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INVARIANT-FREE REPRESENTATIONS OF AUGMENTED RINGS

BY

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ABSTRACT. Let Γ be an augmented ring in the sense of Cartan-Eilenberg, and let there be given a representation of Γ in $\operatorname{End}_k A$, where A is a finite dimensional vector space over the field k. We show that all cohomology of Γ in A is trivial if there are no invariants in A under the action of a suitable commutative subring of Γ. This generalizes a previous result of the author for group cohomology, and is applied to obtain sufficient conditions for the vanishing of the cohomology of Lie algebras and associative algebras.

Introduction. Throughout this paper, Γ will be a (left) augmented ring in the sense of Cartan-Eilenberg [2, Chapter VIII]:

$$0 \to I_\Gamma \to \Gamma \xrightarrow{\varepsilon_\Gamma} Q_\Gamma \to 0,$$

and all subrings will contain the unity element of Γ. (The subscripts Γ will usually be omitted.) Clearly, any subring is also an augmented ring. If A is a (left) Γ-module, then

$$A^\Gamma = \{a \in A: Ia = 0\}$$

is called the set of *invariant elements* of the representation.

The main purpose of this paper is to show that if A is a finite dimensional vector space, or more generally, a module which can be obtained inductively by repeated extensions of such spaces, and $A^\Lambda = 0$ for a suitable commutative subring Λ of Γ, then all cohomology of Γ in A is trivial (Theorem 1). We thus obtain results about the cohomology, and hence about extensions, of groups, Lie algebras and associative algebras. In particular, the results of [3] are subsumed here. The application to Lie algebras (Corollary 1) should be compared with a theorem of D. W. Barnes [1] (and an earlier special case due to Dixmier [4]) to the effect that all cohomology of a finite dimensional nilpotent Lie algebra L in an L-module A vanishes if $A^L = 0$.

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The following sample illustrates the application of the main theorem to extension theory.

THEOREM. *Let E be a group (resp., Lie algebra over a field, resp., associative algebra over a field) and let A be a normal abelian subgroup which is a direct sum of a finite group and a finite dimensional rational vector space (resp., finite dimensional abelian ideal, resp., finite dimensional square zero ideal) in E such that E/A is abelian. If the center of E is disjoint from A, then E splits over A (Corollaries* 2 *and* 5, *and* [3, *Corollary* 3]).

1. **The central case.** The proof of the following lemma is similar to that of statement (b), §1 of [3].

LEMMA 1. *Let Γ be an augmented ring with center Z, and let A be a left Γ-module. If there is an element $\alpha \in Z \cap I$ such that the endomorphism of A given by $a \mapsto \alpha a$ is bijective, then*

$$\mathrm{Ext}_\Gamma^r(Q, A) = 0 \quad \textit{for } r \geqslant 0.$$

PROOF. Start with a Γ-projective resolution of Q of the form

$$\cdots \to X_1 \to \Gamma \xrightarrow{\varepsilon} Q \to 0,$$

and observe that the endomorphism of this complex determined by $x \mapsto \alpha x$ is a map over the zero map on Q. Then proceed as in [3].

PROPOSITION 1. *Let Γ be an augmented ring with center Λ, k a field, and A a Γ-k-bimodule, finite dimensional over k. Suppose that at least one of the following conditions is satisfied:*

(a) *Γ is a k-algebra (in such a way that $(c\gamma)a = c(\gamma a)$ for all $c \in k$, $\gamma \in \Gamma$, $a \in A$).*

(b) *k_0, the prime subfield of k, has order $> \dim_k A$.*

(c) *Every I_Λ-invariant k-subspace of A is Γ-invariant.*

Then, if $A^\Lambda = 0$,

$$\mathrm{Ext}_\Gamma^r(Q, A) = 0 \quad \textit{for } r \geqslant 0.$$

PROOF. (a) We assume first that $\dim_k A = n < q = |k|$, the order of k. For $\lambda \in \Lambda$, let e_λ be the k-endomorphism of A given by: $e_\lambda a = \lambda a$. Let

$$B = \{e_\lambda + 1 : \lambda \in I_\Lambda\} \subset \mathrm{End}_k A.$$

By part (b) of the Lemma, §2 of [3], there exist $\lambda_i \in I_\Lambda$ and $c_i \in k$ such that $\sum c_i e_{\lambda_i}$ is invertible. Hence $\alpha = \sum c_i \lambda_i$ satisfies the hypothesis of Lemma 1, so the desired conclusion follows.

If $n \geqslant q$, let L be an extension of k with $|L| > n$. Then tensor Γ, Q, and A with L (over k) and apply the preceding discussion to conclude that

$$\text{(1)} \qquad \operatorname{Ext}^r_{L\otimes_k\Gamma}(L \otimes_k Q, L \otimes_k A) = 0 \quad \text{for } r \geqslant 0.$$

(One must check that $(L \otimes_k A)^{\Lambda'} = 0$, where

$$\Lambda' = \operatorname{Im}(L \otimes_k \Lambda \to L \otimes_k \Gamma).$$

Suppose

$$\beta = \sum l_i \otimes a_i \in (L \otimes_k A)^{\Lambda'},$$

where we may assume the l_i linearly independent over k. Then for any $\lambda \in I_\Lambda$,

$$0 = (1 \otimes \lambda)\beta = \sum l_i \otimes \lambda a_i.$$

Hence each $\lambda a_i = 0$, so $a_i = 0$.)

Now by [2, VIII, Theorem 3.1],

$$\text{(2)} \qquad \operatorname{Ext}^r_{L\otimes_k\Gamma}(L \otimes_k Q, L \otimes_k A) \approx \operatorname{Ext}^r_\Gamma(Q, L \otimes_k A).$$

(Condition (i) of that theorem follows from [2, II, Proposition 5.1], and (ii) follows from [2, II, Proposition 6.1] with $\phi\colon k \to \Gamma$ the obvious map.)

Finally, let $\sigma\colon L \to k$ be a k-linear map such that the composite $k \to L \to k$ is the identity. Then the composite

$$A \xrightarrow{i} L \otimes_k A \xrightarrow{j} A,$$

where $ia = 1 \otimes a$ and $j(l \otimes a) = \sigma(l)a$, is also the identity, and i and j are Γ-maps, so the composite

$$\operatorname{Ext}^r_\Gamma(Q,A) \to \operatorname{Ext}^r_\Gamma(Q, L \otimes_k A) \to \operatorname{Ext}^r_\Gamma(Q,A)$$

is the identity. Now (1) and (2) imply that $\operatorname{Ext}^r_\Gamma(Q,A) = 0$.

(b) We proceed as in (a) to find a k-automorphism of A of the form $\sum c_i e_{\lambda_i}$ with $c_i \in k$ and $\lambda_i \in I_\Lambda$, and it is clear from the proof of the Lemma, §2 of [3], that the c_i can be chosen as "integers": $c_i = n_i \cdot 1 \in k_0$ $(n_i \in \mathbf{Z})$ since $|k_0| > \dim_k A$. Then $\alpha = \sum n_i \lambda_i$ satisfies the hypothesis of Lemma 1.

(c) Let $A = \sum A_i$ be a direct sum decomposition of A into indecomposable I_Λ-invariant k-subspaces. By hypothesis (c), each A_i is a Γ-submodule, so it suffices to show that for all $r \geqslant 0$,

$$\operatorname{Ext}^r_\Gamma(Q, A_i) = 0 \quad \text{for each } i.$$

By part (a) of the Lemma, §2 of [3], with $V = A_i$ and

$$B = \{e_\lambda + 1 \colon \lambda \in I_\Lambda\} \subset \mathrm{End}_k A_i$$

($e_\lambda a = \lambda a$), there exists $\alpha \in I_\Lambda$ such that e_α is invertible. Now an application of Lemma 1 completes the proof.

2. **The general case.** Let Λ and Γ be augmented rings and let $\phi \colon \Lambda \to \Gamma$ be a ring homomorphism such that $\phi(I_\Lambda) \subset I_\Gamma$. We proceed as in [2, XVI, §6] to obtain a spectral sequence, (4) below, which will be needed to extend Proposition 1.

We call ϕ (*right*) *normal* if the left ideal $\Gamma\phi(I_\Lambda)$ of Γ, denoted $\Gamma \cdot I_\Lambda$, is also a right ideal. Then

$$\Gamma' = \Gamma/\Gamma \cdot I_\Lambda \xrightarrow{\varepsilon'} Q_\Gamma \to 0$$

is an augmented ring if we define ε' by

$$\varepsilon'(\gamma + \Gamma \cdot I_\Lambda) = \varepsilon_\Gamma(\gamma).$$

(Q_Γ is a left Γ'-module since $\Gamma \cdot I_\Lambda$ annihilates Q_Γ, viz., for $\alpha \in \Gamma \cdot I_\Lambda$ and $q = \varepsilon_\Gamma \gamma_1 \in Q_\Gamma$, we have $\alpha q = \varepsilon_\Gamma(\alpha\gamma_1)$, which is 0 since $\Gamma \cdot I_\Lambda$ is a right ideal.)

Now the sequence

$$0 \to I_\Lambda \to \Lambda \to Q_\Lambda \to 0$$

yields

$$\Gamma \otimes_\Lambda I_\Lambda \to \Gamma \to \Gamma \otimes_\Lambda Q_\Lambda \to 0,$$

so

$$\Gamma \otimes_\Lambda Q_\Lambda \approx \Gamma'.$$

Then the spectral sequence $(2)_4$, Case 4, of [2, XVI, §5] becomes in our case:

$$\mathrm{Ext}^p_{\Gamma'}\,(Q_\Gamma = Q_{\Gamma'}, \mathrm{Ext}^q_\Gamma\,(\Gamma', A)) \Rightarrow \mathrm{Ext}^n_\Gamma\,(Q_\Gamma, A) \tag{3}$$

for any left Γ-module A.

Now suppose Γ *is projective as a right* Λ*-module* (*via* ϕ). Then by [2, VI, Proposition 4.1.3] with A and C replaced by Q_Λ and A, resp.,

$$\mathrm{Ext}^q_\Gamma\,(\Gamma', A) \approx \mathrm{Ext}^q_\Gamma\,(\Gamma \otimes_\Lambda Q_\Lambda, A) \approx \mathrm{Ext}^q_\Lambda\,(Q_\Lambda, A),$$

so (3) becomes

$$\mathrm{Ext}^p_{\Gamma'}\,(Q_\Gamma = Q_{\Gamma'}, \mathrm{Ext}^q_\Lambda\,(Q_\Lambda, A)) \Rightarrow \mathrm{Ext}^n_\Gamma\,(Q_\Gamma, A). \tag{4}$$

In what follows, a subring Λ of an augmented ring Γ will be called *normal* if the inclusion $\Lambda \to \Gamma$ is normal and Γ is projective as a right Λ-module. Λ is *subnormal* if there is a finite sequence

$$(5) \qquad \Lambda = \Lambda_0 \subset \Lambda_1 \subset \cdots \subset \Lambda_t = \Gamma$$

with Λ_i normal in Λ_{i+1} for each i. For example, if N is a normal subgroup of the group G, then $\mathbf{Z}N$ is a normal subring of $\mathbf{Z}G$ (see [2, XVI, §6]). Similarly, if J is an ideal in the Lie algebra L (over a field), then J^e, the enveloping algebra of J, is normal in L^e [ibid.].

THEOREM 1. *Let Γ be an augmented ring, k a field, and A a Γ-k-bimodule, finite dimensional over k. Let Λ be a subnormal commutative subring of* Γ, *and suppose at least one of the following conditions is satisfied* (*see Proposition* 1 *for notation*):

(a) Λ *is a k-algebra* (*with* $(c\lambda)a = c(\lambda a)$),

(b) $|k_0| > \dim_k A$,

(c) *Every I_Λ-invariant k-subspace of A is Λ-invariant.*

If $A^\Lambda = 0$, *then*

$$\operatorname{Ext}_\Gamma^r(Q,A) = 0 \quad \textit{for } r \geqslant 0.$$

PROOF. Given the sequence (5), let $Q_i = Q_{\Lambda_i} = \varepsilon_\Gamma(\Lambda_i)$. By Proposition 1, $\operatorname{Ext}_{\Lambda_i}^r(Q_i,A) = 0$ for all r when $i = 0$. Now apply the spectral sequence (4) (with Λ_i and Λ_{i+1} in place of Λ and Γ) to conclude inductively that for all r,

$$\operatorname{Ext}_{\Lambda_i}^r(Q_i,A) = 0 \quad \text{for } i = 0, 1, \ldots, t.$$

REMARK. As in [3] (q.v.), we can extend the class of modules A to which Theorem 1 applies as follows. If Λ is an augmented ring, a V^1-*module* for Λ is a finite direct sum $A' = \sum A_i$, where A_i is a Λ-k_i-bimodule, finite dimensional over the field k_i. The A_i are called *components* of A'. For $j > 1$, a V^j-*module* is a Λ-module which is an extension of a V^1-module by a V^{j-1}-module. A Λ-module is a W-*module* if it is a V^j-module for some j.

Now suppose Γ is an augmented ring, Λ a subnormal commutative subring, and A a W-module for Γ such that each component of the V^1-modules involved in the construction of A satisfies condition (a) or (b) or (c) of Theorem 1. Then if $A^\Lambda = 0$, $\operatorname{Ext}_\Gamma^r(Q, A) = 0$ for $r \geqslant 0$. The proof of this statement is the same, *mutatis mutandis*, as that of Theorem 1 of [3]. Indeed, the latter is a consequence of the former since if Λ is a supplemented $\mathbf{Z}$-algebra, then condition (c) of Theorem 1 is automatically satisfied because $\Lambda = \mathbf{Z} \oplus I_\Lambda$ [2, X, §1].

3. **Applications.** The reader is referred to [2, IX and XIII] for the cohomology theory of associative and Lie algebras, and to [2, XIV], [5], [6], [7], [8] for the terminology of extension theory used below. (All associative algebras are assumed to have unit element.) We merely note here that if L is a Lie algebra, $\Lambda = L^e$ its enveloping algebra, and A an L-module, then

$$A^\Lambda = \{a \in A : la = 0 \text{ for all } l \in L\}$$

since I_Λ is generated by the image of L in L^e [2, p. 268]. Similarly, if R is an associative k-algebra, $\Lambda = R \otimes_k R^*$ its enveloping algebra, and A a two-sided R-module, then

$$A^\Lambda = \{a \in A : ra = ar \text{ for all } r \in R\}$$

because I_Λ is generated by all elements of the form $r \otimes 1 - 1 \otimes r^*$, $r \in R$ [2, IX, Proposition 3.1].

COROLLARY 1. *Let L be a Lie algebra over a field k, J an abelian subideal of L, and A an L-module, finite dimensional over k, such that*

$$A^J \equiv \{a \in A : ja = 0 \textit{ for all } j \in J\} = 0.$$

Then

$$H^r(L, A) = 0 \quad \textit{for } r \geqslant 0.$$

In particular, any L-kernel with center A is extendible, and this extension is unique (up to equivalence).

PROOF. The first conclusion is immediate from Theorem 1 and the remarks preceding that theorem.

The second conclusion follows from the vanishing of $H^3(L, A)$ and $H^2(L, A)$ [7] (or [8, Lemma 5]) and [8, Theorem 3].

COROLLARY 2. *Let E be a Lie algebra over a field, and A a finite dimensional abelian ideal in E such that E/A is abelian. If the center of E is disjoint from A, then E splits over A: $E = A \oplus B$ (vector space direct sum), where B is a Lie subalgebra.*

PROOF. Use Corollary 1 to show that $H^2(E/A, A) = 0$.

COROLLARY 3. *Let R_0 be a commutative ring, R an associative R_0-algebra with center S, k a field, and A a two-sided R-k-bimodule, finite dimensional over k. Suppose at least one of the following conditions holds:*

(a) *R is a k-algebra (in such a way that $(cr)a = c(ra)$ for all $c \in k$, $r \in R$, $a \in A$.*

(b) $|k_0| > \dim_k A$, *where k_0 is the prime field of k.*

(c) *Every k-subspace of A which is invariant under the maps $a \mapsto sa - as$ for all $s \in S$ is R-invariant (as a two-sided module).*

If

$$A^S \equiv \{a \in A : sa = as \textit{ for all } s \in S\} = 0,$$

then

$$H^r(R,A) = 0 \quad \textit{for } r \geqslant 0.$$

PROOF. Use Proposition 1 (not Theorem 1) with $\Gamma = R \otimes_{R_0} R^*$. Condition (a) implies condition (a) of Proposition 1 if we make Γ a k-algebra by defining $c(r \otimes r_1^*) = cr \otimes r_1^*$. Also, it is not hard to see that condition (c) implies condition (c) of Proposition 1.

COROLLARY 4. *Let k be a field, R a k-algebra with center S, A a two-sided R-module, finite dimensional over k, such that $A^S = 0$. Then*

$$H^r(R,A) = 0 \quad \textit{for } r \geqslant 0.$$

In particular, if $[f, K]$ is a representation of R with nucleus A (in the sense of Hochschild [5]), *then there is a unique (up to equivalence) extension of K by R which gives rise to this representation.*

PROOF. Use [5, Theorems 5.2 and 6.2] for the second conclusion.

COROLLARY 5. *Let E be an associative algebra over a field, and A a finite dimensional ideal in E with $A^2 = 0$, such that E/A is commutative. If the center of E is disjoint from A, then E splits over A: $E = A \oplus B$ (vector space direct sum), where B is a subalgebra.*

PROOF. See the proof of Corollary 2 and [2, XIV, Theorem 2.1].

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