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ON A NOTION OF SMALLNESS FOR SUBSETS OF THE BAIRE SPACE

BY

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ABSTRACT. Let us call a set $A \subseteq \omega^\omega$ of functions from ω into ω *σ-bounded* if there is a countable sequence of functions $\{\alpha_n: n \in \omega\} \subseteq \omega^\omega$ such that every member of A is pointwise dominated by an element of that sequence. We study in this paper definability questions concerning this notion of smallness for subsets of ω^ω. We show that most of the usual definability results about the structure of countable subsets of ω^ω have corresponding versions which hold about σ-bounded subsets of ω^ω. For example, we show that every Σ^1_{2n+1} σ-bounded subset of ω^ω has a Δ^1_{2n+1} "bound" $\{\alpha_m: m \in \omega\}$ and also that for any $n \geqslant 0$ there are largest σ-bounded Π^1_{2n+1} and Σ^1_{2n+2} sets. We need here the axiom of projective determinacy if $n \geqslant 1$. In order to study the notion of σ-boundedness a simple game is devised which plays here a role similar to that of the standard *-games (see [My]) in the theory of countable sets. In the last part of the paper a class of games is defined which generalizes the *- and **- (or Banach-Mazur) games (see [My]) as well as the game mentioned above. Each of these games defines naturally a notion of smallness for subsets of ω^ω whose special cases include countability, being of the first category and σ-boundedness and for which one can generalize all the main results of the present paper.

1. **Preliminaries**. 1A. Let $\omega = \{0, 1, 2, \dots\}$ be the set of all natural numbers and $\mathfrak{R} = \omega^\omega$ the set of all functions from ω to ω or, for simplicity, *reals*. Letters $i, j, k, l, m, n, \dots$ denote elements of ω and $\alpha, \beta, \gamma, \delta, \dots$ reals. We study subsets of the *product spaces* $\mathfrak{X} = X_1 \times X_2 \times \cdots \times X_k$, where X_i is ω or $\mathfrak{R}$. We call such subsets *pointsets*. Sometimes we think of them as relations and we write interchangeably $x \in A \Leftrightarrow A(x)$. A *pointclass* is a class of pointsets, usually in all product spaces. We shall be concerned primarily in this paper with the *analytical* pointclasses Σ^1_n, Π^1_n, Δ^1_n and their corresponding *projective* pointclasses $\boldsymbol{\Sigma}^1_n$, $\boldsymbol{\Pi}^1_n$, $\boldsymbol{\Delta}^1_n$. For information about them we refer the reader to [R], [Sh] and [Mo$_1$].

If Γ is a pointclass and $\mathfrak{X}$ a product space ($\mathfrak{X} = \omega$ or $\mathfrak{X} = \mathfrak{R}$ will be enough for this definition), then we say that Γ is *$\mathfrak{X}$-parametrized* if for any product space $\mathfrak{Y}$ there is a $G \in \Gamma$, $G \subseteq \mathfrak{X} \times \mathfrak{Y}$ such that letting $G_x = \{y: (x,y) \in G\}$ we have $\{A \subseteq \mathfrak{Y}: A \in \Gamma\} = \{G_x: x \in \mathfrak{X}\}$. In this case G is called *$\mathfrak{X}$-universal*

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for Γ subsets of $\mathscr{Y}$. If $A = G_x$ we call x a *code* of A. It is well known that Σ^1_n, Π^1_n are ω-parametrized and Σ^1_n, Π^1_n are $\mathfrak{R}$-parametrized with universal sets which are actually in Σ^1_n, Π^1_n respectively.

1B. The notions of games and determinacy are used repeatedly in this paper. For information about them which will be used without explicit reference the reader could consult [My], [Mo$_2$], [Fe] or [Mo$_1$]. For any pointclass Γ,

$$\textit{Determinacy}\,(\Gamma)$$

abbreviates the statement: Every $A \subseteq \mathfrak{R}$, $A \in \Gamma$ is determined. We also abbreviate

$$\text{PD} \Leftrightarrow \textit{every projective set of reals is determined}$$

and

$$\text{AD} \Leftrightarrow \textit{every set of reals is determined.}$$

1C. We shall frequently talk about trees. Given a set X, a *tree* on X is a set of finite sequences from X, closed under subsequences i.e.

$$(x_0, \dots, x_n) \in T \;\&\; k \leqslant n \Rightarrow (x_0 \cdots x_k) \in T.$$

The empty sequence is always a member of a nonempty tree. A *node* of a tree is just a sequence in that tree. A *branch* of a tree T is an infinite sequence $f \in X^\omega$ such that for every n, $(f(0), \dots, f(n)) \in T$. The set of all branches of T is denoted by

$$[T] = \{f \in X^\omega \colon \forall n (f(0) \cdots f(n)) \in T\}.$$

We shall denote by $u \prec v$ the relation of *proper extension* between finite sequences. Thus if $u = (x_0 \cdots x_n)$, $v = (y_0 \cdots y_m)$, then

$$u \prec v \Leftrightarrow n > m \;\&\; \forall i \leqslant m (x_i = y_i).$$

We shall also denote by $u\hat{\ }v$ the *concatenation* of two finite sequences u, v. Thus if $u = (x_0 \cdots x_n)$, $v = (y_0 \cdots y_m)$, then

$$u\hat{\ }v = (x_0 \cdots x_n y_0 \cdots y_m).$$

If $u_0, u_1, u_2, \dots$ is an infinite list of finite sequences we shall also denote by $u_0\hat{\ }u_1\hat{\ }u_2\hat{\ }\cdots$ their concatenation. Thus if $u_0 = (x^0_0 \cdots x^0_{m_0})$, $u_1 = (x^1_0 \cdots x^1_{m_1})$, $\dots$, then

$$u_0\hat{\ }u_1\hat{\ }u_2\hat{\ }\cdots = (x^0_0 x^0_1 \cdots x^0_{m_0} x^1_0 x^1_1 \cdots x^1_{m_1} \cdots)$$

(this in general will be an infinite sequence, unless all u_i after a certain point are empty).

Occasionally we shall have to deal with trees on sets $X = Y \times Z$. A tree T on $Y \times Z$ contains elements of the form $((y_0, z_0), \ldots, (y_n, z_n))$, where $y_i \in Y$, $z_i \in Z$. A branch of such a tree is a sequence $f \in (Y \times Z)^\omega$ which for convenience will be represented by the unique pair (g, h) such that $f(n) = (g(n), h(n))$. The *first projection* of $[T]$, in symbols $p[T]$, is

$$p[T] = \{g: \exists h(g,h) \in [T]\}.$$

Most of the time we shall have $Y = \omega$, $Z = \lambda$ (λ some ordinal). Then $p[T] \subseteq \mathfrak{R}$.

1D. Let X (an arbitrary set containing more than one element) have the discrete topology. We shall always think of X^ω = set of infinite sequences of elements of X as having the product topology. Open sets in this topology are generated by the basic neighborhoods $N_u = \{f \in X^\omega: f \text{ extends } u\}$, where u ranges over the finite sequences from X. One can also visualize easily the closed sets of the space X^ω. Say that a tree T on X has *no finite branches* if for every $u \in T$ there is a proper extension $v \prec u$ in T. Then the map

$$A \mapsto \{(f(0) \cdots f(n)): f \in A, n \in \omega\} \stackrel{\text{def}}{=} T_A$$

gives a 1-1 correspondence between closed sets of X^ω and trees with no finite branches on X such that $A = [T_A]$. It is easy to see that A is perfect iff every $u \in T_A$ has at least two incompatible extensions in T_A (u, v are *incompatible* iff they have no common extension). Also A is compact iff T_A is *finite* splitting i.e. every $u \in T_A$ has only finitely many *immediate* extensions $u^\frown(x)$ in T_A. In case $X = \omega$ note that a closed set $A = [T_A]$ is compact iff there is a real β such that for every $\alpha \in A$, $\forall n(\alpha(n) \leqslant \beta(n))$.

REMARK. In case $X = \omega$ the product spaces will be also equipped with the product topology.

For each countable X, X^ω is a perfect *Polish* (i.e. completely metrizable and separable) space. Topological notions related to category will be used as a tool repeatedly in this paper. In particular the following result will be needed in several places. If one replaces its hypothesis by PD then the result can be already essentially attributed to Banach-Mazur and Oxtoby (see for example **[O]**). It is not clear who noticed first the present stronger version but the key idea of using "witnesses" in various standard games already occurs in early unpublished results of Solovay (see for example **[Bu]**). Martin **[Ma$_2$]** has also used *-games (see **[My]**) with witnesses in order to prove that, under Determinacy (Δ^1_{2n}), every Σ^1_{2n+1} set with no nonempty perfect subset contains only Δ^1_{2n+1} reals.

THEOREM (Folklore). *Assume Determinacy* (Δ^1_{2n}), $n \geqslant 0$. *Let* $\mathfrak{X}$ *be any product space. Then every* Σ^1_{2n+1} (*and thus* Π^1_{2n+1}) *set in* $\mathfrak{X}$ *has the property of*

Baire. Similarly assuming Determinacy (Σ^1_{2n+1}), $n \geqslant 1$, *every* Σ^1_{2n+2} *(and thus* Π^1_{2n+2}*) set has the property of Baire.*

PROOF. We prove the first assertion. The second can be proved in exactly the same way. It is also enough to consider the case $\mathfrak{X} = \mathfrak{R}$.

Assume now $A \subseteq \mathfrak{R}$ and $A \in \Sigma^1_{2n+1}$. Let $B \in \Pi^1_{2n}$ be such that $\alpha \in A \Leftrightarrow \exists\beta B(\alpha,\beta)$. Consider the following modified Banach-Mazur game in which player I plays also witnesses, which we shall denote by $G_p^{**}(B)$ (the $**$-notation for the Banach-Mazur game comes from [My]): Player I plays $k_0 \in \omega$ and a finite sequence s_0 from ω, II plays a nonempty finite sequence s_1 from ω, I plays $k_2 \in \omega$ and a nonempty finite sequence s_2 from ω, II plays a nonempty finite sequence s_3 from ω, etc. At the end of the run of the game let $\beta = (k_0, k_1, k_2, \ldots) \in \mathfrak{R}$, $\alpha = s_0\hat{\ } s_1 \hat{\ } s_2 \hat{\ } \cdots \in \mathfrak{R}$. Then I wins iff $(\alpha,\beta) \in B$. Otherwise II wins. We have now the following

LEMMA. (i) *If* I *has a winning strategy in* $G_p^{**}(B)$, *then for some open nonempty set* G, A *is comeager on* G.

(ii) *If* II *has a winning strategy in* $G_p^{**}(B)$, *then* A *is meager.*

PROOF. (i) If I has a winning strategy with first move (k_0, s_0), then it is easy to see that B is comeager of $N_{s_0} = \{\alpha: s_0 \text{ is an initial segment of } \alpha\}$; see for example [O, p. 28].

(ii)(1) Assume II has a winning strategy τ. Call a sequence $(k_0, s_0, s_1, \ldots, k_{2k}, s_{2k}, s_{2k+1})$ *good* if for all $i \leqslant k$, s_{2i+1} is played according to τ. By convention the empty sequence is good. If $(\alpha,\beta) \in B$ then there must be some good sequence $u = (k_0, s_0, s_1, \ldots, k_{2k}, s_{2k}, s_{2k+1})$ (maybe empty) such that $(k_0, k_2, \ldots, k_{2k}) = (\beta(0) \cdots \beta(k))$ is an initial segment of β and $s_0\hat{\ } s_1\hat{\ } \cdots\ s_{2k+1}$ is an initial segment of α, but no good extension $(k_0, s_0, k_1, \ldots, k_{2k}, s_{2k}, s_{2k+1}, k_{2k+2}, s_{2k+2}, s_{2k+3})$ of u has this property (otherwise (α,β) could be conceived as a play of the game in which II plays according to his strategy τ). Let $\beta(k+1) = m$ and put $M_{u,m} = \{\alpha': s_0\hat{\ } s_1\hat{\ } \cdots\ s_{2k+1}$ is an initial segment of α' and for every s_{2k+2}, if

$$s_{2k+3} = \tau(k_0, s_0, \ldots, k_{2k+1}, s_{2k+1}, m, s_{2k+2}),$$

then $s_0\hat{\ } s_1\hat{\ } \cdots\ s_{2k+3}$ is not an initial segment of $\alpha'\}$. Then $M_{u,m}$ is closed with no interior, i.e. it is nowhere dense and $\alpha \in M_{u,m}$. Since there are only countably many $M_{u,m}$'s, A is meager.

From this lemma it follows that (granting Determinacy (Δ^1_{2n}), which by a result of Martin [Ma$_1$] implies Determinacy (Σ^1_{2n})) every Σ^1_{2n+1} set is meager (thus has the property of Baire) or it is comeager on an open set. Given now $A \in \Sigma^1_{2n+1}$ consider $G = \bigcup\{N_s: A \text{ is comeager on } N_s\}$ (where for s a finite

(1) We would like to thank Sy Friedman for helping simplify considerably the presentation of this part of the proof.

sequence from ω, $N_s = \{\alpha: \alpha \text{ extends } s\}$). Then A is comeager in G, thus $G - A$ is meager. But also $A - G$ is Σ^1_{2n+1} and $A - G$ is not comeager on any open set, therefore $A - G$ is meager. So $(A - G) \cup (G - A) = A \triangle G$ is meager, i.e. A has the property of Baire. □

REMARK. A similar result can be proved about Lebesgue measurability using the Mycielski-Swierczkowski game (see **[My-S]**) with witnesses, which is essentially the game Solovay used in his proof that $AD \Rightarrow BC$ (see **[Bu]**) or the Harrington game (see for example **[Ke$_1$]**) with witnesses.

1E. The whole discussion in this paper takes place in ZF + DC, Zermelo-Fraenkel set theory with dependent choices:

$$\text{(DC)} \qquad \forall u \in x\, \exists v(u,v) \in r \Rightarrow \exists f\, \forall n(f(n), f(n+1)) \in r.$$

Every additional hypothesis is stated explicitly.

Our set theoretic notation and terminology will be standard, when possible. We shall use letters $\xi, \eta, \theta, \lambda, \ldots$ to denote ordinals.

2. **σ-bounded and superperfect sets.** 2A. Let $A \subseteq \mathfrak{R}$. We call A *σ-bounded* iff there is a sequence $\{\alpha_i\}_{i\in\omega}$ of reals such that for each $\alpha \in A$ there is some $i \in \omega$ with $\alpha \leqslant \alpha_i$, where for any two reals α, β

$$\alpha \leqslant \beta \overset{\text{def}}{\Leftrightarrow} \forall n(\alpha(n) \leqslant \beta(n)).$$

We call $\{\alpha_i\}_{i\in\omega}$ a *bound* for A.

Note here that A is σ-bounded *iff* A is contained in a σ-compact (i.e. a countable union of compact sets) subset of $\mathfrak{R}$ *iff* there is a β such that for all $\alpha \in A$, $\alpha \leqslant^* \beta$, where for any two reals α, β

$$\alpha \leqslant^* \beta \overset{\text{def}}{\Leftrightarrow} \exists i\, \forall n \geqslant i(\alpha(n) \leqslant \beta(n)).$$

We think of course of σ-boundedness as a notion of smallness for sets of reals, analogous to countability. Opposite to countable stands the concept of a perfect set. We shall now define the corresponding concept which provides the opposite to σ-boundedness.

DEFINITION. Let T be a tree on ω. We call T *superperfect* iff for every $u \in T$ there is $v \in T$ extending u such that $\{m \in \omega: v^\frown(m) \in T\}$ is infinite i.e. v has infinitely many immediate extensions in T. We shall call a set $A \subseteq \mathfrak{R}$ *superperfect* iff A is closed and the tree of A, T_A is superperfect.

It is easy to see that $A \subseteq \mathfrak{R}$ is superperfect iff A is closed and for every $\alpha \in A$ and any open set G containing α, $G \cap A$ is not contained in a compact set. Also notice that every nonempty superperfect set contains a nonempty superperfect set homeomorphic to $\mathfrak{R}$.

REMARKS. (1) The notion of a superperfect tree is already implicit in Friedman **[Fr$_1$]**, which provided some of the original motivation for studying the notions under consideration here. Lemma 1.3 in **[Fr$_1$]** is in our terminology essentially the statement: Every nonempty Π_1^0 set which contains no Δ_1^1 real contains a nonempty superperfect set.

(2) (The axiom of choice is used in this remark.) We call $A \subseteq \mathfrak{R}$ **κ-bounded,** where κ is an infinite cardinal, if there is a sequence $\{\beta_\xi\}_{\xi<\kappa}$ of reals such that for each $\alpha \in A$ there is a $\xi < \kappa$ with $\alpha \leqslant \beta_\xi$. Clearly A is κ-bounded iff there is a κ-compact (i.e. the union of κ many compact sets) subset of $\mathfrak{R}$ containing A. Put for any $A \subseteq \mathfrak{R}$, ${}^\sharp A$ = least κ such A is κ bounded. Let also $\kappa_0 = {}^\sharp\mathfrak{R}$. Clearly $\kappa_0 > \aleph_0$. Moreover κ_0 = least κ such that κ is the cardinality of a scale on $\mathfrak{R}$, where $S \subseteq \mathfrak{R}$ is a *scale* if S is cofinal with respect to $\leqslant^*$. Thus from results of Solovay (unpublished) and Hechler [H] it follows that the value of κ_0 is consistently with ZFC "essentially independent" of $2^{\aleph_0}$. For example one could have $\kappa_0 = 2^{\aleph_0} > \aleph_1$ or $\kappa_0 = \aleph_1$ and $2^{\aleph_0} = \aleph_{17}$. Notice also that if $A \neq \varnothing$ is superperfect, then ${}^\sharp A = 2^{\aleph_0}$, since $\mathfrak{R}$ is homeomorphic to a closed subset of A.

2B. One of our main purposes in this paper is to establish results about σ-bounded and superperfect sets which are analogous to those about countable and perfect sets. As a simple start let us notice for example that as usual the analog of the Cantor-Bendixson Theorem goes through.

Proposition. *Let $A \subseteq \mathfrak{R}$ be a closed set. Then we can write A uniquely as $A = P \cup C$, where P is superperfect, C is σ-bounded and $P \cap C = \varnothing$. In particular a closed set is either σ-bounded or contains a nonempty superperfect set.*

Proof. Let $A = [T]$, where T is a tree on ω. For any tree J on ω define the derivative:

$$J' = \{s \in J: \exists t(t \prec s \;\&\; t \in J \;\&\; \{m: t\hat{\ }(m) \in J\} \text{ is infinite})\}.$$

Then let by induction: $T_0 = T$, $T_{\xi+1} = (T_\xi)'$, $T_\lambda = \bigcap_{\xi<\lambda} T_\xi$, if λ is limit. If ξ_0 is the least ordinal ξ for which $T_\xi = T_{\xi+1}$, then ξ_0 is countable and $P = [T_{\xi_0}] \subseteq A$ is superperfect, while $C = A - P$ is σ-bounded. To show uniqueness notice that if P, C satisfy the above conditions, then

$$P = A^* = \{\alpha: \forall G(G \text{ open} \;\&\; \alpha \in G \Rightarrow G \cap A \text{ is not } \sigma\text{-bounded})\} \text{ and}$$

$$C = A - A^*. \quad \square$$

Remark. This argument can be used to give an alternative proof of the result of Friedman mentioned in Remark (2) of 2A. Indeed let $A = [T]$ with T recursive. In the notation above, assuming towards a contradiction that A contains no nonempty superperfect subset, we have $T_{\xi_0} = \varnothing$, so ξ_0 is recursive since the derivative $J \to J'$ is arithmetical. Then if $\alpha \in A$ there is $\xi < \xi_0$ such that $\alpha \in T_\xi - T_{\xi+1}$ and thus for some finite sequence s, $N_s \cap [T_\xi]$ is nonempty and compact and thus contains a Δ_1^1 real since $T_\xi \in \Delta_1^1$, a contradiction.

3. The games $\tilde{G}(A)$ and $\tilde{G}_p(B)$. 3A. In order to deal more effectively with the present notions we shall associate with each $A \subseteq \mathfrak{R}$ a game $\tilde{G}(A)$ so that I has a winning strategy in $\tilde{G}(A)$ iff A contains a nonempty superperfect subset and II has a winning strategy in $\tilde{G}(A)$ iff A is σ-bounded. The game $\tilde{G}(A)$ is played as follows:

I	II	
s_0		Player I chooses a finite sequence s_0 from ω, II chooses $k_1 \in \omega$, I chooses a nonempty finite sequence s_1 from ω, II chooses k_2, I chooses a finite non-empty sequence s_2 from ω, etc.
	k_1	
s_1		
	k_2	
s_2		
	$\vdots$	

Then I wins iff (i) $\alpha = s_0{}^\frown s_1{}^\frown s_2{}^\frown \cdots \in A$ and (ii). For all $m \geqslant 1$, the first member of s_m is bigger than k_m. II wins otherwise.

The following result is the analog of the M. Davis Theorem about the *-games, see [D] or [My].

THEOREM 3.1. *Let $A \subseteq \mathfrak{R}$ and let $\tilde{G}(A)$ be the game described above. Then*
(i) I *has a winning strategy in $\tilde{G}(A)$ iff A contains a nonempty superperfect set.*
(ii) II *has a winning strategy in $\tilde{G}(A)$ iff A is σ-bounded.*

PROOF. Only the "only if" part of (ii) is nontrivial. So assume II has a winning strategy τ in $\tilde{G}(A)$. Call a sequence (maybe empty) $(s_0, k_1, s_1, k_2, \ldots, s_n, k_{n+1})$ *good* if for all $1 \leqslant i \leqslant n$ the first member of s_i is bigger than k_i and for all $0 \leqslant j \leqslant n+1$ the k_j's are determined according to τ. By convention the empty sequence is good. If $\alpha \in A$ there must be some good sequence $u = (s_0, k_1, s_1, k_2, \ldots, s_n, k_{n+1})$ (maybe empty) such that $s_0{}^\frown s_1{}^\frown \cdots s_n{}^\frown$ is an initial segment of α, say $s_0{}^\frown s_1{}^\frown \cdots s_n{}^\frown = (\alpha(0) \cdots \alpha(m-1))$, and $\alpha(m) > k_{n+1}$ but no good extension of u has this property. Let $K_u = \{\alpha'\colon s_0{}^\frown s_1{}^\frown \cdots s_n{}^\frown$ is an initial segment of α', say $(\alpha'(0) \cdots \alpha'(m-1))$, and $\alpha'(m) > k_{n+1}$ but for every $s_{n+1} \neq \varnothing$ such that $s_0{}^\frown s_1{}^\frown \cdots s_{n+1}{}^\frown$ is an initial segment of α', say $s_0{}^\frown s_1{}^\frown \cdots s_{n+1}{}^\frown = (\alpha(0), \ldots, \alpha'(l-1))$, we have $\alpha(l) \leqslant k_{n+2} = \tau(s_0, s_1, \ldots, s_{n+1})\}$. It is now easy to see that K_u is a σ-compact set. Also $\alpha \in K_u$. Since there are only countably many such K_u's, A is σ-bounded. $\square$

COROLLARY 3.2. (i) PD $\Rightarrow$ *Every projective set of reals is either σ-bounded or contains a nonempty superperfect set.*

(ii) AD $\Rightarrow$ *Every set of reals is either σ-bounded or contains a nonempty superperfect set.*

3B. In order to get sufficiently strong definability results (having in mind here for example the analog of Martin's Theorem, see [$\mathbf{Ma}_2$] that granting PD every countable Σ^1_{2n+1} set is contained in some $\{\alpha_m\colon m \in \omega\}$ where the sequence $\{\alpha_m\}_{m \in \omega}$ is Δ^1_{2n+1}) we shall have to consider a modified version of the previous game $\tilde{G}(A)$ which allows for player I to play also witnesses. (The same modification of the **-(or Banach-Mazur) games was of course used in the proof of the theorem in §1.)

Let $A \subseteq \mathfrak{R}$ and assume that for some set $B \subseteq \mathfrak{R} \times \lambda^\omega$, where λ is an ordinal,

$$A = p[B] = \{\alpha \colon \exists f \in \lambda^\omega (\alpha, f) \in B\}.$$

Consider the following game $\tilde{G}_p(B)$:

I		II
ξ_0	s_0	
		k_1
ξ_1	s_1	
		k_2
ξ_2	s_2	
	$\vdots$	

I plays $\xi_0 < \lambda$ and s_0 a finite sequence from ω, II plays $k_1 \in \omega$, I plays $\xi_1 < \lambda$ and a nonempty finite sequence s_1 from ω, II plays k_2, I plays $\xi_2 < \lambda$ and a nonempty finite sequence s_2 from ω etc.

Let $\alpha = s_0 {}^\frown s_1 {}^\frown \cdots \in \mathfrak{R}$, $f = (\xi_0, \xi_1, \xi_2, \ldots) \in \lambda^\omega$. Then I wins iff $(\alpha, f) \in B$ and for all $i \geqslant 1$, the first member of s_i is bigger than k_i.

It is now easy to prove the following result, using the ideas of the proof of Theorem 3.1 together with the obvious changes suggested by the proof of the theorem in §1. In its statement we denote by X^+ the smallest admissible set containing X. Also $L[X]$ is the universe constructible from X.

THEOREM 3.3. *Let λ be an infinite ordinal and let $B \subseteq \mathfrak{R} \times \lambda^\omega$. Then*

(i) *If $\mathcal{S}$ is a winning strategy for* I *in $\tilde{G}_p(B)$ then $A = p[B]$ contains a nonempty superperfect set whose tree belongs in $L[\mathcal{S}]$.*

(ii) *If $\mathfrak{T}$ is a winning strategy for* II *in $\tilde{G}_p(B)$ then there is a sequence $\{\alpha_\xi\}_{\xi<\lambda} \in \mathfrak{T}^+$ so that for each $\alpha \in A$ there is $\xi < \lambda$, with $\alpha \leqslant \alpha_\xi$.*

REMARK. In particular, if $\lambda = \omega$ and $\tilde{G}_p(B)$ is determined, then A is σ-bounded $\Leftrightarrow$ II has a winning strategy in $\tilde{G}_p(B)$ and A contains a nonempty superperfect set $\Leftrightarrow$ I has a winning strategy in $\tilde{G}_p(B)$.

4. **Definability results.** Using Theorem 3.3, we can now prove without too much effort the next result every part of which is the analog of a well-known fact about countable and perfect sets.

THEOREM. (i) *If $A \subseteq \mathfrak{R}$ is Σ^1_1, then either A contains a nonempty superperfect set or A is σ-bounded with a Δ^1_1 bound.*

More generally,

(ii) *Determinacy $(\Delta^1_{2n}) \Rightarrow$ Every Σ^1_{2n+1} set either contains a nonempty superperfect set or is σ-bounded with a Δ^1_{2n+1} bound.*

(iii) *If T is a tree on $\omega \times \lambda$, where λ is some ordinal, and $A = p[T] = \{\alpha \colon \exists f \in \lambda^\omega (\alpha, f) \in [T]\}$ then either A contains a nonempty superperfect set* *whose tree is in $L[T]$, or there is a sequence $\{\alpha_\xi\}_{\xi<\lambda}$ in T^+ such that for all $\alpha \in A$ there is $\xi < \lambda$ with $\alpha \leqslant \alpha_\xi$.*

In particular,

(iv) *If $A \subseteq \mathfrak{R}$ is Σ^1_2 either A contains a nonempty superperfect set with tree in L or for every $\alpha \in A$ there is $\beta \in L$ with $\alpha \leqslant \beta$, i.e. $A \subseteq \{\alpha: \exists \beta \in L\, (\alpha \leqslant \beta)\}$.*

PROOF. (ii) If $A \in \Sigma^1_{2n+1}$, then $\alpha \in A \Leftrightarrow \exists \beta\, B(\alpha, \beta)$, where $B \in \Pi^1_{2n}$. The game $\tilde{G}_p(B)$ is then a Π^1_{2n} game, so it is determined assuming Determinacy $(\mathbf{\Delta}^1_{2n})$, by Martin [**Ma**$_1$]. If A contains no nonempty superperfect set then II has a winning strategy in $\tilde{G}_p(B)$, thus by Moschovakis [**Mo**$_3$] II has a Δ^1_{2n+1} winning strategy. Therefore there is a Δ^1_{2n+1} bound $\{\alpha_m\}_{m\in\omega}$ for A (i.e. $\alpha(\langle m, l\rangle) = \alpha_m(l)$ is Δ^1_{2n+1}).

(iii) The game $\tilde{G}_p(B)$, where $B = [T]$, is clearly closed so it is determined. If I has a winning strategy then he has one in $L[T]$, thus A contains a nonempty superperfect set with tree in $L[T]$. If II has a winning strategy then he has one in T^+ and the conclusion follows.

(iv) is clear now since every Σ^1_2 set is of the form $p[T]$ for some $T \in L$. □

5. Largest σ-bounded sets in the analytical pointclasses. 5A. We shall transfer in this section some of the usual results about the structure of countable and thin (i.e. containing no perfect subset) sets in the various analytical pointclasses to the present context. We start with a definition.

DEFINITION. A set $A \subseteq \mathfrak{R}$ is called *weakly thin* iff it contains no nonempty superperfect set.

Note that if $\tilde{G}(A)$ is determined, A is weakly thin iff A is σ-bounded. More generally, if $A = p[B]$, and $\tilde{G}_p(B)$ is determined, then the same conclusion holds. Thus granting Determinacy $(\mathbf{\Delta}^1_{2n})$, a set $A \in \mathbf{\Sigma}^1_{2n+1}$ is weakly thin iff it is σ-bounded and granting Determinacy $(\mathbf{\Sigma}^1_{2n+1})$, a set $A \in \mathbf{\Sigma}^1_{2n+2}$ is weakly thin iff it is σ-bounded. It is consistent with ZFC however that there is a Π^1_1 weakly thin set which is a scale, so it is not σ-bounded. This is the set $\mathfrak{B}_1$ defined below.

5B. Our immediate goal is to show that there are largest weakly thin Π^1_{2n+1} and Σ^1_{2n+2} sets granting appropriate determinacy hypotheses for $n \geqslant 1$. The general abstract approach of §1 of [**Ke**$_2$] is particularly useful here. By Theorem (1A-2) of [**Ke**$_2$] it is enough to prove that the class of weakly thin sets has an appropriate additivity and satisfies some definability conditions. This is accomplished in the following two lemmas.

LEMMA 5.1. *Assume $n \geqslant 0$ and Determinacy $(\mathbf{\Delta}^1_{2n})$. Then, the class of weakly thin sets is $\mathbf{\Sigma}^1_{2n+1}$ and $\mathbf{\Pi}^1_{2n+1}$ additive. Similarly, if Determinacy $(\mathbf{\Sigma}^1_{2n+1})$ holds the class of weakly thin sets in $\mathbf{\Sigma}^1_{2n+2}$ and $\mathbf{\Pi}^1_{2n+2}$ additive.*

PROOF. For the definitions see p. 265 of [**Ke**$_2$]. Consider the first assertion; the other can be proved by exactly the same argument. Let Γ = Boolean algebra generated by Σ^1_{2n+1}. We shall prove (the stronger result) that the class of weakly thin sets is Γ-additive. So assume $\{A_\xi\}_{\xi<\theta}$ is a sequence of sets of reals such that each A_ξ is weakly thin and the prewellordering on $\bigcup_{\xi<\theta}A_\xi$ given by

$$x \leqslant y \Leftrightarrow \mu\xi(x \in A_\xi) \leqslant \mu\xi(y \in A_\xi)$$

is in Γ. We have to prove that $\bigcup_{\xi<\theta}A_\xi$ is also weakly thin. The proof is by induction on $\theta \geqslant 2$ the case $\theta = 2$ being obvious. Consider the successor case first: Let $\theta = \eta + 1$. The prewellordering

$$x \leqslant' y \Leftrightarrow x, y \in \bigcup_{\xi<\eta} A_\xi \;\&\; \mu\xi(x \in A_\xi) \leqslant \mu\xi(y \in A_\xi)$$

is clearly in Γ since

$$x \leqslant' y \Leftrightarrow x \leqslant y \;\&\; \neg(x_0 \leqslant x) \;\&\; \neg(x_0 \leqslant y),$$

where x_0 is a fixed element of $A_\eta - \bigcup_{\xi<\eta}A_\xi$ (if no such x_0 exists the result is obviously true). By induction hypothesis $A = \bigcup_{\xi<\eta}A_\xi$ is weakly thin. Let $B = A_\eta$. Then B is also weakly thin. Assume towards a contradiction that $A \cup B$ is not weakly thin. Then $A \cup B$ contains a superperfect set P homeomorphic to $\mathfrak{R}$. Then $A \cap P$ must be meager in P (with the relative topology). Otherwise $A \cap P$, having as it does the property of Baire, is comeager in an open set G of P. Thus $A \cap P$ contains a G_δ set D dense in G. But D cannot be σ-bounded so D contains a superperfect set and thus so does A, a contradiction. Similarly $B \cap P$ is meager in P. Thus P is meager in P, contradicting the Baire Category Theorem.

For the limit case, assume towards a contradiction, that $P \subseteq \bigcup_{\xi<\theta}A_\xi$ is a superperfect set homeomorphic to $\mathfrak{R}$. Consider the relation

$$x \leqslant'' y \Leftrightarrow x \leqslant y \;\&\; x, y \in P.$$

Then $\leqslant''$ is a prewellordering on P which is in Γ, so has the property of Baire. For each $y \in P$, $\{x: x \leqslant'' y\} = \bigcup\{A_\eta: \eta \leqslant \mu\xi(y \in A_\xi)\} \cap P$, so by the previous arguments and the induction hypothesis $\{y: x \leqslant'' y\}$ is meager in P. So by the Kuratowski-Ulam Theorem (see [O, p. 56]), which is the analog of Fubini for category, we must also have that on a comeager in P set of x's $\{y: x \leqslant'' y\}$ is also meager in P. Thus P is meager in P, a contradiction. $\square$

LEMMA 5.2. *Assume $n \geqslant 0$ and Determinacy (Δ^1_{2n}). Let $G \subseteq \mathfrak{R} \times \mathfrak{R}$ be Π^1_{2n+1} and $\mathfrak{R}$-universal for Π^1_{2n+1} subsets of $\mathfrak{R}$. Then if $G_\alpha = \{\beta: (\alpha, \beta) \in G\}$, the set $W^{2n+1}(\alpha) \Leftrightarrow \mathfrak{R} - G_\alpha$ is weakly thin is Π^1_{2n+1}.*

PROOF. Let $A \subseteq \mathfrak{R}$ be Σ^1_{2n+1}. Let $B \in \Pi^1_{2n}$ be such that $\alpha \in A \Leftrightarrow \exists\beta\, B(\alpha, \beta)$. Then, A is weakly thin $\Leftrightarrow$ II has a winning strategy in $\tilde{G}_p(B) \Leftrightarrow$ I has no winning strategy in $\tilde{G}_p(B)$. Since $\tilde{G}_p(B)$ is a Π^1_{2n} game this statement is

Π^1_{2n+1}, uniformly in a code for B and we are done. □

5C. Using this lemma and Theorem (1A-2) from [$\mathbf{Ke}_2$] we immediately have

THEOREM 5.3. (i) *There is a largest weakly thin* Π^1_1 *set.*

(ii) *Assume* $n > 0$ *and Determinacy* (Δ^1_{2n}). *There exists a largest weakly thin* Π^1_{2n+1} *set.*

In particular, granting PD, there is for each $n \geqslant 0$ a largest σ-bounded Π^1_{2n+1} set.

We also have the corresponding result for even levels.

THEOREM 5.4. (i) *Assume there are only countably many constructible reals. Then there is a largest* σ*-bounded* Σ^1_2 *set, namely* $\{\alpha\colon \exists\beta \in L(\alpha \leqslant \beta)\}$.

(ii) *Assume* $n \geqslant 1$ *and Determinacy* (Σ^1_{2n+1}). *Then there is a largest* σ*-bounded* Σ^1_{2n+2} *set.*

PROOF. (i) follows from the theorem in §4.

(ii) Let $P(n,\alpha,\beta)$ be Π^1_{2n+1} and ω-universal for Π^1_{2n+1} subsets of $\mathfrak{R}\times\mathfrak{R}$. By the uniformization theorem (see [$\mathbf{Mo}_4$]) find $P^*(n,\alpha,\beta)$ uniformizing $P(n,\alpha,\beta)$ on β. Let φ be a Π^1_{2n+1}-norm on P (for the definition see for example [$\mathbf{Mo}_4$, p. 733]). Consider the set B given by

$$\alpha \in B \Leftrightarrow \exists n\, \exists\beta[P^*(n,\alpha,\beta)$$
$$\& \{\alpha'\colon \exists\beta'(P^*(n,\alpha',\beta') \;\&\; \varphi(n,\alpha',\beta') \leqslant \varphi(n,\alpha,\beta))\} \text{ is } \sigma\text{-bounded}].$$

By Lemma 5.2, B is Σ^1_{2n+2} and it is trivial to check that if $A \subseteq \mathfrak{R}$ is Σ^1_{2n+2} and σ-bounded then $A \subseteq B$. So it is enough to prove that B itself is σ-bounded. Let $Q_n(\alpha,\beta) \Leftrightarrow P^*(n,\alpha,\beta) \;\&\; \{\alpha'\colon \exists\beta'(P^*(n,\alpha',\beta') \;\&\; \varphi(n,\alpha',\beta') \leqslant \varphi(n,\alpha,\beta))\}$ is σ-bounded, and $B_n = \{\alpha\colon \exists\beta Q_n(\alpha,\beta)\}$. Since $B = \bigcup_n B_n$ it is enough to show each B_n is σ-bounded. Let $\theta = \sup\{\varphi(n,\alpha,\beta)\colon P^*(n,\alpha,\beta)\}$ and for $\xi < \theta$ let $A_\xi = \{\alpha\colon \exists\beta(Q_n(\alpha,\beta) \;\&\; \varphi(\alpha,\beta) = \xi)\}$. Then $B_n = \bigcup_{\xi<\theta} A_\xi$ and the prewellordering $\alpha_1 \leqslant \alpha_2 \Leftrightarrow \alpha_1, \alpha_2 \in B_n \;\&\; \mu\xi\,(\alpha_1 \in A_\xi) \leqslant \mu\xi\,(\alpha_2 \in A_\xi)$ $\Leftrightarrow \exists\beta_1\, \exists\beta_2(Q_n(\alpha_1,\beta_1) \;\&\; Q_n(\alpha_2,\beta_2) \;\&\; \varphi(n,\alpha_1,\beta_1) \leqslant \varphi(n,\alpha_2,\beta_2))$ is Σ^1_{2n+2}. Since each A_ξ is σ-bounded, we conclude by Lemma 5.1 that $B_n = \bigcup_{\xi<\theta} A_\xi$ is also σ-bounded and we are done. □

NOTATION. Let $\mathfrak{B}_1$ be the largest weakly thin Π^1_1 set, $\mathfrak{B}_2 = \{\alpha\colon \exists\beta \in L(\alpha \leqslant \beta)\}$ and granting PD let $\mathfrak{B}_n =$ largest σ-bounded Σ^1_n or Π^1_n set according as $n \geqslant 3$ is even or odd.

REMARKS. (1) It is easy to see, using PD, that there are no largest σ-bounded Σ^1_{2n+1} or Π^1_{2n} sets, $n \geqslant 0$. This is because a σ-bounded set is meager and every comeager Π^1_{2n+1} or Σ^1_{2n} set contains (by [$\mathbf{Ke}_3$] and [$\mathbf{Mo}_4$]) a Δ^1_{2n+1} or Δ^1_{2n} real, respectively.

(2) Let for $n \geqslant 1$, using PD, $\mathcal{C}_n$ be the largest countable Π^1_n or Σ^1_n set, according as n is odd or even (see [$\mathbf{Ke}_2$]) and $\mathfrak{M}_n =$ largest meager Π^1_n or Σ^1_n set, according as n is again odd or even (see [$\mathbf{Ke}_3$] or [**St**]). Then clearly

$$\mathcal{C}_n \subsetneqq \mathcal{B}_n \subsetneqq \mathfrak{M}_n.$$

5D. We shall concentrate now on the set $\mathcal{B}_1$ = largest weakly thin Π^1_1 set. Clearly $\mathcal{B}_1 \subseteq \mathcal{B}_2$ so for each $\alpha \in \mathcal{B}_1$ there is $\beta \in L$ such that $\alpha \leqslant \beta$. In analogy with the Guaspari-Sacks characterization of $\mathcal{C}_1$ as $\{\alpha: \alpha \in L_{\omega_1^\alpha}\}$ (see **[Gu]** or **[Ke$_2$]**, Theorem (2A-1)), where of course ω_1^α = least ordinal nonrecursive in α, we have:

THEOREM 5.5. *The largest weakly thin* Π^1_1 *set is equal to* $\{\alpha: \exists\beta(\beta \in L_{\omega_1^\beta}\ \&\ \alpha \leqslant \beta)\}$.

PROOF. Use the same idea as in the proof of Theorem (2A-1) in **[Ke$_2$]**, together with the Theorem in §4 (part iii) and the category arguments in Lemma 5.1 to replace the measure theoretic ones used in Theorem (2A-1) of **[Ke$_2$]**. □

Similarly we have the following analog of Theorem (2A-2) in **[Ke$_2$]**, letting $\alpha \leqslant^b_n \beta \Leftrightarrow \exists\gamma \in \Delta^1_n(\beta)\ (\alpha \leqslant \gamma)$.

THEOREM 5.6. *For all* α,

$$\alpha \in \mathcal{B}_1 \Leftrightarrow \forall\beta(\omega_1^\alpha \leqslant \omega_1^\beta \Rightarrow \alpha \leqslant^b_1 \beta).$$

PROOF. The direction $\Rightarrow$ is similar to the corresponding part in the proof of Theorem (2A-2) in **[Ke$_2$]**. For the proof of $\Leftarrow$ it is enough to show that $B = \{\alpha: \forall\beta(\omega_1^\alpha \leqslant \omega_1^\beta \Rightarrow \alpha \leqslant^b_1 \beta)\}$ is Π^1_1 and weakly thin. It is clearly Π^1_1. To show it is weakly thin assume $P \subseteq B$ is a superperfect set homeomorphic to $\mathfrak{R}$. Consider the prewellordering

$$\alpha \leqslant \beta \Leftrightarrow \alpha,\quad \beta \in P\ \&\ \omega_1^\alpha \leqslant \omega_1^\beta$$

on P. For each $\beta \in P$, $\{\alpha: \alpha \leqslant \beta\} = \{\alpha \in P: \omega_1^\alpha \leqslant \omega_1^\beta\}$ is contained in a σ-compact set, namely $\{\alpha: \alpha \leqslant^b_1 \beta\}$. So $\{\alpha: \alpha \leqslant \beta\}$ is meager in P and this contradicts as usual the Kuratowski-Ulam Theorem. □

5E. We shall conclude this section with some remarks on $\mathcal{B}_2 = \{\alpha: \exists\beta \in L\ (\alpha \leqslant \beta)\}$ and some open problems. We have seen in Theorem 5.4 that, granting $\mathfrak{R} \cap L$ is countable, $\mathcal{B}_2$ is the largest σ-bounded Σ^1_2 set. Of course we could only assume that $\mathfrak{R} \cap L$ was σ-bounded. In this case the converse also holds: If there is a largest σ-bounded Σ^1_2 set then $\mathfrak{R} \cap L$ is σ-bounded. This is because every Σ^1_2 set which contains all the countable Σ^1_2 sets must contain also $\mathfrak{R} \cap L$. (Otherwise let α_0 be the least element of L not in the given Σ^1_2 set A which satisfies the above. Then by a result of Friedman **[Fr$_2$]**, α_0 is a Π^1_2 singleton in L, so the set of reals in L preceding or being equal to α_0 is a countable Σ^1_2 set not contained in A, a contradiction.) It seems relevant to mention also that according to unpublished results of Solovay (see **[He]**), in the model M of set theory obtained from L by adding (say) $\aleph_2$ random reals,

$\mathfrak{R} \cap L$ is weakly thin and thus so is $\mathfrak{B}_1$, but $\mathcal{C}_1$ and thus $\mathfrak{B}_1$ is a scale

$$(\text{so } \{\alpha\colon \exists\beta \in \mathcal{C}_1(\alpha \leqslant \beta)\} = \mathfrak{B}_2 = \mathfrak{R}).$$

In particular, $\mathfrak{B}_1$ is not σ-bounded which of course is also true in L. (This supplies a proof for the remark immediately preceding 5B.)

By Theorem 5.4 there is a largest σ-bounded Σ^1_{2n} set $\mathfrak{B}_{2n}$ for each $n \geqslant 1$, granting PD. Since $L \cap \mathfrak{R} = \mathcal{C}_2 =$ largest countable Σ^1_2 set, the fact that $\mathfrak{B}_2 = \{\alpha\colon \exists\beta \in \mathcal{C}_2(\alpha \leqslant \beta)\}$ suggests the following

CONJECTURE. Assuming PD, we have for each $n \geqslant 1$

$$\mathfrak{B}_{2n} = \{\alpha\colon \exists\beta \in \mathcal{C}_{2n}(\alpha \leqslant \beta)\}.$$

We also do not know if there is any relationship of the above type between $\mathcal{C}_{2n+1}$ and $\mathfrak{B}_{2n+1}$ for $n \geqslant 0$. Since $\alpha \in \mathfrak{B}_2 \Leftrightarrow \exists\beta \in \Delta^1_2(\alpha)\,(\beta \in \mathcal{C}_2\ \&\ \alpha \leqslant \beta)$ one might wonder about the validity of the following formula (granting PD for $n \geqslant 1$)

$$\alpha \in \mathfrak{B}_{2n+1} \Leftrightarrow \exists\beta \in \Delta^1_{2n+1}(\alpha)\,(\beta \in \mathcal{C}_{2n+1}\ \&\ \alpha \leqslant \beta).$$

This is however an open problem.

6. **Generalizations; the games $G^{\mathfrak{S}}(A)$.** In this last section we shall present a generalization of the *- and **-games (see [My]) as well as the $\sim$-games considered in the present paper. The generalized games we have in mind give a corresponding notion of smallness for sets of reals whose special cases are countability, being of the first category and σ-boundedness. We shall also see how the main results of the previous sections generalize to this wider context.

Let X be an arbitrary set having more than one element ($X = \{0, 1\}$ or $X = \omega$ are essentially the only cases we are interested in here). Let also R be an arbitrary nonempty set, whose elements we shall call *requirements* and let S be a function which assigns to each $r \in R$ a nonempty set of nonempty finite sequences from X, i.e. $S\colon R \to$ power $(X_*^{<\omega}) - \{\varnothing\}$, where $X_*^{<\omega} =^{\text{def}} \bigcup_{n\geqslant 1} X^n$. If $u \in S(r)$ we shall say that *u satisfies the requirement r* and we shall write for convenience $u \prec_S r$ in this case. We shall assume that $u \prec_S r$ satisfies the following conditions (of which the second is essentially a nontriviality condition that will assure us later that in the notion of smallness generated by $\langle R, S\rangle$ singletons are small):

(1) $v \prec u\ \&\ u \prec_S r \Rightarrow v \prec_S r$ (i.e. any extension of a sequence that satisfies r satisfies r also),

(2) $\forall x \in X\,\exists r \in R\,\forall u \in X_*^{<\omega}\,(u \prec_S r \Rightarrow u(0) \neq x)$ (i.e. for every $x \in X$ there is a requirement which forces the first member of any sequence satisfying it to be different than x).

(3) There is $l\colon R \to \omega$ such that if $r \in R$, $u \in X_*^{<\omega}$ and $\exists u' \leqslant u(u' \prec_S r)$ but $u \nprec_S r$, then there is $q \in R$ with $l(q) < l(r)$ such that $u\hat{\ }v \prec_S r$ iff $v \prec_S q$.

EXAMPLES. (E1) $X = \{0, 1\}$, $R = \{0, 1\}$, $u \prec_S r \Leftrightarrow u(0) = r$ (here $u(0) =$ first member of u). $l(r) \equiv 0$.

(E2) X arbitrary, $R = X^{<\omega}$, $u \prec_S r \Leftrightarrow u \prec r$. $l(r) = \text{length}(r)$.

(E3) $X = \omega$, $R = \omega$, $u \prec_S r \Leftrightarrow u(0) > r$. $l(r) \equiv 0$.

Suppose now a pair $\mathfrak{S} = \langle R, S\rangle$ satisfying the above conditions is given. Abusing language we shall call $\mathfrak{S}$ a *requirement set*. To each $A \subseteq X^\omega$ we associate the following game $G^{\mathfrak{S}}(A)$:

I	II	
u_0		I plays a finite sequence u_0 from X,
	r_1	II plays $r_1 \in R$, I plays a nonempty
u_1		finite sequence u_1 from X, II plays
	r_2	$r_2 \in R$, I plays a nonempty finite
u_2		sequence u_2 from X, etc.
$\vdots$		

Let $f = u_0\hat{}\, u_1\hat{}\, u_2\hat{} \cdots \in X^\omega$. Then I wins iff (i) $f \in A$ and (ii) $\forall i \geqslant 1$, $u_i \prec_S r_i$. Otherwise II wins.

EXAMPLES. In Example E1 before, clearly $G^{\mathfrak{S}}(A)$ is equivalent to the game $G^*(A)$. In E2, $G^{\mathfrak{S}}(A)$ is equivalent to $G^{**}(A)$. In E3, $G^{\mathfrak{S}}(A)$ is equivalent to $\tilde{G}(A)$.

DEFINITION. Let $A \subseteq X^\omega$ be a closed set. We call A *$\mathfrak{S}$-nowhere dense* if for each $u \in T_A$ there is a requirement $r \in R$ such that for all $v \in X_*^{<\omega}$, $u\hat{}v \in T_A \Rightarrow u \not\prec_S r$. We call a set $A \subseteq X^\omega$ *$\mathfrak{S}$-meager* if it is contained in a countable union of closed $\mathfrak{S}$-nowhere dense sets.

EXAMPLES. In E1, $A \subseteq 2^\omega$ is closed $\mathfrak{S}$-nowhere dense iff A is a singleton. Thus $A \subseteq 2^\omega$ is $\mathfrak{S}$-meager iff A is countable. In E2, $A \subseteq X^\omega$ is closed $\mathfrak{S}$-nowhere dense iff A is closed nowhere dense. Thus $A \subseteq X^\omega$ is $\mathfrak{S}$-meager iff A is meager. Finally in E3, $A \subseteq \omega^\omega$ is closed $\mathfrak{S}$-nowhere dense iff A is compact and so $A \subseteq \omega^\omega$ is $\mathfrak{S}$-meager iff A is σ-bounded.

Clearly "$\mathfrak{S}$-meager" is the notion of smallness associated with $\mathfrak{S}$. Notice that because of condition (ii) in the definition of a requirement set every singleton in $\mathfrak{S}$-nowhere dense. So every countable set is $\mathfrak{S}$-meager. Also every $\mathfrak{S}$-nowhere dense closed set is nowhere dense and so any $\mathfrak{S}$-meager set is meager. We shall now define a notion of "extreme largeness" opposite to the notion of "$\mathfrak{S}$-meager".

DEFINITION. Let J be a tree on $X^{<\omega} = \bigcup_n X^n =$ the set of all finite sequences from X. We shall call J *$\mathfrak{S}$-perfect* if the following two conditions are met:

(i) For each $\phi \neq p \in J$, $\{u \in X_*^{<\omega} : p\hat{}(u) \in J\}$ is $\mathfrak{S}$-dense i.e. for each $r \in R$ there is a $u \in X_*^{<\omega}$ and $u' \preccurlyeq u$ such that $p\hat{}(u) \in J$ and $u' \prec_S r$.

(ii) For each $p \in J$ if $p\hat{}(u) \in J$, $p\hat{}(v) \in J$ and $u \neq v$ then u, v are incompatible.

Now for each $\mathfrak{S}$-perfect tree J we let $[J] = \{f \in X^\omega$: for all n there is $s_0, s_1, \ldots, s_n$ such that $(s_0, s_1, \ldots, s_n) \in J$ and $s_0\hat{}s_1\hat{} \cdots s_n\hat{}$ is an initial segment of $f\}$(2). (Note here that $s_0, s_1, \ldots, s_n$ must be necessarily unique.) We call a set $A \subseteq X^\omega$ *$\mathfrak{S}$-perfect* if $A = [J]$ for some $\mathfrak{S}$-perfect tree J.

It is now easy to see that every $\mathfrak{S}$-perfect set is a G_δ in the space X^ω. On the other hand if $\mathfrak{S}$ has the property that

$$u \prec_S r \Leftrightarrow (u(0)) \prec_S r$$

(i.e. satisfaction of a requirement depends only on the first member of a sequence), which is obviously the case in Examples E1 and E3 above, then every $\mathfrak{S}$-perfect set contains a *closed* $\mathfrak{S}$-perfect subset.

Examples. In E1 every nonempty $\mathfrak{S}$-perfect set contains a nonempty perfect subset. In E2 a nonempty $\mathfrak{S}$-perfect set is a dense in some nonempty neighborhood G_δ set. In E3 a nonempty $\mathfrak{S}$-perfect set contains a nonempty superperfect subset.

We now have the analog of Theorem 3.1, provided X is countable and also R is countable, in which case (abusing language again) we shall say that "$\mathfrak{S}$ is countable". It reads as follows: Let $\mathfrak{S}$ be a countable requirement set on a countable set X and $A \subseteq X^\omega$. Then I has a winning strategy in $G^{\mathfrak{S}}(A)$ iff A contains a nonempty $\mathfrak{S}$-perfect set and II has a winning strategy in $G^{\mathfrak{S}}(A)$ iff A is $\mathfrak{S}$-meager. (Again the only nontrivial thing to prove is that if II has a winning strategy $\mathfrak{T}$ then A is $\mathfrak{S}$-meager. For that call a sequence $p = (u_0, r_1, u_1, \ldots, u_n, r_{n+1})$, where $u_0 \in X^{<\omega}$, $u_i \in X_*^{<\omega}$ for $i > 0$ and $r_i \in R$, *good* if for all $i \geq 1$, $u_i \prec_S r_i$ and the r_i are determined following $\mathfrak{T}$. By convention the empty sequence is good. Given $f \in A$ call *p good for f* if there is u_{n+1} such that $u_0\hat{}u_1\hat{} \cdots u_n\hat{}u_{n+1}\hat{}$ is an initial segment of f and $u_{n+1} \prec_S r_{n+1}$. Thus the empty sequence is automatically good for A. Now there must be some p (as above) good for f which has no proper extension good for f. Then $f \in K_p = \{f'$: p is good for f' but for no (u_{n+1}, r_{n+2}) such that $p\hat{}(u_{n+1}, r_{n+2}) = q$ is good, q is also good for $f'\}$. Clearly K_p is a $\mathfrak{S}$-meager set. Since there are only countable many K_p's and A is contained in their union, A must be $\mathfrak{S}$-meager.) We can also get immediately the generalization of Corollary 3.2.

The next step is to consider the analog of Theorem 3.3. This is completely straightforward, using the ideas in the proof of the theorem in §1, after one defines the game $G_p^{\mathfrak{S}}(B)$ for $B \subseteq X^\omega \times X^\omega$ in the obvious fashion. From that the definability results of §4 follow immediately. For simplicity take $X \subseteq \omega$ to be a recursive set now and assume that both R, $\prec_S$ are recursive (coding them appropriately if necessary as in the case of E2 for instance). Then, for example,

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(2) This use of [J] is slightly different from the one in 1C. Which one we use will be clear from the context and the convention that for the rest of this paper J's will be used for trees on $X^{<\omega}$ while T's will be used for trees on X.

the analog of part (i) of the theorem in §4 reads: If $A \subseteq X^{\omega}$ is Σ^1_1 then either A contains a nonempty $\mathfrak{S}$-perfect set or $A \subseteq \bigcup_n [T_n]$ where each $[T_n] = F_n$ is closed $\mathfrak{S}$-nowhere dense and $P(s, n) \Leftrightarrow s \in T_n$ is Δ^1_1. Also (iv) reads: If A is Σ^1_2 then either A contains a nonempty $\mathfrak{S}$-perfect set with tree in L or every $\alpha \in A$ is contained in some closed $\mathfrak{S}$-nowhere dense set with tree in L.

We now proceed to consider the generalizations to the present context of the results in §5.

Definition. Let $X \subseteq \omega$ and $\mathfrak{S}$ be a countable requirement set on X. A set $A \subseteq X^{\omega}$ is called *$\mathfrak{S}$-thin* if it contains no nonempty $\mathfrak{S}$-perfect subset.

Again if $G^{\mathfrak{S}}(A)$ is determined, A is $\mathfrak{S}$-thin iff A is $\mathfrak{S}$-meager. Lemmas 5.1 and 5.2 go again through easily. (For example consider Lemma 5.1. Following the pattern of its proof as well as its notation assume $\{A_{\xi}\}_{\xi<\theta}$ is a sequence of $\mathfrak{S}$-thin sets such that $f \leqslant g \Leftrightarrow \mu\xi\,(f \in A_{\xi}) \leqslant \mu\xi\,(g \in A_{\xi})$ is in Γ (here f, g vary over X^{ω}) and consider for simplicity the first case where $\theta = \eta + 1$. We can conclude exactly as before that $A = \bigcup_{\xi \leqslant \eta} A_{\xi}$ is $\mathfrak{S}$-thin by induction hypothesis. Put $B = A_{\eta}$. If $A \cup B$ was not $\mathfrak{S}$-thin let $P = [J] \subseteq A \cup B$ be an $\mathfrak{S}$-perfect set. It will be enough to show that $A \cap P$, $B \cap P$ are meager in P (then P would be meager in P, which contradicts the fact that P being a G_{δ} of X^{ω} is topologically complete (see [**O**, p. 47]) so satisfies Baire's Theorem). Take A for example. Since $A \cap P$ has the property of Baire in P, if it is not meager in P it must be comeager in some neighborhood $N_u \cap P$ (where $u \in X^{<\omega}$) of P. Then $A \cap P$ contains a dense in $N_u \cap P$ G_{δ} subset D of P. But then I has a winning strategy in the game $G^{\mathfrak{S}}(D)$, so I has a winning strategy in $G^{\mathfrak{S}}(A)$ which contradicts the assumption that A is $\mathfrak{S}$-thin.) Using the generalizations of Lemmas 5.1 and 5.2 and assuming again that X, R, $\prec_S$ are recursive we can show immediately the existence of a largest $\mathfrak{S}$-thin Π^1_1 set and assuming PD, we can also show the existence of largest $\mathfrak{S}$-thin (or equivalently $\mathfrak{S}$-meager) Π^1_n or Σ^1_n sets, according as n is odd or even. We shall denote these sets by $\mathfrak{M}^{\mathfrak{S}}_n$. Clearly $\mathcal{C}_n \subseteq \mathfrak{M}^{\mathfrak{S}}_n \subseteq \mathfrak{M}_n$ for every $\mathfrak{S}$. Finally letting for each M and each $f \in X^{\omega}$,

$$f \mathfrak{S} M \Leftrightarrow \exists T \in M([T] \text{ is } \mathfrak{S}\text{-nowhere dense } \& f \in [T])$$

it is straightforward to check that

$$\mathfrak{M}^{\mathfrak{S}}_2 = \{f \in X^{\omega} : f \mathfrak{S} L\} \quad \text{and} \quad \mathfrak{M}^{\mathfrak{S}}_1 = \{f \in X^{\omega} : f \mathfrak{S} L_{\omega_1^f}\}.$$

Finally letting $f \leqslant^{\mathfrak{S}}_n g \Leftrightarrow \exists T \in \Delta^1_n(g)\,(f \in [T] \,\&\, [T]$ is $\mathfrak{S}$-nowhere dense) we have $f \in \mathfrak{M}^{\mathfrak{S}}_1 \Leftrightarrow \forall g\,(\omega_1^f \leqslant \omega_1^g \Rightarrow f \leqslant^{\mathfrak{S}}_1 g)$.

Added in proof. St. Raymond has independently proved the part of Corollary 3.2(i) which states that a Σ^1_1 set of reals is either σ-bounded or contains a nonempty superperfect set; see J. St. Raymond, *Un théorème d'approximation par l'intérieur*, C. R. Acad. Sci. Paris (Juin 1975). Louveau (*σ-ideaux engendres par des ensembles fermes et théorèmes d'approximation* (to appear) has obtained generalizations of the results in §§2–5 of our paper along directions different than those of §6. We would like to thank A. Louveau for pointing out a number of errors in a preprint of the present paper.

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