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THE REDUCED WITT RING OF A FORMALLY REAL FIELD

BY

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ABSTRACT. The reduced Witt rings of certain formally real fields are computed here in terms of some basic arithmetic invariants of the fields. For some fields, including the rational function field in one variable over the rational numbers and the rational function field in two variables over the real numbers, this is done by computing the image of the total signature map on the Witt ring. For a wider class of fields, including all those with only finitely many square classes, it is done by computing the Witt rings of certain ultracompletions of the field and representing the reduced Witt ring as an appropriate subdirect product of the Witt rings of the ultracompletions. The reduced Witt rings of a still wider class of fields, including for example the fields of transcendence degree one and the rational function field in three variables over the real numbers, are computed similarly, except that the description of the subdirect product no longer involves only local conditions.

1. Introduction. This paper is concerned with the problem of computing the structure of the reduced Witt ring, i.e. the Witt ring modulo its nil radical, of a formally real field in terms of more basic arithmetic invariants of the field. Among the invariants that concern us are the orderings of the field, their associated real places, and the value groups of these real places. Our procedure will be to find the structure of the Witt rings of certain localizations of the field, and then to try to piece these together to obtain the structure of the reduced Witt ring of the field.

The "local-global principle" of A. Pfister [15, Satz 22] implies that an element of the reduced Witt ring of a formally real field is determined by the signatures (with respect to all the orderings of the field) of a quadratic form representing the element. Hence an explicit computation of the image of the total signature map on the reduced Witt ring of a formally real field gives a description of the reduced Witt ring as a subdirect product of the Witt rings

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of its real closures (with respect to all the orderings of the field). For some important classes of fields, the image of the total signature map can be described quite simply (e.g. [12, Corollary 3.21]). However it appears that in general any description of the image of the total signature map must be rather formal and involve complicated invariants associated with the set of orderings of the field. An alternative procedure is to represent the reduced Witt ring as a subdirect product of the Witt rings of certain "ultracompletions" of the field. The Witt rings of these ultracompletions, while not as simple as the Witt rings of real closures, are still fairly transparent. Moreover, their richer structure is used in formulating "compatibility" conditions which describe how the reduced Witt rings of at least some fields are represented as subdirect products of the Witt rings of their ultracompletions. For such fields one also obtains a computation of the image of the total signature map. This is done by computing the images of the total signature maps for the ultracompletions, and then pasting together these local results. The resulting description of the image of the total signature map is rather formal. In some cases it can be specialized to a more useful form. In other cases, the computation of the reduced Witt ring in terms of the Witt rings of the ultracompletions appears to give more insight.

Ultracompletions of a field with respect to places from the field into the real numbers are introduced in §2. We show there how the space of orderings, the Witt ring, and the image of the total signature map of an ultracompletion at a real place can each be computed in terms of the square factor group of the value group of the real place. We then recall (in §3) that the reduced Witt ring of a formally real field can be represented as a subdirect product of the Witt rings of its ultracompletions. We introduce certain parity and continuity conditions which elements of this subdirect product must satisfy; we call those fields "exact" for which these conditions characterize elements of the subdirect product. Some classes of exact fields are presented in §4; particular attention is paid to SAP fields [9, Definition 1.4], superpythagorean fields [9, Definition 4.4], rational function fields, and Henselian valued fields. An apparently weaker condition than exactness, "near exactness", is studied in §5; for these fields one can again describe the reduced Witt ring as a subdirect product of the Witt rings of the ultracompletions, but only by involving less simple invariants of the field. Among the fields shown to be near exact are the formally real fields of transcendence degree one (respectively, two) over the rational (respectively, real) numbers, and the rational function field in one variable over any such field. In §6, fields admitting only finitely many places into the real numbers are shown to be exact. (Any field with only finitely many distinct orderings is of this type.) In this case a very explicit computation of the reduced Witt ring is possible. The image of the total signature map for an exact field is computed in §7. This rather formal computation is then specialized to various classes of fields, including the rational function field in one variable over the rational numbers and the rational function field in two variables over the real numbers. Certain fields of transcendence degree one with prescribed real-valued places are constructed in §8; this construction answers a question raised in §5 and a question of Bröcker [20] and Prestel [21].

F will always denote a formally real field. We let $\mathcal{O}(F)$ denote the set of orderings of F, given the coarsest topology with $V(a) = \{P \in \mathcal{O}(F) \colon a \in P\}$ open for all $a \in F$. (An "ordering" on F is the set of positive elements in a linear order on F making F an ordered field.) $\mathfrak{M}(F)$ will denote the set of places from F into the field R of real numbers; we call such places "real places". (Caution: the term "real place" is used elsewhere in a less restrictive sense; cf. [10], [16].) There is a canonical surjection $\pi\colon \mathcal{O}(F) \to \mathfrak{M}(F)$; we give $\mathfrak{M}(F)$ the finest topology in which π is continuous. (For $P \in \mathcal{O}(F)$, $\pi(P)$ is the unique real place which is never negative on P [2].) For each $\sigma \in \mathcal{O}(F)$, let $\mathcal{O}_\sigma = \pi^{-1}(\sigma)$. The sets $\mathcal{O}_\sigma$ ($\sigma \in \mathfrak{M}(F)$) partition $\mathcal{O}(F)$. For each $a \in F^\cdot$ and $P \in \mathcal{O}(F)$, we let (a, P) denote 1 if $a \in P$ and -1 if $a \notin P$. Let $W(F)$ and $W_{\text{red}}(F)$ denote the Witt ring and reduced Witt ring of F, respectively.

Z, Q, R denote the sets of integers, rational numbers, and real numbers, respectively. For any unitary ring S, we denote its multiplicative group of units by $S^\cdot$. For any set A, we let $|A|$ denote the symbol ∞ if A is infinite and the number of elements in A if A is finite. $A \backslash B$ denotes the set of elements of the set A which are not in the set B. The identity of a multiplicative group and the trivial group which it generates are both denoted by 1.

2. **Ultracompletions.** Throughout this section we let σ denote a fixed element of $\mathfrak{M}(F)$. By the *ultracompletion* of F at σ we mean the pair $(F_\sigma, \bar{\sigma})$ where F_σ denotes the ultracompletion of F at the prime value associated with σ and $\bar{\sigma}$ denotes the canonical extension of σ to a real place on F_σ [5, Definition 1.5 (and Example 1.1C)]. The reader is invited to treat the next lemma as a "short-cut" definition of the ultracompletion.

(2.1) LEMMA. *There is a maximal field extension K/F such that σ extends to a real place τ on K which maps onto R (i.e. has residue class field R) and such that τ has the same value group as σ. For any such pair (K, τ) there exists an F-isomorphism $\psi\colon F_\sigma \to K$ with $\tau\psi = \bar{\sigma}$.*

That σ and τ have the same value group means that the natural map from the value group of the valuation ring $\sigma^{-1}(R)$ to that of $\tau^{-1}(R)$ is bijective. The lemma summarizes part of [5, §1]. The existence and uniqueness (up to "analytic isomorphism") of the extension K/F follows easily from [19, Theorem 5, p. 38 and Lemma 19, p. 218] and [11, proof of Theorem 7], respectively.

(2.2) NOTATION. Let $\tau: F \to K \cup \{\infty\}$ be a place. Let $v_\tau: F^\cdot \to \Lambda_\tau$ denote the map taking each element of $F^\cdot$ to its image in the square factor group of the value group of the valuation ring $\tau^{-1}(K)$. Thus we may identify Λ_τ with $F^\cdot/\tau^{-1}(K^\cdot) \cdot F^{\cdot 2}$ and v_τ with the canonical surjection. We will regard Λ_τ as a discrete abelian group and its character (or "dual") group $\Lambda_\tau^* = \mathrm{Hom}(\Lambda_\tau, Z^\cdot)$ as a compact abelian group [17, 1.7.3(a)].

We now compute the topological space $\mathcal{O}_\sigma = \pi^{-1}(\sigma)$ in terms of Λ_σ.

(2.3) LEMMA. *Let $P, T \in \mathcal{O}_\sigma$. There exists a homomorphism $\langle P, T\rangle \in \Lambda_\sigma^*$ with*

$$\langle P, T\rangle(v_\sigma(a)) = (a, P)(a, T)$$

for all $a \in F^\cdot$. For any $P \in \mathcal{O}_\sigma$, the induced map $\langle P, \cdot\rangle: \mathcal{O}_\sigma \to \Lambda_\sigma^*$ is a homeomorphism.

PROOF. See [2], or see Remark (2.6) below.

We can now compute $\mathfrak{M}(F_\sigma)$ and $\mathcal{O}(F_\sigma)$.

(2.4) LEMMA. *$\bar{\sigma}$ is the only real place on F_σ, i.e., $\mathfrak{M}(F_\sigma) = \{\bar{\sigma}\}$. The map $P \mapsto P \cap F$ gives a homeomorphism from $\mathcal{O}(F_\sigma)$ onto $\mathcal{O}_\sigma$.*

PROOF. Just suppose F_σ admits a real place τ with $\tau \neq \bar{\sigma}$. Then there exists $a \in F_\sigma^\cdot$ with $\tau(a) < 0 < \bar{\sigma}(a) < \infty$ [3, Theorem 2.1A]. Since F_σ is a maximal field, it is Henselian. Thus $a \in F_\sigma^{\cdot 2}$ (cf. Lemma (2.1)). But then $\tau(a) < 0$ is impossible. The second sentence of Lemma (2.4) follows from the first, and the bijections of Lemma (2.3). (Note that there is a natural bijection $\Lambda_\sigma \to \Lambda_{\bar{\sigma}}$, cf. Lemma (2.1).)

One can also prove the second part of (2.4) by arguing directly that if $P \in \mathcal{O}_\sigma$, then $\{ab^2: a \in P, b \in F_\sigma^\cdot\}$ is the unique extension of P to an ordering of F_σ.

We will often write Λ_σ and v_σ in place of $\Lambda_{\bar{\sigma}}$ and $v_{\bar{\sigma}}$, and identify elements of $\mathcal{O}_\sigma$ with the corresponding elements of $\mathcal{O}(F_\sigma)$.

The Witt ring of F_σ is an integral group ring since F_σ is superpythagorean (e.g., see [6, Corollary 8] and [9, 5.13(8)]). Indeed, the isomorphism $W(F_\sigma) \cong Z(\Lambda_\sigma)$ may be chosen "almost" canonically.

(2.5) THEOREM. *Let $P \in \mathcal{O}(F_\sigma)$. There is an isomorphism $\Phi_P: W(F_\sigma) \to Z(\Lambda_\sigma)$ such that for all $a \in F_\sigma^\cdot$,*

$$\Phi_P(\langle a\rangle) = (a, P)v_\sigma(a).$$

The map Φ_P is uniquely determined since the elements $\langle a\rangle$ $(a \in F_\sigma^\cdot)$ generate $W(F_\sigma)$. (For any $a_1, \ldots, a_n \in F_\sigma^\cdot$, we let $\langle a_1, \ldots, a_n\rangle$ denote the element of $W(F_\sigma)$ representing the quadratic form $a_1 x_1^2 + \cdots + a_n x_n^2$.)

PROOF. Let $\phi\colon F_\sigma^{\cdot} \to Z(\Lambda_\sigma)^{\cdot}$ be the group homomorphism

$$\phi(a) = (a,P)v_\sigma(a) \quad (a \in F_\sigma^{\cdot}).$$

To establish the existence of the ring homomorphism Φ_P it suffices to show that the elements $\phi(a)$ ($a \in F_\sigma^{\cdot}$) satisfy the same relations as do the generators $\langle a\rangle$ ($a \in F_\sigma^{\cdot}$) of $W(F_\sigma)$. This boils down to showing that

$$\phi(a) + \phi(b) = \phi(a + b) + \phi(ab(a + b)) \tag{1}$$

whenever a, b, $a + b \in F_\sigma$ [14, Lemma 1.1, p. 85]. Let v be the valuation associated with the place $\bar\sigma$; then v induces v_σ and we may regard v as assigning to each element of $F_\sigma^{\cdot}$ its Archimedean class with respect to P. If $v(a) = v(b) = v(a + b)$, then one easily verifies relation (1). If not, then we might as well assume that $v(a + b) \neq v(a)$. Then either $v(a + b) = v(b)$, in which case $(a + b, P) = (b, P)$, or else $v(a + b) \neq v(b)$, in which case $v(a) = v(b)$ and $(a,P) = (-b,P)$. In either case, (1) is easily verified. The set $\bar\sigma^{-1}(R^{\cdot})F_\sigma^{\cdot 2}$ is in the kernel of the group homomorphism $\psi\colon F_\sigma^{\cdot} \to W(F_\sigma)^{\cdot}$ given by

$$\psi(a) = (a,P)\langle a\rangle \qquad (a \in F_\sigma^{\cdot}).$$

Hence ψ induces a group homomorphism $\Lambda_\sigma \to W(F_\sigma)^{\cdot}$, and thence a ring homomorphism $\Psi\colon Z(\Lambda_\sigma) \to W(F_\sigma)$. Since Ψ and Φ_P are clearly inverse maps, Φ_P is an isomorphism. The theorem is proven.

(2.6) REMARK. For any P, $T \in \mathcal{O}(F_\sigma)$ and $a \in F_\sigma^{\cdot}$ we have

$$\Phi_P\Phi_T^{-1}(v_\sigma(a)) = (a,P)(a,T)v_\sigma(a).$$

This shows that the map $\langle P,T\rangle$ of Lemma (2.3) is well defined, and that it induces in a natural way the automorphism $\Phi_P\Phi_T^{-1}$ of $Z(\Lambda_\sigma)$. The only hard part of Lemma (2.3) to verify is the surjectivity of $\langle P,\cdot\rangle$. This can be done with either the ramification theoretic argument of [2] or by applying [14, Main Lemma (3.5), p. 67] (any $\eta \in \Lambda_\sigma^*$ induces a ring homomorphism $Z(\Lambda_\sigma) \to Z$; the kernel of the composition of this homomorphism with Φ_P corresponds to an ordering T with $\langle P,T\rangle = \eta$).

Let $t_\sigma\colon W(F_\sigma) \to \prod_{\mathcal{O}(F_\sigma)} Z$ denote the total signature map on $W(F_\sigma)$. Thus

$$t_\sigma(\langle a\rangle) = ((a,P))_{P\in\mathcal{O}(F_\sigma)} \qquad (a \in F_\sigma^{\cdot}).$$

t_σ is injective [15, Satz 22]; we now compute its image. (This computation was done in [6, §3], but in a form less convenient for our purposes here.)

Give the discrete group Λ_σ the measure μ with $\mu(\{\lambda\}) = 1$ for all $\lambda \in \Lambda_\sigma$. We will also use μ to denote the unique translation-invariant measure on Λ_σ^* with $\mu(\Lambda_\sigma^*) = 1$ (cf. (2.2) and [17, 1.1.3]). For any topological space X, let

$C(X, Z)$ denote the ring of continuous integer-valued functions on X. (Z has the discrete topology.) We may regard t_σ as mapping into $C(\mathfrak{O}(F_\sigma), Z)$. Let $P \in \mathfrak{O}(F_\sigma)$; P will be fixed for the remainder of this section. We will analyze t_σ by means of a commutative diagram:

$$\begin{array}{ccc} W(F_\sigma) & \xrightarrow{t_\sigma} & C(\mathfrak{O}(F_\sigma), Z) \\ \Phi_P \downarrow & & \downarrow \Psi_P \\ Z(\Lambda_\sigma) & \xrightarrow{t'} & C(\Lambda_\sigma^*, Z). \end{array}$$

Note that $Z(\Lambda_\sigma)$ is simply the ring of continuous integer-valued functions on (the discrete group) Λ_σ with compact support. The map t' in the above diagram is Fourier transformation:

$$t'(1 \cdot \lambda)(\phi) = \phi(\lambda) \qquad (\lambda \in \Lambda_\sigma, \phi \in \Lambda_\sigma^*).$$

The map Ψ_P is the isomorphism induced by the homeomorphism $\langle P, \cdot \rangle$ of Lemma (2.3). The diagram itself is easily checked to commute. By the inversion theorem [17, 1.5.1], an element $f \in C(\Lambda_\sigma^*, Z)$ is in the image of t' if and only if $\int_{\Lambda_\sigma^*} f(\phi)\phi(\lambda)\, d\phi \in Z$ for all $\lambda \in \Lambda_\sigma$ (i.e., its Fourier transform is integer-valued). Hence an element of $C(\mathfrak{O}(F_\sigma), Z)$ is in the image of t_σ if and only if

$$\text{(2)} \qquad \int_{\Lambda_\sigma^*} \Psi_P(f)(\phi)\phi(\lambda)\, d\phi \in Z \quad \text{for all } \lambda \in \Lambda_\sigma.$$

The homeomorphism $\langle P, \cdot \rangle$ allows us to lift the measure on Λ_σ^* back to $\mathfrak{O}(F_\sigma)$. If this is done, then (2) becomes

$$\text{(3)} \qquad \int_{\mathfrak{O}(F_\sigma)} f(S)\langle P, S \rangle(\lambda)\, dS \in Z \quad \text{for all } \lambda \in \Lambda_\sigma.$$

We have proved

(2.7) THEOREM. *The image of t_σ consists of all $f \in C(\mathfrak{O}(F_\sigma), Z)$ which satisfy* (3).

Theorem (2.7) takes an especially simple form when the group Λ_σ is small enough.

(2.8) COROLLARY. *Suppose $|\Lambda_\sigma| \leqslant 4$. Then $\operatorname{Im} t_\sigma$ consists of all the maps $f \in C(\mathfrak{O}(F_\sigma), Z)$ such that $f(S) \equiv f(T) \pmod 2$ for all S, $T \in \mathfrak{O}(F_\sigma)$ and $\sum_{S \in \mathfrak{O}(F_\sigma)} f(S) \equiv 0 \pmod{|\Lambda_\sigma|}$.*

One should note that the measure we have introduced on $\mathfrak{O}(F_\sigma)$ is independent of the choice of P. For, if $T \in \mathfrak{O}(F_\sigma)$ then $\mu \circ \langle P, \cdot \rangle \circ \langle T, \cdot \rangle^{-1} = \mu$ (by the uniqueness of Haar measures), so $\mu \circ \langle P, \cdot \rangle = \mu \circ \langle T, \cdot \rangle$.

(2.9) REMARK. Only minor modifications in the above computation of $\operatorname{Im} t_\sigma$ are required if one wishes to avoid the choice of an ordering P (and use of the map $\langle P, \cdot\rangle$) [6, §3]. One can give conditions equivalent to (3) which not only avoid the use of a chosen ordering P but also avoid the use of any measure theory. We restrict ourselves here to remarking that the following conditions are equivalent (for $f \in C(\mathfrak{O}(F_\sigma), Z)$):

(A) $f \in \operatorname{Im} t_\sigma$.

(B) For every finite subgroup Γ of Λ_σ^* we have $\sum_T f(T) \equiv 0 \pmod{|\Gamma|}$ where the sum is taken over all $T \in \langle P, \cdot\rangle^{-1}(\Gamma)$.

(C) $\int_{\mathfrak{O}(F_\sigma)} f(T)(a, T)\,dT \in Z$ for all $a \in F_\sigma^{\cdot}$.

(D) $\int_{V(a)} f(T)\,dT \in \mu(V(a))Z$ for all $a \in F_\sigma^{\cdot}$.

(Here, μ is also used to denote the measure on $\mathfrak{O}(F_\sigma)$.) See [6, §3] for the equivalence of these statements.

The next two lemmas will be used in §7.

(2.10) LEMMA. *For $f \in C(\mathfrak{O}(F_\sigma), Z)$ and $P \in \mathfrak{O}(F_\sigma)$,*

$$f(P) = \sum_{\lambda \in \Lambda_\sigma} \int_{\mathfrak{O}(F_\sigma)} f(S)\langle P, S\rangle(\lambda)\,dS.$$

PROOF. Apply the inversion theorem [17, 1.5.1] to $\Psi_P(f)$.

(2.11) LEMMA. *Let $P \in \mathfrak{O}(F_\sigma)$ and $q \in W(F_\sigma)$. If $f = t_\sigma(q)$, then*

$$\Phi_P(q) = \sum_{\lambda \in \Lambda_\sigma} \left(\int_{\mathfrak{O}(F_\sigma)} f(S)\langle P, S\rangle(\lambda)\,dS\right)\lambda.$$

PROOF. Apply the inversion theorem to $\Phi_P(q)$.

3. **Exact fields: Definition.** Let $W_{\mathrm{red}}(F)$ denote the reduced Witt ring of F. We define three homomorphisms:

$$t\colon W_{\mathrm{red}}(F) \to \prod_{\mathfrak{O}(F)} Z,$$

$$t_0\colon W_{\mathrm{red}}(F) \to \prod_{\sigma \in \mathfrak{M}(F)} W(F_\sigma),$$

$$t_1\colon \prod_{\sigma \in \mathfrak{M}(F)} W(F_\sigma) \to \prod_{\mathfrak{O}(F)} Z.$$

The map t is induced by the total signature map on $W(F)$. Formally,

$$t(\langle a\rangle_{\mathrm{red}}) = ((a, P))_{P \in \mathfrak{O}(F)} \quad \text{for all } a \in F^{\cdot}$$

where q_{red} is used to denote the image of an element $q \in W(F)$ under the canonical map $W(F) \to W_{\mathrm{red}}(F)$. The map t_0 is induced by the functorial maps $W(F) \to W(F_\sigma)$ ($\sigma \in \mathfrak{M}(F)$). That is,

$$t_0(q_{\text{red}}) = (q \otimes F_\sigma)_{\sigma \in \mathfrak{M}(F)} \quad \text{for all } q \in W(F)$$

where $q \otimes F_\sigma$ denotes the image of an element $q \in W(F)$ under the map $W(F) \to W(F_\sigma)$. t_0 is well defined since each of the rings $W(F_\sigma)$ is reduced, i.e. has no nonzero nilpotent elements (Theorem (2.5)). Finally, t_1 is essentially the product of the maps t_σ ($\sigma \in \mathfrak{M}(F)$). That is,

$$(4) \qquad t_1((q_\sigma)_{\sigma \in \mathfrak{M}(F)}) = (t_{\pi(P)}(q_{\pi(P)})(P))_{P \in \mathcal{O}(F)}$$

for all $(q_\sigma)_{\sigma \in \mathfrak{M}(F)} \in \prod_{\mathfrak{M}(F)} W(F_\sigma)$. (In (4) we have identified each $P \in \mathcal{O}(F)$ with its canonical extension to an ordering in $\mathcal{O}(F_{\pi(P)})$ –cf. Lemma (2.4).)

(3.1) Lemma. $t = t_1 t_0$.

This lemma, whose proof is immediate, shows that t_0 is injective (cf. [5, Theorem 3.3]), and hence that $W_{\text{red}}(F) \cong \operatorname{Im} t_0$. We now turn to the problem of computing $\operatorname{Im} t_0$.

(3.2) Notation. Let $\sigma, \tau \in \mathfrak{M}(F)$. We let $[\sigma, \tau]$ denote the finest place through which σ and τ factor. Thus $[\sigma, \tau]$ is the place associated with the valuation ring $\sigma^{-1}(R) \cdot \tau^{-1}(R)$. We will write $v_{\sigma,\tau}$ and $\Lambda_{\sigma,\tau}$ in place of $v_{[\sigma,\tau]}$ and $\Lambda_{[\sigma,\tau]}$ (cf. (2.2)). The value group of $[\sigma, \tau]$ is a homomorphic image of the value group of σ [18, Chapter C], so we have a canonical map $w_{\sigma,\tau}\colon \Lambda_\sigma \to \Lambda_{\sigma,\tau}$. Specifically, $w_{\sigma,\tau}(v_\sigma(a)) = v_{\sigma,\tau}(a)$ for all $a \in F^{\cdot}$. Thus

$$(5) \qquad w_{\sigma,\tau} \circ v_\sigma = w_{\tau,\sigma} \circ v_\tau.$$

The map $v_{\sigma,\tau}$ induces a homomorphism $v'_{\sigma,\tau}\colon W(F_\sigma) \to Z_2(\Lambda_{\sigma,\tau})$ determined by the rule, for all $a \in F_\sigma^{\cdot}$,

$$v'_{\sigma,\tau}(\langle a \rangle) = (1 + 2Z)v_{\sigma,\tau}(a).$$

(The maps $w_{\sigma,\tau}$ and $Z \to Z_2 = Z/2Z$ together induce a homomorphism $w''_{\sigma,\tau}\colon Z(\Lambda_\sigma) \to Z_2(\Lambda_{\sigma,\tau})$ whose composition with any of the maps Φ_P of Theorem (2.5) gives the map $v'_{\sigma,\tau}$ above. This proves existence.)

We now define the homomorphism

$$v_{**}\colon \prod_{\mathfrak{M}(F)} W(F_\sigma) \to \prod_{\sigma,\tau \in \mathfrak{M}(F)} Z_2(\Lambda_{\sigma,\tau})$$

by the rule: for any $q = (q_\rho)_{\rho \in \mathfrak{M}(F)} \in \prod_{\rho \in \mathfrak{M}(F)} W(F_\rho)$,

$$v_{**}(q) = (v'_{\sigma,\tau}(q_\sigma) + v'_{\tau,\sigma}(q_\tau))_{\sigma,\tau \in \mathfrak{M}(F)}.$$

Notice that the kernel of v_{**} consists of all $(q_\rho)_\rho$ with

$$(6) \qquad v'_{\sigma,\tau}(q_\sigma) = v'_{\tau,\sigma}(q_\tau)$$

for all $\sigma, \tau \in \mathfrak{M}(F)$. (The "compatibility condition" (6) is closely related to those of [3, Theorem 2.1] and [18, Theorem 3, p. 136].)

(3.3) DEFINITION. F is called *exact* if the sequence

$$W_{\mathrm{red}}(F) \to t_1^{-1}(C(\mathfrak{O}(F), Z)) \to \prod_{\sigma,\tau \in \mathfrak{M}(F)} Z_2(\Lambda_{\sigma,\tau}) \tag{7}$$

is an exact sequence. The homomorphisms in (7) are obtained by restricting the codomain and domain of t_0 and v_{**}, respectively.

One should note that t_0 does map into $t_1^{-1}(C(\mathfrak{O}(F), Z))$; this follows from Lemma (3.1) and the fact that t maps into $C(\mathfrak{O}(F), Z)$. Later we shall see that if we give $\bigcup_{\mathfrak{M}(F)} W(F_\sigma)$ the "open-path topology", then $t_1^{-1}(C(\mathfrak{O}(F), Z))$ is precisely the set of continuous maps in $\prod_{\mathfrak{M}(F)} W(F_\sigma)$ (cf. Lemma (5.10)). Thus F is exact if and only if $\operatorname{Im} t_0$ consists of all continuous elements of $\prod_{\mathfrak{M}(F)} W(F_\sigma)$ which satisfy the compatibility condition (6).

(3.4) REMARK. For any field F we have

$$\operatorname{Im} t_0 \subseteq t_1^{-1}(C(\mathfrak{O}(F), Z)) \cap \operatorname{Ker} v_{**} \tag{8}$$

and

$$\operatorname{Im} t \subseteq C(\mathfrak{O}(F), Z) \cap t_1(\operatorname{Ker} v_{**}). \tag{9}$$

F is an exact field precisely when equality holds in one (and hence in both) of (8) or (9).

If F is exact, then $W_{\mathrm{red}}(F)$ is determined up to ring isomorphism by the following basic invariants of F: the spaces $\mathfrak{O}(F)$ and $\mathfrak{M}(F)$, the groups Λ_σ and $\Lambda_{\sigma,\tau}$ $(\sigma, \tau \in \mathfrak{M}(F))$, and the maps $w_{\sigma,\tau}$ and $\langle P, T\rangle$ $(\sigma, \tau \in \mathfrak{M}(F)$ and $P, T \in \mathfrak{O}_\sigma)$. (This is an immediate consequence of Theorem (7.1) below. It also can be easily deduced from Definition (3.3); one uses Lemma (2.3) and Theorem (2.5) to replace the map t_1 in (7) by an appropriate map $\prod_{\mathfrak{M}(F)} Z(\Lambda_\sigma) \to \prod_{\mathfrak{O}(F)} Z$; for a similar argument see Remark (6.2D) below.) The computation of these invariants is more or less in the domain of general valuation theory (and expecially ramification theory).

The next section is devoted to showing that various classes of fields are exact. In the remainder of this section we specialize some of the concepts above to some cases of particular interest.

(3.5) REMARKS. (A) One can easily show that $|\Lambda_\sigma| \leqslant 2$ for all $\sigma \in \mathfrak{M}(F)$ if and only if every $f \in C(\mathfrak{O}(F), Z)$ of constant parity is in $\operatorname{Im} t_1$ (cf. Theorem (2.7) and Corollary (2.8)). ($f \in C(\mathfrak{O}(F), Z)$ has "constant parity" if and only if $f(P) \equiv f(T) \pmod 2$ for all $P, T \in \mathfrak{O}(F)$.) Any formally real algebraic extension of $Q(x)$ has the above property.

(B) It is also easy to show that $\Lambda_{\sigma,\tau} = 1$ for all $\sigma, \tau \in \mathfrak{M}(F)$ with $\sigma \neq \tau$ if and only if $\operatorname{Ker} v_{**}$ consits precisely of those elements of $\prod_{\mathfrak{M}(F)} W(F_\sigma)$ of constant dimension-index. ($g \in \prod_{\mathfrak{M}(F)} W(F_\sigma)$ has "constant dimension-index" when for every $\sigma, \tau \in \mathfrak{M}(F)$, the dimension-index of $g(\sigma)$ equals that of $g(\tau)$.) Any formally real algebraic extension of $R(x)$ has the above property.

(C) If F is a field satisfying the conditions in (A) and (B) above, then F is exact if and only if

$$\operatorname{Im} t = C(\mathcal{O}(F), 2Z) + Z \cdot 1 \tag{10}$$

(cf. (9) above). The fields which satisfy condition (10) are called SAP fields [12, Corollary 3.21], [9, Definition 1.4].

(D) Elman, Lam and Prestel [10, p. 295] have shown that F is a SAP field if and only if F satisfies the following two conditions of Prestel [16, (2.2)]:

(A′) $|\Lambda_\sigma| \leqslant 2$ for any place σ from F into a formally real field.

(B′) $|\mathcal{O}(K)| = 1$ for any place $\sigma\colon F \to K \cup \{\infty\}$ onto a formally real field K with $\Lambda_\sigma \neq 1$.

It is easy to check that (A′) and the conditions in (A) are equivalent, and that when (A′) holds then (B′) and the conditions in (B) are equivalent. The necessity of (A′) for SAP fields follows immediately. One can prove the necessity of (B′) for SAP fields from the observation that if $|\Lambda_\sigma| \leqslant 2$ for all $\sigma \in \mathfrak{M}(F)$ and $t_1(\operatorname{Ker} v_{**}) \supset C(\mathcal{O}(F), 2Z)$, then $\Lambda_{\sigma,\tau} = 1$ for all $\sigma \neq \tau$ in $\mathfrak{M}(F)$. (*Sketch of proof.* Suppose $\Lambda_{\sigma,\tau} \neq 1$. Then there is an open and closed set $U \subset \mathcal{O}(F)$ with $|U \cap (\mathcal{O}_\sigma \cup \mathcal{O}_\tau)| = 3$. The map sending U to 2 and the rest of $\mathcal{O}(F)$ to 0 cannot be in $t_1(\operatorname{Ker} v_{**})$.)

Fields satisfying conditions (A) or (B) above will be studied again in later sections.

4. Exact fields: Examples.

(4.1) PROPOSITION. *All SAP fields are exact.*

Thus, F is exact if it is isomorphic to the field $Q((x))$ of Laurent series, or if it is an algebraic extension of either $R((x))$, Q, or the field of rational functions $R(x)$. To prove the proposition, note that for any field F we have

$$\operatorname{Im} t \subseteq C(\mathcal{O}(F), Z) \cap t_1(\operatorname{Ker} v_{**}) \subseteq Z \cdot 1 + C(\mathcal{O}(F), 2Z). \tag{11}$$

If F is a SAP field, then both inclusions in (11) must be equality, and so F is exact (cf. Remarks (3.4) and (3.5C)).

(4.2) PROPOSITION. *All superordered fields are exact. In particular, all superpythagorean fields are exact.*

Recall that a field F is called *superordered* if every subgroup of $F^{\cdot}$ of index two containing $\sum F^{\cdot 2}$ and excluding -1 is an ordering of F [6]. (Here, $\sum F^{\cdot 2}$

denotes the subgroup of $F^{\cdot}$ of sums of squares.) The superpythagorean fields are precisely the Pythagorean superordered fields. The proposition implies, for example, that $R((x))((y))$ (iterated Laurent series) and $Q[\sqrt{2}]((x))$ are exact. Neither of these fields is a SAP field.

PROOF. Suppose F is superordered. If F admits only one real place, it is clearly exact. Hence we may suppose without loss of generality that F admits exactly two real places σ and τ and that the maps $w_{\sigma,\tau}$ and $w_{\tau,\sigma}$ are isomorphisms [6, Corollary 10]. Suppose $(q,q') \in \operatorname{Ker} v_{**} \subseteq W(F_\sigma) \times W(F_\tau)$. We must show $(q,q') \in \operatorname{Im} t_0$. Without loss of generality we may assume $q' = 0$. (If necessary, consider $(q,q') - t_0(q''_{\text{red}})$ where $q'' \in W(F)$ has $q'' \otimes F_\tau = q'$.) Then $v'_{\sigma,\tau}(q) = 0$, so $q \in 2W(F_\sigma)$ (Theorem (2.5)). thus there exists $q_0 \in W(F)$ with $2(q_0 \otimes F_\sigma) = q$. Pick $c \in F^{\cdot}$ with $\tau(c) < 0 < \sigma(c) < \infty$ [3, Theorem 2.1A]. Then t_0 maps $(\langle 1, c\rangle q_0)_{\text{red}}$ to (q,q').

The next lemma gives a criterion for exactness which will be applied in the proofs of Theorems (4.4) and (4.5).

(4.3) LEMMA. *F is exact if and only if the map* $W_{\text{red}}(F)/2W_{\text{red}}(F) \to \operatorname{Ker} v_{**}/2 \operatorname{Ker} v_{**}$ *induced by t_0 is injective.*

PROOF. ($\Rightarrow$) Suppose $q \in W_{\text{red}}(F)$ and $t_0(q) = 2g$, where $g \in \operatorname{Ker} v_{**}$. Then $t_1(g) = \frac{1}{2}t(q)$ is continuous. Hence $g = t_0(q')$ for some $q' \in W_{\text{red}}(F)$ (F is exact). Since $t_0(2q') = t_0(q)$, we have $q \in 2W_{\text{red}}(F)$ (t_0 is injective).

($\Leftarrow$) Let $g \in \operatorname{Ker} v_{**} \cap t_1^{-1}(C(\mathfrak{S}(F), Z))$. We must show $g \in \operatorname{Im} t_0$. There exists a least integer $m \geqslant 0$ such that for some $q_m \in W_{\text{red}}(F)$,

$$t_1 t_0(q_m) = t(q_m) = 2^m t_1(g) = t_1(2^m g)$$

[12, Theorem 3.18(i)]. Thus $t_0(q_m) = 2^m g$ (t_1 is injective since it is a product of total signature maps on reduced Witt rings). If $m > 0$, then by hypothesis $q_m = 2q_{m-1}$ for some $q_{m-1} \in W_{\text{red}}(F)$. But this implies that $t(q_{m-1}) = 2^{m-1} t_1(g)$ ($W_{\text{red}}(F)$ is torsion-free [15, Satz 22]) contradicting the choice of m. Hence $m = 0$, so $g = t_0(q_m)$.

(4.4) THEOREM. *Suppose all the formally real simple algebraic extensions of F are exact. Then the rational function field $F(x)$ is exact.*

(4.5) APPLICATIONS. (A) $R(x)$ is exact. $R(x)$ also satisfies conditions (A) and (B) of Remark (3.5) and hence is a SAP field. For other proofs of this see [10, Theorem I] and [7, Theorem 15].

(B) $R(x,y)$ (rational functions in two variables) and $Q(x)$ are exact. Neither is a SAP field. More generally, $F(y)$ is exact if F is algebraic over Q or $R(x)$ (see, for example, [10, Theorem I] and [3, Theorem 4.1]). We will study the total signature map on these fields in §7.

(C) Further applications of (4.4) will be made when we have more examples of exact fields (cf. (4.8C) and (6.2B)).

Proof of (4.4). Suppose $q \in W(F(x))$ and $t_0(q_{\text{red}}) \in 2\operatorname{Ker} v_{**}$. It suffices to show that $q_{\text{red}} \in 2W_{\text{red}}(F(x))$ (Lemma (4.3)).

Let $\pi \in F[x]$ be monic and irreducible. Set $F_\pi = F[x]/(\pi)$ and let $\phi = \phi_\pi$: $F(x) \to F_\pi \cup \{\infty\}$ be the canonical π-adic place. The Springer residue homomorphisms $\delta_{i,\pi}$: $W(F(x)) \to W(F_\pi)$ $(i = 1,2)$ are determined by the rule that for each $a \in \phi^{-1}(F_\pi^{\cdot})$ and $j \in Z$ we have $\delta_{i,\pi}(\langle \pi^j a\rangle)$ equal to 0 if $i + j$ is even and equal to $\langle \phi(a)\rangle$ if $i + j$ is odd [14, p. 85, Lemma (1.2)]. Now suppose F_π is formally real. Let $\sigma \in \mathfrak{M}(F_\pi)$. Then $\sigma\phi \in \mathfrak{M}(F(x))$ and ϕ extends to a place $F(x)_{\sigma\phi} \to (F_\pi)_\sigma \cup \{\infty\}$ whose valuation ring is discrete rank one with prime element π. (The place ϕ can be extended to a place $\bar\phi$ on a maximal field K containing $F(x)$ such that $\bar\phi$ has the same value group as ϕ and $\bar\phi$ has residue class field $(F_\pi)_\sigma$ [19, Theorem 5, p. 38 and Lemma 19, p. 218]. But then $(K, \bar\sigma\bar\phi)$ is the ultracompletion of $F(x)$ at $\sigma\phi$ [5, Lemmas 1.16 and 1.17].) Let

$$\delta_{i,\sigma}\colon W(F(x)_{\sigma\phi}) \to W((F_\pi)_\sigma) \qquad (i = 1,2)$$

be the corresponding Springer residue maps. We then have a commutative diagram

$$\begin{tikzcd} W(F(x)) \arrow[r, "\delta_{i,\pi}"] \arrow[d] & W(F_\pi) \arrow[d] \\ W(F(x)_{\sigma\phi}) \arrow[r, "\delta_{i,\sigma}"] & W((F_\pi)_\sigma) \end{tikzcd} \tag{12}$$

where the vertical maps are the functorial maps.

Now let $\sigma, \tau \in \mathfrak{M}(F_\pi)$. Then $[\sigma\phi, \tau\phi] = [\sigma, \tau]\phi$ so we have an exact sequence

$$1 \to \Lambda_{\sigma,\tau} \xrightarrow{\eta} \Lambda_{\sigma\phi,\tau\phi} \to \Lambda_\phi \to 1$$

([18, Chapter C]; the injectivity of the map η follows from the fact that the value group of $[\sigma, \tau]$ corresponds to a convex subgroup of the value group of $[\sigma\phi, \tau\phi]$). Now, every element of $\Lambda_{\sigma\phi,\tau\phi}$ has the form $\eta(\lambda)v_{\sigma\phi,\tau\phi}(\pi^j)$ for some $\lambda \in \Lambda_{\sigma,\tau}$ and $j \in Z$. Thus we have homomorphisms

$$\bar\delta_i\colon Z_2(\Lambda_{\sigma\phi,\tau\phi}) \to Z_2(\Lambda_{\sigma,\tau}) \qquad (i = 1,2)$$

which take each $\eta(\lambda)v_{\sigma\phi,\tau\phi}(\pi^j)$ to 0 if $i + j$ is even and to λ if $i + j$ is odd. The diagram

$$\begin{tikzcd} W(F(x)_{\sigma\phi}) \arrow[r, "\delta_{i,\sigma}"] \arrow[d, "v'_{\sigma\phi,\tau\phi}"'] & W((F_\pi)_\sigma) \arrow[d, "v'_{\sigma,\tau}"] \\ Z_2(\Lambda_{\sigma\phi,\tau\phi}) \arrow[r, "\bar\delta_i"] & Z_2(\Lambda_{\sigma,\tau}) \end{tikzcd} \tag{13}$$

is easily checked to commute for $i = 1, 2$ (cf. (3.2)).

By hypothesis there exist $q_\sigma \in W(F(x))$ (for each $\sigma \in \mathfrak{M}(F(x))$) such that

$$t_0(q_{\mathrm{red}}) = 2(q_\sigma \otimes F(x)_\sigma)_{\sigma \in \mathfrak{M}(F)} \in 2 \operatorname{Ker} v_{**} \tag{14}$$

(the maps $W(F) \to W(F_\sigma)$ are surjective for any $\sigma \in \mathfrak{M}(F)$ since $F_\sigma^{\cdot} = F_\sigma^{\cdot 2} \cdot F^{\cdot}$, cf. Lemma (2.1)). For $i = 1, 2$ we have

$$\begin{aligned} t_0(\delta_{i,\pi}(q)) &= (\delta_{i,\pi}(q) \otimes (F_\pi)_\sigma)_{\sigma \in \mathfrak{M}(F_\pi)} \\ &= (\delta_{i,\sigma}(q \otimes F(x)_{\sigma\phi}))_{\sigma \in \mathfrak{M}(F_\pi)} \\ &= 2(\delta_{i,\sigma}(q_{\sigma\phi} \otimes F(x)_{\sigma\phi}))_{\sigma \in \mathfrak{M}(F_\pi)} \end{aligned}$$

(apply (12) and (14)). Then $t_0(\delta_{i,\pi}(q)) \in 2 \operatorname{Ker} v_{**}$, since for any σ and $\tau \in \mathfrak{M}(F_\pi)$ we have

$$\begin{aligned} v'_{\sigma,\tau}(\delta_{i,\sigma}(q_{\sigma\phi} \otimes F(x)_{\sigma\phi})) &= \bar{\delta}_i v'_{\sigma\phi,\tau\phi}(q_{\sigma\phi} \otimes F(x)_{\sigma\phi}) \\ &= \bar{\delta}_i(v'_{\tau\phi,\sigma\phi}(q_{\tau\phi} \otimes F(x)_{\tau\phi})) \end{aligned}$$

(apply (13) and (14)) which equals $v'_{\tau,\sigma}(\delta_{i,\tau}(q_{\tau\phi} \otimes F(x)_{\tau\phi}))$ (repeat the above computation with σ and τ interchanged). Since F_π is, by hypothesis, exact, there exists $q_{i,\pi} \in W(F_\pi)$ with $\delta_{i,\pi}(q) - 2q_{i,\pi}$ in the nil radical of $W(F_\pi)$, and hence additively torsion (cf. [15, Satz 22] and Lemma (4.3)). We may and we do assume that $q_{i,\pi} = 0$ if $\delta_{i,\pi}(q)$ is itself torsion.

Thus we have chosen elements $q_{i,\pi} \in W(F_\pi)$ for $i = 1, 2$ and for all π with F_π formally real. If F_π is not formally real, let us set $q_{i,\pi} = 0$ ($i = 1,2$).

The Milnor exact sequence [13, Theorem 5.3]

$$0 \to W(F) \xrightarrow{\varepsilon} W(F(x)) \xrightarrow{\delta} \bigoplus_\pi W(F_\pi) \to 0$$

is split by $\delta_{1,x}$. (The map ε is the functorial map; δ is induced by the maps $\delta_{2,\pi}$. The sum extends over all monic irreducible π.) Hence there exists a right inverse ψ of δ with $\psi\delta + \varepsilon\delta_{1,x}$ the identity map on $W(F(x))$. Recall that $q_{2,\pi} = 0$ if $\delta_{2,\pi}(q) = 0$. It follows that $(q_{2,\pi})_\pi \in \oplus_\pi W(F_\pi)$. Hence there exists $q' \in W(F(x))$ with $\delta(q') = (q_{2,\pi})_\pi$. Notice that

$$\delta(q - 2q') = (\delta_{2,\pi}(q) - 2q_{2,\pi})_\pi \tag{15}$$

is torsion. Let us write

$$\begin{aligned} q &= 2q' + (\psi\delta + \varepsilon\delta_{1,x})(q - 2q') \\ &= 2q' + \psi\delta(q - 2q') + \varepsilon(2q_{1,x} + \delta_{1,x}(q) - 2q_{1,x} - 2\delta_{1,x}(q')) \\ &= 2(q' + \varepsilon(q_{1,x} - \delta_{1,x}(q'))) + (\psi\delta(q - 2q') + \varepsilon(\delta_{1,x}(q) - 2q_{1,x})). \end{aligned}$$

Since the second term above is torsion, $q_{\text{red}} \in 2W_{\text{red}}(F(x))$. This completes the proof of Theorem (4.4).

(4.6) REMARK. The reader can easily prove that if $F(x)$ is exact, then F is also exact. We do not know, however, if the converse of (4.4) is true. The converse is implied by the following plausible assertion: if F is the Henselization of an exact field, then F is exact.

(4.7) THEOREM. *Let τ: $F \to K \cup \{\infty\}$ be a surjective place such that $\tau^{-1}(1) \subset \sum F^{\cdot 2}$. Then F is exact if and only if K is exact.*

Note that our hypotheses in (4.7) ensure that K is formally real. As usual, $\sum F^{\cdot 2}$ denotes the group of elements of $F^{\cdot}$ which are sums of squares.

(4.8) APPLICATIONS. (A) If F is Henselian, then it is exact if and only if its residue class field is exact.

(B) F is exact if it is an algebraic extension of either $R((x_1))\cdots((x_n))$ or $Q((x_1))\cdots((x_n))$ (iterated Laurent series). For example, $Q[\sqrt{2},\sqrt{3}]((x))$ is exact; it is neither superordered nor a SAP field.

(C) $F(x)$ is exact if F is any of the fields in (B) (Theorem (4.4)).

(D) If F is superordered, then it admits a place τ: $F \to K \cup \{\infty\}$ with $\tau^{-1}(K) \subseteq \sum F^{\cdot 2}$ and with K a SAP field [6, Theorem 1 and Lemma 2]. Thus, superordered fields are exact. This reproves (4.2).

We begin the proof of Theorem (4.7) with two lemmas.

(4.9) LEMMA. *Let τ be as in* (4.7). *There exists an injective additive homomorphism ψ: $W_{\text{red}}(K) \to W_{\text{red}}(F)$ with $\psi(\langle\tau(a)\rangle) = \langle a\rangle$ for all $a \in \tau^{-1}(K^{\cdot})$.*

PROOF. Let $a_1, \ldots, a_n \in \tau^{-1}(K^{\cdot})$. It suffices to show that $\langle a_1,\ldots,a_n\rangle_{\text{red}} = 0$ if and only if $\langle\tau(a_1),\ldots,\tau(a_n)\rangle_{\text{red}} = 0$. The place τ induces a map $\mathcal{O}(F) \to \mathcal{O}(K)$ (each $P \in \mathcal{O}(F)$ is mapped to $\tau(P) \cap K^{\cdot}$). This map is surjective [2]. Hence the total signature map on $W(F)$ kills $\langle a_1,\ldots,a_n\rangle$ if and only if the total signature map on $W(K)$ kills $\langle\tau(a_1),\ldots,\tau(a_n)\rangle$. Our conclusion now follows from [15, Satz 22].

(4.10) LEMMA. *ψ has a left inverse ψ': $W_{\text{red}}(F) \to W_{\text{red}}(K)$ which is an additive homomorphism having $\psi'(\langle a\rangle) = 0$ for all $a \in F^{\cdot}$ with $v_\tau(a) \neq 1$.*

Note that the map ψ' is uniquely determined since if $v_\tau(a) = 1$ then $a = bc^2$ for some $c \in F^{\cdot}$ and $b \in \tau^{-1}(K^{\cdot})$, whence $\psi'(\langle a\rangle_{\text{red}}) = \langle\tau(b)\rangle_{\text{red}}$.

PROOF OF (4.10). Let $a_1, \ldots, a_n \in F^{\cdot}$. After reindexing and multiplying the a_i by squares, we may assume that for some $m \leqslant n$ we have $a_i \in \tau^{-1}(K^{\cdot})$ for all $i \leqslant m$ and $v_\tau(a_i) \neq 1$ for all $i > m$. Suppose $\langle a_1,\ldots,a_n\rangle_{\text{red}} = 0$. It suffices to show that $\langle\tau(a_1),\ldots,\tau(a_m)\rangle_{\text{red}} = 0$. Hence we must show

$$\langle a_1,\ldots,a_m\rangle_{\text{red}} = 0$$

(Lemma (4.9)). We accomplish this by showing that the signature of $\langle a_1,\dots,a_m\rangle$ at every ordering of F is zero. So let $P \in \mathfrak{O}(F)$. Write $\Lambda_\tau = \Gamma \oplus \Gamma'$ (internal direct sum) where Γ is a finite group containing $v_\tau(a_i)$ for all $i \leqslant n$. Then $\Gamma^* = \{\phi \in \Lambda_\tau^*: \phi(\Gamma') = 1\}$ is finite. There exists a bijection ξ: $\{T \in \mathfrak{O}(F): \tau(T) = \tau(P)\} \to \Lambda_\tau^*$ with $\xi(T)(v_\tau(a)) = (a,T)(a,P)$ for all $a \in F^{\cdot}$ (see [2, Appendix] for this generalization of Lemma (2.3)). Let $\Delta = \xi^{-1}(\Gamma^*)$. For each $a \in v_\tau^{-1}(\Gamma)$ we have

$$\sum_{T\in\Delta}(a,T) = (a,P)\sum_{T\in\Delta}\xi(T)(v_\tau(a)) = (a,P)\sum_{\phi\in\Gamma^*}\phi(v_\tau(a))$$

which equals 0 if $v_\tau(a) \notin \Gamma'$ and which equals $(a,P)|\Gamma^*|$ if $v_\tau(a) \in \Gamma'$. Since the signature of $\langle a_1,\dots,a_n\rangle$ is zero at every $T \in \Delta$, we have

$$0 = \sum_{T\in\Delta}\sum_{i=1}^{n}(a_i,T) = \sum_{i=1}^{m}(a_i,P)|\Gamma^*|.$$

Hence the signature of $\langle a_1,\dots,a_m\rangle$ at P is zero. The lemma is proved.

(4.11) REMARK. Let F' be the Henselization of F at τ (hypotheses as in (4.7)). Then the natural map $W_{\text{red}}(F) \to W_{\text{red}}(F')$ is easily verified to be an isomorphism. The reader is probably aware that $W(F')$ is isomorphic to $W(K)(\Lambda_\tau)$ (not canonically, however). Hence we have an isomorphism $W_{\text{red}}(F) \cong W_{\text{red}}(K)(\Lambda_\tau)$. The two lemmas above can be deduced from the existence of such an isomorphism with appropriate properties.

PROOF OF (4.7). Let $\sigma, \sigma' \in \mathfrak{M}(K)$. Then $[\sigma,\sigma']\tau = [\sigma\tau,\sigma'\tau]$. Hence we may identify $\Lambda_{\sigma,\sigma'}$ with a subgroup of $\Lambda_{\sigma\tau,\sigma'\tau}$. Therefore we have a surjective additive homomorphism $\psi'_{\sigma,\sigma'}$: $Z_2(\Lambda_{\sigma\tau,\sigma'\tau}) \to Z_2(\Lambda_{\sigma,\sigma'})$ (namely, for each $\lambda \in \Lambda_{\sigma\tau,\sigma'\tau}$ map $1\cdot\lambda$ to itself if $\lambda \in \Lambda_{\sigma,\sigma'}$ and to 0 otherwise). Let τ_σ: $E \to K_\sigma \cup \{\infty\}$ be a maximal extension of τ to a place with the same value group as τ and with residue class field K_σ. We may identify E with $F_{\sigma\tau}$ (use [18, Proposition 8, p. 114] to apply Lemma (2.1)). Since $\tau_\sigma^{-1}(1) \subset F_{\sigma\tau}^{\cdot 2}$ (maximal fields are Henselian), we may apply Lemma (4.10) to get a map ψ'_σ: $W_{\text{red}}(F_{\sigma\tau}) \to W_{\text{red}}(K_\sigma)$. We then have a commutative diagram

$$(16)\qquad \begin{array}{ccccc}
W_{\text{red}}(F) & \longrightarrow & W_{\text{red}}(F_{\sigma\tau}) & \xrightarrow{v'_{\sigma\tau,\sigma'\tau}} & Z_2(\Lambda_{\sigma\tau,\sigma'\tau}) \\
\downarrow{\scriptstyle\psi'} & & \downarrow{\scriptstyle\psi'_\sigma} & & \downarrow{\scriptstyle\psi'_{\sigma,\sigma'}} \\
W_{\text{red}}(K) & \longrightarrow & W_{\text{red}}(K_\sigma) & \xrightarrow{v'_{\sigma,\sigma'}} & Z_2(\Lambda_{\sigma,\sigma'})
\end{array}$$

(cf. Lemma (4.10)).

Now suppose K is exact. Let $q \in t_0^{-1}(2\operatorname{Ker} v_{**})$. We must show $q \in 2W_{\text{red}}(F)$ (Lemma (4.3)). We may represent q by an element of $W(F)$ of the form

$$\sum_{i=1}^{n} \langle a_i \rangle \otimes \langle b_{i,1}, b_{i,2}, \ldots, b_{i,s_i} \rangle$$

where the $b_{i,j} \in \tau^{-1}(K^{\cdot})$ and the values $v_\tau(a_1), \ldots, v_\tau(a_n)$ are all distinct. It suffices to show that $\langle b_{i,1}, \ldots, b_{i,s_i} \rangle_{\text{red}}$ is divisible by 2 for all $i \leqslant n$. So pick any $i \leqslant n$. Without loss of generality, $i = 1$ and $a_1 = 1$ (if necessary, reindex and multiply by $\langle a_1 \rangle$). The commutativity of the first square of (16) implies that $t_0(\psi'(q)) \in 2 \prod_{\mathfrak{M}(K)} W(K_\sigma)$. For all $\sigma, \sigma' \in \mathfrak{M}(K)$ we have

$$v'_{\sigma,\sigma'}(\tfrac{1}{2}\psi'(q) \otimes K_\sigma) = v'_{\sigma,\sigma'}(\psi'_\sigma(\tfrac{1}{2}q \otimes F_{\sigma\tau}))$$

which by the second half of (16) equals $\psi'_{\sigma,\sigma'} v'_{\sigma\tau,\sigma'\tau}(\frac{1}{2}q \otimes F_{\sigma\tau})$, which by our hypotheses on q (and (16)) equals

$$\psi'_{\sigma,\sigma'} v'_{\sigma'\tau,\sigma\tau}(\tfrac{1}{2}q \otimes F_{\sigma'\tau}) = v'_{\sigma',\sigma}(\tfrac{1}{2}\psi'(q) \otimes K_{\sigma'}).$$

Thus $t_0(\psi'(q)) \in 2 \operatorname{Ker} v_{**}$ (here, v_{**} is the map $\prod_\sigma W(K_\sigma) \to \prod_{\sigma,\sigma' \in \mathfrak{M}(K)} Z_2(\Lambda_{\sigma,\sigma'})$). Since K is exact, $\psi'(q) \in 2W_{\text{red}}(K)$ (Lemma (4.3)). Thus

$$\langle b_{1,1}, \ldots, b_{1,s_1} \rangle_{\text{red}} = \psi\psi'(q) \in 2W_{\text{red}}(F)$$

(Lemmas (4.9) and (4.10)), which was to be proved.

The proof of the converse is similar but easier; since the converse is not used below we will only sketch the proof. Composition with τ gives a bijection $\mathfrak{M}(K) \to \mathfrak{M}(F)$ [6, Lemma 2]. For each $\sigma, \sigma' \in \mathfrak{M}(K)$ we have a commutative diagram

$$\begin{array}{ccccc}
W_{\text{red}}(K) & \longrightarrow & W(K_\sigma) & \xrightarrow{v'_{\sigma,\sigma'}} & Z_2(\Lambda_{\sigma,\sigma'}) \\
\downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \psi_\sigma} & & \downarrow{\scriptstyle \psi_{\sigma,\sigma'}} \\
W_{\text{red}}(F) & \longrightarrow & W(F_{\sigma\tau}) & \xrightarrow{v'_{\sigma\tau,\sigma'\tau}} & Z_2(\Lambda_{\sigma\tau,\sigma'\tau})
\end{array}$$

where ψ_σ is defined using Lemma (4.9) and $\psi_{\sigma,\sigma'}$ is induced by the canonical map $\Lambda_{\sigma,\sigma'} \to \Lambda_{\sigma\tau,\sigma'\tau}$. One can use this diagram to show that if $t_0(q) \in 2 \operatorname{Ker} v_{**}$ (where $q \in W_{\text{red}}(K)$ and t_0 is the map on $W_{\text{red}}(K)$), then $t_0(\psi(q)) \in 2 \operatorname{Ker} v_{**}$ (where t_0 is now the map on $W_{\text{red}}(F)$). One then applies Lemmas (4.3), (4.9), and (4.10) to show that $q \in 2W_{\text{red}}(K)$, so that K is exact.

5. **Near exact fields.** Let v: $F^{\cdot} \to \prod_{\mathfrak{M}(F)} \Lambda_\sigma$ be the "factorization" map,

$$v(a) = (v_\sigma(a))_\sigma \quad \text{for all } a \in F^{\cdot}.$$

Let $\bar{v}$: $Z(F^{\cdot}) \to \prod_{\mathfrak{M}(F)} Z_2(\Lambda_\sigma)$ be the ring homomorphism induced by v. (Thus $\bar{v}(1 \cdot a) = ((1 + 2Z)v_\sigma(a))$ for all $a \in F^{\cdot}$.) If F has transcendence degree one over Q, then $\operatorname{Im} t$ can be described in terms of $\operatorname{Im} v$ and purely "local" invariants: $\operatorname{Im} t$ consists of all $f \in Z \cdot 1 + C(\mathcal{O}(F), 2Z)$ such that for some $\gamma \in \operatorname{Im} v$,

$$f(P) - f(T) \equiv 1 - \langle P, T\rangle(\gamma(\sigma)) \pmod 4$$

for all $\sigma \in \mathfrak{M}(F)$, P, $T \in \mathcal{O}_\sigma$ (see Application (5.12); the above condition is automatically satisfied unless $P \neq T$, in which case $\langle P, T\rangle(\gamma(\sigma))$ is the image of $\gamma(\sigma)$ under the canonical isomorphism $\Lambda_\sigma \to Z^{\cdot}$). We now study a more general class of fields for which $\operatorname{Im} t_0$ (and hence $\operatorname{Im} t$) can be described rather simply in terms of $\operatorname{Im} \bar{v}$ and "local" invariants.

Let v_*: $\prod_{\mathfrak{M}(F)} W(F_\sigma) \to \prod_{\mathfrak{M}(F)} Z_2(\Lambda_\sigma)$ be the product of the canonical homomorphisms v'_σ: $W(F_\sigma) \to Z_2(\Lambda_\sigma)$ (namely, $\langle a\rangle \mapsto (1 + 2Z)v_\sigma(a)$ for all $a \in F^{\cdot}$, cf. Theorem 2.5).

(5.1) Definition. Call F *near exact* when $\operatorname{Im} t_0$ consists of all $g \in \prod_{\mathfrak{M}(F)} W(F_\sigma)$ with $t_1(g)$ continuous and $v_*(g) \in \operatorname{Im} \bar{v}$.

Our main objective in this section is to show that various classes of fields are near exact.

(5.2) Proposition. *Every exact field is near exact.*

The above proposition is a corollary to Lemma (4.3) characterizing exact fields and the following analogous characterization of near exact fields.

(5.3) Lemma. *F is near exact if and only if every element of $W_{\text{red}}(F)$ which t_0 maps into $4 \prod_{\mathfrak{M}(F)} W(F_\sigma)$ is itself in $2W_{\text{red}}(F)$.*

One has a similar result for SAP fields: F is a SAP field iff every element of $W_{\text{red}}(F)$ which is divisible by 4 in the Witt ring of every real closure of F is divisible by 2 in $W_{\text{red}}(F)$.

Proof of (5.3). ($\Rightarrow$) Suppose $q \in W_{\text{red}}(F)$ and $t_0(q) \in 4 \prod_{\mathfrak{M}(F)} W(F_\sigma)$. Then

$$v_*(\tfrac{1}{2}t_0(q)) \in 2v_*(\textstyle\prod_{\mathfrak{M}(F)} W(F_\sigma)) = \{0\} \subset \operatorname{Im} \bar{v},$$

so by hypothesis $\frac{1}{2}t_0(q) \in \operatorname{Im} t_0$. Thus $q \in 2W_{\text{red}}(F)$ (t_0 is injective).

($\Leftarrow$) Suppose $g \in \prod_{\mathfrak{M}(F)} W(F_\sigma)$ has $t_1(g)$ continuous and $v_*(g) \in \operatorname{Im} \bar{v}$. We must show $g \in \operatorname{Im} t_0$. There exists a least integer $n \geqslant 0$ and $q_n \in W_{\text{red}}(F)$ with $t(q_n) = 2^n t_1(g)$ [12, Theorem 3.18(i)]. Then $t_0(q_n) = 2^n g$. Just suppose $n \neq 0$. Since $\bar{v}$ factors through $v_* t_0$, there exists $q' \in W_{\text{red}}(F)$ with $v_*(g) = v_* t_0(q')$. Then $g - t_0(q') \in 2 \prod_{\mathfrak{M}(F)} W(F_\sigma)$ (Theorem (2.5)). Hence

$$t_0(q_n - 2^n q') = 2^n(g - t_0(q')) \in 4 \prod_{\mathfrak{M}(F)} W(F_\sigma),$$

so by hypothesis $q_n - 2^n q' = 2q$ for some $q \in W_{\text{red}}(F)$. But then

$$t(q + 2^{n-1}q') = \tfrac{1}{2}t(q_n) = 2^{n-1}t_1(g),$$

contradicting the choice of n. Hence $n = 0$, and we are done.

(5.4) THEOREM. *Suppose every formally real simple algebraic extension of F is near exact. Then $F(x)$ is near exact.*

(5.5) THEOREM. *Suppose τ: $F \to K \cup \{\infty\}$ is a surjective place with $\tau^{-1}(1) \subset \sum F^{\cdot 2}$. Then F is near exact if and only if K is near exact.*

The proofs of these two theorems are analogous to those of Theorems (4.4) and (4.7); indeed Lemma (5.3) turns out to be much easier to apply than Lemma (4.3). Applications paralleling those in (4.5) and (4.8) can be made. We will leave the details to the interested reader.

(5.6) THEOREM. *A direct limit of near exact fields is a near exact field.*

(5.7) REMARK. Combining (5.4) and (5.6) we deduce (by transfinite induction) that the following two statements are equivalent:

(A) All formally real fields are near exact.

(B) All formally real simple algebraic extensions of a near exact field are near exact.

We have no counterexample to these statements, or even to the assertion that all formally real fields are exact.

Some applications of (5.6) are made in Remark (5.9).

We now prove (5.6), using the criterion of Lemma (5.3). First recall, however, that $\mathfrak{M}(F)$ is given the finest topology in which π: $\mathcal{O}(F) \to \mathfrak{M}(F)$ is continuous. A subbasis for this topology consists of all the sets

$$V_{\mathfrak{M}}(a) = \{\sigma \in \mathfrak{M}(F)\colon -1 < \sigma(a) < 1\} \qquad (a \in F^{\cdot}). \tag{17}$$

(These sets generate a Hausdorff topology for which π is continuous. Since $\mathcal{O}(F)$ is compact, this topology is precisely that induced by π.)

Let $q = \langle c_1, c_2, \ldots, c_{2r}\rangle \in W(F)$ with $t_0(q_{\text{red}}) \in 4\prod_{\mathfrak{M}(F)} W(F_\sigma)$. We must show that $q_{\text{red}} \in 2W_{\text{red}}(F)$ (Lemma (5.3)).

Let $\sigma \in \mathfrak{M}(F)$. We can find $b_{\sigma,i} \in F^{\cdot}$, say where $1 \leqslant i \leqslant r_\sigma$, with $q \otimes F_\sigma = 4\langle b_{\sigma,1}, \ldots, b_{\sigma,r_\sigma}\rangle$. Let $n_\sigma = r + 2r_\sigma$. The $2n_\sigma$ elements in the sequence $-c_1, \ldots, -c_{2r}, b_{\sigma,1}, \ldots, b_{\sigma,r_\sigma}, b_{\sigma,1}, \ldots, b_{\sigma,1}, \ldots, b_{\sigma,1}, \ldots, b_{\sigma,r_\sigma}$ (each $b_{\sigma,i}$ occurs four times) can be reindexed, say as $d_{\sigma,1}, \ldots, d_{\sigma,2n_\sigma}$, so that for each $i \leqslant n_\sigma$ there exist $e_{\sigma,i} \in F^{\cdot}$ with $0 < \sigma(-e_{\sigma,i}^2 d_{\sigma,i} d_{\sigma,n_\sigma+i}) < 1$ (Theorem (2.5)). Set $f_{\sigma,i} = -e_{\sigma,i}^2 d_{\sigma,i} d_{\sigma,n_\sigma+i}$ and $f_{\sigma,n_\sigma+i} = 1 - f_{\sigma,i}$ for all $i \leqslant n_\sigma$. Then $0 < \sigma(f_{\sigma,i}) < 1$ for all $i \leqslant 2n_\sigma$. Hence there exists a positive integer m_σ with

$$0 < \sigma((1 + 2f_{\sigma,i}^2)^{m_\sigma}(2 + f_{\sigma,i}^2)^{-m_\sigma}) < (2n_\sigma)^{-1}$$

for all $i \leqslant 2n_\sigma$. Hence if we set

$$h_\sigma = \sum_{i=1}^{2n_\sigma} (1 + 2f_{\sigma,i}^2)^{m_\sigma}(2 + f_{\sigma,i}^2)^{-m_\sigma},$$

then σ is in

$$V_{\mathfrak{M}}(h_\sigma) \cap V_{\mathfrak{M}}(1 - h_\sigma). \tag{18}$$

As we let σ range over $\mathfrak{M}(F)$, the sets (18) form an open cover of the compact space $\mathfrak{M}(F)$. Hence there exists a finite subset I of $\mathfrak{M}(F)$ with

$$\mathfrak{M}(F) = \bigcup_{\sigma \in I} (V_{\mathfrak{M}}(h_\sigma) \cap V_{\mathfrak{M}}(1 - h_\sigma)).$$

Let $n = |I|$. For each integer $i > 0$ and $\sigma \in I$, set

$$U_{\sigma,i} = V_{\mathfrak{M}}(1 - h_\sigma) \cap V_{\mathfrak{M}}(nh_\sigma^i).$$

For $i < j$ we have $U_{\sigma,i} \subseteq U_{\sigma,j} \subseteq V_{\mathfrak{M}}(1 - h_\sigma) \cap V_{\mathfrak{M}}(h_\sigma)$. Further, for fixed σ, the sets $U_{\sigma,i}$ $(i > 0)$ cover $V_{\mathfrak{M}}(1 - h_\sigma) \cap V_{\mathfrak{M}}(h_\sigma)$. Hence there exists an integer $s > 0$ with

$$\mathfrak{M}(F) = \bigcup_{\sigma \in I} U_{\sigma,s}.$$

Now set

$$\beta = n\left(\prod_{\sigma \in I} h_\sigma^s\right)\left(\sum_{\tau \in I}\left(\prod_{\sigma \in I \setminus \{\tau\}} h_\sigma^s\right)\right)^{-1}.$$

Then β is totally positive (since each h_σ is). We now show that $1 - \beta$ is totally positive. Let $\rho \in \mathfrak{M}(F)$. Then $\rho \in U_{\sigma_0,s}$ for some $\sigma_0 \in I$. Also, $\rho(h_\tau)$ is finite and positive for all $\tau \in I$. Indeed, any real place on any field E will take a finite positive value on an element of the form $(1 + 2a^2)(2 + a^2)^{-1}$ $(a \in E)$. Hence

$$0 < \rho(\beta) < \prod_{\sigma \neq \sigma_0} \rho(h_\sigma^s) \Big/ \prod_{\sigma \neq \sigma_0} \rho(h_\sigma^s) = 1.$$

Thus for all real places $\rho \in \mathfrak{M}(F)$, $0 < \rho(1 - \beta) < 1$. This shows that $1 - \beta$ is totally positive. Hence there exist $\alpha_1, \ldots, \alpha_t \in F^\cdot$ with $1 - \beta = \alpha_1^2 + \cdots + \alpha_t^2$.

Since F is a direct limit of near exact fields, there exists a near exact field E with

$$F \supseteq E \supseteq \{\alpha_1, \ldots, \alpha_t\} \cup \left(\bigcup_{\sigma \in I, i \leqslant n_\sigma} \{e_{\sigma,i}, d_{\sigma,i}, d_{\sigma,i+n_\sigma}\}\right)$$

Let $\rho \in \mathfrak{M}(E)$. We claim that $|\rho(h_\sigma)| < 1$ for some $\sigma \in I$. Just suppose not. Since β and $1 - \beta$ are totally positive in E, we have

$$1 > \rho(\beta) \geqslant n \prod_{\sigma \in I} \rho(h_\sigma)^s \Big/ \sum_{\tau \in I} \left(\prod_{\sigma \in I} \rho(h_\sigma)^s \right) = 1.$$

(Recall that $\rho(h_\sigma)$ is finite and positive for each σ.) This contradiction shows that we can pick $\sigma \in I$ with $0 < \rho(h_\sigma) < 1$. Hence $0 \leqslant 1 + 2f_{\sigma,i}^2/2 + f_{\sigma,i}^2 < 1$ for all $i \leqslant 2n_\sigma$. Hence $-1 < \rho(f_{\sigma,i}) < 1$ $(i \leqslant 2n_\sigma)$. Indeed, $0 < \rho(f_{\sigma,i}) < 1$ for $i \leqslant n_\sigma$ (since for all such i, $f_{\sigma,i+n_\sigma} = 1 - f_{\sigma,i}$). That is,

$$0 < \rho(-e_{\sigma,i}^2 d_{\sigma,i} d_{\sigma,i+n_\sigma}) < 1 \qquad (i \leqslant n_\sigma). \tag{19}$$

If $\alpha_1, \ldots, \alpha_m \in E$, let $\langle \alpha_1, \ldots, \alpha_m \rangle_E$ denote the corresponding element of $W(E)$. Then (19) implies that $\langle d_{\sigma,i}, d_{\sigma,i+n_\sigma} \rangle_E \otimes E_\rho = 0$ for all $i \leqslant n_\sigma$, whence

$$\langle c_1, \ldots, c_{2r} \rangle_E \otimes E_\rho = 4\langle b_1, \ldots, b_{r_\sigma} \rangle_E \otimes E_\rho \in 4W(E_\rho).$$

Since E is near exact, we conclude (Lemma (5.3)) that there exists $q' \in W(E)$ with $(\langle c_1, \ldots, c_{2r} \rangle_E)_{\text{red}} = 2q'_{\text{red}}$. Then

$$q_{\text{red}} = (\langle c_1, \ldots, c_{2r} \rangle_E \otimes F)_{\text{red}} \in 2W_{\text{red}}(F),$$

which was to be proven.

(5.8) THEOREM. *Suppose $V_{\mathfrak{M}}(a)$ (cf. (17)) is closed for all $a \in F^{\cdot}$. Then F is near exact.*

(5.9) APPLICATIONS. (A) If $\mathfrak{M}(F)$ is finite, then F is near exact. We will apply this result in §6 to show that if $\mathfrak{M}(F)$ is finite, then F is exact. Among the fields with $\mathfrak{M}(F)$ finite are the superordered fields (and hence, in particular, the superpythagorean fields) and the fields with only finitely many orderings (and hence, in particular, the fields with only finitely many square classes).

(B) Suppose $\mathfrak{M}(F)$ is finite. Then $\mathfrak{M}(E)$ is finite for every simple algebraic extension E/F. Hence every formally real algebraic extension of F is near exact (apply (A) above and Theorem (5.6)) and the rational function field $F(x)$ is near exact (Theorem (5.4)).

(C) Suppose F/Q is algebraic. Then F is near exact (cf. (B) above). Note that $\operatorname{Im} \bar{v}$ consists only of 0 and 1 (since $\Lambda_\sigma = 1$ for all $\sigma \in \mathfrak{M}(F)$). Also note that $\pi: \mathfrak{O}(F) \to \mathfrak{M}(F)$ is a bijection and that we may identify the map $W(F) \to W(F_\sigma)$ with the signature map of the ordering $\pi^{-1}(\sigma)$ (for all $\sigma \in \mathfrak{M}(F)$). Hence $\operatorname{Im} t = C(\mathfrak{O}(F), 2Z) + Z \cdot 1$ (Definition (5.1)). Thus F is a SAP field. This result is also proved in [3, §4], [14, Example 2.10, p. 64], and [9, §3, Example 1].

(D) Suppose that F is an algebraic extension of $Q(x)$ such that the map $\sigma \mapsto \sigma(x)$ carries $\mathfrak{M}(F)$ bijectively onto a subset C of $R \cup \{\infty\}$. Then F satisfies the hypotheses of Theorem (5.8) if and only if all algebraic elements of C are isolated. (We regard $R \cup \{\infty\}$ as the one-point compactification of R. ∞ is regarded as an algebraic element. In §8 we will show that for any compact subset C of $R \cup \{\infty\}$, there exists an algebraic extension F of $Q(x)$ with $\mathfrak{M}(F)$ mapping bijectively onto C.)

PROOF OF (D). ($\Leftarrow$) Let $a \in F^{\cdot}$. We show $V_{\mathfrak{M}}(a)$ is closed. The complement of $V_{\mathfrak{M}}(a)$ is the disjoint union

$$V_{\mathfrak{M}}(a^{-1}) \cup \{\sigma\colon \sigma(a+1) = 0\} \cup \{\sigma\colon \sigma(a-1) = 0\}.$$

Hence it suffices to show that $\{\sigma\colon \sigma(b) = 0\}$ is open for all $b \in F^{\cdot}$. So let $b \in F^{\cdot}$. Since b is algebraic over $Q(x)$, there exist polynomials $f_i \in Q[x]$ with $f_0 = \sum r_i x^i \neq 0$ and $\sum_{i=0}^{n} f_i b^i = 0$. If $\sigma(b) = 0$, then either $\sigma(x) = \infty$ or $\sum r_i \sigma(x)^i = 0$. In either case $\sigma(x)$ is algebraic. Thus $\{\sigma\colon \sigma(b) = 0\}$ is open (all of its points are isolated).

($\Rightarrow$) Just suppose $\tau(x)$ is algebraic and not isolated in C for some $\tau \in \mathfrak{M}(F)$. Then there exist distinct $\tau_1, \tau_2, \ldots \in \mathfrak{M}(F)$ with $\tau_i(x) \to \tau(x)$ as $i \to \infty$. There exists $f \in Q(x)^{\cdot}$ with $\tau(f) = 0$ and f either equal to $1/x$ or to a polynomial. Our hypothesis implies that $V = \{\sigma \in \mathfrak{M}(F)\colon \sigma(f) = 0\}$ is open (it is the intersection of $V_{\mathfrak{M}}(f)$ with the complement of

$$V_{\mathfrak{M}}(1+f) \cup V_{\mathfrak{M}}(f-1)).$$

Since V is a neighborhood of τ it must contain infinitely many of the τ_i. On the other hand, if $f = x^{-1}$ then $V = \{\tau\}$ and if $f \in Q[x]$ then $V = \{\sigma\colon \sigma(x)$ is a zero of $f\}$. In either case V is finite, a contradiction.

We begin the proof of (5.8) with a lemma of interest in itself. As usual, we regard $\prod_{\sigma \in \mathfrak{M}(F)} W(F_\sigma)$ as a set of maps from $\mathfrak{M}(F)$ to the disjoint union $\bigcup_{\sigma \in \mathfrak{M}(F)} W(F_\sigma)$. We give this disjoint union the coarsest topology in which the "paths"

$$\text{path}(q) = \{q \otimes F_\sigma\colon \sigma \in \mathfrak{M}(F)\}$$

are open for all $q \in W(F)$.

(5.10) LEMMA. *Let $g \in \prod_{\mathfrak{M}(F)} W(F_\sigma)$. Then g is continuous if and only if $t_1(g)\colon \mathcal{O}(F) \to Z$ is continuous.*

PROOF. First suppose g is continuous. Let $P \in \mathcal{O}(F)$. We must find a neighborhood W of P such that $t_1(g)(W) = t_1(g)(P)$. Since $W(F) \to W(F_{\pi(P)})$ is surjective, there exists $a_1, \ldots, a_n$ with $g(\pi(P)) = \langle a_1, \ldots, a_n\rangle \otimes F_{\pi(P)}$. It then suffices to set W equal to

$$(g\pi)^{-1}(\text{path}\langle a_1,\dots,a_n\rangle) \cap \left(\bigcap_{i=1}^{n} V((a_i,P)a_i)\right).$$

Now suppose $t_1(g)$ is continuous. Let $\sigma \in \mathfrak{M}(F)$, and let $q \in W(F)$ be such that $g(\sigma) = q \otimes F_\sigma$. We must find a neighborhood of σ which g maps into path(q). There exists an integer $n > 0$ and $q' \in W(F)$ with $t(q'_{\text{red}}) = 2^n t_1(g)$ [12, Theorem 3.18(i)]. Then $t_0(q'_{\text{red}}) = 2^n g$. Pick $a_1,\dots,a_{2m} \in F^{\cdot}$ with $2^n q - q' = \langle a_1,\dots,a_{2m}\rangle$. Since $(2^n q - q') \otimes F_\sigma = 0$, we may assume the a_i are indexed so that $-a_i a_{i+m} \in F_\sigma^{\cdot 2}$ for all $i \leqslant m$ (Theorem (2.5)). Hence there exist $c_i \in F^{\cdot}$, $i \leqslant m$, with $-1 < \sigma(c_i^2 a_i a_{i+m}) < 0$ $(i \leqslant m)$. Hence

$$W = \bigcup_{d_1,\dots,d_m \in F^{\cdot}} \bigcap_{i=1}^{m} \left(V_{\mathfrak{M}}(a_i a_{i+m} d_i^2) \cap V_{\mathfrak{M}}(1 + a_i a_{i+m} d_i^2)\right)$$

is a neighborhood of σ. Suppose $\tau \in W$. Then for some $d_1,\dots,d_m \in F^{\cdot}$ we have $-1 < \tau(a_i a_{i+m} d_i^2) < 0$ for all $i \leqslant m$. Hence $\langle a_i, a_{i+m}\rangle \otimes F_\tau = 0$ for all $i \leqslant m$. Thus $(2^n q - q') \otimes F_\tau = 0$. Hence $2^n(q \otimes F_\tau) = t_0(q'_{\text{red}})(\tau) = 2^n g(\tau)$. Thus $g(\tau) = q \otimes F_\tau \in \text{path}(q)$. Hence g is continuous.

We now prove Theorem (5.8). Suppose there exists $g \in \prod_{\mathfrak{M}(F)} W(F_\sigma)$ and $q \in W_{\text{red}}(F)$ with $t_0(q) = 4g$. It suffices to show that $q \in 2W_{\text{red}}(F)$ (Lemma (5.3)). g is continuous since $t_1(g) = \frac{1}{4}t(q)$ (Lemma (5.10)). For each $\sigma \in \mathfrak{M}(F)$ pick $q_\sigma \in W(F)$ with $g(\sigma) = q_\sigma \otimes F_\sigma$. The open sets $g^{-1}(\text{path}\ (q_\sigma))$ cover $\mathfrak{M}(F)$. Our hypotheses ensure that $\mathfrak{M}(F)$ is totally disconnected. Hence there exists a disjoint open cover $W_1,\dots,W_n$ of $\mathfrak{M}(F)$ and $q_1,\dots,q_n \in W(F)$ with $g(W_i) \subseteq \text{path}\, q_i$ for all $i \leqslant n$ (this requires a purely topological argument). For each $i \leqslant n$ choose $u_i \in F^{\cdot}$ with $\sigma(u_i) < 0$ for all $\sigma \notin W_i$ and $0 < \sigma(u_i) < 1$ for all $\sigma \in W_i$ [3, Theorem 2.1A]. Then for any $\sigma \in \mathfrak{M}(F)$, say $\sigma \in W_j$, we have

$$\sum_{i=1}^{n} (\langle 1,u_i\rangle q_i) \otimes F_\sigma = 2q_j \otimes F_\sigma = 2g(\sigma) = \tfrac{1}{2}t_0(q)(\sigma).$$

Thus

$$t_0(q) = t_0\left(2 \sum_{i=1}^{n} \langle 1,u_i\rangle q_i\right)$$

Since t_0 is injective, $q \in 2W_{\text{red}}(F)$. The theorem is proved.

(5.11) Theorem. *Suppose $W_{\text{red}}(F)$ is 2-stable, i.e., $I^3 = 2I^2$ where I denotes the set of elements of $W_{\text{red}}(F)$ of dimension-index* 0. *Then F is near exact.*

(5.12) Applications. (A) If F has transcendence degree one over Q or two over R then F is near exact [20, §3]. Hence $F(x)$ is also near exact (Theorem 5.4). Thus, for example, $Q(x,y)$ and $R(x,y,z)$ are near exact.

(B) Suppose $|\Lambda_\sigma| \leqslant 2$ for all $\sigma \in \mathfrak{M}(F)$. Then $\operatorname{Im} t$ consists of all $f \in Z \cdot 1 + C(\mathcal{O}(F), 2Z)$ such that for some $\gamma \in \operatorname{Im} v$,

$$(*) \qquad f(P) - f(T) \equiv 1 - \langle P, T\rangle(\gamma(\sigma)) \pmod 4$$

for all $\sigma \in \mathfrak{M}(F)$ and P, $T \in \mathcal{O}_\sigma$. (Note that $\langle P, T\rangle(\gamma(\sigma)) = 1$ if and only if $P = T$ or $\gamma(\sigma) = 1$.) *Proof.* Suppose $f = t(q)$ where $q \in W(F)$ has discriminant d and dimension-index n (where we regard n as an element of $\{0, 1\} \subset Z$). Then $q^* = q - n\langle 1\rangle + \langle 1, -d\rangle$ has discriminant 1 and dimension-index 0, so q^* is in the square of the augmentation ideal of $W(F)$. Hence $t(q^*) \in C(\mathcal{O}(F), 4Z)$. Thus for all $\sigma \in \mathfrak{M}(F)$ and P, $T \in \mathcal{O}_\sigma$,

$$f(P) - f(T) \equiv (d, P) - (d, T) \equiv 1 - \langle P, T\rangle(v_\sigma(d)) \pmod 4.$$

This proves the necessity of $(*)$ for elements of $\operatorname{Im} t$. Now suppose $f \in Z \cdot 1 + C(\mathcal{O}(F), 2Z)$ satisfies $(*)$, say with $\gamma = v(d)$. Then $f = t_1(g)$ for some $g \in \prod_{\mathfrak{M}(F)} W(F)$ (Corollary (2.8)). Let $\sigma \in \mathfrak{M}(F)$ and P, $T \in \mathcal{O}_\sigma$. The above calculation shows

$$\langle P, T\rangle(v_\sigma(d)) \equiv \langle P, T\rangle(v_\sigma(\operatorname{disc} g(\sigma))) \pmod 4.$$

Hence $v_\sigma(d) = v_\sigma(\operatorname{disc} g(\sigma))$ (Lemma(2.3). Using the fact that $|\Lambda_\sigma| \leqslant 2$, one checks that $v'_\sigma(g(\sigma)) = 1 + n + 2Z + v_\sigma(\operatorname{disc} g(\sigma))$ (note first that $v_\sigma(a) + v_\sigma(b) = 1 + v_\sigma(ab)$ in $Z_2(\Lambda_\sigma)$ for all $a, b \in F^{\cdot}$). Hence $v_*(g) = \bar{v}(1 + n + 1 \cdot d)$. Since $W_{\text{red}}(F)$ is 2-stable [20, §3], our result now follows from Theorem (5.11) and Definition (5.1).

Proof of Theorem (5.11). Suppose $q \in t_0^{-1}(4 \prod_{\mathfrak{M}(F)} W(F_\sigma))$ and $f = t(q)/2$. It suffices to show $f \in \operatorname{Im} t$ (Lemma (5.3)). Suppose P, $T \in \mathcal{O}(F)$ induce the same place $\sigma \in \mathfrak{M}(F)$. We can write $q \otimes F_\sigma = 4 \sum n_c\langle c\rangle$ where the sum is taken over all c in some set of elements of $F^{\cdot}$ whose values in Λ_σ are distinct. Note

$$f(P) \equiv 2(\textstyle\sum n_c(c, P)) \equiv \sum 2n_c \equiv f(T) \pmod 4.$$

Since $\operatorname{Im} t \supset C(\mathcal{O}(F), 4Z)$ [20, §3], we may suppose $\operatorname{Im} f \subset \{0, 2\}$. Then we can write $f = g\pi$ where $g \in C(\mathfrak{M}(F), Z)$. It therefore suffices to find $a \in F^{\cdot}$ with $\sigma(a) < 0$ if $g(\sigma) = 0$ and $\infty > \sigma(a) > 0$ if $g(\sigma) = 2$. This is possible by the next lemma.

(5.13) Lemma. *Let* $A = \bigcap_{\sigma \in \mathfrak{M}(F)} \sigma^{-1}(R)$. *Define* $\phi\colon A \to C(\mathfrak{M}(F), R)$ *by*

$$\phi(a)(\sigma) = \sigma(a) \text{ for all } a \in A \text{ and } \sigma \in \mathfrak{M}(F).$$

Then $\phi(A)$ *is dense in* $C(\mathfrak{M}(F), R)$ (*in the sup norm*).

PROOF. A is a ring containing Q. Hence the closure of $\phi(A)$ contains the constant functions. If $\sigma \neq \tau \in \mathfrak{M}(F)$, there exists $a \in F^{\cdot}$ with $\infty > \sigma(a) > 0 > \tau(a)$. We may suppose $a \in A$ (replace a by $a/a^2 + 1$). Hence $\phi(A)$ separates points. Now apply the Stone-Weierstrass Theorem.

6. Fields with only finitely many real places.

(6.1) THEOREM. *Suppose $\mathfrak{M}(F)$ is finite. Then*

$$0 \to W_{\mathrm{red}}(F) \xrightarrow{t_0} \prod_{\mathfrak{M}(F)} W(F_\sigma) \xrightarrow{v_{**}} \prod_{\sigma,\tau \in \mathfrak{M}(F)} Z_2(\Lambda_{\sigma,\tau})$$

is exact. That is, F is exact.

(6.2) APPLICATIONS. (A) A field with only finitely many orderings is exact.

(B) If $\mathfrak{M}(F)$ is finite, then $\mathfrak{M}(E)$ is finite for all simple algebraic extensions of F. Hence $F(x)$ is exact (Theorem (4.4)).

(C) Theorem (6.1) gives an alternate method for showing that some familiar fields are exact. For example it implies that superordered fields are exact (cf. (4.2)) and that all formally real simple algebraic extensions of $Q((x_1))\cdots((x_n))$ are exact (cf. (4.8B)).

(D) If $\mathfrak{M}(F)$ is finite, then $W_{\mathrm{red}}(F)$ is determined up to ring isomorphism by the groups Λ_σ ($\sigma \in \mathfrak{M}(F)$) and the maps $w_{\sigma,\tau}\colon \Lambda_\sigma \to \Lambda_{\sigma,\tau}$ ($\sigma,\tau \in \mathfrak{M}(F)$). For, let $w''_{\sigma,\tau}\colon Z(\Lambda_\sigma) \to Z_2(\Lambda_{\sigma,\tau})$ be the map induced by $w_{\sigma,\tau}$ ($\sigma,\tau \in \mathfrak{M}(F)$). Then $W_{\mathrm{red}}(F)$ is isomorphic to the subring of $\prod_{\sigma\in\mathfrak{M}(F)} Z(\Lambda_\sigma)$ consisting of all $(q_\sigma)_{\sigma\in\mathfrak{M}(F)}$ such that $w''_{\sigma,\tau}(q_\sigma) = w''_{\tau,\sigma}(q_\tau)$ for all $\sigma, \tau \in \mathfrak{M}(F)$. (*Proof.* Define u: $\prod_{\mathfrak{M}(F)} Z(\Lambda_\sigma) \to \prod_{\sigma,\tau\in\mathfrak{M}(F)} Z_2(\Lambda_{\sigma,\tau})$ by

$$u((q_\sigma)_\sigma) = (w''_{\sigma,\tau}(q_\sigma) + w''_{\tau,\sigma}(q_\tau))_{\sigma,\tau}.$$

For each $\sigma \in \mathfrak{M}(F)$ pick $P_\sigma \in \mathcal{O}_\sigma$. Then $\prod_{\mathfrak{M}(F)} \Phi_{P_\sigma}$ is an isomorphism whose composition with u is v_{**}. Now apply Theorem (6.1).) For example, if the real places on $\mathfrak{M}(F)$ are pairwise independent, then $W_{\mathrm{red}}(F)$ is isomorphic to the subring of $\prod_{\mathfrak{M}(F)} Z(\Lambda_\sigma)$ consisting of elements of constant degree-index. (The "degree-index" of an element of $Z(\Lambda_\sigma)$ is the sum of its coefficients mod 2.)

We now begin the proof of (6.1). Write $\mathfrak{M}(F) = \{\sigma_1, \ldots, \sigma_n\}$. For each $i, j \leqslant n$ let $A_i = \sigma_i^{-1}(R)$, $\Lambda_i = \Lambda_{\sigma_i}$, $v_i = v_{\sigma_i}$, $\Lambda_{i,j} = \Lambda_{\sigma_i,\sigma_j}$, $v'_{i,j} = v'_{\sigma_i,\sigma_j}$, and $w_{i,j} = w_{\sigma_i,\sigma_j}$. We may suppose that the σ_i are indexed so that

$$A_i A_{i+1} \subseteq A_i A_j \quad \text{for all } i < j \leqslant n. \tag{20}$$

(Such an indexing is easily constructed inductively. These indexings arise when the tree of all valuation rings of F which are products of the A_i is represented by a plane graph.) Since $A_i A_j \subseteq A_i A_{i+1} A_j \subseteq A_i A_j$ for all $i < j \leqslant n$, we can deduce by an easy induction that

$$A_i A_j = A_i A_{i+1} \cdots A_j \quad (\text{if } i < j \leqslant n). \tag{21}$$

We have a homomorphism $w\colon \prod_{i \leqslant n} \Lambda_i \to \prod_{i<n} \Lambda_{i,i+1}$ given by the rule

$$w((\lambda_i)_{i \leqslant n}) = (w_{i,i+1}(\lambda_i) w_{i+1,i}(\lambda_{i+1}))_{i<n}$$

for all $(\lambda_i)_{i \leqslant n} \in \prod_{i \leqslant n} \Lambda_i$. We now compute $\operatorname{Im} v$.

(6.3) LEMMA. *The sequence*

$$F^{\cdot} \xrightarrow{v} \prod_{i \leqslant n} \Lambda_i \xrightarrow{w} \prod_{i<n} \Lambda_{i,i+1} \to 1$$

is exact. In particular, the image of v is

$$\left\{ (\lambda_i)_{i \leqslant n} \in \prod_{i \leqslant n} \Lambda_i \colon w_{i,i+1}(\lambda_i) = w_{i+1,i}(\lambda_{i+1}) \textit{ for all } i < n \right\}.$$

PROOF. w is surjective since each of the maps $w_{i+1,i}$ is surjective. (To find a preimage $(\lambda_1, \ldots, \lambda_n)$ for an element of $\prod_{i<n} \Lambda_{i,i+1}$, set $\lambda_1 = 1$ and solve for $\lambda_2, \ldots, \lambda_n$ in turn.) Formula (5) of §3 implies that $\operatorname{Im} v \subseteq \operatorname{Ker} w$. Now suppose $(\lambda_i)_{i \leqslant n} \in \operatorname{Ker} w$. We shall construct $b_1, \ldots, b_n \in F^{\cdot}$ with $v_i(b_i) = \lambda_i$ and $b_i A_i A_j = b_j A_i A_j$ for all $i, j \leqslant n$. That $(\lambda_i)_{i \leqslant n} \in \operatorname{Im} v$ will then follow from [3, Theorem 2.1B]. For each $i \leqslant n$, pick $a_i \in v_i^{-1}(\lambda_i)$. Let $b_1 = a_1$. Suppose inductively that for some m, $1 \leqslant m < n$, we have found $b_1, \ldots, b_m$ with $v_i(b_i) = \lambda_i$ and $b_i A_i A_j = b_j A_i A_j$ whenever $i, j \leqslant m$. By hypothesis b_m and a_{m+1} have the same image in $\Lambda_{m,m+1}$. Hence there exists $c \in F^{\cdot}$ with $a_{m+1} b_m^{-1} c^2$ a unit in the valuation ring $A_m A_{m+1}$. Let $b_{m+1} = a_{m+1} c^2$. Then $v_{m+1}(b_{m+1}) = \lambda_{m+1}$ and $b_m A_m A_{m+1} = b_{m+1} A_m A_{m+1}$. Further, for all $i < m$ we have (by hypothesis and (21))

$$\begin{aligned} b_i A_i A_{m+1} &= b_i A_i A_{i+1} \cdots A_{m+1} = A_i b_{i+1} A_{i+1} \cdots A_{m+1} \\ &= A_i A_{i+1} b_{i+2} A_{i+2} \cdots A_{m+1} = \cdots \\ &= A_i \cdots A_m b_{m+1} A_{m+1} = b_{m+1} A_i A_{m+1}. \end{aligned}$$

This completes the induction and the proof of Lemma (6.3).

Let $w'_{i,j}\colon Z_2(\Lambda_i) \to Z_2(\Lambda_{i,j})$ be the ring homomorphism induced by $w_{i,j}\colon \Lambda_i \to \Lambda_{i,j}$. We have a homomorphism

$$\bar{w}\colon \prod_{i \leqslant n} Z_2(\Lambda_i) \to \prod_{i<n} Z_2(\Lambda_{i,i+1})$$

given by the rule (for any $a_i \in Z_2(\Lambda_i)$)

$$\bar{w}((a_i)_{i \leqslant n}) = (w'_{i,i+1}(a_i) + w'_{i+1,i}(a_{i+1}))_{i<n}.$$

We now compute the image of $\bar{v}$.

(6.4) LEMMA. *The following sequence is exact*:

$$Z(F^{\cdot}) \xrightarrow{\bar{v}} \prod_{i \leqslant n} Z_2(\Lambda_i) \xrightarrow{\bar{w}} \prod_{i<n} Z_2(\Lambda_{i,i+1}).$$

PROOF. That $\operatorname{Ker} \bar{w} \supseteq \operatorname{Im} \bar{v}$ follows from (5) of §3. We now prove the reverse inclusion.

For each $\sigma \in \mathfrak{M}(F)$, the group homomorphism $\Lambda_\sigma \to 1$ induces the "augmentation homomorphism" $Z_2(\Lambda_\sigma) \to Z_2$. The product of these ring homomorphisms is a map

$$\prod_{\mathfrak{M}(F)} Z_2(\Lambda_\sigma) \to \prod_{\mathfrak{M}(F)} Z_2 \tag{22}$$

which carries each element of $\operatorname{Ker} \bar{w}$ to either 0 or 1. Let B_0 be the set of elements of $\operatorname{Ker} \bar{w}$ which the map (22) carries to 0. Since $\operatorname{Ker} \bar{w} = B_0 \cup (\bar{v}(1) + B_0)$, it suffices to show that $B_0 \subseteq \operatorname{Im} \bar{v}$. For each $i \leqslant n$ let

$$B_i = \{(\lambda_i)_{i \leqslant n} \in B_0 : \lambda_1 = \lambda_2 = \cdots = \lambda_i = 0\}.$$

Clearly $B_n \subseteq \operatorname{Im} \bar{v}$. Suppose inductively that $B_{k+1} \subseteq \operatorname{Im} \bar{v}$ for some $k \geqslant 0$. It will suffice to show that $B_k \subseteq \operatorname{Im} \bar{v}$. So let $\lambda = (\lambda_i)_{i \leqslant n} \in B_k$. We can write $\lambda_{k+1} = \sum_{i=1}^{2s} v_{k+1}(a_i)$ (where the $a_i \in F^{\cdot}$). Since $w'_{k+1,k}(\lambda_{k+1}) = w'_{k,k+1}(\lambda_k) = 0$ we may assume the a_i's are indexed so that for all $i \leqslant s$,

$$w_{k+1,k} v_{k+1}(a_{i+s}) = w_{k+1,k} v_{k+1}(a_i) = w_{k,k+1} v_k(a_i)$$

(cf. (5) of §3). But this says that for all $i \leqslant s$,

$$(v_1(a_i), v_2(a_i), \ldots, v_k(a_i), v_{k+1}(a_{i+s}), v_{k+2}(a_{i+s}), \ldots, v_n(a_{i+s})) \tag{23}$$

is in Ker w. Hence by Lemma (6.3) there exist $b_i \in F^{\cdot}$ with (23) equal to $v(b_i)$ (for all $i \leqslant s$). By construction we have

$$\lambda + \sum_{i=1}^{s} (v(b_i) + v(a_i)) \in B_{k+1} \subseteq \operatorname{Im} \bar{v}$$

so $\lambda \in \operatorname{Im} \bar{v}$. This completes the induction and the proof of the lemma.

We now give the proof of Theorem (6.1). First, F is near exact (Application (5.9A)), so

$$\operatorname{Im} t_0 = t_1^{-1}(C(\mathcal{O}(F), Z)) \cap v_*^{-1}(\operatorname{Im} \bar{v})$$

(Definition (5.1)). Also $\operatorname{Im} \bar{v} = \operatorname{Ker} \bar{w}$ (Lemma (6.4)) and $t_1^{-1}(C(\mathcal{O}(F), Z)) = \prod_{\mathfrak{M}(F)} W(F_\sigma)$ (each $\mathcal{O}_\sigma$ is open in $\mathcal{O}(F)$). Moreover $v_*^{-1}(\operatorname{Ker} \bar{w}) \supset \operatorname{Ker} v_{**}$ since $\bar{w} \cdot v_*$ factors through v_{**}. Putting all this together we have

$$\operatorname{Im} t_0 = v_*^{-1}(\operatorname{Ker} \overline{w}) \supseteq \operatorname{Ker} v_{**} \supseteq \operatorname{Im} t_0$$

(cf. (8)). The theorem is proved.

Lemma (6.4) permits a slightly nicer calculation of $W_{\mathrm{red}}(F)$ than Theorem (6.1) alone.

(6.5) COROLLARY. *Suppose* $\mathfrak{M}(F)$ *is finite and that its elements* $\sigma_1, \ldots, \sigma_n$ *are indexed so that* (20) *holds. Then* $\operatorname{Im} t_0$ *consists of all* $(q_1, \ldots, q_n) \in \prod_{i=1}^n W(F_{\sigma_i})$ *with*

$$v'_{i,i+1}(q_i) = v'_{i+1,i}(q_{i+1}) \quad \textit{for all } i < n.$$

(6.6) COROLLARY. *Suppose* $|\mathfrak{M}(F)| < \infty$. *The following are equivalent*:

(A) $\operatorname{Im} t_0$ *consists of all elements of* $\prod_{\mathfrak{M}(F)} W(F_\sigma)$ *of constant degree-index.*

(B) $\Lambda_{\sigma,\tau} = 1$ *for all* $\sigma, \tau \in \mathfrak{M}(F)$.

(C) $v\colon F^{\cdot} \to \prod_{\mathfrak{M}(F)} \Lambda_\sigma$ *is surjective.*

PROOF. The equivalence of (A) and (B) follows from Theorem (6.1) and Remark (3.5B). That (B) implies (C) follows from Lemma (6.3). Now suppose (C) holds; we will deduce (B). Lemma (6.3) implies that $\Lambda_{i,i+1} = 1$ for all $i < n$. But for any $i < j$, we have $A_i A_j \supseteq A_i A_{i+1}$ (cf. (20)), so that $\Lambda_{i,j}$ is a homomorphic image of $\Lambda_{i,i+1}$. Thus $\Lambda_{i,j} = 1$ for all $i \neq j$.

(6.7) REMARK. Suppose $\mathfrak{M}(F) = \{\sigma_1, \ldots, \sigma_n\}$, the σ_i satisfying (20). In Lemma (6.3) we showed that the cokernel of v is $\sum_{i<n} \Lambda_{i,i+1}$. We sketch here how to compute the kernel of v. We also give another computation of the image of v.

We have an exact sequence

$$1 \to \sum F^{\cdot 2} \xrightarrow{\alpha} \operatorname{Ker} v \xrightarrow{\beta} \prod_{\mathfrak{M}(F)} R^{\cdot}/R^{\cdot 2} \to 1. \tag{22}$$

The group $R^{\cdot}/R^{\cdot 2}$ is cyclic of order two, so the exactness of (22) says that $|\operatorname{Ker} v/\sum F^{\cdot 2}| = 2^n$. The map α above is inclusion, and β is defined by

$$\beta(a) = (\sigma(ac_\sigma^2)R^{\cdot 2})_{\sigma \in \mathfrak{M}(F)}$$

where for each $a \in \operatorname{Ker} v$ and $\sigma \in \mathfrak{M}(F)$ we choose $c_\sigma \in F^{\cdot}$ so that $\sigma(ac_\sigma^2) \in R^{\cdot}$. $\beta(a)$ is clearly independent of the choice of the c_σ. The surjectivity of β follows from [3, Theorem 2.1B]. That $\operatorname{Im}\alpha \subseteq \operatorname{Ker}\beta$ is obvious. The reverse inclusion follows since any $a \in \operatorname{Ker}\beta$ is positive under every ordering of F, and hence is totally positive.

We now give an isomorphism from $\operatorname{Im} v$ to $\Lambda_1 \times \prod_{i=1}^n \operatorname{Ker} w_{i,i-1}$. The groups $\operatorname{Ker} w_{i,i-1}$ may be canonically identified with the square factor groups of the value groups of certain valuation rings [18, Chapter C]. Let $\Lambda_{1,0} = 1$

and let $w_{1,0}\colon \Lambda_1 \to \Lambda_{1,0}$ be the zero map. For each i, $1 \leqslant i \leqslant n$, let $K_i = \operatorname{Ker} w_{i,i-1}$ and let $\gamma_i\colon K_i \to \Lambda_i$ be the inclusion map. Since the sequences

$$1 \to K_i \xrightarrow{\gamma_i} \Lambda_i \xrightarrow{w_{i,i-1}} \Lambda_{i,i-1} \to 1$$

are split exact, we have exact sequences

$$1 \leftarrow K_i \overset{\gamma_i'}{\leftarrow} \Lambda_i \overset{\eta_i}{\leftarrow} \Lambda_{i,i-1} \leftarrow 1$$

with $\gamma_i'\gamma_i$ and $w_{i,i-1}\eta_i$ the respective identity maps. Define $\Upsilon\colon \prod_{i\leqslant n} K_i \to \prod_{i \leqslant n} \Lambda_i$ inductively by the condition that $\Upsilon(\lambda_1,\ldots,\lambda_n) = (\mu_1,\ldots,\mu_n)$ if and only if $\lambda_1 = \mu_1$ and for $n > i \geqslant 1$,

$$\mu_{i+1} = \lambda_{i+1}\eta_{i+1}w_{i,i+1}(\mu_i).$$

One checks that the sequence

$$1 \to \prod_{i\leqslant n} K_i \xrightarrow{\Upsilon} \prod_{i\leqslant n} \Lambda_i \xrightarrow{w} \prod_{i<n} \Lambda_{i,i+1} \to 1$$

is exact. That Υ maps isomorphically onto $\operatorname{Im} v$ now follows from Lemma (6.3).

7. **The total signature map of an exact field.** Recall that for each $\sigma \in \mathfrak{M}(F)$, $\mathcal{O}_\sigma$ is homeomorphic to the space $\mathcal{O}(F_\sigma)$, which carries a canonical Borel measure (cf. §2, especially (2.4)). Hence $\mathcal{O}_\sigma$ carries a canonical Borel measure μ with $\mu(\mathcal{O}_\sigma) = 1$; one can define it more directly by using any of the homeomorphisms $\langle P, \cdot\rangle$ of Lemma (2.3) to lift the canonical Haar measure on Λ_σ^* back to $\mathcal{O}_\sigma$.

(7.1) THEOREM. *F is exact if and only if* $\operatorname{Im} t$ *consists of precisely those maps $f \in C(\mathcal{O}(F), Z)$ such that for all $P, T \in \mathcal{O}(F)$ (say with $\sigma = \pi(P)$ and $\tau = \pi(T)$) and $\beta \in \Lambda_{\sigma,\tau}$,*

$$(23) \qquad \sum_{\lambda \in w_{\sigma,\tau}^{-1}(\beta)} \int_{\mathcal{O}_\sigma} f(S)\langle P,S\rangle(\lambda)\,dS + \sum_{\lambda \in w_{\tau,\sigma}^{-1}(\beta)} \int_{\mathcal{O}_\tau} f(S)\langle T,S\rangle(\lambda)\,dS$$

is an even integer.

PROOF. Let $f \in C(\mathcal{O}(F), Z)$. Let $P, T \in \mathcal{O}(F)$ with $\sigma = \pi(P)$, $\tau = \pi(T)$. For each $\beta \in \Lambda_{\sigma,\tau}$ let $a_{\beta,P,T}$ be the number displayed in (23) above.

Suppose $\sigma = \tau$. Then $w_{\sigma,\tau}$ and $w_{\tau,\sigma}$ are identity maps. For each $\beta \in \Lambda_\sigma$ with $\langle P,T\rangle(\beta) = -1$ we have

$$\langle P,S\rangle(\beta) + \langle T,S\rangle(\beta) = \langle P,S\rangle(\beta) + \langle P,T\rangle(\beta)\langle P,S\rangle(\beta) = 0$$

for all $S \in \mathcal{O}_\sigma$, so $a_{\beta,P,T} = 0$. For each $\beta \in \Lambda_\sigma$ with $\langle P,T\rangle(\beta) = 1$ we have

$$a_{\beta,P,T} = 2\int_{\mathfrak{O}_\sigma} f(S)\langle P,S\rangle(\beta)\,dS.$$

In particular, for all $\beta \in \Lambda_\sigma$ we have

$$a_{\beta,P,P} = 2\int_{\mathfrak{O}_\sigma} f(S)\langle P,S\rangle(\beta)\,dS.$$

We conclude that $a_{\beta,P,T}$ will be an even integer for all P and T with $\pi(P) = \pi(T)$ and all β if and only if $f \in \operatorname{Im} t_1$ (Theorem (2.7)).

Now suppose $\sigma \neq \tau$ and that $f \in \operatorname{Im} t_1$. Write $f = t_1((q_\rho)_\rho)$, where $q_\rho \in W(F_\rho)$ for all $\rho \in \mathfrak{M}(F)$. Let $w''_{\sigma,\tau}\colon Z(\Lambda_\sigma) \to Z_2(\Lambda_{\sigma,\tau})$ be the map induced by $w_{\sigma,\tau}$; then $w''_{\sigma,\tau}\Phi_P = v'_{\sigma,\tau}$ (cf. (3.2)). By Lemma (2.11),

$$\begin{aligned} v'_{\sigma,\tau}(q_\sigma) &= w''_{\sigma,\tau}\Big(\sum_{\lambda\in\Lambda_\sigma}\Big(\int_{\mathfrak{O}_\sigma} f(S)\langle P,S\rangle(\lambda)\,dS\Big)\lambda\Big) \\ &= \sum_{\beta\in\Lambda_{\sigma,\tau}}\Big(\sum_{\lambda\in w_{\sigma,\tau}^{-1}(\beta)}\int_{\mathfrak{O}_\sigma} f(S)\langle P,S\rangle(\lambda)\,dS + 2Z\Big)\beta. \end{aligned}$$

A similar calculation holds for $v'_{\tau,\sigma}(q_\tau)$. Hence $v_{**}((q_\rho)_\rho)$ takes at (σ,τ) the value $\sum_{\beta\in\Lambda_{\sigma,\tau}}(a_{\beta,P,T} + 2Z)\beta$. Hence $a_{\beta,P,T}$ is an even integer for all P, T and β with $\pi(P) \neq \pi(T)$ if and only if $v_{**}((q_\rho)_\rho) = 0$, i.e. $(q_\rho)_\rho \in \operatorname{Ker} v_{**}$.

Thus the set of $f \in C(\mathfrak{O}(F), Z)$ with $a_{\beta,P,T} \in 2Z$ for all P, T, β is exactly $t_1(\operatorname{Ker} v_{**}) \cap C(\mathfrak{O}(F), Z)$. But F is exact if and only if this set is $\operatorname{Im} t$ (cf. Remark (3.4)). The theorem is proved.

(7.2) Remarks. Let $f \in C(\mathfrak{O}(F), Z)$ have constant parity.

(A) Our proof of (7.1) shows that if $f \in \operatorname{Im} t$, then

$$a_{\beta,P,T} \in 2Z \quad \text{for all } P,\ T \in \mathfrak{O}(F),\ \beta \in \Lambda_{\sigma,\tau}. \tag{24}$$

(As above, $a_{\beta,P,T}$ equals the number displayed in (23), $\sigma = \pi(P)$, and $\tau = \pi(T)$.) The proof of (7.1) also shows that condition (24) is highly redundant: (24) holds if and only if for all σ, $\tau \in \mathfrak{M}(F)$ and $\beta \in \Lambda_{\sigma,\tau}$ there exists $P \in \mathfrak{O}_\sigma$ and $T \in \mathfrak{O}_\tau$ (with $P = T$ if $\sigma = \tau$) with $a_{\beta,P,T} \in 2Z$.

(B) If σ, $\tau \in \mathfrak{M}(F)$ have $\Lambda_{\sigma,\tau} = 1$ (i.e. σ and τ are "independent modulo squares"), then $a_{\beta,P,T} = f(P) + f(T)$ for all $\beta \in \Lambda_{\sigma,\tau}$ (Lemma (2.10)). Thus, for example, the condition "$a_{\beta,P,T} \in 2Z$" is automatically satisfied when P and T induce independent real places. It is also automatically satisfied when P and T induce the same real place σ and $|\Lambda_\sigma| \leqslant 2$. Thus, in particular, (24) holds for all functions $f \in C(\mathfrak{O}(F), Z)$ of constant parity if $|\Lambda_\sigma| \leqslant 2$ and $\Lambda_{\sigma,\tau} = 1$ for all σ, $\tau \in \mathfrak{M}(F)$ ($\sigma \neq \tau$). But we knew that this had to be the case since such fields have $\operatorname{Im} t = C(\mathfrak{O}(F), 2Z) + Z \cdot 1$ (cf. Remark (3.5)).

We now specialize Theorem (7.1) to those fields in which at most four orderings can induce any one real place. Such fields include, for example, all of the formally real algebraic extensions of the exact fields $Q(x)$, $R(x,y)$, $R((x))(y)$, and $Q(x)((y))$. We will use the next theorem to compute $\operatorname{Im} t$ for the fields $Q(x)$ and $R(x,y)$.

(7.3) THEOREM. *Suppose* $|\Lambda_\sigma| \leqslant 4$ *for all* $\sigma \in \mathfrak{M}(F)$. *Then* F *is exact if and only if* $\operatorname{Im} t$ *consists of all* $f \in C(\mathcal{O}(F), Z)$ *of constant parity with*

$$\sum_{\mathcal{O}_\sigma} f(P) + \sum_{\mathcal{O}_\tau} f(P) \equiv 0 \pmod 8 \tag{25}$$

for all $\sigma, \tau \in \mathfrak{M}(F)$ *with* $|\Lambda_{\sigma,\tau}| = 4$, *and*

$$\sum_{i=1}^{4} f(P_i) \equiv 0 \pmod 4 \tag{26}$$

for all distinct P_1, P_2, P_3, P_4 *in* $\mathcal{O}(F)$ *such that for some* $\sigma, \tau \in \mathfrak{M}(F)$ $(\sigma \neq \tau)$ *and* $\eta \in \Lambda^*_{\sigma,\tau}$ *we have* $P_1, P_3 \in \mathcal{O}_\sigma$, $P_2, P_4 \in \mathcal{O}_\tau$, *and*

$$\langle P_1, P_3 \rangle = \eta w_{\sigma,\tau} \quad \textit{and} \quad \langle P_2, P_4 \rangle = \eta w_{\tau,\sigma}. \tag{27}$$

Before proving Theorem (7.3) we will give two interpretations of condition (27) and some applications of the theorem.

(7.4) REMARK. Suppose $|\Lambda_\sigma| \leqslant 4$ for all $\sigma \in \mathfrak{M}(F)$. Suppose $\sigma, \tau \in \mathfrak{M}(F)$ $(\sigma \neq \tau)$, and that $P_1, P_3 \in \mathcal{O}_\sigma$ and $P_2, P_4 \in \mathcal{O}_\tau$ are all distinct (cf. Theorem (7.3)).

(A) We first give a case-by-case discussion of when there exists $\eta \in \Lambda^*_{\sigma,\tau}$ with (27) valid. First suppose $|\Lambda_{\sigma,\tau}| = 4$. Then (27) holds for some $\eta \in \Lambda^*_{\sigma,\tau}$ if and only if $w_{\sigma,\tau}(\operatorname{Ker}\langle P_1, P_3\rangle) = w_{\tau,\sigma}(\operatorname{Ker}\langle P_2, P_4\rangle)$ (i.e., $\langle P_1, P_3\rangle = \langle P_2, P_4\rangle$, treating the isomorphisms $w_{\sigma,\tau}$ and $w_{\tau,\sigma}$ as identifications). Next note that if $\Lambda_{\sigma,\tau} = 1$, then (27) can never hold. This is because if (27) holds for some $\eta \in \Lambda^*_{\sigma,\tau}$, then $\eta \neq 1$ (otherwise $\langle P_1, P_3\rangle = 1$, contradicting that $P_1 \neq P_3$, cf. Lemma (2.3)). Finally suppose $|\Lambda_{\sigma,\tau}| = 2$. Then there is only one nontrivial map $\eta \in \Lambda^*_{\sigma,\tau}$. We will have $\langle P_1, P_3\rangle = \eta w_{\sigma,\tau}$ (for this η) if and only if either $|\Lambda_\sigma| = 2$, or else $|\Lambda_\sigma| = 4$ and $\operatorname{Ker} w_{\sigma,\tau} = \operatorname{Ker}\langle P_1, P_3\rangle$. (A similar remark holds with τ in place of σ.)

The case when $|\Lambda_\sigma| = |\Lambda_\tau| = 4$ and $|\Lambda_{\sigma,\tau}| = 2$ is of particular interest (cf. (7.5C), below). Then $w_{\sigma,\tau}$ and $\langle P_1, P_3\rangle$ have the same kernel if and only if P_1 and P_3 induce the same ordering in the residue class field of $[\sigma, \tau]$.

(B) There exists $\eta \in \Lambda^*_{\sigma,\tau}$ with (27) valid if and only if $\Lambda_{\sigma,\tau} \neq 1$, $\{P_1, P_3\} = \mathcal{O}_\sigma \cap V(b)$, and $\{P_2, P_4\} = \mathcal{O}_\tau \cap V(b)$ for some $b \in F^{\cdot}$ with $v_{\sigma,\tau}(b) = 1$ if $|\Lambda_{\sigma,\tau}| = 2$ [3, Theorem 2.1A].

(7.5) APPLICATIONS. (A) The image of the total signature map on $W_{\text{red}}(Q(x))$ consists of all $f \in C(\mathcal{O}(F), Z)$ of constant parity such that

$$\sum_{P \in \mathcal{O}_\sigma} f(P) = \sum_{P \in \mathcal{O}_\tau} f(P) \pmod 4$$

for all pairs of nontrivial equivalent real places $\sigma, \tau \in \mathfrak{M}(Q(x))$. (Two real places σ, τ on $Q(x)$ are nontrivial and equivalent if and only if $\Lambda_{\sigma,\tau} \neq 1$; this is because every nontrivial real place on $Q(x)$ is discrete rank one. Now use Application (4.5B) and Remark (7.4A) to apply Theorem (7.3) to $Q(x)$.) It is perhaps worth noting here that the spaces $\mathcal{O}(Q(x))$, $\mathfrak{M}(Q(x))$, and the equivalence relation on $\mathfrak{M}(Q(x))$ are all fairly transparent. $\mathfrak{M}(Q(x))$ is discussed in §8 below. Tom Craven has shown me a computation of the space $\mathcal{O}(Q(x))$ (unpublished, but see [7]).

(B) If $|\Lambda_\sigma| \leqslant 2$ for all $\sigma \in \mathfrak{M}(F)$, then F is exact if and only if $\operatorname{Im} t$ consists of all $f \in C(\mathcal{O}(F), Z)$ with $\sum_{P \in \mathcal{O}_\sigma} f(P) \equiv \sum_{P \in \mathcal{O}_\tau} f(P) \pmod 4$ for all $\sigma, \tau \in \mathfrak{M}(F)$ with $\Lambda_{\sigma,\tau} \neq 1$. (This is a generalization of (A) above.)

(C) The image of the total signature map on $R(x,y)$ consists of all $f \in C(\mathcal{O}(F), Z)$ of constant parity such that $\sum_{i=1}^4 f(P_i) \equiv 0 \pmod 4$ whenever P_1, P_2, P_3, P_4 are distinct orderings of $R(x,y)$ which either all induce the same real place on $R(x,y)$ or else induce two real places σ and τ with $P_1, P_3 \in \mathcal{O}_\sigma$ and $P_2, P_4 \in \mathcal{O}_\tau$, and $\operatorname{Ker} w_{\sigma,\tau} = \operatorname{Ker}\langle P_1, P_3\rangle$ and $\operatorname{Ker} w_{\tau,\sigma} = \operatorname{Ker}\langle P_2, P_4\rangle$. (All pairs of distinct real places on $R(x,y)$ are incomparable (Lemma (2.4)), and all real places on $R(x,y)$ are either rank one or discrete rank two [1, Proposition 2.46]. Hence $|\Lambda_{\sigma,\tau}| = 4$ only if $\sigma = \tau$, and $|\Lambda_{\sigma,\tau}| = 2$ only if $\sigma = \tau$ or $|\Lambda_\sigma| = |\Lambda_\tau| = 4$. Now use Application (4.5B) and Remark (7.4A) to apply Theorem (7.3) to $R(x,y)$.) A similar computation of $\operatorname{Im} t$ holds for exact finite dimensional extensions of $R(x,y)$. We note in passing that $\mathfrak{M}(R(x,y))$ and $\mathcal{O}(R(x,y))$ have been computed as sets; the key fact is that the set of real places on $R(x,y)$ which extend a given real place on $R(x)$ is bijective with an easily described set of (R, Z)-signatures [4, especially §6]. The dependency relation on $\mathfrak{M}(R(x,y))$ can also be discussed in terms of signatures.

We now give the proof of Theorem (7.3). Let $f \in C(\mathcal{O}(F), 2Z) + Z \cdot 1$. Let $\sigma, \tau \in \mathfrak{M}(F)$ with $\Lambda_{\sigma,\tau} \neq 1$. We can write $\mathcal{O}_\sigma = \{P_1, P_3, P_5, P_7\}$ and $\mathcal{O}_\tau = \{P_2, P_4, P_6, P_8\}$ where $P_1 = P_2$ if $\sigma = \tau$, $P_1 = P_5$ and $P_3 = P_7$ if $|\mathcal{O}_\sigma| = 2$, and $P_2 = P_6$ and $P_4 = P_8$ if $|\mathcal{O}_\tau| = 2$. By Remark (7.2) it suffices to show the equivalence of the following two statements:

(I) $a_{\beta, P_1, P_2} \in 2Z$ for all $\beta \in \Lambda_{\sigma,\tau}$.

(Here, $a_{\beta,P,T}$ denotes the number displayed in (23).)

(II)

$$\sum_{i=1}^{3} f(P_i) \equiv 0 \pmod 8 \quad \text{if } |\Lambda_{\sigma,\tau}| = 4,$$

and

$$\sum_{i=1}^{4} f(P_i) \equiv 0 \pmod 4 \quad \text{if } \sigma \neq \tau$$

and (27) holds for some $\eta \in \Lambda^*_{\sigma,\tau}$.

Case 1. $|\Lambda_{\sigma,\tau}| = 4$. We use the isomorphisms $w_{\sigma,\tau}$ and $w_{\tau,\sigma}$ to identify Λ_σ, Λ_τ, and $\Lambda_{\sigma,\tau}$. We have

$$a_{1,P_1,P_2} = \frac{1}{4}\left(\sum_{i=1}^{8} f(P_i)\right), \tag{28}$$

and whenever

$$1 \neq \beta \in \operatorname{Ker}\langle P_1, P_3\rangle \cap \operatorname{Ker}\langle P_2, P_4\rangle, \tag{29}$$

then

$$\begin{aligned} a_{\beta,P_1,P_2} &= \frac{1}{4}\left(\sum_{i=1}^{4} f(P_i) - \sum_{i=5}^{8} f(P_i)\right) \\ &= \frac{1}{2}\sum_{i=1}^{4} f(P_i) - a_{1,P_1,P_2}. \end{aligned} \tag{30}$$

Now suppose (I) holds. Then $\sum_{i=1}^{8} f(P_i) \equiv 0 \pmod 8$ (by (28)). If $\sigma \neq \tau$ and (27) holds for some η, then (29) holds (cf. Remark (7.4A)) with β a generator of $\operatorname{Ker}\eta$. Hence (30) shows that $\sum_{i=1}^{4} f(P_i) \equiv 0 \pmod 4$. Now let us suppose that (II) holds. Then $a_{1,P_1,P_2} \in 2Z$ by (28). Let $1 \neq \beta \in \Lambda_{\sigma,\tau}$. We may assume that $P_3, P_4, \ldots, P_8$ are indexed so that (29) holds (Lemma (2.3)). Then (27) is satisfied (with $\eta \in \Lambda^*_{\sigma,\tau}$ the unique map with kernel $\{1,\beta\}$). That $a_{\beta,P_1,P_2} \in 2Z$ now follows from (28) and (30) (if $\sigma = \tau$, then (29) implies that $P_3 = P_4$). This completes our discussion of Case 1.

We henceforth assume that $|\Lambda_{\sigma,\tau}| = 2$. If $1 \neq \beta \in \Lambda_{\sigma,\tau}$, then we have (for $P \in \mathcal{O}_\sigma$, $T \in \mathcal{O}_\tau$)

$$a_{\beta,P,T} = f(P) + f(T) - a_{1,P,T} \equiv a_{1,P,T} \pmod 2.$$

Hence it suffices to show the equivalence of the following two statements:

(I′) $a_{1,P_1,P_2} \in 2Z$.

(II′)

$$\sum_{i=1}^{4} f(P_i) \equiv 0 \pmod 4 \quad \text{if } \sigma \neq \tau,$$

$\langle P_1, P_3\rangle = w_{\sigma,\tau}$, and $\langle P_2, P_4\rangle = w_{\tau,\sigma}$.

In the statement of (II′) we have made the canonical identification of $\Lambda_{\sigma,\tau}$ and $Z^{\cdot}$.

Case 2. $|\Lambda_\sigma| = |\Lambda_\tau| = 2$. Then $a_{1,P_1,P_2} = \frac{1}{2}\sum_{i=1}^4 f(P_i)$. Also $\langle P_1, P_3\rangle = w_{\sigma,\tau}$ and $\langle P_2, P_4\rangle = w_{\tau,\sigma}$ (each map has trivial kernel). The equivalence of (I′) and (II′) is now immediate.

We may now assume without loss of generality that $|\Lambda_\sigma| = 4$. Hence $\sigma \neq \tau$.

Case 3. $|\Lambda_\tau| = 4$. Let α_1 generate $\operatorname{Ker} w_{\sigma,\tau}$ and α_2 generate $\operatorname{Ker} w_{\tau,\sigma}$. Now, $\langle P_1, P_3\rangle = w_{\sigma,\tau}$ and $\langle P_2, P_4\rangle = w_{\tau,\sigma}$ if and only if

$$\langle P_1, P_3\rangle(\alpha_1) = 1 = \langle P_2, P_4\rangle(\alpha_2). \tag{31}$$

If (31) holds, then $a_{1,P_1,P_2} = \frac{1}{2}\sum_{i=1}^4 f(P_i)$ (since a_{1,P_1,P_2} equals

$$\frac{1}{4}\sum_{P\in\mathcal{O}_\sigma} f(P)(1 + \langle P, P_1\rangle(\alpha_1)) + \frac{1}{4}\sum_{P\in\mathcal{O}_\tau} f(P)(1 + \langle P, P_2\rangle(\alpha_2))).$$

That (I′) implies (II′) is now immediate; the converse follows from the fact that we can index $P_3, P_4, \ldots, P_8$ so that (31) holds (Lemma (3.2)).

Case 4. $|\Lambda_\tau| = 2$. The proof for Case 3 carries over to this case with only minor modification.

8. Some fields of transcendence degree one. We thank Tom Craven for his collaboration on this section. Our objective is the following example, which was promised in Application (5.9D).

(8.1) EXAMPLE. Let C be a compact subset of $R \cup \{\infty\}$ (the one-point compactification of R). There exists an algebraic extension F of $Q(x)$ such that the map $\sigma \mapsto \sigma(x)$ carries $\mathfrak{M}(F)$ homeomorphically onto C and such that F is SAP if and only if C contains no pair of distinct conjugate algebraic numbers. (We regard ∞ as "algebraic", but not a number.)

For example, if $C = \{\sqrt{2}, -\sqrt{2}, \pi\}$, then the corresponding field F is not a SAP field. However, the quadratic extension $K = F[\sqrt{(x-2)}]$ is a SAP field. (The only real places on K are the two homomorphisms carrying $\sqrt{(x-2)}$ to $\sqrt{(\pi-2)}$ and to $-\sqrt{(\pi-2)}$.) Thus a non-SAP field can have a finite algebraic extension which is SAP. This answers a question of Bröcker [20, p. 251] and Prestel [21, p. 151].

We begin the proof of (8.1) with two lemmas.

(8.2) LEMMA. *Let F be a field and $a \in F^{\cdot}$. Let $K = F[a_0, a_1, \ldots]$ and $E = F[b_0, b_1, \ldots]$ be algebraic extensions of F with $a_0 = b_0 = a$ and $a_{i+1}^2 = a_i$ and $b_{i+1}^2 = b_i$ for all $i \geqslant 0$. If $a \notin Z^{\cdot} \cdot F^{\cdot 2}$ then there is an F-isomorphism from K to E carrying each a_i to b_i.*

PROOF. Let $F_n = F[a_0, a_1, \ldots, a_n]$ for $n \geqslant 0$. Suppose inductively that $a_n \notin Z^{\cdot} \cdot F_n^{\cdot 2}$. Then $a_{n+1} \notin Z^{\cdot} \cdot F_{n+1}^{\cdot 2}$, since otherwise for some $c, d \in F_n$, $(ca_{n+1} + d)^2 = \pm a_{n+1}$. But then $2cd \neq 0$ and $a_n = -(d/c)^2 \in -F_n^{\cdot 2}$, a contradiction. The required isomorphism is now easily constructed inductively.

When the hypotheses of (8.2) hold we write $a^{1/2^n}$ for a_n. The only other use we make of this notation is when a is a positive real number; then, $a^{1/2^n}$ denotes its unique positive 2^nth root.

(8.3) LEMMA. *Let $\sigma \in \mathfrak{M}(F)$ and $a \in \sigma^{-1}(R^{\cdot 2})$. Suppose $a \notin F^{\cdot 2}$. Let $E = F[a^{1/2}, a^{1/4}, \ldots]$ (cf. Lemma 8.2). Then σ has a unique extension to a real place σ' on E. The value groups of σ and σ' are the same, and the residue class field of σ' is $K = F'[\sigma(a)^{1/2}, \sigma(a)^{1/4}, \ldots]$ (cf. the paragraph after Lemma 8.2) where $F' = R \cap \sigma(F)$.*

PROOF. There exists an extension σ_H of σ to a field F_H which is Henselian, has the same value group as σ, and has residue class field K. By Lemma 8.2 we may regard E as a subfield of F_H (apply Hensel's Lemma [18, p. 185] to the polynomials $x^2 - a^{1/2^n}$). Let $\sigma' = \sigma_H | E$. Then σ' has the required value group and residue class field. Suppose σ'' is an extension of σ to a real place on E. Suppose inductively that σ' and σ'' agree on $F_n = F[a^{1/2}, a^{1/4}, \ldots, a^{1/2^n}]$ for some $n \geqslant 0$. Then there exists an F_n-automorphism ρ of F_{n+1} and a $\sigma(F_n)$-isomorphism θ: $\sigma'(F_{n+1}) \to \sigma''(F_{n+1})$ with $\sigma''\rho = \theta\sigma'$ on F_{n+1} [p. 152, Chapter 6 of Bourbaki, *Algèbre commutative*, Hermann, Paris, 1964]. If θ is not the identity map, then ρ is the identity map [18, p. 228]. Hence

$$0 < \sigma''(a_{n+2})^2 = \sigma''(a_{n+1}) = \theta\sigma'(a_{n+1}) = -\sigma'(a_{n+1}) < 0,$$

a contradiction. Hence $\sigma''\rho = \sigma'$. If ρ is not the identity, then again $0 < \sigma'(a_{n+1}) = -\sigma''(a_{n+1}) < 0$. Hence $\sigma' = \sigma''$ on F_{n+1}. The result now follows by induction.

We now prove (8.1). Assume $C \neq R \cup \{\infty\}$ (otherwise take $F = Q(x)$). We can write the complement of C in $R \cup \{\infty\}$ as a countable union of intervals $\bigcup_{i=1}^{\infty}(a_i, b_i)$ where the a_i and b_i are all distinct rational numbers in the complement of C (thus, $a_i \neq a_j \neq b_k \neq b_m$ if $i \neq j$, $k \neq m$). We allow $a_i > b_i$; in this case $(a_i, b_i) = \{c \in R \cup \{\infty\}$: $c > a_i$ or $c < b_i\}$ is an interval about ∞. Let $f_n = \pm e_n(x - a_n)(x - b_n)(x^2 + 1)^{-1}$ for all $n \geqslant 1$, where the sign $\pm$ is chosen so that f_n is negative on (a_n, b_n). The e_n are positive integers, which for the moment can be assigned arbitrarily. Let $F_0 = Q(x)$ and set $F_{n+1} = F_n[f_{n+1}^{1/2}, f_{n+1}^{1/4}, \ldots]$ for all $n \geqslant 0$. (For each $n \geqslant 0$, $f_{n+1} \notin Z^{\cdot} \cdot F_n^{\cdot 2}$ since f_{n+1} has minimal positive value in any extension of the $(x - a_{n+1})$-adic valuation on $Q(x)$ to F_n [Remark 5.2C of R. Brown and H. D. Warner, *Quadratic extensions of linearly compact fields*, Trans. Amer. Math. Soc. **163** (1972), 379–399]. Hence the F_n are well defined by Lemma 8.2.) Let $F = \bigcup_{n \geqslant 0} F_n$. By construction the (continuous) map $\sigma \mapsto \sigma(x)$ must carry $\mathfrak{M}(F)$ into C. Lemma (8.3) shows that $\mathfrak{M}(F)$ must map bijectively onto C (each f_n takes finite positive value at any real place on $Q(x)$ mapping x into C).

Next suppose that C contains no pair of distinct conjugate algebraic numbers. Let σ be a place from F onto a formally real field E with $\Lambda_\sigma \neq 1$. Then $|\Lambda_\sigma| \leqslant 2$ and E/Q is algebraic. If τ and ρ are embeddings of E into R then $\tau\sigma(x) = \rho\sigma(x)$ ($\tau\sigma$ and $\rho\sigma$ restrict to equivalent nontrivial places on $Q(x)$ and hence assign conjugate values to x). Thus $\tau = \rho$. Thus E has a unique ordering. The criterion of Elman, Lam and Prestel (cf. Remark 3.5) shows that F is a SAP field. Conversely, suppose C does contain a pair of conjugate algebraic numbers $\sigma(x)$ and $\tau(x)$ (where $\sigma, \tau \in \mathfrak{M}(F)$). We claim we can pick the e_n so that $\sigma(f_{n+1}) \notin Z^{\cdot} \cdot K_n^{\cdot 2}$, where $K_n = Q[\sigma(x)][\{\sigma(f_i)^r : i \leqslant n, r = \frac{1}{2}, \frac{1}{4}, \ldots\}]$ is the residue class field of $\sigma | F_n$ (Lemma 8.3). Suppose such e_m have been chosen for all $m \leqslant n$. There exists an odd rational number p which is unramified in $Q(\sigma(x))$ and for which all the nonzero coefficients of all the irreducible polynomials over Q of the numbers $\sigma(f_1), \ldots, \sigma(f_n)$ are p-adic units. Then K_n is obtained from $Q(\sigma(x))$ by a sequence of adjunctions of the square root of units (with respect to any extension v of the p-adic valuation to K_n). Hence p has minimal positive value in $v(K_n^{\cdot})$. Thus $p \notin Z^{\cdot} \cdot K_n^{\cdot 2}$. It therefore suffices to let $e_{n+1} = p$ if $(x - a_{n+1})(x - b_{n+1})(x^2 + 1)^{-1} \in Z^{\cdot} \cdot K_n^{\cdot 2}$ and let $e_{n+1} = 1$ otherwise. The claim is proved. Now apply Lemma 8.3 (inductively) to show that $|\Lambda_\sigma| = 2$ and that the real place on $Q(\sigma(x))$ sending $\sigma(x)$ to $\tau(x)$ extends to the residue class field of σ. This shows F cannot be a SAP field (Remark (3.5)). Finished,

References

1. S. Abhyankar, *Ramification theoretic methods in algebraic geometry*, Ann. of Math. Studies, no. 43, Princeton Univ. Press, Princeton, N.J., 1959. MR **21** #4158.

2. Ron Brown, *Real places and ordered fields*, Rocky Mountain J. Math. **1** (1971), 633–636. MR **44** #2730.

3. ———, *An approximation theorem for extended prime spots*, Canad. J. Math. **24** (1972), 167–184. MR **45** #1884.

4. ———, *Valuations, primes and irreducibility in polynomial rings and rational function fields*, Trans. Amer. Math. Soc. **174** (1972), 451–488.

5. ———, *Extended prime spots and quadratic forms*, Pacific J. Math. **51** (1974), 379–395.

6. ———, *Superpythagorean fields*, J. Algebra **42** (1976), 483–494.

7. T. C. Craven, *The topological space of orderings of a rational function field*, Duke Math. J. **41** (1974), 339–347. MR **50** #2132.

8. ———, *The Boolean space of orderings of a field*, Trans. Amer. Math. Soc. **209** (1975), 225–235.

9. R. Elman and T. Y. Lam, *Quadratic forms over formally real fields and Pythagorean fields*, Amer. J. Math. **94** (1972), 1155–1194. MR **47** #3427.

10. R. Elman, T. Y. Lam and A. Prestel, *On some Hasse principles over formally real fields*, Math. Z. **134** (1973), 291–301. MR **48** #8384.

11. I. Kaplansky, *Maximal fields with valuations*, Duke Math. J. **9** (1942), 303–321. MR **3**, 264.

12. M. Knebusch, A. Rosenberg and R. Ware, *Signatures on semilocal rings*, J. Algebra **26** (1973), 208–250. MR **48** #6103.

13. J. Milnor, *Algebraic K-theory and quadratic forms*, Invent. Math. **9** (1970), 318–344.

14. J. Milnor and D. Husemoller, *Symmetric bilinear forms*, Ergebnisse Math. Grenzgebiete, Band 73, Springer-Verlag, New York, 1973.

15. A. Pfister, *Quadratische Formen in beliebigen Körpern*, Invent. Math. **1** (1966), 116–132. MR **34** #169.

16. A. Prestel, *Quadratische Semi-Ordnungen und Quadratische Formen*, Math. Z. **133** (1973), 319–342. MR **49** #2682.

17. W. Rudin, *Fourier analysis on groups*, Interscience, New York, 1962. MR **27** #2808.

18. P. Ribenboim, *Théorie des valuations*, 2nd ed., Séminaire de Mathématiques Supérieures, No. 9 (Été, 1964), Les Presses de l'Université de Montréal, Montréal, 1968. MR **40** #2670.

19. O. F. G. Schilling, *The theory of valuations*, Math. Surveys, no. 4, Amer. Math. Soc., Providence, R.I., 1950. MR **13**, 315.

20. L. Bröcker, *Zur Theorie der quadratischen Formen über formal reellen Körpern*, Math. Ann. **210** (1974), 233–256. MR **50** #7027.

21. A. Prestel, *Lectures on formally real fields*, Monogr. Math. 22, Inst. Mat. Pura Apl., Rio de Janeiro, 1975.

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