$, and when η is contracted to

$$W, \qquad [V/z]U, \qquad [U_1/v_1] \cdots [U_k/v_k]M^*,$$

the residuals of ξ, ζ occur in too-close pairs as required above. Hence

$$\zeta / (\eta + \xi // \eta) = \varnothing.$$

This completes the proof of Lemma 4.

THEOREM 9.1. *For $\lambda\beta\eta$-reduction with extra axiom-schemes of form* (a), *satisfying* (D1)–(D9); *every well separated set α of redexes in a term T has a complete development, and all complete developments of α are strongly equivalent.*

PROOF. First, a complete development of α does exist; just take the MCD.

Second, to prove (E^+) for well separated α we can either use the direct Church-Rosser method, with Lemmas 3 and 4 at appropriate places, or use the abstract method in §5. Following §5: checking the proof of [3] Lemma 8 shows that all MCDs of a finite well separated set α of redexes are strongly equivalent. (Of course this needs Lemmas 3 and 4.) Then, again using Lemmas 3 and 4, Lemma 9(ii) on [3, p. 552] can be strengthened to say that if the set of redexes given in the lemma is well separated, then the reductions in the lemma's conclusion will be strongly equivalent. Finally, the proof of (1) in §3 above will also give strong equivalence. This completes the proof.

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DEPARTMENT OF PURE MATHEMATICS, UNIVERSITY OF SWANSEA, SWANSEA, WALES, UNITED KINGDOM