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MODULAR FORMS FOR $\Gamma_0(N)$ AND DIRICHLET SERIES

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ABSTRACT. A criterion is given for a function to be a modular form for $\Gamma_0(N)$. It is similar to the criterion given by Weil in his 1967 Math. Ann. paper *Über die Bestimmung Dirichletscher Reihen durch Funktionalgleichungen* in that it involves checking that certain twists of the associated Dirichlet series satisfy functional equations. It differs in the number and type of such equations which need to be satisfied.

1. In his 1967 paper *Über die Bestimmung Dirichletscher Reihen durch Funktionalgleichungen*, Weil gives a criterion for a function to be a modular form for $\Gamma_0(N)$. The criterion to be met is that a collection of Dirichlet series associated with the function satisfy functional equations of a specified type. This (among other reasons) led Weil to make a conjecture to the effect that every elliptic curve E over the rationals is isomorphic to a factor of the Jacobian of the modular curve $H^*/\Gamma_0(N)$ where N is the conductor of E. The point is that to check whether a given E is parameterized by forms for $\Gamma_0(N)$ it is only necessary to check that its zeta-function suitably twisted by various characters satisfies a functional equation of prescribed type.

In this paper a criterion is given for a function to be a modular form for $\Gamma_0(N)$. It is similar to Weil's in that it involves checking that certain twists of the associated Dirichlet series satisfy functional equations. It differs in the number and type of such equations which need be satisfied.

From the point of view of elliptic curves, Weil's criterion involves the proving of functional equations for factors of the zeta-function of the curve over various extensions of the rationals whose discriminants are relatively prime to the conductor. The criterion in this paper requires the proving of functional equations for factors of the zeta-function over extensions of the rationals whose discriminants are not relatively prime to the conductor. In the important special case of prime conductor p, it is only necessary to check the functional equation for factors of the zeta-function of the curve over the cyclotomic field $Q(1^{1/p})$.

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Another application is to the study of the eigenvalues of Hecke operators for $\Gamma_0(N)$ at the bad primes (those which divide N). These have been discussed by Ogg [2] and by Atkin and Lehner [1] in the case where ψ is the unit character and by Ogg [2] when ψ is real. It is possible to recover their results from the criterion of this paper together with the assumption that the associated Dirichlet series has an Euler product. Some additional results (for more general ψ) can also be obtained. The results will be published at a later time if there seems to be any interest.

2. Let c be a positive integer. Let S_c be the subset of SL(2, $\mathbf{Z}$) consisting of all matrices of the form $\left(\begin{smallmatrix} * & * \\ c & * \end{smallmatrix}\right)$. For each integer a which is relatively prime to c, choose an element $\sigma_a \in S_c$ of the form $\left(\begin{smallmatrix} a & * \\ c & * \end{smallmatrix}\right)$. Let $f(z)$ be a holomorphic function on the upper half-plane H such that $f(z+1) = f(z)$. For each character χ mod c define

$$f_\chi(z) = \sum_{a \pmod c} \chi(a) f\left(\frac{z+a}{c}\right). \tag{1}$$

Proposition 1. *Let ψ be a character* mod c *and let k be an integer. Then the following are equivalent.*

A. *For all* $\sigma_a = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in S_c$,

$$f\left(\frac{az+b}{cz+d}\right) = \psi(d)(cz+d)^k f(z). \tag{2}$$

B. *For all characters* χ mod c,

$$f_\chi\left(-\frac{1}{z}\right) = \chi(-1)\psi(-1)z^k f_{\bar\chi\psi}(z). \tag{3}$$

Proof. A $\Rightarrow$ B. First note that

$$\sigma_a\left(\frac{z-d}{c}\right) = \frac{-z^{-1}+a}{c} \quad \text{and} \quad c\left(\frac{z-d}{c}\right) + d = z. \tag{4}$$

Thus, by A,

$$f\left(\frac{-z^{-1}+a}{c}\right) = \psi(d) z^k f\left(\frac{z-d}{c}\right)$$

and

$$f_\chi(-z^{-1}) = \sum_{a \bmod c} \chi(a)\psi(d) z^k f\left(\frac{z-d}{c}\right).$$

Since $\chi(a) = \bar\chi(d)$,

$$f_\chi(-z^{-1}) = \chi(-1)\psi(-1)z^k \sum_{d \bmod c} \bar\chi\psi(-d) f\left(\frac{z-d}{c}\right)$$

or,

$$f_\chi(-z^{-1}) = \chi(-1)\psi(-1)z^k f_{\bar\chi\psi}(z).$$

B $\Rightarrow$ A. We reverse the steps. By character summation and the definition (1) of f_χ,

$$f\left(\frac{z+a}{c}\right) = \frac{1}{\phi(c)} \sum_{\chi \bmod c} \bar\chi(a) f_\chi(z). \tag{5}$$

Therefore,

$$\begin{aligned} f\left(\frac{-z^{-1}+a}{c}\right) &= \frac{1}{\phi(c)} \sum_{\chi \bmod c} \bar\chi(a) f_\chi(-z^{-1}) \\ &= \frac{1}{\phi(c)} \sum_{\chi \bmod c} \bar\chi(a)\chi(-1)\psi(-1)z^k f_{\bar\chi\psi}(z) \\ &= \frac{\psi(d)z^k}{\phi(c)} \sum_{\chi \bmod c} \chi\bar\psi(-d) f_{\bar\chi\psi}(z) \\ &= \psi(d)z^k f\left(\frac{z-d}{c}\right). \end{aligned}$$

The conclusion now follows by (4).

The above proposition is formal. Suppose that $f(z)$ is further required to be regular at $i\infty$ in the sense that it has a Fourier expansion

$$f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z}, \qquad a_n = O(n^\kappa) \quad \text{as } n \to \infty. \tag{6}$$

Then the $f_\chi(z)$ also have Fourier expansions

$$f_\chi(z) = \sum_{n=0}^{\infty} \tau_n(\chi; c) a_n e^{2\pi i n z/c} \tag{7}$$

where the Gauss sums $\tau_n(\chi; c)$ are given by

$$\tau_n(\chi; c) = \sum_{t \bmod c} \chi(t) e^{2\pi i n t/c}. \tag{8}$$

In particular, the constant term in the expansion (7) is $\phi(c)a_0\delta a_\chi$ where

$$\delta_\chi = \begin{cases} 1 & \text{if } \chi \text{ is principal mod } c, \\ 0 & \text{if } \chi \text{ is not.} \end{cases}$$

To each $f_\chi(z)$ associate a Dirichlet series

$$\phi(s, \chi) = \sum_{n=1}^{\infty} \tau_n(\chi, c) a_n n^{-s} \tag{9}$$

and let

$$\Phi(s, \chi) = \left(\frac{2\pi}{c}\right)^{-s} \Gamma(s)\phi(s, \chi). \tag{10}$$

Then the usual Mellin transform argument (as in [3, p. I-5, Theorem 1]) applies to give the following equivalent formulation of condition B of Proposition 1.

C. For all characters χ mod c,

$$\Phi(s,\chi) = \chi(-1)\psi(-1)i^k\Phi(k-s,\bar{\chi}\psi)$$

and

$$\Phi(s,\chi) + \phi(c)a_0\left(\frac{\delta_\chi}{s} + \frac{\chi(-1)\psi(-1)i^k\delta_{\bar{\chi}\psi}}{k-s}\right) \tag{11}$$

is entire and bounded in vertical strips.

REMARK. For C (or equivalently, B) to be satisfied nontrivially it is necessary to assume that $\psi(-1) = (-1)^k$. This assumption will not be used explicitly.

We can now restate Proposition 1 as

PROPOSITION 2. *If $f(z)$ is regular at $i\infty$ (in the sense of (6)) then f satisfies* A *if and only if it satisfies* C.

3. If condition A is satisfied by a function f, it is of some interest to know what group is generated by the elements of S_c. A partial answer is given in the following proposition.

PROPOSITION 3. *For each positive integer c, choose a finite subset S_c' of S_c consisting of one element $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ for each congruence class of a (or d)* mod c. *The elements of S_c' as c runs from 1 to N together with $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ generate $\Gamma_0(N)$.*

PROOF. Let $c > 0$ and $(c, d) = 1$. Consider the system of inequalities

$$|Ncx + dy| < c, \quad |y| < c. \tag{12}$$

If $c^2 > Nc$, that is, if $c > N$, then there must be a solution in integers $(x, y) \neq (0, 0)$ to these inequalities. Replacing (x, y) by $(-x, -y)$ if necessary, we may assume $-c < Ncx + dy < 0$ whence,

$$0 < c(Nx+1) + dy < c \quad \text{and} \quad |y| < c.$$

Let $\delta = (Nx+1, y)$, $\alpha = (Nx+1)/\delta$ and $\gamma = y/\delta$. Then,

$$(\alpha, N\gamma) = 1, \qquad 0 < c\alpha + d\gamma < c \text{ and } |\gamma| < c. \tag{13}$$

Now let $\left(\begin{smallmatrix} * & * \\ Nc & d \end{smallmatrix}\right)$ be any element of $\Gamma_0(N)$ with $c > N$. Find α and γ as in (13) and choose any matrix $\left(\begin{smallmatrix} \alpha & * \\ N\gamma & * \end{smallmatrix}\right) \in \Gamma_0(N)$. Then

$$\begin{pmatrix} * & * \\ Nc & d \end{pmatrix}\begin{pmatrix} \alpha & * \\ N\gamma & * \end{pmatrix} = \begin{pmatrix} * & * \\ N(c\alpha + d\gamma) & * \end{pmatrix}$$

or,

$$\begin{pmatrix} * & * \\ Nc & d \end{pmatrix} = \begin{pmatrix} * & * \\ N(c\alpha + d\gamma) & * \end{pmatrix}\begin{pmatrix} * & * \\ -N\gamma & \alpha \end{pmatrix}.$$

Thus any $\left(\begin{smallmatrix} * & * \\ Nc & * \end{smallmatrix}\right)$ with $c > N$ can be expressed as a product of two matrices with smaller values of c (possibly after multiplying by $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$). Iterating this procedure gives the desired result once one notes that the presence of $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ permits a and d to be taken mod Nc in each case.

Obviously, this proposition is not in general (or at least in specific cases) best possible. One might conjecture that the values of c up to about $\log N$ would suffice to generate $\Gamma_0(N)$. In the important case $N = p^r$, a prime power, the result can be strengthened as follows.

PROPOSITION 4. *If $N = p^r$, a prime power, then the elements*

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} a & b \\ N & d \end{pmatrix} \in \Gamma_0(N)$$

generate $\Gamma_0(N)$ as a runs through a complete system of residues mod N.

PROOF. The idea is to start with $\left(\begin{smallmatrix} a & b \\ Nc & d \end{smallmatrix}\right) \in \Gamma_0(N)$ with $c > 1$ and find $\left(\begin{smallmatrix} \alpha & * \\ N\gamma & * \end{smallmatrix}\right) \in \Gamma_0(N)$ such that

$$|c\alpha + d\gamma| < c \quad \text{and} \quad |\gamma| < c. \tag{14}$$

With $c > 1$, such nonzero α and γ must exist, but for general N there is no guarantee that $(\alpha, N) = 1$. If N is a prime power and $(\alpha, N) = 1$, then $(\alpha \pm 1, N) = 1$. But if (α, γ) satisfies (14) so does one of $(\alpha + 1, \gamma)$ and $(\alpha - 1, \gamma)$. Thus, as in the proof of Proposition 3, $\left(\begin{smallmatrix} a & b \\ Nc & d \end{smallmatrix}\right)$ can be factored appropriately.

Lest it be thought that Proposition 4 might hold when N is not a prime power a counterexample is provided.

Let $N = pq$, p, q distinct primes and choose integers μ and λ such that $\mu p - \lambda q = 1$. Then

$$\begin{pmatrix} * & * \\ 2pq & \mu p + \lambda q \end{pmatrix}\begin{pmatrix} \delta & * \\ \pm pq & * \end{pmatrix} = \begin{pmatrix} * & * \\ pq(2\delta \pm (\mu p + \lambda q)) & * \end{pmatrix}.$$

Suppose we had $2\delta \pm (\mu p + \lambda q) = \pm 1$. Then, since $\mu p - \lambda q = 1$, $\delta = \pm \mu p$ or $\pm \lambda q$. But this means that

$$\begin{pmatrix} \delta & * \\ \pm pq & * \end{pmatrix} \in SL(2, \mathbf{Z}).$$

On the other hand, parity considerations rule out $2\delta \pm (\mu p + \lambda q) = 0$. Thus the matrix

$$\begin{pmatrix} * & * \\ 2pq & \mu p + \lambda q \end{pmatrix}$$

cannot be factored into matrices from $\Gamma_0(pq)$ with 0 or $\pm pq$ in the lower left corner.

4. By combining the results of the two preceding sections we get a criterion for $f(z)$ to be a modular form for $\Gamma_0(N)$. Let $\mathfrak{M}_k(\Gamma_0(N), \psi)$ denote the space of entire modular forms for $\Gamma_0(N)$ with character ψ. Thus f is in $\mathfrak{M}_k(\Gamma_0(N), \psi)$ if and only if f is holomorphic on H and satisfies the conditions

(i) $f((az + b)/(cz + d)) = \psi(d)(cz + d)^k f(z)$ for all $\sigma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$.

(ii) f is regular at all cusps. That is, f has a Fourier expansion (as in (6)) at $i\infty$ and for all $\sigma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma = \mathrm{SL}(2, \mathbf{Z})$, $(cz + d)^{-k} f((az + b)/(cz + d))$ has a Fourier expansion $\sum_{n=0}^{\infty} a_n e^{2\pi i n z/N}$ at $i\infty$. The *value* of f at the cusp a/c is a_0.

As usual, we say f is a cusp form if f has value 0 at every cusp. For convenience, we state the following criterion for f to be an entire modular form. (See [3, p. V-14].)

LEMMA. *Let $f(z)$ be holomorphic in H and suppose f has the Fourier expansion at $i\infty$*

$$f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z/N}.$$

If $f((az + b)/(cz + d)) = (cz + d)^k f(z)$ for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(N)$ and if $f(x + iy) = O(y^{-\sigma})$ as $y \to 0$ uniformly in x then f is regular at all cusps. If, in addition, $a_0 = 0$ and $\sigma < k$, then f is a cusp form.

We can now prove the main theorem of this paper.

THEOREM. *Let $f(z)$ be holomorphic in H and have the Fourier expansion at $i\infty$*

$$f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z}, \quad a_n = O(n^{-\kappa}).$$

Then $f \in \mathfrak{M}_k(\Gamma_0(N), \psi)$ if and only if condition C *holds for all characters χ* (mod Nc) *for all c such that $1 \leqslant c \leqslant N$. If, in addition, $\sum_{n=1}^{\infty} a_n n^{-s}$ converges absolutely for some $\sigma < k$, then f is a cusp form.*

PROOF. The condition C holding for all χ (mod Nc) $(1 \leqslant c \leqslant N)$ is equivalent to condition A holding for all $\sigma \in S_{Nc}$ $(1 \leqslant c \leqslant N)$. But by Proposition 3, this is equivalent to f satisfying the property (i) in the definition of $\mathfrak{M}_k(\Gamma_0(N), \psi)$. Since $a_n = O(n^{-\kappa})$ implies that $f(x + iy) = O(y^{-\kappa-1})$ as $y \to 0$ (uniformly in x), the lemma applies. Thus f also satisfies the regularity condition (ii). Finally, the last statement follows precisely as in Weil's proof of his theorem. (See [3, p. V-16] or [5, p. 155].)

REMARKS. (1) If N is a prime power, Proposition 4 can be used instead of Proposition 3. Then one need only check condition C for characters mod N.

(2) It would probably be preferable if the theorem only referred to primitive characters (mod Nc). This is possible at least if f is an eigenfunction for the appropriate Hecke algebra–that is, if the associated Dirichlet series has an Euler product. The precise result will be published at a later time.

(3) Similar theorems may be possible for integrals of modular forms of weight 2 and even for higher integrals. An example of this may be found in the author's paper [4] dealing with the transformation of log η.

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