

## SOME PROPERTIES OF FAMILIES OF CONVEX CONES<sup>(1)</sup>

BY

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**ABSTRACT.** The purpose of this paper is to study properties of finite families of convex cones in  $n$ -dimensional Euclidean space  $R^n$ , whose members all have the origin as a common apex.

Of special interest are such families of convex cones in  $R^n$  which have the following property: Each member of the family is of dimension  $n$ , the intersection of any two members is at least  $(n - 1)$ -dimensional, . . . , the intersection of any  $n$  members is at least 1-dimensional and the intersection of all the members is the origin.

**1. Introduction.** The purpose of this paper is to study properties of families of convex cones in  $n$ -dimensional Euclidean space in  $R^n$ , whose members have the origin as a common apex.

For a set  $A$  in  $R^n$ ,  $\dim A$  denotes the dimension of the minimal flat containing  $A$ . For a family  $A_T = \{A_i; i \in T\}$  of sets in  $R^n$ ,  $A(S)$  denotes  $\bigcap \{A_i; i \in S\}$  and  $\bar{A}(S) = A(T \setminus S)$ . We use the convention that  $A(\emptyset) = \bar{A}(T) = R^n$ .

Unless stated otherwise a family is a finite family and a cone is a *convex cone with apex 0*.

Of special interest are nonempty families of cones in  $R^n$  which are nondegenerate in the following sense:

*Each member of the family is of dimension  $n$ , the intersection of any two members of the family is of dimension  $n - 1$  at least, . . . , the intersection of any  $n$  members of the family is of dimension 1 at least and the intersection of all members of the family is the origin.* Such families are called *nondegenerate families* or *N.D.F.s*.

Equivalently: A family  $A_T$  of cones in  $R^n$  is an N.D.F. if  $A(T) = \{0\}$  and  $\dim A(S) \geq n - |S| + 1$  for any nonempty  $S \subset T$ . Three properties of N.D.F.s are given in Theorems 1-3:

**THEOREM 1 (PERLES).** *If  $A_T$  is an N.D.F. in  $R^n$  then  $A_T$  covers  $R^n$  (i.e.,  $\bigcup \{A_i; i \in T\} = R^n$ ).*

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**THEOREM 2.** *If  $A_T$  is an N.D.F. in  $R^n$  then for each  $j \in T$  the intersection of all members of  $A_T$  excluding  $A_j$  is contained in  $-A_j$  (i.e.,  $\overline{A}(\{j\}) \subset -A_j$ ).*

**THEOREM 3.** *If  $A_T$  is an N.D.F.,  $M$  is a subset of  $T$  and  $\overline{A}(M) = A(T \setminus M)$  contains an  $M$ -dimensional subspace then  $M$  is the empty set.*

Theorem 1 was originally proved by M. A. Perles (private communication). Perles' proof is algebraic in nature.

A geometric proof of Theorem 2 is given in §2.

It is possible to prove Theorems 1 and 3 using the same methods described in §2.

Let  $A_T$  be a family of cones in  $R^n$ . Subsets  $B$  of  $T$  for which  $\overline{A}(B)$  is a subspace will be called *faces of  $A_T$* .  $B$  will be called a *k-face of  $A_T$*  if  $B$  is a face of  $A_T$  and  $\dim \overline{A}(B) = |B| - k$ .

It is natural to ask the following question:

Given a family  $A_T$  of convex cones in  $R^n$  and given  $\dim A(S)$  for each  $S \subset T$ , can we determine the subsets  $B$  of  $T$  which are faces of  $A_T$ ?

In general, as easy examples can show, the answer is negative.

However, Theorem 1 yields a sufficient condition:

By Theorem 1, if  $B \neq \emptyset$  and  $\{\overline{A}(B) \cap A_i; i \in B\}$  is an N.D.F. in span  $\overline{A}(B)$ , then  $\cup \{\overline{A}(B) \cap A_i; i \in B\} = \text{span } \overline{A}(B)$  and therefore  $\overline{A}(B)$  is a subspace. Thus we have

**THEOREM 4.** *If  $B = \emptyset$  and  $A(T) = \{0\}$  or if the family  $\{\overline{A}(B) \cap A_i; i \in B\}$  is an N.D.F. in span  $\overline{A}(B)$ , then  $B$  is a face of  $A_T$ .*

A subset  $B$  of  $T$  will be called a *nondegenerate subset of  $T$  relative to  $A_T$*  or an *N.D.S. of  $A_T$*  if  $B = \emptyset$  and  $A(T) = \{0\}$  or if the family  $\{\overline{A}(B) \cap A_i; i \in B\}$  is an N.D.F. in span  $\overline{A}(B)$ .

Equivalently: a subset  $B$  of  $T$  will be called an *N.D.S. of  $A_T$*  if  $B = \emptyset$  and  $A(T) = \{0\}$  or if  $\overline{A}(\emptyset) = A(T) = 0$  and  $|S| - \dim \overline{A}(S) < |B| - \dim \overline{A}(B)$  for any proper subset  $S$  of  $B$ .

$B$  will be called a *k-N.D.S. of  $A_T$*  if it is an N.D.S. of  $A_T$  and if  $\dim \overline{A}(B) = |B| - k$ .

If  $A_T$  is an N.D.F. then the sufficient condition in Theorem 4 is also a necessary condition:

**THEOREM 5.** *If  $A_T$  is an N.D.F. then a subset  $B$  of  $T$  is a  $[k]$ -face of  $A_T$  iff it is a  $[k]$ -N.D.S. of  $A_T$ .*

The proof of Theorem 5 is established in §3.

Theorem 5 yields an algorithm for finding all the faces of an N.D.F. when  $\dim A(S)$  is known for all  $S \subset T$ .

**ALGORITHM.** If  $A_T$  is an N.D.F. then a subset  $B$  of  $T$  is a face of  $A_T$  iff

$|S| - \dim \bar{A}(S) < |B| - \dim \bar{A}(B)$  for any proper subset  $S$  of  $B$ .

In the last section we attempt to 'justify' nondegenerate families of convex cones by discussing some results which may be obtained by using N.D.F.s.

**2. Proof of Theorem 2.** The essence of the proof is the use of a suitable separating hyperplane and induction on  $n$ .

The following lemma will be used:

**LEMMA 1.** *If  $A$  and  $B$  are polyhedral cones in  $R^n$ ,  $A$  is pointed and  $A \cap B = 0$ , then there is a hyperplane  $H$  which separates  $A$  and  $B$  and strictly supports  $A$  (i.e.,  $A$  and  $B$  are on different sides of  $H$  and  $H \cap A = \{0\}$ ).*

Let  $A_T = \{A_i: i \in T\}$  be an N.D.F. in  $R^n$ . It is enough to prove that  $\cup \{A_i: i \in T\} = R^n$  under the additional assumption that  $A_T$  is a family of polyhedral cones.

The proof is by induction on  $n$  and for fixed  $n$  by induction on  $t = |T| - n$  (since  $A_T$  is an N.D.F.,  $t \geq 1$ ).

If  $\bar{A}(j) = A(T \setminus \{j\}) = 0$ , the proof is trivial (this includes the case  $n = 0$ ).

Otherwise. Suppose  $0 \neq x \in \bar{A}(j)$ . We have to show that  $-x \in A_j$ . There are two cases to consider.

1. For each  $i \in T \setminus \{j\}$ ,  $\bar{A}(i) \neq \{0\}$ . (This includes the case  $t = 1$ .)

If  $\bar{A}(i)$  contains a line for each  $i \in T \setminus \{j\}$  we would have  $A_j = R^n$  (since  $A_j$  contains the sum of these lines, the lines are linearly independent ( $A(T) = \{0\}$ ), and there are at least  $n$  lines).

This leads to  $x \in A_j \cap \bar{A}(j) = A(T) = \{0\}$ , a contradiction.

Suppose  $i \in T \setminus \{j\}$  and  $\bar{A}(i)$  is a pointed cone. Using Lemma 1, let  $H$  be a hyperplane which separates  $\bar{A}(i)$  and  $A_i$  (both of them are polyhedral by our assumptions).

Define  $A'_{T \setminus \{i\}}$ , an N.D.F. in  $R^{n-1} = H$  by

$$A'_l = A_l \cap H \quad \text{for } l \in T \setminus \{i\}.$$

(It is not difficult to prove that  $A'_{T \setminus \{i\}}$  is an N.D.F.)

Let  $H^+$  and  $H^-$  be the two closed halfspaces determined by  $H$ , and suppose that  $\bar{A}(i) \subseteq H^+$ . Since  $x \in \bar{A}(j) \subseteq A_i \subseteq H^-$  and  $\bar{A}(i) \cap \text{int } H^+ \neq \emptyset$ , there is a  $y \in \bar{A}(i)$  such that  $x + y \in H$ . Therefore  $x + y \in \bar{A}(\{i \cup j\}) \cap H = \bar{A}'(j)$ .

By the induction assumption (on  $n$ ),  $-x - y \in A'_i \subseteq A_i$ . Consequently,  $-x = (-x - y) + y \in A_j$  ( $y \in \bar{A}(i) \subseteq A_i$ ).

2. Suppose  $i \in T \setminus \{j\}$  and  $\bar{A}(i) = \{0\}$ . Define  $A'_{T \setminus \{i\}} = \{A'_f = A_f: f \in T \setminus \{i\}\}$ . Then  $A'_{T \setminus \{i\}}$  is an N.D.F. of  $n + (t - 1)$  cones in  $R^n$  and  $x \in$

$\bar{A}(\{i, j\}) = \bar{A}'(j)$ . By the induction assumption (on  $t = |T| - n$ ),  $-x \in A'_j = A_j$ .

The proof is now complete.

**3. N.D.S.s and faces.** The main object of this section is a proof of Theorem 5.

The proof of Theorem 5 relies on Lemmas 2, 3 and 4. Lemma 5 states that distinct 1-N.D.S.s are disjoint; it is presented on its own merit.

In the following, let  $A_T$  be a family of  $|T| = n + t$  cones in  $R^n$  such that  $A(T) = \{0\}$ .

**LEMMA 2.** *If  $\dim \bar{A}(B) < |B| - k$  and  $0 < k' < k$  then  $B$  contains a  $k'$ -N.D.S. of  $A_T$  and  $B$  contains a  $k'$ -face of  $A_T$ .*

**PROOF OF LEMMA 2.** By the conditions of Lemma 2,  $\dim \bar{A}(B) < |B| - k'$ . Let  $B'$  be a minimal subset of  $B$  which satisfies  $\dim \bar{A}(B) < |B'| - k'$ . It is easily verified that  $B'$  is a  $k'$ -N.D.S. of  $A_T$  and, by Theorem 4,  $B'$  is a  $k'$ -face of  $A_T$ .

**LEMMA 3. A.** *If  $B$  is an N.D.S. of  $A_T$ ,  $M \subset B$  and  $\bar{A}(M)$  contains an  $|M|$ -dimensional subspace, then  $M = \emptyset$ .*

**B.** *If  $A_T$  is an N.D.F.,  $B \subset M \subset T$ ,  $B$  is a  $k$ -face of  $A_T$  and  $\bar{A}(M)$  contains an  $(|M| - k)$ -dimensional subspace, then  $M = B$ .*

**C.** *If  $A_T$  is an N.D.F.,  $B_1 \subsetneq B_2 \subset T$  and  $B_i$  is a  $k_i$ -face of  $A_T$  for  $i = 1, 2$ , then  $k_1 < k_2$ .*

**PROOF OF LEMMA 3.** Part A follows from Theorem 3 applied to the family  $\{\bar{A}(B) \cap A_i : i \in B\}$  in  $\text{span } \bar{A}(B)$ . *Proof of B:* The case  $B = T$  is trivial, assume therefore that  $B$  is a proper subset of  $T$ . Let  $Y$  be a subspace complementary to  $\bar{A}(B)$  relative to  $R^n$ .

Define  $C_{T \setminus B} = \{C_i : A_i \cap Y : i \in T \setminus B\}$ , a family of cones in  $Y$ .

For each  $S \subset T \setminus B$ ,  $C(S) = A(S) \cap Y$ ,  $\bar{C}(S) = \bar{A}(S \cup B) \cap Y$  and  $\dim C(S) = \dim A(S) - \dim \bar{A}(B)$ .

It follows that  $C_{T \setminus B}$  is an N.D.F. in  $Y$ . Since  $\bar{C}(M \setminus B) = \bar{A}(M) \cap Y$  and  $\bar{A}(M)$  contains an  $(|M| - k)$ -dimensional subspace it follows that  $\bar{C}(M \setminus B)$  contains an  $[(|M| - k) - \dim \bar{A}(B) = |M| - k - (|B| - k) = |M - B|]$ -dimensional subspace. By Theorem 3 applied to  $C_{T \setminus B}$ ,  $M \setminus B = \emptyset$ , proving part B.

Part C is an immediate result of B.

**LEMMA 4.** *If  $A_T$  is an N.D.F. and  $B$  is a  $k$ -face of  $A_T$ , then  $0 < k < t$ ,  $k = 0$  iff  $B = \emptyset$ ,  $k = t$  iff  $B = T$ .*

Lemma 4 is easily derived from the definitions of a  $k$ -face, the definition of an N.D.F. and Theorem 3.

LEMMA 5. *If  $B_1$  and  $B_2$  are distinct 1-N.D.S.s of  $A_T$  then  $B_1$  and  $B_2$  are disjoint.*

PROOF. By Theorem 4,  $B_1$  and  $B_2$  are 1-faces of  $A_T$ .  $B_1 \cap B_2$  is a face of  $A_T$  since  $\overline{A}(B_1 \cap B_2) = \overline{A}(B_1) \cap \overline{A}(B_2)$  is a subspace.

Since  $B_1$  and  $B_2$  are distinct,  $B_1 \cap B_2$  is either a proper subset of  $B_1$  or a proper subset of  $B_2$ . Suppose that  $B_1 \cap B_2$  is a proper subset of  $B_1$ . Since  $B_1$  is a 1-N.D.S.,

$$|B_1 \cap B_2| - \dim \overline{A}(B_1 \cap B_2) < |B_1| - \dim \overline{A}(B_1) = 1.$$

Therefore  $\dim \overline{A}(B_1 \cap B_2) > |B_1 \cap B_2|$  and, by Lemma 3.A,  $B_1 \cap B_2 = \emptyset$ , completing the proof.

PROOF OF THEOREM 5. Suppose that  $A_T$  is an N.D.F. in  $R^n$ . We have to prove that for any subset  $B$  of  $T$ ,  $B$  is an N.D.S. of  $A_T$  iff  $B$  is a face of  $A_T$ .

If  $B$  is a  $k$ -N.D.S. then  $B$  is a  $k$ -face by Theorem 4.

We assume that  $B$  is a  $k$ -face and not a  $k$ -N.D.S. and derive a contradiction.

If  $B$  is not a  $k$ -N.D.S. of  $A_T$  then there exists a proper subset  $B'$  of  $B$  such that  $|B'| - \dim \overline{A}(B') > |B| - \dim \overline{A}(B) = k$ . Therefore  $\dim \overline{A}(B') < |B'| - k$  and  $k > 0$  by Lemma 4. By Lemma 2,  $B'$  contains a  $k$ -face  $C$ . Since  $C$  is a proper subset of  $B$  and  $A_T$  is an N.D.F., we have by Lemma 3.C that  $k < k$ , a contradiction. The proof of Theorem 5 is now complete.

**4. Remarks.** We will briefly discuss some results which may be obtained using properties of N.D.F.s:

1. Reconstructing dimensions of intersections of convex sets.

We can prove that if  $K_T$  and  $K'_T$  are two finite families of convex sets in  $R^n$  and  $\dim \cap \{K_i; i \in S\} = \dim \cap \{K'_i; i \in S\}$  for each  $S \subset T$  with  $|S| \leq n + 1$ , then  $\dim \cap \{K_i; i \in T\} = \dim \cap \{K'_i; i \in T\}$  (see [3]).

2. In [4] we use properties of N.D.F.s and Gale diagrams (see [2, Chapter 5, §4], for Gale diagrams) to establish connections between N.D.F.s and convex polytopes:

For each N.D.F.,  $A_T = \{A_i; i \in T\}$  in  $R^n$  there is a  $(|T| - n - 1)$ -polytope  $P$  such that the lattice of faces of  $A_T$  ordered by the inclusion relation is isomorphic to the lattice of faces of  $P$  ordered by the inclusion relation.

This result enables one to obtain properties of families of cones by using well-known theorems on convex polytopes. An illustration is [5]:

3. Using N.D.F.s, [4], and properties of neighborly polytopes we can generalize a result of M. J. C. Baker [1] and prove a theorem which is equivalent to the following:

Let  $F$  be a finite family of at least  $n + 1 + t$  'convex' sets on  $S_n$  ( $t \geq 0$ ). If

every  $n + 1$  members of  $F$  have nonempty intersection then there are  $n + 1 + \lfloor t/2 \rfloor$  members of  $F$  whose intersection is nonempty. ( $S_n$  is the  $n$ -dimensional unit sphere and a set is 'convex' if it is the intersection of a convex cone with apex 0 in  $R^{n+1}$  with  $S_n$ .)

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