

# THE MACKEY BOREL STRUCTURE ON THE SPECTRUM OF AN APPROXIMATELY FINITE-DIMENSIONAL SEPARABLE $C^*$ -ALGEBRA

BY

GEORGE A. ELLIOTT

**ABSTRACT.** It is shown that the Mackey Borel structures on the spectra of any two approximately finite-dimensional separable  $C^*$ -algebras not of type I are isomorphic.

**1. Main result.** In a lecture in Bâton Rouge in 1967, J. Dixmier raised the question of comparing the Mackey Borel structures on the spectra of different separable  $C^*$ -algebras. Specifically, he asked whether for a simple anti-liminal separable  $C^*$ -algebra the Mackey Borel structure is always the same. He suggested the study of approximately finite-dimensional  $C^*$ -algebras; we have studied this case, and can report that the Mackey Borel structure on the spectrum is the same for any approximately finite-dimensional separable  $C^*$ -algebra which is not postliminary. In other words, there is only one nonstandard Borel space which can arise as the spectrum of an approximately finite-dimensional separable  $C^*$ -algebra.

We recall that a  $C^*$ -algebra is said to be approximately finite-dimensional if any finite number of elements can be approximated arbitrarily closely in norm by elements of a finite-dimensional sub- $C^*$ -algebra.

**2. THEOREM** (cf. [6]). *Let  $A$  be an approximately finite-dimensional separable  $C^*$ -algebra, and let  $B$  be a separable  $C^*$ -algebra which is not postliminary. Then there exist a sub- $C^*$ -algebra  $C$  of  $B$  and a surjective morphism  $\varphi: C \rightarrow A$  with the following properties.*

- (i) *For every state  $f$  of  $A$ , the state  $f \circ \varphi$  of  $C$  has a unique extension to a state of  $B$ , say  $f^\sim$ .*
- (ii) *The map  $f \mapsto f^\sim$  is an affine homeomorphism from the state space of  $A$  (with the weak dual space topology) onto a locally compact face of the state space of  $B$ . (So  $f^\sim$  is pure if and only if  $f$  is.)*

---

Received by the editors August 20, 1975.

*AMS (MOS) subject classifications* (1970). Primary 46L05, 54H05; Secondary 02K30, 04A15, 28A05.

*Key words and phrases.* Uniqueness of Mackey Borel structure, approximately finite-dimensional  $C^*$ -algebras.

© American Mathematical Society 1977

(iii) If  $f$  and  $f'$  are pure states of  $A$  then  $f^{\sim}$  and  $f'^{\sim}$  are equivalent if and only if  $f$  and  $f'$  are.

PROOF. In the case  $A = M_{2^{\infty}}$ , this theorem was essentially proved by Glimm in [6] (see also [2, §9]). (A similar proof would deal with the case  $A = M_n$  for any generalized integer  $n$ , i.e., the case that  $A$  is a Glimm algebra—see [3].) In the general case we shall need to refine Glimm's methods to construct the subalgebra  $C$  and the morphism  $\varphi$ , although the proof of properties (i) to (iii) remains the same.

It is clearly sufficient to prove the theorem with  $A$  replaced by the  $C^*$ -algebra obtained by adjoining a unit to  $A$ . We may, then, assume that  $A$  has a unit, and for convenience we shall do so. Then (see 1.6 of [3])  $A$  is the closure of the union of an increasing sequence  $A_1 \subset A_2 \subset \dots$  of finite-dimensional sub- $C^*$ -algebras of  $A$  each containing the unit of  $A$ .

Since  $B$  has a nonzero antiliminary quotient ([2, 4.3.6]), we may suppose that  $B$  itself is antiliminary. We may then apply 9.3.7 of [2] to  $B$  and a dense sequence  $(s_0, s_1, \dots)$  in the hermitian part of  $B$ . We remark that the presence of a unit stipulated in 9.3.7 of [2] is not necessary; in the proof of that lemma the elements  $v(\emptyset)$  and  $b(0)$  need only be chosen to be nonzero, positive, and such that  $v(\emptyset)b(0) = b(0)$ , and this is possible in any nonzero  $C^*$ -algebra. (Of course, 9.3.7(iv) must be changed, or just deleted.)

We shall choose  $C$  to be the sub- $C^*$ -algebra of  $B$  generated by a certain subset of the elements

$$v(a_1, \dots, a_n)v(b_1, \dots, b_n)^*, \quad a_i, b_j \in \{0, 1\}, n = 0, 1, \dots,$$

where the  $v(a_1, \dots, a_n)$  have the properties of 9.3.7 of [2].

To describe the construction of  $C$  more precisely, we shall use the notation of 9.2 of [2]. The linear span of the  $v(a_1, \dots, a_n)v(b_1, \dots, b_n)^*$  will be denoted by  $M(n)$  for fixed  $n = 0, 1, \dots$ , and we shall denote by  $e(n)$  the sum

$$\sum_{a_1, \dots, a_n \in \{0, 1\}} v(a_1, \dots, a_n)v(a_1, \dots, a_n)^*.$$

Realizing  $B$  as a  $C^*$ -algebra of operators on a Hilbert space  $H$ , we shall denote the range projection of  $e(n)$  by  $E(n)$ .

By 9.2.8, 9.2.9 and 9.2.11 of [2], for all  $k = 1, 2, \dots$ ,  $E(n+k)$  is permutable with  $M(n)$ , and  $M(n)E(n+k)$  is a simple finite-dimensional  $C^*$ -algebra of order  $2^n$ . In particular, the restriction map  $M(n) \rightarrow M(n)E(n+k)$  is injective.

We shall construct a sequence  $n_1 < n_2 < \dots$  and a sequence  $C_1 \subset M(n_1)$ ,  $C_2 \subset M(n_2)$ ,  $\dots$  such that for each  $j = 1, 2, \dots$ , the restriction  $C_j E(n_j + 1)$  is a sub- $C^*$ -algebra of  $M(n_j)E(n_j + 1)$  and the unit of  $C_{j+1}E(n_{j+1} + 1)$  is permutable with  $C_j E(n_j + 1)$ , and such that the sequence

$$C_1 E(n_1 + 1) \rightarrow C_2 E(n_2 + 1) \rightarrow \dots$$

defined by the canonical restrictions is isomorphic to  $A_1 \rightarrow A_2 \rightarrow \dots$

We shall choose  $C_1, C_2, \dots$  so that, moreover, for each  $j = 1, 2, \dots$  there exists  $c_j \in C_j$  with

$$\|f_{j+1}(s_j - c_j)f_{j+1}\| \leq (j+1)^{-1},$$

where  $f_{j+1}$  is the preimage in  $C_{j+1}$  of the unit of  $C_{j+1}E(n_{j+1} + 1)$ .

Label the minimal direct summands of each  $A_i$  from 1 to  $r_i$ . Then  $A_i$  is determined by the column vector  $m_i = (m_{ki})_{1 \leq k \leq r_i}$  in which  $m_{ki}$  is the order of the  $k$ th minimal direct summand of  $A_i$ . Moreover, the inclusion of  $A_i$  in  $A_{i+1}$  may be described by the matrix of multiplicities

$$P_i = (p_{jk}^{(i)})_{1 \leq j \leq r_{i+1}, 1 \leq k \leq r_i},$$

in which  $p_{jk}^{(i)} = 0, 1, \dots$  and is the multiplicity with which the  $k$ th minimal direct summand of  $A_i$  is mapped into the  $j$ th minimal direct summand of  $A_{i+1}$ . (Cf. [1].) Then  $P_i m_i = m_{i+1}$ ,  $i = 1, 2, \dots$

Now choose a strictly increasing sequence  $1 < n_{01} < n_1 < n_{02} < n_2 < \dots$  in  $\mathbf{N}$  such that

$$\begin{aligned} 2^{n_{01}} &> \sum_{k=1}^{r_1} m_{k1}, \quad 2^{n_1 - n_{01}} > r_1, \\ 2^{n_{02} - n_1} &> \max_{1 \leq k \leq r_1} \sum_{j=1}^{r_2} p_{jk}^{(1)}, \quad 2^{n_2 - n_{02}} > r_2, \\ 2^{n_{03} - n_2} &> \max_{1 \leq k \leq r_2} \sum_{j=1}^{r_3} p_{jk}^{(2)}, \dots \end{aligned}$$

Since

$$2^{n_{01}} > \sum_{k=1}^{r_1} m_{k1},$$

there is a sub- $C^*$ -algebra of  $M(n_{01})E(n_{01} + 1)$  isomorphic to  $A_1$ . We may choose such a  $C^*$ -algebra to be generated by minimal projections of  $M(n_{01})E(n_{01} + 1)$ ; denote its preimage in  $M(n_{01})$  by  $C_{01}$ . We may suppose that  $C_{01}$  is the linear span of certain of the  $v(a_1, \dots, a_{n_{01}})v(b_1, \dots, b_{n_{01}})^*$ .

The relative commutant of  $M(n_{01})E(n_1 + 1)$  in  $M(n_1)E(n_1 + 1)$  is simple of order  $2^{n_1 - n_{01}}$ . Since  $2^{n_1 - n_{01}} > r_1$ , we may choose  $r_1$  mutually orthogonal minimal projections  $E_1, \dots, E_{r_1}$  in this relative commutant. Denote by  $C_{01}^{(1)}, \dots, C_{01}^{(r_1)}$  the preimages in  $M(n_1)$  of the minimal direct summands of  $C_{01}E(n_1 + 1)$ . Then the sum

$$\sum_{k=1}^{r_1} C_{01}^{(k)} E_k$$

is a sub- $C^*$ -algebra of  $M(n_1)E(n_1 + 1)$  isomorphic to  $C_{01}E(n_1 + 1)$ ; denote its preimage in  $M(n_1)$  by  $C_1$ . We may suppose that  $C_1$  is the linear span of certain of the  $v(a_1, \dots, a_{n_1})v(b_1, \dots, b_{n_1})^*$  (choosing each  $E_k$  to be a sum of certain  $v(a_1, \dots, a_{n_1})v(a_1, \dots, a_{n_1})^*E(n_1 + 1)$ ).

Denote by  $f_1$  the preimage in  $M(n_1)$  of the unit of  $C_1E(n_1 + 1)$ . Then

$$C_1E(n_1 + 1) = f_1M(n_{01})f_1E(n_1 + 1).$$

To see this, denote by  $e_k$  the preimage in  $M(n_{01})$  of the unit of the minimal direct summand  $C_{01}^{(k)}E(n_1 + 1)$  of  $C_{01}E(n_1 + 1)$ , so that  $C_{01}^{(k)}E(n_1 + 1) = e_kM(n_{01})e_kE(n_1 + 1)$ . Then

$$f_1E(n_1 + 1) = \sum_{k=1}^{r_1} e_kE_k,$$

and

$$\begin{aligned} f_1M(n_{01})f_1E(n_1 + 1) &= \sum_{j,k=1}^{r_1} e_jM(n_{01})e_kE_jE_k \\ &= \sum_{k=1}^{r_1} e_kM(n_{01})e_kE_k = C_1E(n_1 + 1). \end{aligned}$$

The relative commutant of  $M(n_1)E(n_{02} + 1)$  in  $M(n_{02})E(n_{02} + 1)$  is simple of order  $2^{n_{02}-n_1}$ . Since  $2^{n_{02}-n_1} > \max_{1 \leq k \leq r_1} \sum_{j=1}^{r_2} p_{jk}^{(1)}$ , there exist, for each  $k = 1, \dots, r_1$ , mutually orthogonal projections  $E_{jk}$ ,  $j = 1, \dots, r_2$ , each the sum of  $p_{jk}^{(1)}$  minimal projections in this relative commutant. With  $e_1, \dots, e_{r_1}$ ,  $E_1, \dots, E_{r_1}$  as in the preceding paragraph, set

$$\sum_{k=1}^{r_1} E_{jk}e_kE_k = F_j, \quad j = 1, \dots, r_2.$$

Then  $F_1, \dots, F_{r_2}$  are mutually orthogonal projections in  $M(n_{02})E(n_{02} + 1)$ . ( $F_iF_j = \sum_k E_{ik}E_{jk}e_kE_k = \delta_{ij}F_j$ .)

The sub- $C^*$ -algebra  $\sum_{j=1}^{r_2} F_jM(n_{02})F_j$  of  $M(n_{02})E(n_{02} + 1)$  is isomorphic to  $A_2$ ; denote its preimage in  $M(n_{02})$  by  $C_{02}$ . Moreover, each  $F_j$  is permutable with  $C_1$ , and the map  $C_1E(n_1 + 1) \rightarrow C_{02}E(n_{02} + 1)$  consisting of multiplication by  $\sum_{j=1}^{r_2} F_j$  is isomorphic to the morphism  $A_1 \rightarrow A_2$ . We may suppose that  $C_{02}$  is the linear span of certain of the  $v(a_1, \dots, a_{n_{02}})v(b_1, \dots, b_{n_{02}})^*$  (choosing the  $E_{jk}$  to be sums of certain  $v(a_1, \dots, a_{n_{02}})v(a_1, \dots, a_{n_{02}})^*E(n_{02} + 1)$ ).

Since  $2^{n_2-n_{01}} > r_2$ , we may construct  $C_2$  starting from  $C_{02}$  in the same way as  $C_1$  was constructed from  $C_{01}$ . Thus,  $C_2 \subset M(n_2)$ —moreover,  $C_2$  is the linear span of certain of the  $v(a_1, \dots, a_{n_2})v(b_1, \dots, b_{n_2})^*$ — $C_2E(n_2 + 1)$  is a sub- $C^*$ -algebra of  $M(n_2)E(n_2 + 1)$ , and if  $f_2$  denotes the preimage in  $M(n_2)$  of the unit of  $C_2E(n_2 + 1)$ , then

$$C_2E(n_2 + 1) = f_2M(n_{02})f_2E(n_2 + 1).$$

In addition,  $f_2E(n_2 + 1)$  is permutable with  $C_1$ , and the canonical morphism  $C_1E(n_1 + 1) \rightarrow C_2E(n_2 + 1)$  is isomorphic to  $A_1 \rightarrow A_2$ .

By 9.3.7 of [2] there exists  $t_1 \in M(1)$  such that  $\|e(2)(s_1 - t_1)e(2)\| \leq 2^{-1}$ . To define  $c_1$  such that  $c_1$  belongs to  $C_1$  and satisfies the inequality

$$\|f_2(s_1 - c_1)f_2\| \leq 2^{-1},$$

we must recall some more properties of the  $v(a_1, \dots, a_n)v(b_1, \dots, b_n)^*$ ,  $n = 0, 1, \dots$ .

It follows from 9.2.8, 9.2.11 of [2] that if  $n = 0, 1, \dots$  and  $k = 1, 2, \dots$ ,  $M(n)M(n + k) \subset M(n + k)$ . Hence it follows that if  $e \in M(n)$  and  $f \in M(n + k)$  are such that  $eE(n + k + 1)$  and  $fE(n + k + 1)$  are projections with the second smaller than the first, then  $ef = f$ . Indeed,

$$ef \in M(n + k),$$

and

$$efE(n + k + 1) = eE(n + k + 1)fE(n + k + 1) = fE(n + k + 1).$$

In particular,  $M(1)M(n_{01}) \subset M(n_{01})$ , since  $n_{01} > 1$ , so  $t_1f_{01} \in M(n_{01})$ , where  $f_{01}$  is the preimage in  $M(n_{01})$  of the unit of  $C_{01}E(n_{01} + 1)$ . Also  $f_{01}f_1 = f_1$ , since  $f_{01} \in M(n_{01})$ ,  $f_1 \in M(n_1)$ , and  $f_{01}E(n_1 + 1)$  and  $f_1E(n_1 + 1)$  are both projections with the second smaller than the first. This proves that  $f_1t_1f_1 \in f_1M(n_{01})f_1$ . Since  $f_1M(n_{01})f_1E(n_1 + 1) = C_1E(n_1 + 1)$ , there exists  $c_1 \in C_1$  such that

$$f_1t_1f_1E(n_1 + 1) = c_1E(n_1 + 1).$$

Multiplying on the right by  $f_2E(n_2 + 1)$ , we have

$$f_1t_1f_1f_2E(n_2 + 1) = c_1f_2E(n_2 + 1).$$

Since  $f_1t_1f_1f_2 \in M(n_1)M(n_{01})M(n_1)M(n_2)$ , and  $n_{01} < n_1 < n_2$ , we have  $f_1t_1f_1f_2 \in M(n_2)$ . Since  $c_1f_2 \in M(n_1)M(n_2)$  and  $n_1 < n_2$ , we have  $c_1f_2 \in M(n_2)$ . Since multiplication by  $E(n_2 + 1)$  is injective on  $M(n_2)$ ,

$$f_1t_1f_1f_2 = c_1f_2.$$

Since  $f_1E(n_2 + 1)$  and  $f_2E(n_2 + 1)$  are projections with the second smaller than the first,

$$f_1f_2 = f_2.$$

Since  $f_1$  and  $f_2$  are both selfadjoint ( $M(n_1)$  and  $M(n_2)$  are both selfadjoint, and the restriction maps are injective),

$$f_2f_1 = f_2.$$

Combining the preceding three equations gives

$$f_2t_1f_2 = f_2c_1f_2.$$

Since  $n_2 > 2$ , and  $e(2)$ ,  $f_2$  are selfadjoint, we have

$$e(2)f_2 = f_2, \quad f_2e(2) = f_2.$$

Since  $C_2$  is the linear span of certain of the  $v(a_1, \dots, a_{n_2})v(b_1, \dots, b_{n_2})^*$ ,  $f_2$  must be the sum of certain of the  $v(a_1, \dots, a_{n_2})v(a_1, \dots, a_{n_2})^*$ . Since these are selfadjoint, with mutually orthogonal supports ([2, 9.3.7(i)]), and of norm one ([2, 9.3.7]), it follows that  $f_2$  has norm one. Therefore, from

$$\|e(2)(s_1 - t_1)e(2)\| \leq 2^{-1}$$

follows  $\|f_2(s_1 - c_1)f_2\| \leq 2^{-1}$ .

It is clear that the construction may be continued to obtain a sequence  $C_1, C_2, \dots$  of finite-dimensional linear subspaces of  $B$  with the stipulated properties.

Denote the sub- $C^*$ -algebra of  $B$  generated by  $C_1 \cup C_2 \cup \dots$  by  $C$ . We shall construct a morphism  $\varphi$  from  $C$  onto  $A$  such that conditions (i) to (iii) of the theorem are satisfied.

Denote by  $D_j$  the sub- $C^*$ -algebra of  $C$  generated by  $C_1 \cup \dots \cup C_j, j = 1, 2, \dots$ . Then  $D_1 \subset D_2 \subset \dots$ , and  $\bigcup D_j$  is dense in  $C$ . Since  $f_j E(n_j + 1)$  is permutable with  $C_j$ , it is permutable with  $D_j$ . Moreover, for each  $j = 1, 2, \dots$ ,

$$D_j f_j E(n_j + 1) = C_j E(n_j + 1).$$

Since the sequence  $C_1 E(n_1 + 1) \rightarrow C_2 E(n_2 + 1) \rightarrow \dots$  is isomorphic to  $A_1 \rightarrow A_2 \rightarrow \dots$ , the inductive limit of which is  $A$ , there is defined a compatible sequence of morphisms  $D_1 \rightarrow A, D_2 \rightarrow A, \dots$ . The common extension of these to  $\bigcup D_j$  is a continuous morphism onto a dense subalgebra of  $A$ ; denote the extension to  $C$  by  $\varphi$ .

Let us first prove (i). Let  $f$  be a state of  $A$ , and let  $g$  be a state of  $B$  such that the restriction of  $g$  to  $C$  is  $f \circ \varphi$ . To show that  $g$  is unique it is enough to show that for  $j = 1, 2, \dots, |g(s_j - c_j)| \leq (j + 1)^{-1}$ . Also, we may (at this moment) suppose that  $B$  has a unit, and that this belongs to  $C$  (if necessary, adjoining one, and extending  $\varphi$  to map it into the unit of  $A$ ). Fix  $j = 1, 2, \dots$ . Since  $f_{j+1} \in C$  and  $\varphi f_{j+1} = 1$ ,

$$g((1 - f_{j+1})^2) = f(\varphi(1 - f_{j+1})^2) = 0.$$

Since  $f_{j+1} = f_{j+1}^*$ , it follows by the Cauchy-Schwartz inequality that

$$g(b) = g(f_{j+1} b f_{j+1}), \quad \text{all } b \in B.$$

In particular this holds for  $b = s_j - c_j$ . Using

$$\|f_{j+1}(s_j - c_j)f_{j+1}\| \leq (j + 1)^{-1},$$

we have

$$|g(s_j - c_j)| \leq (j + 1)^{-1},$$

the desired inequality.

Next, let us prove (ii). Continuity of  $f \mapsto f^\sim$  follows from the inequalities

$$|f^\sim(s_j) - f(c_j)| \leq (j+1)^{-1}, \quad j = 1, 2, \dots$$

Since the state space of  $A$  is compact (in the case that  $A$  has a unit, which it suffices to consider), it follows that the image of  $f \mapsto f^\sim$  is compact. Suppose that  $f$  is a state of  $A$  and that  $f', f''$  are states of  $B$  such that for some real  $\lambda > 0$ ,  $f^\sim = \lambda f' + (1 - \lambda) f''$ . Since  $f^\sim(f_1) = f(\varphi f_1) = f(1) = 1$ , and also  $\|f_1\| = 1$  and  $\|f'\| = \|f''\| = 1$ , we have  $f'(f_1) = f''(f_1) = 1$ , whence the restrictions of  $f'$  and  $f''$  to  $C$  are states of  $C$ . Since, moreover,

$$\lambda f'|_C \leq f \circ \varphi, \quad (1 - \lambda) f''|_C \leq f \circ \varphi,$$

there exist states  $g'$  and  $g''$  of  $A$  such that

$$f'|_C = g' \circ \varphi, \quad f''|_C = g'' \circ \varphi.$$

Hence by (i),

$$f' = g'^\sim, \quad f'' = g''^\sim.$$

This shows that the image of  $f \mapsto f^\sim$  is a face.

Now let us prove (iii). If  $f$  and  $f'$  are equivalent pure states of  $A$ , in other words, if the associated representations are unitarily equivalent, then by Corollary 8 of [7] there is a unitary  $u \in A$  such that  $f = u^* f' u$ . Choose  $w \in C$  of norm one such that  $\varphi w = u$ . Then

$$f \circ \varphi = w^* (f' \circ \varphi) w.$$

Hence by (i),

$$f^\sim = w^* f'^\sim w,$$

whence  $f^\sim$  and  $f'^\sim$  are equivalent. Conversely, suppose that  $f^\sim$  and  $f'^\sim$  are equivalent, that is, that for some unitary  $v \in B$ ,

$$f^\sim = v^* f'^\sim v.$$

As shown in the proof of (i) (see also [6, p. 586]),

$$f^\sim = f_j f^\sim f_j, \quad f'^\sim = f_j f'^\sim f_j, \quad j = 1, 2, \dots$$

Hence

$$f^\sim = (f_j v^* f_j) f'^\sim (f_j v f_j), \quad j = 1, 2, \dots$$

Choose  $k$  and  $j = 1, 2, \dots$  so that  $k < j$ ,  $j > 5$ , and

$$\|v - (s_k + i s_j)\| \leq (j+1)^{-1}, \quad \|s_k + i s_j\| \leq 1.$$

Since  $f_{k+1} f_{j+1} = f_{j+1}$ —see the proof of the inequality  $\|f_{j+1}(s_j - c_j) f_{j+1}\| \leq (j+1)^{-1}$ —and also  $\|f_{j+1}\| = 1$ , we have

$$\|f_{j+1}(s_k - c_k) f_{j+1}\| \leq (j+1)^{-1}.$$

It follows that

$$\|f_{j+1}(v - (c_k + ic_j))f_{j+1}\| \leq 3(j+1)^{-1}.$$

Since

$$\|a_1 g b_1 - a_2 g b_2\| \leq \|g\|(\|a_1\| \|b_1 - b_2\| + \|b_2\| \|a_1 - a_2\|),$$

we have

$$\|f - (f_{j+1}(c_k + ic_j)f_{j+1})^* f' (f_{j+1}(c_k + ic_j)f_{j+1})\| \leq 6(j+1)^{-1} < 1;$$

in particular,  $\|f - a^* f' a\| < 1$ , where  $a = \varphi(f_{j+1}(c_k + ic_j)f_{j+1})$ . The proof of Corollary 8 of [7] shows that if  $f$  and  $g$  are pure positive linear functionals on  $A$  such that  $\|f - g\| < \|f\| + \|g\|$  then  $f$  and  $g$  are equivalent. Since  $f'$  is pure, so is  $a^* f' a$ . Since  $\|f\| = 1$  we have

$$\|f - a^* f' a\| < 1 \leq \|f\| + \|a^* f' a\|.$$

Hence  $f$  is equivalent to  $a^* f' a$ , and therefore to  $f'$ .

3. COROLLARY. *Let  $A$  be an approximately finite-dimensional separable  $C^*$ -algebra and let  $B$  be a separable  $C^*$ -algebra which is not postliminary. Then there exists a homeomorphism from the set of pure states of  $A$  (with the weak dual topology) onto the set of pure states contained in a locally compact subset of the state space of  $B$  which preserves equivalence in both directions.*

4. LEMMA. *Let  $B$  be a separable  $C^*$ -algebra and let  $E$  be the set of pure states contained in a locally compact subset  $S$  of the state space of  $B$ . Then the saturation of  $E$  with respect to equivalence is Borel.*

PROOF. In the case  $E = S$ , the result is due to G. K. Pedersen; the general case requires no new ideas. By Corollary 8 of [7], the saturation of  $E$  is  $\bigcup_{u \in U} u^* E u$ , where  $U$  denotes the unitary group of  $B$  (with unit adjoined, if necessary). By Corollary 9 of [7], which says that two pure states strictly closer in norm than 2 are equivalent, this is the same as the set of pure states in  $\bigcup_{u \in U} (u^* E u + B_1^*)$ , where  $B_1^*$  denotes the closed unit ball of the dual of  $B$ . In fact, it is the same as the set of pure states in  $\bigcup_{u \in U} (u^* S u + B_1^*)$ —a modification of the proof of Corollary 9 of [7] shows that if  $f$  is a pure state,  $g \in u^* S u$ , and  $\|f - g\| \leq 1$ , then  $g$  is pure, i.e.,  $g \in u^* E u$ . Choose a dense sequence  $(u_n)$  in  $U$ . Then

$$\bigcup_{u \in U} u^* E u \subset \bigcup_{n=1,2,\dots} (u_n^* S u_n + B_1^*) \subset \bigcup_{u \in U} (u^* S u + B_1^*);$$

hence the saturation of  $E$  is the set of pure states in  $\bigcup_{n=1,2,\dots} (u_n^* S u_n + B_1^*)$ . This set, like  $S$ , is a countable union of compacts.

5. LEMMA. *If two Borel spaces are each isomorphic to a Borel subset of the other, they are isomorphic.*



PROOF. The bijection constructed in the proof of the Cantor-Bernstein theorem (see, e.g., [8]) is easily seen to be Borel in each direction.

**6. Proof of the main result.** Let  $A$  and  $B$  be approximately finite-dimensional separable  $C^*$ -algebras which are not postliminary. By 3 (applied to the pairs  $A, B$  and  $B, A$ ), together with 4, the set of pure states of each of  $A$  and  $B$ , with the Borel structure determined by the weak dual topology, is Borel isomorphic to a Borel subset of the pure states of the other, in such a way that equivalence is respected (by both the isomorphisms and their inverses), and the saturations of the images are Borel.

If a given Borel subset  $S$  of pure states has a Borel saturation, then its quotient Borel structure is equal to the restriction of the whole quotient Borel structure. To see this, note that the saturation of any Borel set of pure states is Souslin, being the second coordinate projection of the Borel set (cf. 4) of pairs  $(f, u^*fu)$ ,  $f$  in the set. Also, recall that if the union of two disjoint Souslin subsets of a standard Borel space is Borel, each of the sets is Borel. What is to be verified is that a relatively saturated subset of the given Borel set  $S$  is Borel if and only if its saturation in the space of all pure states is Borel. One implication is immediate. In the other direction, if a relatively saturated subset of  $S$  is Borel, the relative complement in  $S$  is both relatively saturated and Borel. The full saturations of these two sets are therefore both disjoint and Souslin; hence, since their union is Borel, they are Borel.

It follows from the preceding two paragraphs that the quotients of the Borel spaces of pure states of  $A$  and  $B$  are isomorphic to Borel subsets of each other. By [5], these quotient structures are the Mackey Borel structures on the spectra of  $A$  and  $B$ . By 5, the two spaces are isomorphic.

**7. Remark.** Combining the present result with the classification of the biduals of approximately finite-dimensional separable  $C^*$ -algebras given in [4], one obtains the following statements for this class of separable  $C^*$ -algebras.

The bidual determines the Mackey Borel structure on the spectrum.

The Mackey Borel structure on the spectrum, together with the number of extreme tracial states of type  $I_n$ ,  $n = 1, 2, \dots$ , and of type  $II_1$ , determines the bidual.

## REFERENCES

1. O. Bratteli, *Inductive limits of finite-dimensional  $C^*$ -algebras*, Trans. Amer. Math. Soc. **171** (1972), 195–234. MR **47** #844.
2. J. Dixmier, *Les  $C^*$ -algèbres et leurs représentations*, Cahiers Scientifiques, no. 29, Gauthier-Villars, Paris, 1964. MR **30** #1404.
3. ———, *On some  $C^*$ -algebras considered by Glimm*, J. Functional Analysis **1** (1967), 182–203. MR **35** #4740.

4. G. A. Elliott, *On approximately finite-dimensional von Neumann algebras*, *Math. Scand.* **39** (1976), 91–101.
5. L. T. Gardner, *On the Mackey Borel structure*, *Canad. J. Math.* **23** (1971), 674–678. MR **44** #4532.
6. J. G. Glimm, *Type I  $C^*$ -algebras*, *Ann. of Math. (2)* **73** (1961), 572–612. MR **23** #A2066.
7. J. G. Glimm and R. V. Kadison, *Unitary operators in  $C^*$ -algebras*, *Pacific J. Math.* **10** (1960), 547–556. MR **22** #5906.
8. P. R. Halmos, *Naive set theory*, Van Nostrand, Princeton, N. J., 1960. MR **22** #5575.

MATHEMATICS INSTITUTE, UNIVERSITY OF COPENHAGEN, COPENHAGEN, DENMARK