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WEAK CONVERGENCE OF THE AREA OF NONPARAMETRIC L_1 SURFACES

BY

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ABSTRACT. The main purpose of this work is to obtain an analogue to a theorem of L. C. Young on the behavior of the nonparametric surface area of continuous functions. The analogue is for L^1 functions of generalized bounded variation. By considering arbitrary Borel vector measures and kernels other than the area kernel, results concerning the weak behavior of measures induced by a class of sublinear functionals are obtained.

0. **Introduction.** The classes BVC and ACC of continuous functions over the open unit cube Q in R^m have been extended to the classes BV and AC of L^1 functions by use of the existence of certain types of distribution derivatives. Namely, if the distribution derivative is given by a finite Borel vector measure, the function is of bounded variation. If in fact the derivative is given by a function (i.e., the measure is absolutely continuous with respect to Lebesgue measure) the function is absolutely continuous. A natural question then becomes: Which properties do these wider classes share with the continuous classes?

Here we consider a theorem concerning the behavior of the continuous, nonparametric surface area of surfaces in BVC given by L. C. Young, [6], in 1944; and develop an analogue behavior in the class BV for the generalized surface area as in [6].

In the classic area formula for $ACC(R^2)$ functions, if the partial derivatives are replaced with difference quotients, a formula results which can be applied to any function in BVC:

$$A\alpha\beta = \iint \left[1 + \left[\frac{f(x+\alpha,y) - f(x,y)}{\alpha} \right]^2 + \left[\frac{f(x,y+\beta) - f(x,y)}{\beta} \right]^2 \right]^{1/2} dy\, dx.$$

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The question then becomes: How does $A\alpha\beta$ approximate the area? Is the area $\lim_{\alpha,\beta\to 0} A\alpha\beta$?

L. C. Young shows the answer is no, and gives necessary and sufficient conditions for the answer to be yes. (Compare with Saks [1, p. 182].) His condition is that there must be a Borel partition of Q into two sets A_1 and A_2 so that off A_i the function is absolutely continuous in the variable x_i for almost all fixed values of the other variables.

For the generalized class AC there is a similar area formula

$$\int_Q \left(1 + \frac{d\alpha_1^2}{dL} + \frac{d\alpha_2^2}{dL}\right)^{1/2} dL$$

where (α_1, α_2) is the distribution derivative measure of f and $d\alpha_i/dL$ is the general derivative [1, p. 106] of α_i with respect to Lebesgue measure L.

Following Young, we want to replace the derivatives with "difference quotients" for these derivatives and find necessary and sufficient conditions for the limit to be the area when applied to a function known only to be in the class BV.

A precise statement of the result is Theorem 6 in §5 of the paper and requires a lot of development of notation. However the necessary and sufficient condition simply stated is a direct analogue of Young's condition.

The area will be given as the limit if and only if the cube Q can be partitioned into sets $A_1, \ldots, A_m$ so that off A_i the ith component of the distribution derivative measure is absolutely continuous.

By considering arbitrary Borel vector measures and kernels other than the area kernel, the results of §1 through 5 are results concerning the weak behavior of measures induced from vector measures by a class of sublinear functionals.

1. The integral average of a measure. In this section we make precise the idea of "difference quotient" for a measure by introducing an averaging process for measures which is an analogue to taking the integral average of a function.

For notational convenience we introduce the following definitions and conventions.

(1) L will always denote Lebesgue measure.

(2) M will always denote the collection of all finite Borel measures on the open unit cube Q in R^m.

(3) For each $h > 0$ let $Q_h = \times_{i=1}^m [0, h]$ be the closed h cube in R^m, and let $K_h(x) = \chi_{Q_h}(x) \div h^m$ (where χ indicates the characteristic function).

(4) By $\kappa[\ ,\ ]$ we denote a mapping from $M \times R^+$ into the continuous functions from R^m into R given by

$$\kappa[\sigma, h](x) = \int_{R^m} K_h(y - x)\, d\sigma(y).$$

Note that $\kappa[\sigma, h](x) = \sigma I_h(x)/LI_h(x)$ where $I_h(x)$ is a cube of side length h containing x. Thus it is $\kappa[\sigma, h](x)$ which corresponds to the difference quotients for functions. Further, as h tends to zero, by Lebesgue's theorem [1, p. 115], we have that for almost all X, $\lim_{h\to 0}\kappa[\sigma, h](x) = d\sigma/dL$.

(5) $\mu[\ ,\ ]$ will denote a mapping from $M \times R^+$ into M given by

$$\mu[\sigma, h](E) = \int_E \kappa[\sigma, h](x)\, dL(x)$$

for each Borel set E.

Similar constructions can be found in [4, p. 167], with continuous kernels K_h.

If σ is supported inside Q then $\mu[\sigma, h]$ is supported inside an h-neighborhood of Q. Further, for $h > 0$, $\mu[\sigma, h](R^m) = \sigma(R^m)$; since

$$\begin{aligned}\mu[\sigma, h](R^m) &= \int_{R^m}\int_{R^m} K_h(y - x)\, d\sigma(y)\, dL(x)\\ &= \int_{R^m}\int_{R^m} K_h(x - y)\, dL(x)\, d\sigma(y)\\ &= \int_{R^m} h^m h^{-m}\, d\sigma(y) = \sigma(R^m).\end{aligned}$$

As h tends to zero, supports for $\mu[\sigma, h]$ tend to Q and we have

$$\lim_{h\to 0} \mu[\sigma, h](Q) = \sigma(Q).$$

A sequence of measures σ_k converges weakly to a measure σ if for each real valued continuous function f of compact support we have $\int f\, d\sigma_k \to \int f\, d\sigma$. For probability measures this is equivalent to $\lim\inf_k \sigma_k(G) \geqslant \sigma(G)$ for each open set G [2, p. 11]. This equivalence will also hold if σ_k are positive and $\lim_k \sigma_k(Q) = \sigma(Q)$.

THEOREM 1. *As h tends to zero in R^+, $\mu[\sigma, h]$ converges weakly to σ.*

PROOF. See [4] and replace "uniform convergence" with "dominated convergence" in the proof of Theorem 4.

This averaging process is extended to n-dimensional vector measures in $\times_{i=1}^n M$ as follows: $\mu[\ ,\ ]$: $\times_{i=1}^n M \times \times_{i=1}^n R_+ \to \times_{i=1}^n M$ is defined for each $\bar{\sigma} = (\sigma^1, \ldots, \sigma^n)$ in $\times_{i=1}^n M$ and vector $\bar{h} = (h^1, \ldots, h^n)$ in R_+^n by $\mu[\bar{\sigma}, \bar{h}] = \sum_{i=1}^n \mu[\sigma^i, h^i]\bar{e}_i$ where $\{\bar{e}_i\}$ is the standard basis for R^n and $\mu[\sigma^i, h^i]$ is as in definition (5) above.

COROLLARY TO THEOREM 1. *As $\bar{h}$ tends to zero in R_+^n, $\mu[\bar{\sigma}, \bar{h}]$ converges weakly to $\bar{\sigma}$.*

2. The $\mathfrak{T}$-variation of a measure. Given a vector measure $\bar{\sigma} = (\sigma^1, \ldots, \sigma^n)$ in $\times_{i=1}^n M$ and a sublinear functional $\mathfrak{T}$ on R^n, Goffman and Serrin [4] obtain a scalar measure $\mathfrak{T}\bar{\sigma}$ in much the same manner as the total variation is defined. We shall state the basic definitions and list three useful theorems from [4] as lemmas.

(6) Definition. $\mathfrak{T}: R^n \to R$ is a type-A functional on R^n if it satisfies:

(a) $\mathfrak{T}(x + y) \leqslant \mathfrak{T}(x) + \mathfrak{T}(y)$.

(b) $\mathfrak{T}(ax) = a\mathfrak{T}(x)$ for positive scalars a.

(c) There is $C > 0$ so that for every p in R^n we have $|\mathfrak{T}(p)| \leqslant C\|p\|$; $\| \ \|$ = Euclidean norm.

Note $\mathfrak{T}$ is continuous.

(7) Definition. For a type-A functional $\mathfrak{T}$ on R^n and $\bar{\sigma} \in \times_{i=1}^n M$ we define the $\mathfrak{T}$-variation measure of $\bar{\sigma}$ to be given by

$$\mathfrak{T}\bar{\sigma}(E) = \sup\left\{ \sum_{F \in \pi} \mathfrak{T} \circ \bar{\sigma}(F) \colon \pi \text{ is a finite Borel partition of } E \right\}.$$

Lemma 1. *For $\bar{\alpha} \perp \bar{\beta}$ and $\bar{\mu} = \bar{\alpha} + \bar{\beta}$ we have $\mathfrak{T}\bar{\mu} = \mathfrak{T}\bar{\alpha} + \mathfrak{T}\bar{\beta}$.*

Lemma 2. *Let $\bar{\sigma}(E) = \int_E a \, d\nu + \bar{\beta}(E)$ be the Lebesgue decomposition of $\bar{\sigma}$ with respect to the positive measure ν; then $\int_E \mathfrak{T} \circ a \, d\nu + \mathfrak{T}\bar{\beta}(E)$ is the Lebesgue decomposition of $\mathfrak{T}\bar{\sigma}$ with respect to ν.*

Lemma 3. *If $\mathfrak{T}$ is a positive type-A functional and $\{\bar{\sigma}_k\}$ converges weakly to $\bar{\sigma}$, then for each open set G we have* $\liminf_k \mathfrak{T}\bar{\sigma}_k(G) \geqslant \mathfrak{T}\bar{\sigma}(G)$.

Corollary to Theorem 1. *As h tends to zero in R^+, $\mu[\sigma, h]^+$ and $\mu[\sigma, h]^-$ tend weakly to σ^+ and σ^- respectively.*

Proof. Since $\mu[\sigma, h]$ converges weakly to σ, it suffices to show that $|\mu[\sigma, h]|$ converges weakly to $|\sigma|$. But by Lemma 3 we have for each open G

$$\liminf |\mu[\sigma, h]|(G) \geqslant |\sigma|(G);$$

hence we need only show $\limsup |\mu[\sigma, h]|(R^m) \leqslant |\sigma|(R^m)$. But,

$$|\mu[\sigma, h]|(R^m) = \int_{R^m} |\kappa[\sigma, h](x)| \, dL(x)$$

$$\leqslant \int_{R^m} \kappa[|\sigma|, h](x) \, dL(x) = \mu[|\sigma|, h](R^m).$$

Given $\bar{\sigma} \in \times_{i=1}^n M$, $\bar{h} \in R_+^n$ and $\mathfrak{T}$ as above we denote by $\mu[\mathfrak{T}; \bar{\sigma}, \bar{h}]$ the measure given by $\int_E \mathfrak{T} \circ \kappa[\bar{\sigma}, \bar{h}](x) \, dL(x)$.

Note that the two "different" measures $\mu[\mathfrak{T}; \bar{\sigma}, \bar{h}]$ and $\mathfrak{T}\mu[\bar{\sigma}, \bar{h}]$ are in fact the same. To see this note that the Lebesgue decomposition of $\mu[\bar{\sigma}, \bar{h}]$ is given by $\int_E \kappa[\bar{\sigma}, \bar{h}](x) \, dL(x) + \mathcal{O}(E)$ where $\mathcal{O}$ = zero measure. Hence by Lemma 2 we have that the decomposition of $\mathfrak{T}\mu[\bar{\sigma}, \bar{h}]$ is given by

$$\int_E \mathfrak{T} \circ \kappa[\bar{\sigma}, \bar{h}](x)\, dL(x) + \mathfrak{T}\Theta(E) = \mu[\mathfrak{T}; \bar{\sigma}, \bar{h}](E).$$

In general the measures $\mathfrak{T}\mu[\bar{\sigma}, \bar{h}]$ do not converge weakly to $\mathfrak{T}\bar{\sigma}$ in higher dimensions; however, for scalar measures we do have weak convergence. More precisely we state:

THEOREM 2. *Let $\sigma \in M$ and (h_k) be a sequence of positive real numbers converging to zero; then for any nonnegative type-A functional $\mathfrak{T}$ on R we have $\mathfrak{T}\mu[\sigma, h_k]$ converges weakly to $\mathfrak{T}\bar{\sigma}$.*

PROOF. The only such $\mathfrak{T}$ are characterized as

$$\mathfrak{T}(p) = \begin{cases} p\mathfrak{T}(1) & \text{for } p \geqslant 0, \\ -p\mathfrak{T}(-1) & \text{for } p < 0. \end{cases}$$

Put $\sigma_k = \mu[\sigma, h_k]$. By our Corollary to Theorem 1 we have that $\mathfrak{T}(\sigma_k)^+ = \mathfrak{T}(1)(\sigma_k)^+$ and $\mathfrak{T}(-\sigma_k)^- = \mathfrak{T}(-1)(-\sigma_k)^-$ converge weakly to $\mathfrak{T}(1)\sigma^+ = \mathfrak{T}\sigma^+$ and $\mathfrak{T}(-1)(-\sigma^-) = \mathfrak{T}(-\sigma^-)$ respectively. But $\sigma_k^+ \perp \sigma_k^-$ and $\sigma^+ \perp \sigma^-$, so by Lemma 1 we have

$$\mathfrak{T}\sigma_k = \mathfrak{T}(\sigma_k^+ - \sigma_k^-) = \mathfrak{T}\sigma_k + \mathfrak{T}(-\sigma_k^-)$$

converges weakly to $\mathfrak{T}\sigma^+ + \mathfrak{T}(-\sigma^-) = \mathfrak{T}(\sigma^+ - \sigma^-) = \mathfrak{T}\sigma$.

3. **The lim sup formula.** Our main concern with the measures $\mathfrak{T}\mu[\bar{\sigma}, \bar{h}]$ is their weak behavior as $\bar{h}$ tends to zero. To this end we shall develop a formula for the lim sup of $\mathfrak{T}\mu[\bar{\sigma}, \bar{h}]$ as $\bar{h}$ tends to zero. This will require a refinement in the class of functionals considered.

DEFINITION. Each type-A functional $\mathfrak{T}$ on R^n induces a type-A functional $\mathfrak{T}^i$ on the ith coordinate axis (i.e., on R) *via* the following:

$$\mathfrak{T}^i(p) = \mathfrak{T}(0, \ldots, 0, p, 0, \ldots, 0)$$

where p is in the ith place.

By the sublinearity of $\mathfrak{T}$, for each $(p^1, \ldots, p^n)$ we have

$$\mathfrak{T}(p^1, \ldots, p^n) \leqslant \sum_{i=1}^{n} \mathfrak{T}^i(p^i).$$

We shall say that $\mathfrak{T}$ is a *type-A** functional if in addition for all $\bar{p} = (p^1, \ldots, p^n)$ and $i = 1, \ldots, n$ we have $\mathfrak{T}(\bar{p}) \geqslant \mathfrak{T}^i(p^i)$. Denote by $\mathcal{Q}^*(R^n)$ the nonnegative, type-A* functionals on R^n.

THEOREM 3. *Let $\bar{\sigma} \in \times_{i=1}^{n} M$, $\bar{\sigma} = (\sigma^1, \ldots, \sigma^n)$ and $\mathfrak{T} \in \mathcal{Q}^*(R^n)$. For each i, let β_i be the singular part of the decomposition of σ_i with respect to L. Then the following formula holds*:

$$\limsup_{\bar{h} \to \bar{0}} \mathfrak{T}\mu[\bar{\sigma}, \bar{h}](Q) = \int_Q \mathfrak{T} \circ \frac{d\bar{\sigma}}{dL}(x)\, dL + \sum_{i=1}^{n} \mathfrak{T}^i \beta_i(Q).$$

PROOF. Let $(\bar{h}_k)$ be an arbitrary sequence tending to zero in R^n_+. By using the summability of $\mathfrak{T} \circ d\bar{\sigma}/dL$, Eggroff's theorem and the singularity of $\mathfrak{T}^i\beta_i$ and L, we can define inductively a sequence of sets $\{E_j\}$ having the following properties:

(1) $L(E_j) < 1/j$.

(2) $\int_{E_j} \mathfrak{T} \circ (d\bar{\sigma}/dL)\, dL < 1/j$.

(3) $E_{j+1} \subset E_j$ for $j = 1, 2, \ldots$.

(4) On $Q - E_j$, $\kappa[\sigma^i, h^i_k]$ converges uniformly to $d\sigma^i/dL$ for each $i = 1, \ldots, n$.

(5) For every Borel set S and all j we have $\mathfrak{T}^i\beta_i(S) = \mathfrak{T}^i\beta_i(S \cap E_j)$.

PART I (lim sup $\leqslant$ formula). For each j we have

$$\mathfrak{T}\mu\left[\bar{\sigma}, \bar{h}_k\right](Q) = \int_{Q-E_j} \mathfrak{T} \circ \kappa\left[\bar{\sigma}, \bar{h}_k\right](x)\, dL + \int_{E_j} \mathfrak{T} \circ \kappa\left[\bar{\sigma}, \bar{h}_k\right](x)\, dL.$$

Let I_1 and I_2 denote the first and second terms respectively. First consider the integral I_2. By the A* property and definitions of $\mathfrak{T}^i$ we get

$$(*) \qquad I_2 \leqslant \sum_{i=1}^{n} \int_{E_j} \kappa\left[\mathfrak{T}^i\alpha_i, h^i_k\right](x)\, dL + \sum_{i=1}^{n} \int_{E_j} \kappa\left[\mathfrak{T}^i\beta_i, h^i_k\right](x)\, dL$$

where $\alpha_i \ll L$, $\beta_i \perp L$ is the Lebesgue decomposition of σ_i. Since $\alpha_i \ll L$, the expression $\kappa[\mathfrak{T}^i\alpha_i, h^i_k](x)$ is the integral average of the summable function $d\mathfrak{T}^i\alpha_i/dL$. Thus as $k \to \infty$ this converges in L^1 to $d\mathfrak{T}^i\alpha_i/dL$. Hence

$$\limsup \int_{E_j} \kappa\left[\mathfrak{T}^i\alpha_i, h^i_k\right] dL = \mathfrak{T}^i\alpha_i(E_j).$$

Applying Theorem 2 from $(*)$ we conclude that

$$(**) \qquad \limsup I_2 \leqslant \sum \mathfrak{T}^i\alpha_i(E_j) + \sum \mathfrak{T}^i\beta_i(Q).$$

On $Q - E_j$, $\kappa[\bar{\sigma}, \bar{h}_k]$ converges uniformly to $d\bar{\sigma}/dL$ and $\mathfrak{T}$ is bounded, so by the bounded convergence theorem we get:

$$(***) \qquad \limsup I_1 \leqslant \int_Q \mathfrak{T} \circ \frac{d\bar{\sigma}}{dL}(x)\, dL.$$

Letting $j \to \infty$ and noting that $\mathfrak{T}^i\alpha_i(E_j) \to 0$ $(**)$ and $(***)$ give Part I.

PART II (lim sup $\geqslant$ formula). Only the case $n = 3$ will be shown, the general case being the same.

Let $\bar{h}_k = (1/k, 1/k, 1/k)$. Given an integer m and $\varepsilon > 0$, there exist integers j_1, j_2 and j_3, k_1, k_2, and k_3 so that $m < j_1 < j_2 < j_3$, $m < k_1 < k_2 < k_3$ and

$$\int_{E_{j_1}-E_{j_2}} \mathfrak{T}^1 \circ \kappa[\sigma_1, 1/k_1]\, dL > \mathfrak{T}^1\beta_1(Q) - \varepsilon,$$

$$\int_{E_{j_2}-E_{j_3}} \mathfrak{T}^2 \circ \kappa[\sigma_2, 1/k_2]\, dL > \mathfrak{T}^2\beta_2(Q) - \varepsilon,$$

$$\int_{E_{j_3}} \mathfrak{T}^3 \circ \kappa[\sigma_3, 1/k_3]\, dL > \mathfrak{T}^3\beta_3(Q) - \varepsilon.$$

To establish this recall that we know that $\int \mathfrak{T}^i \circ \kappa[\sigma_i, 1/k_i]\, dL$ converges weakly to $\mathfrak{T}^i\sigma_i$. Whence on the open sets E_j, we have

$$\liminf \int \mathfrak{T}^i \circ \kappa[\sigma_i, 1/k_i]\, dL \geqslant \mathfrak{T}^i\sigma_i(E_j) \geqslant \mathfrak{T}^i\beta_i(E_j) = \mathfrak{T}^i\beta_i(Q).$$

There is an integer $k_i(E_j)$ so that $k > k_i$ implies that

$$\int_{E_j} \mathfrak{T}^i \circ \kappa[\sigma_i, 1/k]\, dL > \mathfrak{T}^i\beta_i(Q) - \varepsilon.$$

Given m take $j_1 \geqslant m + 1$, choose $k_1 = k_1(E_{j_1}) + m$. Then

$$\int_{E_{j_1}} \mathfrak{T}^1 \circ \kappa[\sigma_1, 1/k_1]\, dL > \mathfrak{T}^1\beta_1(Q) - \varepsilon.$$

But for this fixed k_1, the integrand is summable so there is an integer $j_2 > j_1$ for which $L(E_{j_2})$ is sufficiently small to make

$$\int_{E_{j_1}-E_{j_2}} \mathfrak{T}^1 \circ \kappa[\sigma_1, 1/k_1]\, dL > \mathfrak{T}^1\beta_1(Q) - \varepsilon.$$

Choose $k_2 > k_2(E_{j_2}) + m + k_1$; then

$$\int_{E_{j_2}} \mathfrak{T}^2 \circ \kappa[\sigma_2, 1/k_2]\, dL > \mathfrak{T}^2\beta_2(Q) - \varepsilon.$$

There is an integer $j_3 > j_2$ for which $L(E_{j_2})$ is sufficiently small to ensure that

$$\int_{E_{j_2}-E_{j_3}} \mathfrak{T}^2 \circ \kappa[\sigma_2, 1/k_2]\, dL > \mathfrak{T}^2\beta_2(Q) - \varepsilon.$$

Fix $k_3 > k_3(E_{j_3}) + m + k_2$. Now put $\bar{k}_m^* = (1/k_1, 1/k_2, 1/k_3)$,

$$\mathfrak{T}\mu[\bar{\sigma}, \bar{k}_m^*](Q) = \int_{Q-E_{j_1}} + \int_{E_{j_1}-E_{j_2}} + \int_{E_{j_2}-E_{j_3}} + \int_{E_{j_3}}.$$

But $\mathfrak{T}^i(p^i) \leqslant \mathfrak{T}(p^1, \ldots, p^n)$ for $i = 1, \ldots, n$ so replacing $\mathfrak{T} \circ \kappa[\bar{\sigma}, \bar{h}_m^*]$ by smaller integrands in each integral we obtain:

$$\mathfrak{T}\mu\left[\bar{\sigma}, \bar{k}_m^*\right](Q) \geqslant \int_{Q-E_j} \mathfrak{T} \circ \kappa\left[\bar{\sigma}, \bar{k}_m^*\right] dL + \int_{E_{j_1}-E_{j_2}} \mathfrak{T}^1 \circ \kappa\left[\sigma_1, 1/k_1\right] dL$$
$$+ \int_{E_{j_2}-E_{j_3}} \mathfrak{T}^2 \circ \kappa\left[\sigma_2, 1/k_2\right] dL + \int_{E_{j_3}} \mathfrak{T}^3 \circ \kappa\left[\sigma_3, 1/k_3\right] dL$$
$$\geqslant \sum_{i=1}^{3} \mathfrak{T}^i\beta_i(Q) + \int_{Q-E_j} \mathfrak{T} \circ \kappa\left[\bar{\sigma}, \bar{k}_m^*\right] dL - 3\varepsilon.$$

Again on $Q - E_j$ we have uniform convergence, and we get

$$\liminf_m \mathfrak{T}\mu\left[\bar{\sigma}, \bar{k}_m^*\right](Q) \geqslant \text{formula} - 3\varepsilon - 1/j_1.$$

The existence of such a sequence $\{\bar{k}_m^*\}$ establishes Part II.

4. **The necessary and sufficient conditions.** For $\mathfrak{T} \in \mathfrak{C}^*(R^n)$ we can now establish necessary and sufficient conditions for $\lim \mathfrak{T}\mu[\bar{\sigma}, \bar{h}]$ to exist and give $\mathfrak{T}\bar{\sigma}$. We first give an easy condition on $\mathfrak{T}$; and then, restricting our attention to the Euclidean norm, obtain a deeper condition on the measure $\bar{\sigma}$.

THEOREM 4. *Let $\mathfrak{T} \in \mathfrak{C}^*(R^m)$ and $\bar{\sigma} \in \times_{i=1}^n M$. Then $\mathfrak{T}\mu[\bar{\sigma}, \bar{h}](Q)$ converges to $\mathfrak{T}\bar{\sigma}(Q)$ if and only if $\mathfrak{T}\bar{\beta}(Q) = \sum_{i=1}^n \mathfrak{T}^i\beta_i(Q)$, where $\bar{\beta} = (\beta_1, \ldots, \beta_n)$ is the singular part of $\bar{\sigma}$.*

PROOF. By Theorem 3, Lemma 2, and Lemma 3, we have:

$$\int_Q \mathfrak{T} \circ \frac{d\bar{\sigma}}{dL} dL + \mathfrak{T}\bar{\beta}(Q) = \mathfrak{T}\bar{\sigma}(Q) \leqslant \liminf_{\bar{h}\to\bar{0}} \mathfrak{T}\mu\left[\bar{\sigma}, \bar{h}\right](Q)$$
$$\leqslant \limsup_{\bar{h}\to\bar{0}} \mathfrak{T}\mu\left[\bar{\sigma}, \bar{h}\right](Q)$$
$$= \int_Q \mathfrak{T} \circ \frac{d\sigma}{dL} dL + \sum_{i=1}^{n} \mathfrak{T}^i\beta_i(Q).$$

If in Theorem 4 we replace $\mathfrak{T}$ with the Euclidean n-norm, $\|\cdot\|$, we then have: $\int_Q \|k[\bar{\sigma}, \bar{h}](x)\| \, dL$ converges to $\|\bar{\sigma}\|(Q)$ if and only if $\|\bar{\beta}\|(Q) = \sum_{i=1}^n |\beta_i|(Q)$.

This yields the equivalent condition on $\bar{\beta}$:

THEOREM 5. *Let $\bar{\beta} \in \times_{i=1}^n M$; then for every Borel set E, $\|\bar{\beta}\|(E) = \sum|\beta_i|(E)$ if and only if for $i \neq j$, $\beta_i \perp \beta_j$.*

REMARK. Since $\|\bar{\beta}\|$ and $\sum|\beta_i|$ are both measures, equality on each E and on Q is the same in light of the inequality $\|\bar{\beta}\| \leqslant \sum|\beta_i|$.

PROOF. Suppose for every Borel set E, $\|\bar{\beta}\|(E) = \sum|\beta_i|(E)$.

(*Special Case*) $n = 2$, $\beta_i \geqslant 0$. Since $\bar{\beta} \ll \|\bar{\beta}\|$, there is a Radon-Nikodým derivative $d\bar{\beta}/d\|\bar{\beta}\| = (d\beta_1/d\|\bar{\beta}\|, d\beta_2/d\|\bar{\beta}\|)$ which has modulus 1 everywhere [3]. Hence $d\|\bar{\beta}\|/d\|\bar{\beta}\| = \|(d\beta_1/d\|\bar{\beta}\|, d\beta_2/d\|\bar{\beta}\|)\| = 1$. But

$\|\bar{\beta}\|(E) = \beta_1(E) + \beta_2(E)$ implies that except on a set S_0 of $\|\bar{\beta}\|$ measure zero,

$$\frac{d\beta_1}{d\|\bar{\beta}\|}(x) + \frac{d\beta_2}{d\|\bar{\beta}\|}(x) = \left\| \left(\frac{d\beta_1}{d\|\bar{\beta}\|}(x), \ \frac{d\beta_2}{d\|\bar{\beta}\|}(x) \right) \right\|$$

which can happen if and only if one term or the other is zero.

Let $A_1 = \{x | d\beta_1/d\|\bar{\beta}\| = 0\} - S_0$, $A_2 = \{x | d\beta_2/d\|\bar{\beta}\| = 0\} \cup S_0$. Since $\beta_1(A_1) = \int_{A_1}(d\beta_1/d\|\bar{\beta}\|)\, d\|\beta\| = 0$ and

$$\beta_2(A_2) = \int_{A_2 - S_0} \frac{d\beta_2}{d\|\bar{\beta}\|}\, d\|\beta\| + \int_{S_0} \frac{d\beta_2}{d\|\bar{\beta}\|}\, d\|\beta\| = 0 + 0$$

and $Q = A_1 \cup A_2$ we conclude that $\beta_1 \perp \beta_2$.

For β_i signed we note that singularity of β_i and β_j is equivalent to $|\beta_i|$ and $|\beta_j|$ reducing to the nonnegative case. For $n > 2$, note that $\Sigma|\beta_i|(Q) = \|\bar{\beta}\|(Q) \leq [\|(\beta_1, \beta_2)\| + \Sigma_3|\beta_i|(Q)]$ reducing to $n = 2$ case.

That the singularity of the components implies the equality of the measures is an easy consequence of the Jordan decomposition of Q relative to the β_i's and the triangle inequality.

5. **The area of a nonparametric surface.** Let f be an L_1 function on the open unit cube in R^n of type BVT. Then we associate with f two vector measures $\bar{\sigma}$ and $\bar{\sigma}^*$, where $\bar{\sigma}$ is the n-dimensional derivative measure of f and $\bar{\sigma}^*$ is the $(n + 1)$-dimensional measure formed by adjoining Lebesgue measure L as the first component [i.e., $\bar{\sigma} = (\sigma_1, \ldots, \sigma_n)$ implies that $\bar{\sigma}^* = (L, \sigma_1, \ldots, \sigma_n)$].

It is known [4] that the generalized surface area of f over Q is given by $\|\bar{\sigma}^*\|(Q)$ and that the partial area is given by $\|\bar{\sigma}\|(Q)$. By Lemma 2 we have that $\|\bar{\sigma}\|(Q) = \int_Q \|\bar{\sigma}'(\underline{x})\|\, dL + \|\bar{\beta}\|(Q)$, $\|\bar{\sigma}^*\|(Q) = \int_Q \|\bar{\sigma}^{*\prime}(x)\|\, dL(x) + \|\bar{\beta}^*\|(Q)$ where $\bar{\beta}$ and $\bar{\beta}^*$ are the singular parts of $\bar{\sigma}$ and $\bar{\sigma}^*$, respectively. We are now in a position to prove the analogue to L. C. Young's theorem.

THEOREM 6. *Let f be an L_1 function on the open unit cube Q in R^n of type BVT and let $\bar{\sigma} = (\sigma_1, \ldots, \sigma_n)$ be its distribution derivative measure. For each i, let β_i be the singular part of the Lebesgue decomposition of σ_i. Then* Area(f) = $\lim_{\bar{h} \to \bar{0}} \| \mu[\bar{\sigma}^*, \bar{h}] \|(Q)$ *iff for $i \neq j$ we have $\beta_i \perp \beta_j$.*

PROOF. This follows from Theorem 5 and our discussion above.

REMARK. Note that Partial Area$(f) = \lim_{\bar{h} \to \bar{0}} \| \mu[\bar{\sigma}, \bar{h}] \|(Q)$ under the same conditions.

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