

ON EVERYWHERE-DEFINED INTEGRALS⁽¹⁾

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ABSTRACT. Hardly any finite integrals can be defined for all real-valued functions. In contrast, if infinity is admitted as a possible value for the integral, then every finite integral can be extended to all real-valued functions.

Introduction. An *S-integral* is an order-preserving, linear functional defined for an ordered linear space, S . A natural problem would seem to be: for a given S , find all the S -integrals. Of course, in the special case in which S is the set of continuous, real-valued, functions on a compact space, then a well-known theorem of F. Riesz, and its generalizations, provides some answer. In this paper, I study another special case, namely that in which S is the space of all finite, real-valued functions defined on a set, and conclude that there are very few such integrals (Lemma 1, Theorem 1, and Corollary 1). Since it is convenient to have an integral defined for all functions, the question then arises as to the possibility of extending an integral to the space L of all functions if infinity is admitted as a possible value. This question has an affirmative answer, as is the purpose of §2 to demonstrate.

1. Throughout this paper, Ω is a fixed, nonempty set, \mathcal{U} is a σ -field of its subsets, L is the vector lattice of all finite, real-valued, \mathcal{U} -measurable functions defined on Ω , and S is a linear subspace of L . Of course, L is ordered by the cone of all nonnegative elements of L .

If, for some $\omega \in \Omega$, $P(f) = f(\omega)$ for all $f \in L$, then P is an *evaluation*. If P_1, \dots, P_n are evaluations, and t_1, \dots, t_n are nonnegative, real numbers, then $\sum_1^n t_i P_i$ is an *elementary integral*. As is the purpose of this section to show, for interesting \mathcal{U} , there are no L -integrals other than elementary integrals.

Call a $U \in \mathcal{U}$ *trivial* for P if $P(U) = 0$, and *very trivial* for P if, for every $f \in L$ which vanishes outside U , $Pf = 0$. Let U^c be the complement of U . Call $T \in \mathcal{U}$ *reducible* if $\exists U \in \mathcal{U}$, $U \subset T$, such that neither U nor TU^c is *very trivial*. If T is neither reducible nor very trivial, then T is an *atom* for P . If, for an atom T , $Pf = P(fT)$ for all $f \in L$, then P is *atomic*. (The

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convenient notational device introduced by Bruno de Finetti [1], of identifying a set such as T with its indicator function, that is, the function which assumes the value 1 on T and 0 off T , is adapted here, so, fT is the function which equals f on T and 0 off T .)

If P_1, \dots, P_n are atomic integrals and t_1, \dots, t_n are nonnegative, real numbers, then $\sum t_i P_i$ is a *molecular* integral.

LEMMA 1. *Every L -integral is molecular.*

PROOF. If an L -integral P is identically zero, it is plainly molecular. So suppose Pf is unequal to zero for some $f \in L$. By considering a positive multiple of $|f|$, it may be supposed that $f \geq 0$ and that $P(f) = 1$. As will now be verified, P possesses at least one atom T . For otherwise, there exists an infinite sequence of pairwise disjoint elements of \mathcal{U} , say U_1, U_2, \dots , none of which is very trivial. There necessarily also exist f_1, f_2, \dots , each of which is nonnegative and which satisfy $f_i = 0$ off U_i and $P(f_i) = 1$. Plainly, $f = \sum_i^\infty f_i$ is an element of L and $P(f) = +\infty$, which contradicts the assumption that P is finite on L . Hence there is an atom T for P . If T^c is very trivial for P , then P is plainly atomic and, hence, molecular. If T^c is not very trivial for P , then, as the above argument shows, there is an atom T_2 for P which is disjoint from $T_1 = T$. Proceeding thus, there are two possibilities. Either P possesses an infinite sequence of pairwise disjoint atoms T_1, T_2, \dots , or, for some finite sequence of pairwise disjoint atoms T_1, \dots, T_n , $(UT_i)^c$ is very trivial for P . In the latter case, P is plainly molecular. And, as has been shown, there cannot exist an infinite sequence of pairwise disjoint elements of \mathcal{U} none of which is very trivial. This completes the proof.

LEMMA 2. *Let P and P' be L -integrals such that $P(f) = P'(f)$ for all bounded f . Then P is identical with P' .*

PROOF. For each $g \geq 0$, let $Q(g)$ be the supremum of $P(f)$ over all bounded $f \leq g$. Plainly, $Q(g) < \infty$ for $Q(g) \leq P(g) < \infty$. As is easily verified, if $g_i \geq 0$, $i = 1, 2$, then $Q(g_1 + g_2) = Q(g_1) + Q(g_2)$. For any $g \in L$, let $Q(g)$ be $Q(g^+) - Q(g^-)$ where g^+ is the maximum of g and 0, and $g^- = g - g^+$. Plainly, as extended, Q is additive, and hence is an L -integral. Moreover, since $Q(g) \leq P(g)$ for all nonnegative g , $P - Q$ is actually an L -integral which vanishes on B , that is, on the set of bounded f .

Similarly, $P' - Q'$ is an L -integral which vanishes on B , and, since Q and Q' depend only on the values that P and P' assume on B , Q is identical with Q' . Summarizing, $P - Q$ and $P' - Q$ are L -integrals which vanish on B . Therefore, the proof will be complete once it is shown that *every L -integral R which vanishes on B is identically zero*. Suppose that R was a nonidentically zero L -integral. Then, plainly, for some nonnegative f , $R(f) = 1$. Let U_n be

the event, $n < f < n + 1$. Of course, $U_n \in \mathcal{Q}$. Designate the sum over all n of $(n + 1)^2 U_n$ by g . Then verify that g is everywhere finite, and hence an element of L . Verify, too, that, for all $n \geq 0$,

$$(1) \quad g \geq (n + 1)f - n(n + 1) \sum_{j < n} U_j,$$

so

$$(2) \quad Rg \geq (n + 1)Rf - n(n + 1) \sum_{j < n} RU_j.$$

Since R vanishes on bounded elements of L , RU_j is 0, so

$$(3) \quad Rg \geq (n + 1)Rf = n + 1$$

for all n , which is incompatible with the assumed finiteness of R on L . This completes the proof.

Call a measure P *very countably additive* if, for every decreasing sequence of events, $U_1 \supset U_2 \supset \dots$ whose intersection is empty, $P(U_i)$ is zero for some i .

LEMMA 3. *Let P be an L -integral. Then the restriction of P to \mathcal{Q} is very countably additive.*

PROOF. If P were not very countably additive, then there exists a strictly decreasing sequence U_i of elements of \mathcal{Q} with $U_1 = \Omega$ and with $P(U_i) > 0$ for all i . Let x_i be $i/P(U_i)$, and define f to be x_i on $U_i - U_{i+1}$. Plainly, $f \geq 0$ everywhere and $f \geq x_i$ on U_i . Therefore,

$$(4) \quad Pf \geq x_i P U_i = i \quad \text{for all } i,$$

which is incompatible with the finiteness of Pf .

A probability measure P defined on \mathcal{Q} is *two-valued* if $P(A)$ is 0 or 1 for all $A \in \mathcal{Q}$. Call \mathcal{Q} *normal* if every countably additive, two-valued, probability measure defined on \mathcal{Q} is an evaluation.

LEMMA 4. *If \mathcal{Q} is normal, and P is an atomic L -integral, then P is a multiple of an evaluation.*

PROOF. Plainly, the conclusion holds if P is identically zero. If P is not identically zero, then P restricted to \mathcal{Q} , say P_r , is a positive multiple of a probability measure, as is implied by Lemma 2. For simplicity of exposition, it may be assumed that P_r is a probability measure. By Lemma 3, P_r is seen to be countably additive, and, because P was assumed atomic, P_r is two-valued. Since \mathcal{Q} is normal, P_r is necessarily an evaluation; that is, for some $\omega \in \Omega$ and all $U \in \mathcal{Q}$, $P_r(U) = 1$ or 0 according as $\omega \in U$ or $\omega \in U^c$. Equivalently, for every bounded $f \in L$, $Pf = f(\omega)$. Lemma 2 now applies to show that $Pf = f(\omega)$ for all f , which completes the proof.

Which \mathcal{U} are normal? For each $\omega \in \Omega$, the \mathcal{U} -atom containing ω , say $\mathcal{U}(\omega)$, is the intersection of all $U \in \mathcal{U}$ for which $\omega \in U$. Say that the atoms of \mathcal{U} are *countably determined* if, for some countably generated σ -field $\mathcal{V} \subset \mathcal{U}$, and for every $\omega \in \Omega$, $\mathcal{V}(\omega) = \mathcal{U}(\omega)$.

THEOREM 1. *Each of the following conditions implies its successor:*

- (a) \mathcal{U} is countably generated;
- (b) The atoms of \mathcal{U} are countably determined;
- (c) \mathcal{U} is normal;
- (d) Every L -integral is elementary.

PROOF. That (a) \rightarrow (b) is trivial. To see that (b) \rightarrow (c), let P be a countably additive, two-valued probability measure defined on \mathcal{U} , and let \mathcal{V} be a countably generated sub sigma field of \mathcal{U} which determines the atoms of \mathcal{U} . Then, as is easily verified, P restricted to \mathcal{V} , say P_V , is an evaluation; that is, for some $\omega \in \Omega$, and all $V \in \mathcal{V}$ for which $\omega \in V$, $P(V) = 1$. In particular, $P(\mathcal{V}(\omega)) = 1$. Now let $\omega \in U \in \mathcal{U}$. Since $\mathcal{V}(\omega) = \mathcal{U}(\omega) \subset U$, one concludes that $P(U) = 1$. Hence \mathcal{U} is normal. To see that (c) \rightarrow (d), use Lemmas 1 and 4.

COROLLARY 1. *For \mathcal{U} the field of Borel, analytic, or all, subsets of a separable metric space, M , every L -integral is elementary.*

In contrast to L , the set C of continuous functions defined on M plainly admits many nonelementary integrals except, of course, if the cardinality of M is finite.

For if X is any compact subset of M , q is a (countably additive) nonnegative finite measure on X , and $\pi: C(M) \rightarrow C(X)$ is defined by

$$(5) \quad (\pi(f))(x) = f(x) \quad \text{for } f \in C(M), x \in X,$$

and

$$(6) \quad P(f) = \int \pi(f) dq,$$

then P is an integral defined on C . That there might be no other integrals on C is a suggestion which was made to me by Jacob Feldman, and is confirmed, here, thus.

PROPOSITION 1. *Let M be a locally compact, σ -compact, space, and let P be an integral defined on $C(M)$, the space of all real-valued, continuous functions defined on M . Then there is a unique, countably additive measure, q , with compact support, X , such that*

$$(7) \quad P = Q \circ \pi$$

where π is the projection of $C(M)$ onto $C(X)$, and $Q(f) = \int f dq$.

PROOF. As will first be shown, there is a compact subset K of M such that, for every $f \in C(M)$ which vanishes on K , $P(f) = 0$. For assume otherwise, and let K_i be a sequence of compact sets whose union is M and for which K_i is a subset of the interior of K_{i+1} for all positive integers i . Now let f_i vanish on K_i with $P(f_i)$ nonzero. By considering the absolute value of f_i , it may be assumed that f_i is nonnegative, and, by considering an appropriate positive multiple of f_i , that $P(f_i)$ exceeds i . Let f be the sum of the f_i and verify that f is an element of $C(M)$ and that $P(f)$ exceeds i for all i , which is a contradiction. Consequently, K exists as asserted. Therefore, if f and f' agree on K , then $P(f)$ equals $P(f')$. Of course, the Tietze extension theorem implies that the projection map, π , of $C(M)$ into $C(K)$ is onto. For each $g \in C(K)$, define $Q(g)$ to equal $P(f)$ where $f \in C(M)$ satisfies $\pi(f) = g$. The remarks above show that Q is well defined. Plainly, (7) holds. There remains the verification that Q is an integral. To see that Q is order preserving, let g and g' be in $C(K)$, with $g \leq g'$, and notice that, among the f and f' which project onto g and g' , there are those for which $f \leq f'$. To see this, simply consider the minimum and maximum of any preimages of g and g' . Since $P(f)$ is at most $P(f')$, $Q(g)$ is at most $Q(g')$. Because the projection map π is linear, it is even simpler to verify that Q is linear. Hence, Q is an integral. The proof is completed by a standard application of Riesz's representation theorem.

2. In contrast to Theorem 1 which implies that L -integrals are rare objects, wide-sense integrals, that is, those which admit ∞ as a possible value, exist in abundance. Indeed, as is the purpose of this section to show, every integral can be extended to a wide-sense integral defined on L .

For f and g in L , let $[f, g]$ denote the set of $h \in L$ for which $f \leq h \leq g$, and call $[f, g]$ an interval. If, for every pair of points in S , S includes the interval they determine, then S is interval-closed.

THEOREM 2. *Let S be a linear subspace of L and P an S -integral. Then P can be extended to a wide-sense integral, P_w , defined on L , and, among such P_w , there is one for which $-\infty < P_w f < +\infty$ if, and only if, f belongs to the interval closure of S .*

Though there can be little doubt as to what is intended by the notion of a wide-sense integral, a precise definition requires some preliminaries. Let R^e designate the real line, R , with $+\infty$ and $-\infty$ adjoined. The binary operation of multiplication on R can be unambiguously extended to be a binary operation on R^e by setting $0 \cdot \infty$ and $0 \cdot (-\infty)$ equal to 0. Though the binary operation of addition on R cannot usefully be extended to a binary operation on R^e , it does extend in a natural way to a triadic relation on R^e to be designated as PLUS, and defined thus. For x, y and z in R^e , PLUS obtains for (x, y, z) and for (y, x, z) if one of these four conditions hold.

- (a) x, y , and z are in R and $x + y = z$;
- (b) $x = +\infty = z$ and $y > -\infty$;
- (c) $x = -\infty = z$ and $y < +\infty$;
- (d) $x = +\infty$ and $y = -\infty$.

If PLUS obtains for (x, y, z) , write $x + y = z$, but it is necessary to remember that $+$ is no longer a binary operation.

A *wide-sense* linear functional on a linear subspace S of L is a mapping P of S into R^e which satisfies:

$$(1) \quad P(x + y) = P(x) + P(y),$$

and

$$(2) \quad P(tx) = tP(x).$$

A *wide-sense integral* is an order-preserving, wide-sense linear functional, P .

LEMMA 2.1. *Let S be a linear subspace of L . Then these two conditions are equivalent:*

- (a) S is interval closed;
- (b) The set, K , of all nonnegative elements of the complement of S , is a convex cone.

PROOF. (a) \rightarrow (b). Let $f > 0, g > 0$, with f and g in S^c , the complement of S . Then $f/2 \in S^c$ and $(f + g)/2 > 0$. Because of (a), $(f + g)/2 \in S^c$, which establishes (b). Now assume (b) and let $f < h < g$ with f and g in S . Plainly, $h - f$ and $g - h$ are nonnegative, and their sum, $g - f$, is in S . In view of (b), one of the summands, $h - f$ or $g - h$, is in S . In either case, $h \in S$, which establishes (a).

PROOF OF THEOREM 2. As is easily verified, the interval closure of S is a linear space to which P extends so as to be an integral. Therefore, it may henceforth be assumed that S itself is interval-closed. If $S = L$, there is nothing further to prove. Assume, therefore, that S is a proper subspace of L , which implies that K , the set of nonnegative elements of the complement of S , is nonempty. According to Lemma 2.1, K is a convex cone. Let K' be $S + K$. As is easily verified, K' is a convex cone which satisfies these three conditions:

- (i) $S + K' \subset K'$;
- (ii) $f > 0$ and $f \in S^c \rightarrow f \in K'$;
- (iii) $K', -K'$, and S are pairwise disjoint.

Among all convex cones K' which satisfy (i), (ii) and (iii), there is a maximal one, as Zorn's Lemma implies. As can be verified without difficulty, for any such maximal K' , the set-theoretic union of the three disjoint sets $K', -K'$, and S is L . If P_w is defined to be $+\infty$ on K' , $-\infty$ on $-K'$, and P on S , then, as is obvious, P_w extends P and is finite only on S . That P_w is a wide-sense

integral is straightforward to verify. This completes the proof.

Let B be the set of bounded elements of L . An integral, or a wide-sense integral, P , is *regular* if, for all nonnegative f in its domain, Pf is the supremum of Pg over all $g \in B$ which are in the domain of P and satisfy $g < f$.

If P is a B -integral, then its inner integral P_* is not only superadditive, as is every inner integral, but, on the cone of nonnegative f , it is even additive, as is easily verified. Therefore, P extends in one and only one way so as to be a regular integral defined on the linear space generated by the cone of all nonnegative f for which P_*f is finite. In view of this remark one obtains

COROLLARY 2. *Let P be a B -integral. Then P can be extended to be a regular, wide-sense, integral defined on L . On the set of nonnegative elements of L , all regular extensions agree.*

In accord with [1], an integral P is a *prevision* if, for all f in the domain of P

$$(3) \quad P(f) \leq \sup f(\omega) \quad (\omega \in \Omega).$$

Plainly, even in the present setting in which countable additivity is not assumed, in order that f possess a distribution function F which, as usual, converges to 0 at $-\infty$ and to 1 at $+\infty$, it is necessary and sufficient that

$$(4) \quad P(-n < f < n) \rightarrow 1 \quad \text{as } n \rightarrow \infty.$$

As noted in [1, Chapter 6], for $f \geq 0$,

$$(5) \quad \int x \, dF(x) \leq P(f),$$

but equality need not hold.

COROLLARY 3. *Let P be a B -integral and let P_w be a (wide-sense) extension of P (to L). Then P is regular if, and only if, equality holds in (5) for all nonnegative f in the domain of P_w .*

A caution: In contrast to usual teachings in integration theory, it is possible for an f to assume the value zero except on a set of P -measure zero and yet $P(f)$ be greater than zero. If P is regular, however, $P(f)$ is necessarily zero. Therefore, if P is regular and two functions are equal almost surely, then their P -integrals are equal. But this conclusion need not hold for nonregular P . Because of the possible ambiguity of $\infty - \infty$, and because such an ambiguity, even on a set of probability zero, can affect the value of $P(f - g)$ for nonregular P , the functions f integrated in this paper never assume infinite values, though their integrals may.

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the utility one attributes to his own life is finite, one need only observe that one risks it to cross the street to see a friend. Second, he expressed doubts about the possibility of a consistent theory for unbounded utilities. It was in the spirit of investigating whether these doubts could not be dissipated that we were led to a primitive version of Theorem 2. Another stimulus to this paper is a study with William Sudderth of certain gambling problems with not necessarily bounded utilities. Finally, I am indebted to David Margolies, and other students, for their participation in a seminar where early versions of this material were presented.

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