TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 229, 1977

QUASI-ANOSOV DIFFEOMORPHISMS AND HYPERBOLIC MANIFOLDS

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ABSTRACT. Let f be a diffeomorphism of a smooth manifold N and $M \subset N$ a compact boundaryless submanifold such that it is a hyperbolic set for f. The diffeomorphism f/M is characterized and it is proved that it is Anosov if and only if M is an invariant isolated set of f (i.e. the maximal invariant subset of some compact neighborhood). Isomorphisms of vector bundles with the property that the zero section is an isolated subset are studied proving that they can be embedded in hyperbolic vector bundle isomorphisms.

Let N be a closed C^∞ manifold $f: N \hookleftarrow$ a C^1 diffeomorphism and $M \subset N$ a hyperbolic set for f [1]. When M is a closed C^1 submanifold we say that M is a hyperbolic manifold. In [2] Hirsch posed the following question: If M is a hyperbolic manifold for f, is f/M an Anosov diffeomorphism? To study this problem we shall give a characterization of the diffeomorphisms that can arise as a restriction to a hyperbolic manifold. In Theorem A below we shall prove that these diffeomorphisms are those given by the following definition:

DEFINITION. A C^1 diffeomorphism of a closed C^∞ manifold M is quasi-Anosov if for all $0 \neq v \in TM$ the set $\{\|(Tf)^n v\| \mid n \in \mathbf{Z}\}$ is unbounded.

In the statement of the next theorem we shall use the following notation: M will be a closed C^∞ manifold, $\mathrm{Diff}^r(M)$ the set of C^r diffeomorphisms of M, $r \geq 1$, $\Gamma^0(TM)$ the Banach space of continuous sections of TM with the norm $\|\eta\| = \sup\{\|\eta(x)\| \mid x \in M\}$, and $f_*: \Gamma^0(TM) \hookleftarrow$ the isomorphism defined by:

$$f_*(\eta) = Tf \circ \eta \circ f^{-1}.$$

Moreover, for $x \in M$ and $f \in \mathrm{Diff}^r(M)$, the stable and unstable sets of x are defined as:

$$W^s(x) = \{y \in M | d(f^n(x) \cdot f^n(y)) \to 0 \text{ when } n \to +\infty\},$$

$$W^u(x) = \{y \in M | d(f^n(x) \cdot f^n(y)) \to 0 \text{ when } n \to -\infty\}$$

where $d(\cdot, \cdot)$ is the Riemannian distance on M. Recall that when f satisfies Axiom A, $W^s(x)$ and $W^u(x)$ are immersed submanifolds [1].

Received by the editors December 15, 1975.
AMS (MOS) subject classifications (1970). Primary 58F15; Secondary 58F10.
Key words and phrases. Hyperbolic set, Anosov diffeomorphism, stability.

THEOREM A. *For* $f \in \mathrm{Diff}^r(M)$ *the following conditions are equivalent*:

(a) f *is quasi-Anosov.*

(b) f *satisfies Axiom* A *and for all* $x \in M$

$$T_x W^s(x) \cap T_x W^u(x) = \{0\}.$$

(c) *There exist a closed* C^∞ *manifold* N, *a* C^∞ *embedding* $i: M \to N$ *and* $g \in \mathrm{Diff}^r(N)$ *such that* $g \circ i = i \circ f$ *and* $i(M)$ *is a hyperbolic manifold for* g.

(d) *If* I *is the identity map on* $\Gamma^0(TM)$, $I - f_*$ *is injective and has closed range.*

REMARK. In [3] is proved that the following condition is also equivalent to (a):

(e) There exists a neighborhood $\mathfrak{U}$ of f such that g is expansive for all $g \in \mathfrak{U}$.

The equivalence between (a) and (c) proves that the question posed at the beginning is equivalent to the following one: Does quasi-Anosov imply Anosov? This question has a negative answer as shown by Franks and Robinson in [4]. They exhibited a quasi-Anosov, non-Anosov diffeomorphism of a 3-dimensional manifold. Obviously (b) proves that in 2-dimensional manifolds all quasi-Anosovs are Anosov. However condition (b) proves that when the nonwandering set of f is all of M a quasi-Anosov is Anosov, a result also proved by Selgrade [5] and Sacker and Sell [6].

COROLLARY 1. *The following conditions are equivalent*:

(a) f *is Anosov.*

(b) f *is quasi-Anosov and structurally stable.*

(c) f *is quasi-Anosov and* $W^s(x)$ *has the same dimension for all periodic point* x.

If $K \subset N$ is an invariant set for $f \in \mathrm{Diff}^1(N)$ we say that K is isolated if there exists a compact neighborhood U of K such that $\bigcap_{n \in \mathbf{Z}} f^n(U) = K$.

COROLLARY 2. *If* M *is a hyperbolic manifold for* $f \in \mathrm{Diff}^1(N)$, f/M *is Anosov if and only if* M *is isolated.*

A related and still unsolved problem is whether the restriction of an Anosov diffeomorphism to an invariant compact submanifold is Anosov. Partial results are given in [7] and [8].

The theorem and its corollaries will be proved in §2. §1 is devoted to quasi-Anosov isomorphisms of vector bundles. We show that any quasi-Anosov vector bundle can be embedded in a hyperbolic one (see §1 for definitions). This property is applied to the proof of the (a) $\Rightarrow$ (c) part of the theorem. For related results see [6].

This paper is essentially contained in the author's doctoral thesis at the IMPA under the guidance of J. Palis. I wish to thank Professor Palis for

many helpful conversations and general perspective and orientation during the preparation of this material.

1. Let K be a compact metric space. We shall denote by $\mathcal{E}(K)$ the set of all finite dimensional continuous vector bundles on K with a Finsler structure. If $F \in \mathcal{E}(K)$, F_x will be the fiber of F over x. We shall always consider K identified with the zero section of F, and $\mathcal{E}(F)$ endowed with the topology defined by the metric

$$d(\Phi_1, \Phi_2) = \sup\{d_0(\Phi_1(v), \Phi_2(v)) \mid \|v\| \leq 1\}$$

where $d_0(\cdot, \cdot)$ is a metric inducing the topology of F.

DEFINITION 1. $\Phi \in \mathcal{E}(F)$ is hyperbolic if there exist continuous subbundles F^s, F^u of F and constants $K > 0$, $0 < \lambda < 1$, such that

(a) $F^s \oplus F^u = F$,

(b) $\Phi(F^s) = F^s$, $\Phi(F^u) = F^u$,

(c) $\|\Phi^n/F^s_x\| \leq K\lambda^n$, $\|\Phi^{-n}/F^u_x\| \leq K\lambda^n$ for all $x \in K$, $n \in \mathbf{Z}^+$.

DEFINITION 2. If $\Phi \in \mathcal{E}(F)$, the stable subspace of Φ at x, denoted $E^s_x(\Phi)$, is the set of vectors $v \in F_x$ such that the set $\{\|\Phi^n(v)\| \mid n \in \mathbf{Z}^+\}$ is bounded. The unstable subspace of Φ at x, denoted $E^u_x(\Phi)$, is the set of vectors $v \in F_x$ such that the set $\{\|\Phi^{-n}(v)\| \mid n \in \mathbf{Z}^+\}$ is bounded.

DEFINITION 3. $\Phi \in \mathcal{E}(F)$ is a quasi-Anosov vector bundle isomorphism if $E^s_x(\Phi) \cap E^u_x(\Phi) = \{0\}$ for all $x \in K$.

In this section we shall prove the following results:

PROPOSITION 1.1. *If $\Phi_0 \in \mathcal{E}(F)$ is a quasi-Anosov vector bundle isomorphism, there exist a neighborhood $\mathcal{U}$ of Φ_0 in $\mathcal{E}(F)$ and constants $K > 0$, $0 < \lambda < 1$, such that for all $x \in K$, $n \in \mathbf{Z}^+$ and $\Phi \in \mathcal{U}$*

$$\|\Phi^n/E^s_x(\Phi)\| \leq K\lambda^n, \tag{1}$$

$$\|\Phi^{-n}/E^u_x(\Phi)\| \leq K\lambda^n. \tag{2}$$

COROLLARY 1.2. *$\Phi \in \mathcal{E}(F)$ is hyperbolic if and only if:*

$$F_x = E^s_x(\Phi) \oplus E^u_x(\Phi)$$

for all $x \in K$.

Let $L^+(\Phi)$ be the set of ω-limit points of Φ/K [10] and $\bar{L}^+(\Phi)$ its closure.

COROLLARY 1.3. *If $\Phi \in \mathcal{E}(F)$ is quasi-Anosov $\Phi/(F/\bar{L}^+(\Phi))$ is hyperbolic.*

PROPOSITION 1.4. *Let $\Phi \in \mathcal{E}(F)$ and $\Lambda \subset K$ a compact invariant (i.e. $\Phi \in \mathcal{E}(F_0 \oplus F_1)$ such that it is hyperbolic and $\Phi/F_0 = \Phi_0$. Moreover, given $\varepsilon > 0$, we can find Φ satisfying $\|(\pi_1 \circ \Phi)/F_{1,x}\| \leq \varepsilon$ for all $x \in K$, where π_1: $F_0 \oplus F_1 \to F_1$ is the canonical projection.*

Proposition 1.5. *If $\Phi_0 \in \mathfrak{L}(F_0)$ is quasi-Anosov there exists $F_1 \in \mathfrak{L}(K)$ and $\Phi \in \mathfrak{L}(F_0 \oplus F_1)$ such that is hyperbolic and $\Phi/F_0 = \Phi_0$. Moreover, given $\varepsilon > 0$, we can find Φ satisfying $\|(\pi_1 \circ \Phi)/F_{1,x}\| \leqslant \varepsilon$ for all $x \in K$, where π_1: $F_0 \oplus F_1 \to F_1$ is the canonical projection.*

For the proof of these proposition we shall need the following lemmas.

Lemma 1.6. *The set of quasi-Anosov isomorphisms is open.*

Proof. If $F \in \mathfrak{L}(K)$ and $\Phi \in \mathfrak{L}(F)$ is quasi-Anosov there exists $n_0 \in \mathbf{Z}^+$ such that for all $v \in F$ there exists $-n_0 \leqslant n \leqslant n_0$ such that $\|\Phi^n(v)\| \geqslant 3\|v\|$. Hence there exists a neighborhood $\mathfrak{U}$ of Φ in $\mathfrak{L}(F)$ such that if $\Phi_1 \in \mathfrak{U}$ and $v \in F$ there exists $-n_0 \leqslant n \leqslant n_0$ satisfying $\|\Phi_1^n(v)\| \geqslant 2\|v\|$. Applying this property to $\Phi_1(v)$ it follows that for all $m \in \mathbf{Z}^+$ there exists $-mn_0 \leqslant N \leqslant mn_0$ such that $\|\Phi_1^N(v)\| \geqslant 2^m\|v\|$.

Lemma 1.7. *If $\Phi \in \mathfrak{L}(F)$ is quasi-Anosov and $v \in E_x^s(\Phi)$,*

$$\liminf_{n\to+\infty}\|\Phi^n(v)\| = 0.$$

Proof. If $\liminf_{n\to+\infty}\|\Phi^n(v)\| \neq 0$ there exists $c > 0$ satisfying $\|\Phi^n(v)\| \geqslant c\|v\|$ for $n \geqslant 0$. Moreover $v \in E_x^s(\Phi)$ implies that for some $K > 0$ we have $\|\Phi^n(v)\| \leqslant K\|v\|$ for all $n \in \mathbf{Z}^+$. Take a sequence of positive integers $\{n_j | j \in \mathbf{Z}^+\}$ such that $n_j \to +\infty$ when $j \to +\infty$ and $0 \neq w = \lim_{j\to+\infty}\Phi^{n_j}(v)$ exists. Then for all $n \in \mathbf{Z}$:

$$\Phi^n(w) = \lim_{j\to+\infty} \Phi^{n+n_j}(v).$$

Hence $\|\Phi^n(w)\| \leqslant K\|v\|$ for all $n \in \mathbf{Z}$ contradicting the definition of quasi-Anosov.

Lemma 1.8. *If $\Phi \in \mathfrak{L}(F)$ is quasi-Anosov there exist a neighborhood $\mathfrak{U}$ of Φ and a constant $K > 0$ that for all $\Phi_1 \in \mathfrak{U}$, $v \in F$ and integers $0 \leqslant n \leqslant m$ satisfy:*

$$\|\Phi_1^n(v)\| \leqslant K\big(\|v\| + \|\Phi_1^m(v)\|\big).$$

Proof. As we observed in the proof of Lemma 1.3 we can find a neighborhood $\mathfrak{U}$ of Φ and an integer $n_0 \in \mathbf{Z}^+$ such that for all $w \in F$, $\Phi_1 \in \mathfrak{U}$ there exist $-n_0 \leqslant n \leqslant n_0$ satisfying $\|\Phi_1^n(w)\| \geqslant 2\|w\|$. Hence if $v \in F$, $\Phi_1 \in \mathfrak{U}$, $m \in \mathbf{Z}^+$ and we take $0 \leqslant j_0 \leqslant m$ defined by:

$$\|\Phi_1^{j_0}(v)\| = \sup\{\|\Phi_1^j(v)\| \,|0 \leqslant j \leqslant m\}$$

it follows that $\|\Phi_1^n(\Phi_1^{j_0}(v))\| \leqslant \|\Phi_1^{j_0}(v)\|$ for all $-j_0 \leqslant n \leqslant m - j_0$. Then $j_0 < n_0$ or $m - j_0 < n_0$. The definition of j_0 implies that for all $0 \leqslant n \leqslant m$

$$\|\Phi_1^n(v)\| \leqslant \|\Phi_1^{j_0}(v)\|.$$

Then if $j_0 < n_0$

$$\|\Phi_1^n(v)\| \leqslant \left(\sup\left\{\|\Psi^j/F_x\| \,|\Psi \in \mathcal{U}, x \in K, 0 \leqslant j \leqslant n_0\right\}\right)\|v\|$$

and if $m - j_0 < n_0$

$$\|\Phi_1^n(v)\| \leqslant \left\|\Phi_1^{-(m-j_0)}(\Phi_1^m(v))\right\|$$
$$\leqslant \sup\left\{\|\Psi^{-j}/F_x\| \,|\Psi \in \mathcal{U}, x \in K, 0 \leqslant j \leqslant n_0\right\}\|\Phi_1^m(v)\|.$$

Taking

$$K = \sup\left\{\|\Psi^j/F_x\| \,|\Psi \in \mathcal{U}, x \in K, -n_0 \leqslant j \leqslant n_0\right\}$$

the lemma is proved.

Proof of Proposition 1.1. Let $\mathcal{U}$ and K be given by Lemma 1.8. We shall prove only property (1) of the proposition. Property (2) follows in a similar way. Observe that (1) is equivalent to the existence of $n_0 \in \mathbf{Z}^+$ such that

$$\|\Phi_1^{n_0}/E_x^s(\Phi_1)\| \leqslant \tfrac{1}{2}$$

for all $\Phi_1 \in \mathcal{U}$, $x \in K$. To find n_0, take $N \in \mathbf{Z}^+$ such that for all $v \in F$, $\Phi_1 \in \mathcal{U}$ (taking a smaller $\mathcal{U}$ if necessary)

$$\sup\left\{\|\Phi_1^j(v)\| \,|-N \leqslant j \leqslant N\right\} \geqslant 10K^2\|v\|. \tag{3}$$

This N exists because Φ is quasi-Anosov. Now observe that $\Phi_1 \in \mathcal{U}$, by Lemma 1.7, for all $n \in \mathbf{Z}^+$, there exists $m_n \geqslant n$ such that $\|\Phi_1^{m_n}(v)\| \leqslant \|v\|$. Hence, by Lemma 1.8:

$$\|\Phi_1^n(v)\| \leqslant 2K\|v\|. \tag{4}$$

Define $n_0 = 2N + 1$. If there exists $\Phi_1 \in \mathcal{U}$, $x \in K$, $0 \neq v \in E_x^s(\Phi_1)$ satisfying $\|\Phi_1^{n_0}(v)\| \geqslant \frac{1}{2}\|v\|$, by (4) we have for all $0 \leqslant j \leqslant n_0$

$$\|\Phi_1^{n_0}(v)\| = \left\|\Phi_1^{n_0-j}(\Phi_1^j(v))\right\| \leqslant 2K\|\Phi_1^j(v)\|.$$

Hence:

$$\|v\|/4K \leqslant \|\Phi_1^j(v)\| \tag{5}$$

for all $0 \leqslant j \leqslant n_0$. Then if $-N \leqslant k$ by (4)

$$\left\|\Phi_1^k(\Phi_1^N(v))\right\| \leqslant 2K\|v\|$$

and by (5)

$$\left\|\Phi_1^k(\Phi_1^N(v))\right\| \leqslant 2K\|v\| \leqslant 8K^2\|\Phi_1^N(v)\|$$

contradicting (3).

Proof of Corollary 1.2. Define F^s and F^u as the subbundle with fibers $F_x^s = E_x^s(\Phi)$, $F_x^u = E_x^u(\Phi)$. If we prove that these subbundles are continuous we are done because the inequalities of Definition 1 will follow from Proposition 1.1. To prove the continuity consider a sequence $\{x_n | n \in \mathbf{Z}^+\} \subset K$, with $x_n \to x$ when $n \to +\infty$ and such that there exist subspaces E^s and E^u of F_x

satisfying $E^s_{x_n}(\Phi) \to E^s$, $E^u_{x_n}(\Phi) \to E^u$ when $n \to +\infty$. Then, applying Lemma 1.6

$$\|\Phi^n/E^s\| \leq K_1\lambda^n, \qquad \|\Phi^{-n}/E^u\| \leq K_1\lambda^n$$

for all $n \in \mathbf{Z}^+$. Therefore

$$E^s_x \subset E^s_x(\Phi), \qquad E^u_x \subset E^u_x(\Phi)$$

but:

$$\begin{aligned}\dim E^s_x + \dim E^u_x &= \dim E^s_{x_n}(\Phi) + \dim E^u_{x_n}(\Phi)\\ &= \dim E^s_{x_n}(\Phi) \oplus E^u_{x_n}(\Phi) = \dim F_x;\end{aligned}$$

hence $E^s_x(\Phi) = E^s_x$, $E^u_x(\Phi) = E^u_x$.

Proof of Corollary 1.3. Let $y \in \overline{L^+(\Phi)}$. By Corollary 1.2 it is sufficient to prove $E^s_y(\Phi) \oplus E^u_y(\Phi) = F_y$. Let $\{x_n | n \in \mathbf{Z}^+\}$ be a sequence in $L^+(\Phi)$ such that $x_n \to y$ and $E^s_{x_n}(\Phi)$, $E^u_{x_n}(\Phi)$ converge to subspaces F^s, F^u of F_y. By Proposition 1.1 $F^s \subset E^s_y(\Phi)$, $F^u \subset E^u_y(\Phi)$. Moreover $\dim F^s = \dim E^s_{x_n}(\Phi)$, $\dim F^u = \dim E^u_{x_n}(\Phi)$. Hence if we prove $E^s_y(\Phi) \oplus E^u_y(\Phi) = F_y$ for all $y \in L^+(\Phi)$ we are done. Let $x \in K$ and $y \in \omega(x)$. Let $\{n_k | k \in \mathbf{Z}^+\}$ be a sequence of positive integers, $\lim_{k\to+\infty} n_k = +\infty$, $\lim_{k\to+\infty} \Phi^{n_k}(x) = y$. Let $x_k = \Phi^{n_k}(x)$. Let $\hat{E}^u$ be a subspace of F_x satisfying:

(1) $$\hat{E}^u \supset E^u_x(\Phi),$$

(2) $$\hat{E}^u \oplus E^s_x(\Phi) = F_x.$$

We claim that there exists $C > 0$ such that for all $n_1 > n_2 > 0$ we have:

(3) $$\|\Phi^{-n_2}/\Phi^{n_1}(\hat{E}^u)\| < C.$$

If this is false for all $C > 0$ there exist $n_1 = n_1(C) > n_2 = n_2(C) > 0$ and $v = v(C) \in \Phi^{n_1}(\hat{E}^u)$ such that:

(4) $$\|\Phi^{-n_2}(v)\| > C.$$

By Lemma 1.8

(5) $$\|\Phi^{-n_2}(v)\| = \|\Phi^{n_1-n_2}(\Phi^{-n_1}(v))\| \leq (\|\Phi^{-n_1}(v)\| + \|v\|);$$

hence:

(6) $$\|\Phi^{-n_1}(v)\| \geq \|\Phi^{-n_2}(v)\|/K - 1 \geq C/K - 1.$$

Applying Lemma 1.8 again, if $0 < n < n_1 - n_2$

$$\|\Phi^n(\Phi^{-n_1}(v))\| \leq K(\|\Phi^{-n_1}(v)\| + \|\Phi^{-n_2}(v)\|)$$

by (5) and (6)

$$\begin{aligned} \|\Phi^n(\Phi^{-n_1}(v))\| &\leq (K+K^2)\|\Phi^{-n_1}(v)\| + K^2 \\ &\leq \big((K+K^2) + K^2/(C/K-1)\big)\|\Phi^{-n_1}(v)\| \end{aligned} \tag{7}$$

for all $0 < n < n_1 - n_2$.

For some positive sequence $\{C_j | j \in \mathbf{Z}^+\}$ such that $\lim_{j\to+\infty} C_j = 0$ we can suppose that if $v_j = v(C_j)$, $u_j = \Phi^{-n_1(C_j)}(v(C_j))$ the sequence $u_j/\|u_j\|$ converges to $u \in \hat{E}^u$ and

$$\lim_{j\to+\infty} \big(n_1(C_j) - n_2(C_j)\big) = +\infty.$$

By (7) $\{\|\Phi^n(u)\| \, | n \in \mathbf{Z}^+\}$ is bounded thus contradicting (2) and proving the claim. Now suppose that the sequences $\Phi^{n_k}(E^s_x(\Phi))$ and $\Phi^{n_k}(\hat{E}^u)$ converge to subspaces G^s and G^u of F_y when $k \to +\infty$. By Proposition 1.1 and (3)

$$G^s \subset E^s_y(\Phi), \qquad G^u \subset E^u_y(\Phi);$$

hence

$$\begin{aligned} \dim E^s_y(\Phi) + \dim E^u_y(\Phi) &\geq \dim G^s + \dim G^u \\ &= \dim E^s_x(\Phi) + \dim \hat{E}^u = \dim F_x = \dim F_y. \end{aligned}$$

Proof of Proposition 1.4. Let $F/\Lambda = F^s \oplus F^u$ be the hyperbolic splitting of $\Phi/(F/\Lambda)$. Take a neighborhood U of Λ and continuous subbundles F_1, F_2 of F/U satisfying $F_1 \oplus F_2 = F/U$, $F_1/\Lambda = F^s$, $F_2/\Lambda = F^u$. Let π_i: $F/U \to F_i$ be the canonical projections associated to this splitting. For $x \in U$ let

$$C_\varepsilon(x) = \{v \in F_x | \; \|\pi_2 v\| \leq \varepsilon\|\pi_1 v\|\},$$

$$S_\varepsilon(x) = \{v \in F_x | \; \|\pi_1 v\| \leq \varepsilon\|\pi_2 v\|\}.$$

By [1, Lemma 3.7], we can suppose that there exists $0 < \lambda < 1$ satisfying $\|\Phi/F^s_x\| \leq \lambda$, $\|\Phi^{-1}/F^u_x\| \leq \lambda$ for all $x \in \Lambda$. It follows that if ε is small enough we can take $\lambda < \bar{\lambda} < 1$ such that for all $x \in \Lambda$, $v \in C_\varepsilon(x)$, $w \in S_\varepsilon(x)$ satisfies:

$$\|\Phi^n(v)\| \leq \bar{\lambda}^n\|v\|, \qquad \|\Phi^{-n}(w)\| \leq \bar{\lambda}^n\|w\|,$$

$$\Phi^{-1}(C_\varepsilon(x)) \subset C_{\bar{\lambda}^2\varepsilon}(\Phi^{-1}(x)), \qquad \Phi(S_\varepsilon(x)) \subset S_{\bar{\lambda}^2\varepsilon}(\Phi(x)).$$

By continuity we can take U such that there exists $\bar{\lambda} < \mu < 1$ with the following properties:

$$\|\Phi^n(v)\| \leq \mu^n\|v\|, \tag{1}$$

$$\|\Phi^{-n}(w)\| \leq \mu^n\|w\|, \tag{2}$$

$$\Phi^{-1}(C_\varepsilon(x)) \subset C_{\mu^2\varepsilon}(\Phi^{-1}(x)), \tag{3}$$

$$\Phi(S_\varepsilon(x)) \subset S_{\mu^2\varepsilon}(\Phi(x)), \tag{4}$$

for all $x \in U$, $v \in C_\varepsilon(x)$, $w \in S_\varepsilon(x)$. Let $\Lambda^s = \bigcap_{n \geq 0} \Phi^{-n}(U)$ and $F^{(n)} = \Phi^{-n}(F_1)/\Lambda^s$. By Tichonoff's theorem there exists a sequence $\{n_j | j \in \mathbf{Z}^+\} \subset \mathbf{Z}^+$ and a (possibly not continuous) subbundle $\tilde{F}_1$ of F/Λ^s such that for all $x \in \Lambda^s$:

$$\tilde{F}_{1,x} = \lim_{j \to +\infty} F_x^{(n)}$$

for all $x \in \Lambda^s$. Observe that pointwise:

$$\Phi(\tilde{F}_1) = \lim_{j \to +\infty} \Phi^{-n_j+1}(F_1)/\Lambda^s$$

but:

$$\Phi^{-n_j+1}(F_1)/\Lambda^s \subset \bigcup_{x \in \Lambda^s} C_\varepsilon(x);$$

hence:

$$\Phi(\tilde{F}_1) \subset \bigcup_{x \in \Lambda^s} C_\varepsilon(x)$$

and by (1):

$$\|\Phi^n(v)\| \leq \mu^n \|v\|$$

for all $v \in \tilde{F}_1$, $n \in \mathbf{Z}^+$. Therefore $E_x^s(\Phi) \subset \tilde{F}_{1,x}$ for all $x \in \Lambda^s$. If $0 \neq v \in E_x^s(\Phi)$ and $v \not\in \tilde{F}_{1,x}$ we can write $v = v_1 + v_2$ where $v_1 \in \tilde{F}_{1,x}$, $v_2 \in S_\varepsilon(x)$ and $v_2 \neq 0$. By (1) and (2)

$$\|\Phi^n(v)\| \geq \big|\, \|\Phi^n(v)\| - \|\Phi^n(v_1)\| \,\big| \geq (1/\mu)^n \|v_1\| - \mu^n \|v_2\|. \tag{5}$$

Since $v_2 \neq 0$ this implies $\|\Phi^n(v)\| \to +\infty$ when $n \to +\infty$ contradicting $v \in E_x^s(\Phi)$. Therefore for all $x \in \Lambda^s$

$$\tilde{F}_{1,x} = E_x^s(\Phi). \tag{6}$$

To prove the continuity of the map $x \to E_x^s(\Phi)$ in Λ^s consider a sequence $x_n \to x \in \Lambda^s$ when $n \to +\infty$, such that $E_{x_n}^s(\Phi)$ converges to a subspace $E \subset F_x$. By (5) and (6) it follows that $\|\Phi^n/E\| \leq \mu^n$ for all $n \in \mathbf{Z}^+$. Hence $E \subset E_x^s(\Phi)$. But $\dim E_x^s(\Phi) = \dim \tilde{F}_{1,x} = \dim \tilde{F}_{1,x_n} = \dim E$. Hence $E = E_s^x(\Phi)$.

For the proof of Proposition 1.5 we shall need the following lemma that proves the existence of certain invariant extension of the unstable subbundle.

LEMMA 1.9. *Let F, Λ and Φ be as in Proposition* 1.6. *Moreover let us suppose that there exists a neighborhood U_0 of Λ such that*

$$\bigcap_{n \in \mathbf{Z}} \Phi^n(U_0) = \Lambda \tag{0}$$

and a continuous subbundle G of $F/(\Lambda_0^s - \Lambda)$ where $\Lambda_0^s = \bigcap_{n \geq 0} \Phi^{-n}(U_0)$, with $\Phi(G) \subset G$, $G_x \oplus E_x^s(\Phi) = F_x$ for all $x \in \Lambda_0^s - \Lambda$ and satisfying the following property: If $\{x_n | n \in \mathbf{Z}^+\}$ is a sequence of $\Lambda_0^s - \Lambda$ such that $x_n \to x \in \Lambda$ when $n \to +\infty$ and G_{x_n} converges to a subspace $G \subset F_x$, then $G \subset E_x^u(\Phi)$. Then there exists a continuous subbundle $\tilde{E}^u$ of F/Λ_0^s such that $\tilde{E}_x^u = E_x^u(\Phi)$ for all $x \in \Lambda$, $\Phi(\tilde{E}^u) \subset \tilde{E}^u$, $\tilde{E}_x^u \oplus E_x^s(\Phi) = F_x$ for all $x \in \Lambda_0^s$ and $\tilde{E}_x^u \supset G_x$ for all $x \in \Lambda_0^s - \Lambda$.

PROOF. If $\Lambda \subset U \subset U_0$ is a compact neighborhood of Λ, it follows from (0) that $\Lambda = \bigcap_{n \in \mathbf{Z}} \Phi^n(U)$. Let $\Lambda^s = \bigcap_{n \geq 0} \Phi^{-n}(U)$. If we find a continuous subbundle E^u of F/Λ^s satisfying the desired properties for $x \in \Lambda^s$, then, observing that there exists $n_0 \in \mathbf{Z}^+$ such that $\Lambda_0^s \subset \Phi^{-n_0}(\Lambda^s)$, the subbundle defined in Λ^s by $E_x^u = \Phi^{-n_0}(E_y^u)$, $y = \Phi^{n_0}(x)$ satisfies the lemma. Take U, F_1, F_2, $S_\varepsilon(x)$, $C_\varepsilon(x)$ and μ as in the previous proof. Taking U small enough we can suppose $G_x \subset S_{\varepsilon/2}(x)$ for all $x \in \Lambda^s - \Lambda$. Let $N = \Lambda^s - \Phi(\Lambda^s)$ and $\bar{N}$ its closure. We leave to the reader the verification of the following relations:

$$\bigcup_{n \geq 0} \Phi(\bar{N}) = \Lambda^s - \Lambda, \tag{1}$$

$$\bar{N} \cap \Lambda = \varnothing, \tag{2}$$

$$\bar{N} \cap \Phi^j(\bar{N}) = \varnothing \quad \text{for } j \geq 2. \tag{3}$$

Now take a continuous subbundle $\tilde{E}^u$ of $F/(\bar{N} \cap \Phi^{-1}(\bar{N}))$ such that

$$S_{\varepsilon/2}(x) \supset \tilde{E}_x^u \supset G_x, \qquad \tilde{E}_x^u \oplus E_x^s(\Phi) = F_x \tag{4}$$

for all $x \in \bar{N} \cap \Phi^{-1}(\bar{N})$. Define for $x \in \Phi(\bar{N} \cap \Phi^{-1}(\bar{N})) = \Phi(\bar{N}) \cap \bar{N}$:

$$\tilde{E}_x^u = \Phi(E_{\Phi^{-1}(x)}^u).$$

By (3) $(\Phi(\bar{N}) \cap \bar{N}) \cap (\bar{N} \cap \Phi^{-1}(\bar{N}))$ is empty, hence this definition is correct. Now we claim that there exists an extension of $\tilde{E}^u$ to $\bar{N}$ satisfying

$$S_\varepsilon(x) \supset \tilde{E}_x^u \supset G_x, \qquad \tilde{E}_x^u \oplus E_x^s(\Phi) = F_x \tag{5}$$

for all $x \in \bar{N}$. To prove this claim take a continuous extension E of E^u to a neighborhood V of $P = (\bar{N} \cap \Phi(\bar{N}))$ in $\bar{N}$ and a continuous vector bundle homomorphism α: $F_2/P \to F^s$, where F^s is the subbundle of F/Λ^s defined by the map $x \to E_x^s(\Phi)$, covering the identity and such that

$$\operatorname{graph}(\alpha/F_{2,x}) = E_x, \qquad \|\alpha/F_{2,x}\| \leq \varepsilon \tag{6}$$

for all $x \in V$. Condition (6) follows from (4) if V is small enough. Take a continuous function Ψ: $\bar{N} \to \mathbf{R}$ such that $\Psi(x) = 1$ for $x \in P$, $\Psi(x) = 0$ for $x \in \bar{N} - V$ and $\Psi(x) \leq 1$ for all $x \in \bar{N}$. Define for $x \in \bar{N}$:

$$\tilde{E}_x^u = \operatorname{graph}(\Psi x/F_{2,x}). \tag{7}$$

This defines a subbundle satisfying (5) and (6). Finally, define (recalling (2)) for $x \in \Lambda^s - \Lambda$

$$\tilde{E}^u_x = \Phi^n(\tilde{E}^u_y)$$

where $y = \Phi^{-n}(x) \in \overline{N}$ and $\tilde{E}^u_x = E^u_x(\Phi)$ for $x \in \Lambda$. If $m \in \mathbf{Z}$, $m > n$ and $\Phi^{-m}(x) = z \in \overline{N}$ we have:

$$y \in \overline{N} \cap \Phi^{m-n}(\overline{N}).$$

By (3) $m - n = 1, y \in \overline{N} \cap \Phi(\overline{N})$ and then

$$\Phi^n(\tilde{E}^u_y) = \Phi^n(\Phi(\tilde{E}^u_z)) = \Phi^{n+1}(\tilde{E}^u_z) = \Phi^m(\tilde{E}^u_z);$$

hence our definition is correct. The continuity of $\tilde{E}^u$ at points of $\Lambda^s - \Lambda$ is easy to verify. At points of Λ it is continuous because if $\{x_n | n \in \mathbf{Z}^+\}$ is a sequence in Λ^s such that $x_n \to x \in \Lambda$ when $n \to +\infty$ and $\tilde{E}^u_{x_n}$ converges to a subspace $E \subset F_x$, then, observing that $\bigcap_{n \geq 0} \Phi^n(\Lambda^s) = \Lambda$, we can find a sequence $\{m_n | n \in \mathbf{Z}^+\} \subset \mathbf{Z}^+$, $m_n \to \infty$ when $n \to +\infty$, and $x_n \in \Phi^{m_n}(\Lambda^s)$. Hence, if $y_n = \Phi^{-m_n}(x_n) \in \Lambda^s$, from (1) and (2) in the proof of Proposition 1.4 follows that $\tilde{E}^u_{x_n} = \Phi^{m_n}(\tilde{E}^u_{y_n}) \subset S_{\mu_n}(x_n)$ where $\mu_n = \mu^{2m_n}$. Therefore $E \subset E^u_x(\Phi)$. But $\dim E = \dim E^u_x(\Phi)$. Hence $E = E^u_x(\Phi)$.

Proof of Proposition 1.5. Define

$$\Sigma = \{x \in K | E^s_x(\Phi) \oplus E^u_x(\Phi) = F_x\},$$

$$\Sigma_j = \{x \in \Sigma | \dim E^s_x(\Phi) = j\},$$

$$W^s(\Sigma_j) = \{y \in K | \omega(y) \subset \Sigma_j\},$$

$$W^u(\Sigma_j) = \{y \in K | \alpha(y) \subset \Sigma_j\},$$

$$\Lambda_j = \bigcup_{i \leq j} W^u(\Sigma_i).$$

By Corollary 1.3 $\alpha(x) \cup \omega(x) \subset \Sigma$ for all $x \in K$. Follows from Proposition 1.1 that the set Σ_j are closed and obviously they are disjoint. Using the well-known fact that limit sets of points cannot be decomposed in invariant closed disjoint sets it follows that for all $x \in K$, $\alpha(x) \in \Sigma_i$, $\omega(x) \in \Sigma_j$ for some $i \geq 0, j \geq 0$. The set Λ_j can be defined by:

$$\Lambda_j = \{x \in K | \dim E^u_x(\Phi) \geq n - j\}$$

because if $x \in \Lambda_j$ then $\alpha(x) \subset \Sigma_j$, $i \leq j$, and by Proposition 1.4 this implies $\dim E^u_x(\Phi) = n - i \geq n - j$. If $\dim E^u_x(\Phi) \geq n - j$, by Proposition 1.1 $\dim E^u_y(\Phi) \geq n - j$ for all $y \in \alpha(x)$. Hence $x \in \Lambda_j$. This definition of Λ_j proves that it is closed. We claim that for all j such that $\Sigma_j \neq \emptyset$ there exists a compact neighborhood M_j of Λ_j with the following properties:

(1) $$\Phi(M_j) \subset \operatorname{Int}(M_j),$$

(2)
$$\bigcap_{n>0} \Phi^n(M_j) = \Lambda_j,$$

(3)
$$\bigcap_{n>0} \Phi^{-n}(\Lambda_k - M_j) = \Sigma_k$$

where k is such that $k > j$, $\Sigma_k \neq \varnothing$ and $\Sigma_i = \varnothing$ for all $j < i < k$. To prove this take a compact neighborhood U of Λ_j such that for all $i > j$, $U \cap \Sigma_i = \varnothing$. Such a neighborhood exists because Λ_j is closed and disjoint of Σ_i for all $i > j$. If $x \in \cap_{n>0}\Phi^n(U)$ it follows that $\alpha(x) \subset U$. But $\alpha(x) \subset \Omega(\Phi) \subset \Sigma$ and since $\Sigma \cap U = \cup_{i \leqslant j}\Sigma_i$ we have $x \in \Lambda_j$. If $x \in \Lambda_j$ obviously $x \in \cap_{n>0}\Phi^n(U)$. Then by [10, Lemma 3.5] there exists M_j satisfying (1) and (2). To prove (3) take $x \in \cap_{n>0}\Phi^{-n}(\Lambda_k - M_j)$. Then $\omega(x) \subset \Lambda_k - M_j$. But $\omega(x) \subset \Omega(\Phi)$. Hence $\omega(x) \subset (\Lambda_k - M_j) \cap \Sigma = \Sigma_k$. Moreover $\Phi^{-1}(\Lambda_k - M_j) = \Phi^{-1}(\Lambda_k) - \Phi^{-1}(M_j) \subset \Lambda_k - M_j$. Therefore $\alpha(x) \subset \Lambda_k - M_j$. Again $\alpha(x) \subset \Omega(\Phi)$ and then $\alpha(x) \subset \Sigma_k$. By Proposition 1.2, dim $F_x^s \geqslant k$ because $\omega(x) \subset \Sigma_k$ and dim $F_x^u \geqslant n - k$ because $\alpha(x) \subset \Sigma_k$. Then $x \in \Sigma_k$. Conversely it is immediate that $\Sigma_k \subset \cap_{n>0}\Phi^{-n}(\Lambda_k - M_j)$. Moreover, it is easy to see that we can suppose $M_j \subset \mathrm{Int}(M_k)$ for $k > j$. Let $0 \leqslant j_1 < j_2 < \cdots < j_k$ such that $\Sigma_j \neq \varnothing$ if and only if $j \in \{j_1, \ldots, j_k\}$. We shall change the subindex denoting Σ_i, Λ_i, M_i, the sets Σ_{j_i}, Λ_{j_i}, M_{j_i}. Clearly $M_k = K$. Define for $j = 2, \ldots, k$:

$$F_j = F \oplus \overset{c_j}{\cdots} \oplus F$$

where $c_2 = 1$, $c_{j+1} = 2c_{j+1}$. We shall construct by recurrence a family of compact neighborhoods U_j, V_j, $j = 1, \ldots, k$ of Λ_j such that $V_j \subset \mathrm{Int}(U_j) \subset U_j \subset \mathrm{Int}(M_j)$ and continuous isomorphisms:

$$\Phi_j\colon (F \oplus F_j)/U_j \to (F \oplus F_j)/\Phi(U_j)$$

such that

(a) $\Phi_j/(F/U_j) = \Phi$.

(b) $\Phi_j/((F \oplus F_j)/U_j)$ is hyperbolic.

(c) For $j < k$,

$$\Phi_j/\big((F \oplus F_j)/(K - V_j)\big) = \Phi \oplus \overset{c_j}{\cdots} \oplus \Phi.$$

(d) If $\pi_1\colon F \oplus F_j \to F$ is the canonical projection, $\|(\pi_1 \circ \Phi_j)/F_x\| \leqslant \varepsilon$ for all $x \in U_j$.

Taking $\Phi = \Phi_k$ the theorem will be proved. For the construction of these isomorphisms we shall use the following lemma.

LEMMA. *Let K_0 be a compact metric space, $F_0 \in \mathcal{L}(K_0)$, $\Phi_0 \in \mathcal{L}(F_0)$ and $\Gamma \subset K_0$ a closed subset such that $\Phi_0(\Gamma) = \Gamma$, $\Phi_0/(F_0/\Gamma)$ is quasi-Anosov and there exists a closed subset $S \subset K_0$ satisfying*

(1) $\Phi_0(S) \subset \mathrm{Int}(S)$.

(2) *If* $\Gamma_0 = \bigcap_{n\geq 0}\Phi_0^n(S \cap \Gamma)$, $\Gamma_1 = \bigcap_{n\geq 0}\Phi_0^{-n}(\Gamma - (S \cap \Gamma))$ *the isomorphisms* $\Phi_0/(F_0/\Gamma_0)$ *and* $\Phi_0/(F_0/\Gamma_1)$ *are hyperbolic.*

Then given a closed neighborhood U *of* Γ *and* $\varepsilon > 0$, *there exists a quasi-Anosov isomorphism* $\Phi \in \mathfrak{L}(F \oplus F)$ *such that* (*identifying* $F \oplus \{0\}$ *with* F):

$$\Phi/F_0 = \Phi_0, \tag{3}$$

$$\Phi/\left((F_0 \oplus F_0)/K - U\right) = \Phi_0 \oplus \Phi_0, \tag{4}$$

$$\Phi/\left((F_0 \oplus F_0)/\Gamma\right) \textit{ is hyperbolic}, \tag{5}$$

$$\|(\pi_2 \circ \Phi)/F_{0,x}\| \leq \varepsilon, \tag{6}$$

for all $x \in K_0$, *where* π_2: $F_0 \oplus F_0 \to F_0 \oplus \{0\}$ *is the canonical projection.*

Let us suppose that we have defined U_j, V_j, Φ_j satisfying (a), (b), (c). Take a continuous function λ: $K \to \mathbf{R}$ such that $\lambda(x) > 0$ for all $x \in K$, $\lambda(x) = 1$ if $x \in K - W$, where W is a compact neighborhood of Σ_{j+1} such that $W \cap U_j = \varnothing$ and $W_j \subset \operatorname{Int}(M_{j+1})$, and $\lambda(x) > C$ when $x \in \Sigma_{j+1}$ where C satisfies $C\|(Tf)v\| \geq 2\|v\|$ for all $v \in F$. Let π: $F \oplus F_j \to K$ be the canonical projection. Define $\tilde{\Phi}_{j+1} \in \mathfrak{L}(F \oplus F_j)$ by:

$$\tilde{\Phi}_{j+1}(v_0, v_1, \ldots, v_{c_j}) = \left(\Phi(v_0), \lambda(p)\Phi(v_1), \ldots, \lambda(p)\Phi(v_{c_j})\right) \tag{7}$$

when $p = \pi(v_0, v_1, \ldots, v_{c_j}) \in K - U_j$, and $\tilde{\Phi}_{j+1}(v) = \Phi_j(v)$ when $\pi(v) \in U_j$. $\tilde{\Phi}_{j+1}/(F \oplus F_j/\Lambda_{j+1})$ is quasi-Anosov because if for $v \in (F \oplus F_j)/\Lambda_{j+1}$ the set $\{\|\tilde{\Phi}_{j+1}^n(v)\| \,|n \in \mathbf{Z}\}$ is bounded then $\pi(v) \not\in \Lambda_j$ because

$$\tilde{\Phi}_{j+1}/\left((F \oplus F_j)/\Lambda_j\right) = \Phi_j$$

is quasi-Anosov. Hence $\pi(v) \in \Lambda_{j+1} - \Lambda_j$ and $\alpha(\pi(v)) \subset \Sigma_{j+1}$. But by (6) and the property $\lambda(p) > C$ for $p \in \Sigma_{j+1}$ it follows that $v = (v_0, 0, \ldots, 0)$. Since $\tilde{\Phi}_{j+1}/F = \Phi$ is quasi-Anosov we obtain $v_0 = 0$. Now define U_{j+1} being a compact neighborhood of $U_j \cup \Lambda_{j+1} \cup W$ satisfying $U_{j+1} \subset \operatorname{Int}(M_{j+1})$ and as V_{j+1} take a compact neighborhood of $U_j \cup \Lambda_{j+1} \cup W$ contained in $\operatorname{Int}(U_{j+1})$. Apply the lemma putting $K_0 = K$, $F_0 = F \oplus F_j$, $\tilde{\Phi}_{j+1} = \Phi_0$, $\Gamma = \Lambda_{j+1}$, $S = M_j$, $\Gamma_0 = \Lambda$, $\Gamma_1 = \Sigma_{j+1}$, $U = U_{j+1}$. If $\Phi \in \mathfrak{L}(F_0 \oplus F_0)$ is the isomorphism given by the lemma define $\Phi_{j+1} = \Phi/(F_0 \oplus F_0)/U_{j+1}$ and identify $F_0 \oplus F_0 = F \oplus F_j \oplus F \oplus F_j$ with $F \oplus F_{j+1}$. Properties (a), (b), (c), (d) of Φ_{j+1} follows from properties (3), (5), (4), (6) of the lemma respectively.

Proof of the Lemma. By Proposition 1.4 the map $x \to E_x^s(\Phi)$ defines a continuous subbundle of $F_0/(\Gamma \cap \Phi_0^{-2}(S))$ that we shall call E^s. Applying Lemma 1.9 to $\Lambda = \Gamma_1$, $U_0 = \Gamma \cap \Phi_0^{-2}(S)$, $G_x = E_x^u(\Phi)$ (the semicontinuity property of G follows from Proposition 1.1) we obtain a subbundle G^+ on $\Gamma \cap \Phi_0^{-2}(S)$ satisfying:

(a) $E^s \oplus G^+ = F_0/(\Gamma \cap \Phi_0^{-2}(S))$,

(b) $\Phi_0(E^s) \subset E^s$,
(c) $\Phi_0(G^+) \subset G^+$,
(d) E^s/Γ_1 is the stable subbundle of $\Phi_0/(F_0/\Gamma_1)$,
(e) G^+/Γ_1 is the unstable subbundle of $\Phi_0/(F_0/\Gamma_1)$,
(f) $G_x^+ \supset E_x^u(\Phi_0)$ for all $x \in (\Phi_0^{-2}(S) - \Phi_0^3(S)) \cap \Gamma$
and subbundles E^u, G^- of $F_0/(\Gamma - \Phi_0^3(S))$ satisfying:
(a′) $E^u \oplus G^- = F_0/(\Gamma - \Phi_0^3(S))$,
(b′) $\Phi_0^{-1}(E^u) \subset E^u$,
(c′) $\Phi_0^{-1}(G^-) \subset G^-$,
(d′) E^u/Γ_0 is the unstable subbundle of $\Phi_0/(F_0/\Gamma_0)$,
(e′) G^- is the unstable subbundle of $\Phi_0/(F_0/\Gamma_0)$,
(f′) $G_x^- \supset E_x^s(\Phi_0)$ for all $x \in (\Phi_0^{-2}(S) - \Phi_0^3(S)) \cap \Gamma$;
hence if $G^0 = G^+ \cap G^-$:

$$E_x^s \oplus G_x^0 = G_x^+, \tag{1}$$

$$E_x^u \oplus G_x^0 = G_x^-, \tag{2}$$

$$E_x^s \oplus E_x^u \oplus G_x^0 = F_{0,x}, \tag{3}$$

for all $x \in (\Phi_0^{-2}(S) - \Phi_0^3(S)) \cap \Gamma$. We can suppose that there exists a compact neighborhood U_0 of Γ contained in $\operatorname{Int}(U)$ and such that E^s, G^+ are defined in $U_0 \cap \Phi_0^{-2}(S)$ and E^u, G^- in $U_0 \cap (K_0 - \Phi^3(S))$ satisfying (1), (2), (3) in $(\Phi_0^{-2}(S) - \Phi_0^3(S)) \cap U_0$. Take another compact neighborhood V of Γ contained in $\operatorname{Int}(U_0)$ and constants $0 < C_1 < C_2$ satisfying

$$C_1\|\Phi_0(V)\| \leqslant \|v\| \leqslant C_2\|\Phi_0(v)\|$$

for all $v \in F$. Let λ_i: $K \to \mathbf{R}$ be positive continuous functions such that:

$$\lambda_1(x) < \tfrac{1}{2}C_1, \qquad x \in \Phi_0^2(S) \cap V, \tag{4}$$

$$\lambda_2(x) > 2C_2, \qquad x \in \Phi_0^2(S) \cap V, \tag{5}$$

$$\lambda_1(x) = \lambda_2(x) = 1, \qquad x \in K_0 - (\Phi_0(S) \cap V), \tag{6}$$

$$\lambda_3(x) < \tfrac{1}{2}C_1, \qquad x \in V - \Phi_0^{-1}(S), \tag{7}$$

$$\lambda_4(x) = \lambda_3(x) = 1, \qquad x \in K_0 - (U_0 - S). \tag{8}$$

Define $\Phi_1 \in \mathfrak{L}(F_0)$ by:

$$\Phi_1(v) = \lambda_1(x)\Phi_0(v_+) + \lambda_2(x)\Phi_0(v_s)$$

for $\pi(v) = x \in S \cap U_0$ and $v = v_+ + v_s$, $v_+ \in G_x^+$, $v_s \in E_x^s$,

$$\Phi_1(v) = \lambda_4(x)\Phi_0(v_-) + \lambda_3(x)\Phi_0(v_u)$$

for $\pi(v) = x \in U_0 - \Phi_0(S)$, $v = v_- + v_+$, $v_- \in G_x^-$, $v_u \in E_x^u$, and

$$\Phi_1(v) = \Phi_0(v)$$

when $\pi(v) \in K_0 - U_0$. Then, if $x \in \Phi_0^{-1}(S) \cap \Gamma$:

(9) $$E_x^s(\Phi_1) = G_x^+$$

and if $x \in \Gamma - \Phi_0^2(S)$

(10) $$E_x^u(\Phi_1) = G_x^-.$$

Finally define $\Phi \in \mathcal{L}(F_0 \oplus F_0)$ by:

(11) $$\Phi(v_1, v_2) = (\Phi_0(v_1), \Phi_1(v_2))$$

when $\pi(v_1, v_2) \in K_0 - (\Phi_0^{-1}(S) - \Phi_0^2(S)) \cap U_0$, $v_1 \in F_0$, $v_2 \in F_0$ and if $x \in (\Phi^{-1}(S) - \Phi_0^2(S)) \cap U_0$, $\pi(v) = x$ and $v = (v_1, v_s + v_0 + v_u)$ where $v_1 \in F_u$, $v_s \in E_x^s$, $v_u \in E_x^u$, $v_0 \in G_x^0$ define:

(12) $$\Phi(v) = (\Phi_0(v_1) + \Psi(x)v_0, \Phi_1(v_s + v_u + v_0))$$

being $\Psi\colon K_0 \to \mathbf{R}$ a continuous positive function such that

(13) $$\Psi(x) = 1, \qquad x \in S - \Phi_0(S),$$

(14) $$\Psi(x) = 0, \qquad x \notin \Phi_0^{-1}(S) - \Phi_0^2(S).$$

We shall prove that Φ satisfies the lemma. To avoid confusion, in what follows if $x \in K$ and $E \subset F_x$ is a subspace we shall denote by $\tilde{E}$ the subspace $\{0\} \oplus E \subset F_{0,x} \oplus F_{0,x}$. If $x \in \Gamma_0$ we have by (4), (5), (7), (8) and (9):

$$E_x^s(\Phi) = E_x^s(\Phi_0) \oplus E_x^s(\Phi_1) = E_x^s(\Phi_0) \oplus \tilde{G}_x^+,$$

$$E_x^u(\Phi) = E_x^u(\Phi_0) \oplus E_x^u(\Phi_1) = E_x^u(\Phi_0) \oplus \tilde{E}_x^s(\Phi)$$

then, using (a)

(15) $$E_x^s(\Phi) \oplus E_x^u(\Phi) = F_{0,x} \oplus F_{0,x}.$$

The same argument proves (15) for $x \in \Gamma_1$. Observing that the orbit of a point in $\Gamma - (\Gamma_0 \cup \Gamma_1)$ always intersects $(\Phi_0^2(S) - \Phi_0^3(S)) \cap \Gamma$, we can suppose, if we want to verify the condition $E_x^s(\Phi) \oplus E_x^u(\Phi) = F_{0,x} \oplus F_{0,x}$ for $x \in \Gamma - (\Gamma_0 \cup \Gamma_1)$, that $x \in (\Phi_0^2(S) - \Phi_0^3(S)) \cap \Gamma$. Moreover, from (9) and (14) it follows, for $x \in \Phi_0^2(S) \cap \Gamma$

(16) $$E_x^s(\Phi) = E_x^s(\Phi_0) \oplus E_x^s(\Phi_1) = E_x^s \oplus \tilde{G}_x^+ = E_x^s \oplus \tilde{E}_x^u \oplus \tilde{G}_x^0$$

and from (10) and (14):

(17) $$E_x^u(\Phi) = E_x^u(\Phi) \oplus E_x^u(\Phi_1) = E_x^u \oplus \tilde{G}_x^- = E_x^u \oplus \tilde{E}_x^s \oplus \tilde{G}_x^0$$

for $x \in \Gamma - S$. Since the stable and unstable subspaces are invariant under Φ we obtain

$$\dim E_x^s(\Phi) + \dim E_x^u(\Phi) = 2 \dim F_{0,x}$$

for all $x \in \Gamma - (\Gamma_0 \cup \Gamma_1)$. Then it is sufficient to prove:

$$E_x^s(\Phi) \cap E_x^u(\Phi) = \{0\}$$

in $(\Phi_0^2(S) - \Phi_0^3(S)) \cap \Gamma$. Let $x \in (\Phi_0^2(S) - \Phi_0^3(S)) \cap \Gamma$ and $y = \Phi_0^{-3}(x)$. From (17) follows that

$$E_x^u(\Phi) = \Phi^3(E_y^u(\Phi)) = \Phi^3\big(E_y^u \oplus \tilde{E}_y^s \oplus \tilde{G}_y^0\big) = E_x^u \oplus \tilde{E}_x^s \oplus \Phi^3\big(\tilde{G}_y^0\big).$$

Hence, by (16):

$$E_x^s(\Phi) \cap E_x^u(\Phi) = \tilde{G}_y^0 \cap \Phi^3\big(\tilde{G}_y^0\big).$$

Let us calculate $\tilde{G}_y^0 \cap \Phi^3(\tilde{G}_y^0)$. If $0 \neq v \in \tilde{G}_y^0$ we have

$$\Phi(v) = (\Psi(y)v, \lambda_4(y)\Phi_0(v)),$$

$$\Phi_2(v) = \big(\Psi(y), \Phi_0(v) + \lambda_4(y)\Psi(\Phi_0(y))\Phi_0(v), \lambda_4(y)\Phi_0^2(v)\big),$$

$$\Phi_3(v) = \big(\Psi(y)\Phi_0^2(v) + \lambda_4(y)\Psi(\Phi(y))\Phi_0^2(v) + \lambda_4(y)\Psi\big(\Phi_0^2(v)\big)\Phi_0^2(v), \lambda_4(y)\Phi_0^3(v)\big).$$

If $\Phi^3(v) \in \tilde{G}_y^0$ we obtain

$$\Psi(y) + \lambda_4(y)\Psi(\Phi_0(y)) + \lambda_4(y)\Psi\big(\Phi_0^2(y)\big) = 0$$

thus contradicting the fact that $\lambda_4(y) > 0$, and $\Psi(\Phi_0(y)) = 1$ because $\Phi_0(y) \in (S - \Phi_0(S)) \cap \Gamma$.

2. **Quasi-Anosov diffeomorphisms.** In this section we shall apply the results in §1 to prove Theorem A and its corollaries.

PROOF OF THEOREM A. (a) $\Rightarrow$ (b). By Corollary 1.3 the closure of the set of α-limit points of f, $L^-(f)$, is a hyperbolic set. Let $L^-(f) = L_1 \cup \cdots \cup L_k$ be the spectral decompositon given in [11], and using the notation in [11] let $\gamma(L_j) = \dim W^s(x)$, $x \in L_j$. If $z \in W^s(x_i) \cap W^u(x_j)$, $x_i \in L_i$, $x_j \in L_j$ we have

$$E_{x_i}^s(Tf) = T_{x_i}W^s(x_i), \qquad E_{x_j}^u(Tf) = T_{x_j}W^u(x_j)$$

by the definition of quasi-Anosov diffeomorphism:

$$T_{x_i}W^s(x_i) \cap T_{x_j}W^u(x_j) = \{0\} \tag{1}$$

hence:

$$\gamma(L_j) \leqslant \gamma(L_i).$$

If $L_{i_1}, \ldots, L_{i_k}$ is a cycle, i.e. if

$$W^s(L_{i_n}) \cap W^u(L_{i_{n+1}}) \neq \varnothing, \qquad 1 \leqslant n \leqslant k-1,$$
$$W^s(L_{i_k}) \cap W^u(L_{i_1}) \neq \varnothing$$

it follows that:

$$\gamma(L_{i_1}) \leqslant \gamma(L_{i_2}) \leqslant \cdots \leqslant \gamma(L_{i_k}) \leqslant \gamma(L_{i_1})$$

and this together with (1) implies that all the intersections of the stable and unstable manifolds of basic sets of the cycle are transversal. Therefore these points of intersection are also in $L^-(f)$. This proves that f satisfies the no cycles condition. By [11] it follows Axiom A and $L^-(f) = \Omega(f)$.

(b) $\Rightarrow$ (a). Follows easily from the property: $E_x^s(Tf) = T_x W^s(x)$, $E_x^u(Tf) = T_x W^u(x)$ for all $x \in M$.

(a) $\Rightarrow$ (c). By Proposition 1.5 there exists a continuous vector bundle F on M and a hyperbolic isomorphism Φ: $TM \oplus F \to TM \oplus F$ such that

$$\Phi(v, w) = ((Tf)v + Pw, \Phi_1 w)$$

where Φ_1 is a continuous isomorphism and P: $F \to TM$ a continuous homomorphism, both covering f. Moreover we can suppose that P/F_x is near to the zero map for all $x \in M$, and that F is a C^∞ vector bundle because any continuous vector bundle on a C^∞ manifold is isomorphic to a smooth one. Take Φ': $TM \oplus F \to TM \oplus F$ defined by:

$$\Phi'(v, w) = ((Tf)v + P'(w), \Phi_1'(w))$$

where Φ_1' and P' are C^∞ approximations of Φ_1 and P. Then Φ' is hyperbolic because it is near in the topology of $\mathfrak{L}(TM \oplus F)$ to a hyperbolic isomorphism. Moreover there exists a C^∞ sphere bundle N on M with a C^∞ section i: $M \to N$ and a C^r diffeomorphism g_1: $N \to N$ such that $g_1 \circ i = i \circ f$ and, identifying $TN/i(M)$ with $TM \oplus F$,

$$(Tg)(v, w) = ((Tf)v, \Phi_1'(w))$$

for $(v, w) \in TN/i(M)$. Then it is easy to find a C^r diffeomorphism g: N such that $g \circ i = i \circ f$ and:

$$(Tg)(v, w) = ((Tf)v + P'(w), \Phi_1'(w))$$

for $(v, w) \in TN/i(M)$.

(c) $\Rightarrow$ (d). Let $M' = i(M)$ and i_*: $\Gamma^0(TM) \to \Gamma^0(TM')$, g_*: $\Gamma^0(TN/M') \to \Gamma^0(TN/M')$ defined as

$$i_*(\eta) = Ti \circ \eta \circ i^{-1}, \qquad g_*(\eta) = Tg \circ \eta \circ g^{-1}.$$

Since M' is a hyperbolic set $I - g_*$ is an isomorphism, hence $(I - f_*)\Gamma^0(TM) = i_*^{-1}(I - g_*)\Gamma^0(TM')$ is a closed subspace.

(d) $\Rightarrow$ (a). We claim that (d) implies that the set of periodic points of f, $\mathrm{Per}(f)$, has empty interior. If U is an open set, $U \subset \mathrm{Per}(f)$, and S_n is the set of fixed points of f^n, we have

$$U = \bigcup_{n>0} (S_n \cap U).$$

Hence there exists n_0 such that $S_n \cap U$ has nonemtpy interior $\mathrm{Int}(S_{n_0} \cap U)$. But:

$$\mathrm{Int}(S_{n_0} \cap U) = \bigcup_{n=1}^{n_0-1} (S_{n+1} - S_n) \cap \mathrm{Int}(S_{n_0} \cap U)$$

therefore there exists $1 \leqslant m \leqslant n_0 - 1$ such that

$$(S_{m+1} - S_m) \cap \mathrm{Int}(S_{n_0} \cap U)$$

has nonempty interior. Choose an open set $V \subset (S_{m+1} - S_m) \cap \mathrm{Int}(S_{n_0} \cap U)$ such that $f^j(V) \cap V = \emptyset$ for $1 \leqslant j \leqslant m$ and $f^{m+1}(x) = x$ for all $x \in V$. Take $\eta \in \Gamma^0(TM)$ with support in V. Define $\xi \in \Gamma^0(TM)$ as

$$\xi(x) = 0, \qquad x \not\in \bigcup_{j=0}^{m} f^j(V),$$

$$\xi(x) = (Tf)^j \xi(f^{-j}(x)), \qquad x \in f^j(V).$$

It is easy to see that $(I - f_*)\xi = 0$ contradicting (d). Now let us suppose that there exists $K > 0$, $p \in M$ and $0 \neq v \in T_pM$ such that $\sup\{\|(Tf)^n v\| \,|n \in \mathbf{Z}\} \leqslant K$. Given $\varepsilon > 0$ take $N \in \mathbf{Z}^+$ such that:

$$N + 1 \geqslant 3K\|f_*\|/\varepsilon\|v\|.$$

Take a nonperiodic point q near to p (recall that $\mathrm{Per}(f)$ has empty interior) and $0 \neq w \in T_qM$ with $\|w\| = \|v\|$ and $\|(Tf)^n w\| \leqslant 2K$ for $|n| \leqslant N$. It follows that there exists a neighborhood U of q such that $f^j(U) \cap U = \emptyset$ for $|j| \leqslant N$ and $0 \neq \eta \in \Gamma^0(TM)$ with support in U and satisfying $\eta(q) = w$, $\|(Tf)^n\eta(x)\| \leqslant 3K$ for all $x \in U$, $|n| \leqslant N$. Define $\xi \in \Gamma^0(TM)$ by:

$$\xi(x) = 0, \qquad x \not\in U,$$

$$\xi(x) = \left(1 - \frac{|j|}{N+1}\right)\eta(f^{-j}(U)), \qquad x \in f^j(U), |j| \leqslant N.$$

Then $\|\xi\| \geqslant \|\xi(q)\| = \|w\| = \|v\|$. Moreover, if $x \in f^j(U)$, $-N + 1 \leqslant j \leqslant N$,

$$\begin{aligned}\left\|\xi(x) - (f_*\xi)(x)\right\| &= \left\|\left(1 - \frac{|j|}{N+1}\right)(Tf)^j\eta(f^{-j}(x))\right.\\ &\qquad \left. - \left(1 - \frac{|j-1|}{N+1}\right)(Tf)^j\eta(f^{-j}(x))\right\|\\ &\leqslant \frac{3K}{N+1};\end{aligned}$$

if $x \in f^{N+1}(U)$

$$\|\xi(x) - (f_*\xi)(x)\| = \|(f_*\xi)(x)\|$$
$$\leqslant \|f_*\| \left\|\left(1 - \frac{N}{N+1}\right)(Tf)^N \eta(f^{-N}(x))\right\| < \frac{3K\|f_*\|}{N+1};$$

if $x \in f^{-N}(U)$

$$\|\xi(x) - (f_*\xi)(x)\| \leqslant \|\xi(x)\| < \frac{3K}{N+1}$$

in all other points $\|\xi(x) - (f_*\xi)(x)\| = 0$. Hence

$$\|(I - f_*)\xi\| < \frac{3K}{N+1}\|f_*\| \leqslant \varepsilon\|v\| \leqslant \varepsilon\|\xi\|$$

(recall that for all diffeomorphisms $\|f_*\| \geqslant 1$) and this implies that $I - f_*$ is not an isomorphism into.

Proof of Corollary 1. (a) $\Rightarrow$ (b) is trivial.

(b) $\Rightarrow$ (c). By the previous theorem f satisfies Axiom A. A structurally stable Axiom A diffeomorphism satisfies also the strong transversality condition [13] that together with property (a) of the theorem gives:

(1) $$T_x W^s(x) \oplus T_x W^u(x) = T_x M$$

for all $x \in M$ (see the introduction for definitions of $W^s(x)$, $W^u(x)$). To prove (c) it is enough to prove that if Ω_1, Ω_2 are basic sets of f [9] and $W^s(\Omega_1) \cap W^u(\Omega_2) \neq \varnothing$ then $\dim W^s(x) = \dim W^s(y)$ for some $x \in \Omega_1, y \in \Omega_2$. Take $x \in \Omega_1, y \in \Omega_2$ such that $W^s(x) \cap W^u(y) \neq \varnothing$.

If $z \in W^s(x) \cap W^u(y)$ we have by (1)

$$\dim W^s(x) = \dim W^s(z) = \dim M - \dim W^u(z)$$
$$= \dim - W^u(y) = \dim W^s(y).$$

(c) $\Rightarrow$ (a). By Corollary 1.2 to prove that f is Anosov it is enough to prove (using the notation of §1):

$$E_x^s(Tf) \oplus E_x^u(Tf) = T_x M$$

for all $x \in M$. By Theorem A this is true for all $x \in \Omega(f)$. For $x \notin \Omega(f)$ observe that

$$E_x^s(Tf) = T_x W^s(x), \qquad E_x^u(Tf) = T_x W^u(x);$$

therefore $E_x^s(Tf) \cap E_x^u(Tf) = \{0\}$. If we prove $\dim E_x^s(Tf) + \dim E_x^u(Tf) = \dim M$ we are done. But if x_1 and x_2 are periodic points in the basic sets containing the α and ω limits of x respectively we have

$$\dim E_x^s(Tf) = \dim E_{x_1}^s(Tf),$$
$$\dim E_x^u(Tf) = \dim E_{x_2}^u(Tf) = \dim M - \dim E_{x_2}^s(Tf).$$

Proof of Corollary 2. To prove that M is isolated it is sufficient, by [11], to verify that there exists $\varepsilon > 0$ such that $W_\varepsilon^s(x) \cap W_\varepsilon^u(y) \subset M$ for all x,

$y \in M$ (see [11] for notation). Let $\delta_0 > 0$ be such that for all $x, y \in M$, $0 < \varepsilon < \delta_0$ the set $W_\varepsilon^s(x) \cap W_\varepsilon^u(y)$ contains at most one point. Let, for $x \in M$, $\tilde{W}_\varepsilon^s(x)$, $\tilde{W}_\varepsilon^u(x)$ be the local stable and unstable manifolds of f/M at x [1]. We can suppose that $\tilde{W}_\varepsilon^s(x) = W_\varepsilon^s(x) \cap M$, $\tilde{W}_\varepsilon^u(x) = W_\varepsilon^u(x) \cap M$. Take $c > 0$ such that if $d(x, y) < c$ then $\tilde{W}_{\delta_0}^s(x) \cap \tilde{W}_{\delta_0}^u(y)$ is exactly one point. Choice $\delta_0 > \varepsilon > 0$ such that $W_\varepsilon^s(x) \cap W_\varepsilon^u(y) \neq \varnothing$ implies $d(x, y) < c$. Hence if $W_\varepsilon^s(x) \cap W_\varepsilon^u(y) \neq \varnothing$, the intersection is one point because $\varepsilon < \delta_0$ and must be the unique point in the intersection $\tilde{W}_{\delta_0}^s(x) \cap \tilde{W}_{\delta_0}^u(y)$. Conversely, let us suppose that there exists a compact neighborhood U of M such that

$$\bigcap_{n \in \mathbf{Z}} f^n(U) = M.$$

By Corollary 1 it is enough to show that f/M is structurally stable. Let $g \in \operatorname{Diff}^r(M)$ be C^r near to f. We can suppose that $g = g_0/M$ where $g \in \operatorname{Diff}^1(M)$ is C^1 near to f. By [1] there exists a homeomorphism h: $M \to \bigcap_{n \in \mathbf{Z}} g_0^n(U)$ such that $(g_0 \circ h)(x) = (h \circ f)(x)$ for all $x \in M$. But $g(M) = M$ implies $M \subset \bigcap_{n \in \mathbf{Z}} g_0^n(U)$, then $\varphi = h^{-1}/M$ is injective and C^0 near to the identity, therefore is surjective. Then

$$h(M) = h(\varphi(M)) = M$$

and f is structurally stable.

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