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THE WEAKLY COUPLED YUKAWA$_2$ FIELD THEORY: CLUSTER EXPANSION AND WIGHTMAN AXIOMS(1)

BY

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ABSTRACT. We prove convergence of the Glimm-Jaffe-Spencer cluster expansion for the weakly coupled Yukawa model in two dimensions, thereby verifying the Wightman axioms including a positive mass gap.

I. **Introduction.** In [9], [10] Glimm, Jaffe and Spencer developed a powerful method for studying the weakly coupled $P(\phi)_2$ model. In this paper we shall apply their "vacuum cluster expansion" of [9] to the Yukawa model in two dimensions. This method is largely model independent and should be applicable to any weakly coupled, superrenormalizable model based on functional integration in boson "Q-space". (For recent applications to the ϕ_3^4 model, see Feldman and Osterwalder [6] and Magnen and Sénéor [13].) In the next section we present a brief review of the cluster expansion, but the basic idea is as follows: For a Euclidean field model with interaction in the region $\Lambda \subset \mathbf{R}^d$, one expects the interacting measure to have the form

$$d\nu_\Lambda = \rho_\Lambda d\mu. \tag{I.1}$$

Here $d\mu$ is the free boson measure with mass $m_0 > 0$, i.e. the Gaussian measure on $Q = \mathcal{S}'(\mathbf{R}^d)$ with mean 0 and covariance $C = (-\Delta + m_0^2)^{-1}$; and ρ_Λ is a function of the fields in Λ that embodies the interaction. For example, for $P(\phi)_2$, $\rho_\Lambda = e^{-U_\Lambda}$ where $U_\Lambda = \int_\Lambda : P(\phi) : d^2x$. One wishes to show that $d\nu_\Lambda$ decouples between distant regions exponentially in the distance. If the model is local in the sense that

$$\rho_{\Lambda_1 \cup \Lambda_2} = \rho_{\Lambda_1}\rho_{\Lambda_2} \tag{I.2}$$

for disjoint Λ_1 and Λ_2, then the coupling in (I.1) is due only to the Gaussian measure $d\mu$ ($C = (-\Delta + m_0^2)^{-1}$ is a nonlocal operator). Instead of $d\mu$, suppose we consider the Gaussian measure $d\mu_{\mathfrak{B}}$ with covariance $C_{\mathfrak{B}} = (-\Delta_{\mathfrak{B}} + m_0^2)^{-1}$; here $\Delta_{\mathfrak{B}}$ is the Laplacian ($d = 2$ henceforth) with Dirichlet boundary conditions on $\mathfrak{B} = (\mathbf{Z}^2)^*$, the set of unit line segments (= bounds)

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joining lattice points in $\mathbf{Z}^2$. The measure $d\mu_{\mathfrak{B}}$ completely decouples across lines in $\mathfrak{B}$ and hence by (I.2) so does $d\nu_{\Lambda,0} \equiv \rho_\Lambda\, d\mu_{\mathfrak{B}}$. The cluster expansion consists of relating the measure of interest, $d\nu_\Lambda$, to this exactly decoupling measure, $d\nu_{\Lambda,0}$, by means of a perturbation expansion in the bonds in $\mathfrak{B}$.

We consider now the Yukawa model in two dimensions (Y_2) for interacting bosons and fermions. At present there is not available a fermion integration theory analogous to the boson functional integration on which to base the cluster expansion. However, Seiler [22] recently established a rigorous version of a formula of Matthews and Salam in which the fermions have been "integrated out" with the result that the Euclidean Green's functions or Schwinger functions of the Y_2 model can be represented by a purely boson integration with a measure of the form (I.1). The Matthews-Salam-Seiler formula for the Schwinger functions for n bosons and m fermion-antifermion pairs is as follows:

(I.3) $$S_\Lambda(f, g, h) = Z^{-1}\int \Phi(h) \det S'(f_i, g_j; \phi)\rho_\Lambda\, d\mu$$

where

(a) $f = (f_1, \ldots, f_m)$, $g = (g_1, \ldots, g_m)$ and $h = (h_1, \ldots, h_n)$ are suitable test functions (e.g. h_j in $\mathcal{S}(\mathbf{R}^2)$, and f_j, g_j in $\mathcal{S}(\mathbf{R}^2) \oplus \mathcal{S}(\mathbf{R}^2)$);

(b) $d\mu$ is the free boson measure on $\mathcal{S}'(\mathbf{R}^2)$ with mass $m_b > 0$ and $\Phi(h) = \Pi_{i=1}^n \phi(h_i)$ where ϕ is the free boson field;

(c) $S'(f_i, g_j; \phi) = (f_i, (1 - \lambda K)^{-1} S_0 g_j)_0$ where $(\cdot, \cdot)_0$ denotes the inner product on $\mathcal{H}_0 = L^2(\mathbf{R}^2) \oplus L^2(\mathbf{R}^2)$; $\lambda \in \mathbf{R}$ is the coupling constant; S_0 is the free two point Schwinger function for the fermions with mass $m_f > 0$,

(I.4) $$S_0(x - y) = \frac{1}{(2\pi)^2}\int \frac{\not{p} + m_f}{p^2 + m_f^2} e^{ip\cdot(x-y)}\, d^2p$$

where $\not{p} = p_0\gamma_0 + p_1\gamma_1$ in terms of $p = (p_0, p_1)$ and the 2×2 γ-matrices γ_0 and γ_1 (see [22]). $K = K_\Lambda$ is the operator with kernel

(I.5) $$K_\Lambda(x, y) = S_0(x - y)\phi(y)\chi_\Lambda(y)$$

where χ_Λ is the characteristic function of Λ. The determinant in (I.3) is the $m \times m$ determinant of the matrix with elements $S'(f_i, g_j; \phi)$.

(d)

(I.6) $$\rho_\Lambda = \det_{\text{ren}}(1 - \lambda K_\Lambda) = \det_3(1 - \lambda K_\Lambda)\exp(-\lambda^2 B_\Lambda),$$

where

(I.7) $$\det_3(1 - A) = \exp\left[\operatorname{Tr}\left(\ln(1 - A) + A + A^2/2\right)\right]$$

and

(I.8) $$B_\Lambda = \tfrac{1}{2} :\operatorname{Tr}(K^2 + K^\dagger K):,$$

where $K^\dagger$ is the adjoint of K as an operator on $\mathcal{H}_0$ and : : denotes Wick ordering with respect to $d\mu$;

(e) the partition function

$$Z = \int \rho_\Lambda \, d\mu. \tag{I.9}$$

The above description (a)–(e) of (I.3) is only formal. For instance, it is not clear what $\det_3(1 - \lambda K_\Lambda)$ or

$$R = (1 - \lambda K_\Lambda)^{-1} \tag{I.10}$$

means, inasmuch as K_Λ depends on the field ϕ which takes values in $\mathcal{S}'(\mathbf{R}^2)$. What Seiler did was to introduce momentum cutoffs so that each component of (I.3) made sense, to perform appropriate renormalization cancellations, and then to take the limit as the cutoff was removed. In this way he showed that the integrand in (I.3) was indeed an integrable function of ϕ. Refinements of Seiler's results and further developments based on (I.3) have been obtained independently by McBryan [15], [16], [17] and by Seiler and Simon [23], [24], [25]. We mention two of their results, but first some notation. Let $D = (p^2 + m_f^2)^{1/2} = (-\Delta + m_f^2)^{1/2}$ and let $\mathcal{H}_\alpha$ be the Sobolev space $\mathcal{H}_\alpha = L^2(D^{2\alpha} d^2p)$. Define $\mathcal{H} = \mathcal{H}_{1/2} \oplus \mathcal{H}_{1/2}$ and $\bigwedge^m \mathcal{H}$ to be the m-fold antisymmetric tensor product of $\mathcal{H}$ with itself. It is not hard to show that

$$\det S'(f_i, g_j; \phi) = m! \operatorname{Tr}_{\bigwedge^m \mathcal{H}}(\bigwedge^m R \cdot P) \tag{I.11}$$

where $\bigwedge^m R$ is the m-fold antisymmetric tensor product of the operator R (see (I.10)) with itself and P is the projection operator on $\bigwedge^m \mathcal{H}$ defined by

$$P = (\eta, \cdot)_{\bigwedge^m \mathcal{H}} \psi \tag{I.12}$$

where $\eta = D^{-1} f_1 \wedge \cdots \wedge D^{-1} f_m$ and $\psi = S_0 g_1 \wedge \cdots \wedge S_0 g_m$. (In the Appendix we have collected the various definitions and facts about $\bigwedge^m \mathcal{H}$ that we shall use.) Now define

$$w_{m,\Lambda}(\phi) = \|\bigwedge^m R\| \rho_\Lambda \tag{I.13}$$

where $\|\bigwedge^m R\|$ is the norm of $\bigwedge^m R$ as an operator on $\bigwedge^m \mathcal{H}$. Then it is shown in [16] and [24] that for any $p < \infty$ there are constants c_1 and c_2 independent of m and Λ such that ("Linear Lower Bound")

$$\|w_{m,\Lambda}\|_{L^p(\mu)} \leqslant c_1^m e^{c_2|\Lambda|} \tag{I.14}$$

where $|\Lambda|$ is the volume of Λ. Note that the estimate (I.14) involves a cancellation between poles of $\bigwedge^m R$ as a function of λ and zeros of $\det_3(1 - \lambda K)$. This estimate is an important ingredient in obtaining bounds on the Schwinger functions: McBryan [17] and Seiler and Simon [25] show that if $\Lambda \to \infty$ through a sequence of rectangles then

$$\varlimsup_{\Lambda \to \infty} |S_\Lambda(f, g, h)| < (n!)^{1/2} |f|_0 |g|_0 |h|_1 \tag{I.15}$$

where $|\cdot|_0$ and $|\cdot|_1$ are Schwartz space norms with $|f|_0 = \prod_{i=1}^m |f_i|_0$ etc.

At first sight the formula (I.3) for S_Λ seems inappropriate for a cluster expansion since the "interaction" is nonlocal: the price paid for integrating out the fermions is that the "effective interaction", $-\log \det_{\text{ren}}(1-\lambda K)$, involves products of the boson field at different points in $\mathbf{R}^2$. Put differently, (I.2) fails for ρ_Λ defined by (I.6):

$$\rho_{\Lambda_1 \cup \Lambda_2} \neq \rho_{\Lambda_1}\rho_{\Lambda_2}. \tag{I.16}$$

However, if we reinstate the fermi fields by going to the Osterwalder-Schrader representation [18], then the interaction is certainly local and it makes good sense (at least at the pre-estimate level) to base a cluster expansion on inserting Dirichlet barriers in both the boson and fermion two point Green's functions. One is therefore confident at the outset that the nonlocality (I.16) cannot be a serious problem for a cluster expansion, provided of course that one inserts Dirichlet barriers into the fermion "covariance" (I.4). We do so in the following elementary way:

$$S_0(s; x, y) = C_f(s; x, y)(\not{p} + m_f) = (-i\not{\nabla}_y + m_f)C_f(s; x, y), \tag{I.17}$$

where s is a multiparameter in $[0, 1]^{\mathfrak{B}}$ measuring the strength of Dirichlet barriers (see §II) and $C_f(s; x, y)$ is defined in precisely the same way as the boson covariance (see (II.15)) except that m_b is replaced by m_f. Since $\not{p} = -i\not{\nabla}$ is a local operator $S_0(s)$ decouples across a Dirichlet barrier just as $C_f(s)$ does. With this choice, the Y_2 theory does "decouple at $s = 0$", as we explain in §III where we formally derive the Y_2 cluster expansion.

In §IV we prove convergence of the Y_2 cluster expansion. As is to be expected, some vestige of nonlocality remains in the sense that the cluster expansion produces nonlocal polynomials of the form $\int w(x_1, \ldots, x_j)\phi(x_1)\ldots\phi(x_j)\,dx$; however the nonlocality is exponentially small in the sense that the kernels $w(x_1, \ldots, x_j)$ are exponentially small in the distance between the x_i's. The estimates required for convergence are established in §VI and §VII. For example, in §VII.4 we extend the linear lower bound (I.14) to the s-dependent theory. For $P(\phi)_2$, estimates on L^p norms of $\partial^\gamma C(s; x, y)$ are critical [9], where, for γ a finite set of bonds in $\mathfrak{B}$, $\partial^\gamma = \prod_{b\in\gamma}\partial/\partial s_b$. It is clear from definition (I.17) that we must consider analogous bounds involving spatial derivatives as well. It turns out that all the estimates in this paper can be reduced to estimates on the second mixed partial $(\partial^2/\partial x_i \partial y_j)\partial^\gamma C$. §VI is devoted to such estimates and in a certain sense is the technical heart of the paper. Instead of a Wiener integral representation as in [9], [28], we employ elementary techniques from the theory of partial differential equations.

As in the case of $P(\phi)_2$, the Wightman axioms [30] for weakly coupled Y_2 (except asymptotic completeness) are an immediate consequence of the convergence of the cluster expansion. In §V, we sketch the proofs of the following results:

THEOREM I.1. *If* $|\lambda|/m_b$ *and* $|\lambda|/m_f$ *are sufficiently small then*:

(a) *The infinite volume Schwinger functions* $S(f, g, h) = \lim_{\Lambda \to \mathbf{R}^2} S_\Lambda(f, g, h)$ *exist.*

(b) *The* $S(f, g, h)$ *satisfy all of the Osterwalder-Schrader axioms* [19] *including exponential decoupling. Hence the corresponding relativistic theory satisfies the Wightman axioms including a positive mass gap.*

In our development of the Y_2 cluster expansion we have tried to conform as closely as possible to the version of the expansion in the Erice lectures [9]. However there are some additional features worth noting here:

(a) The integrand, as well as the Gaussian measure, depends on s. Derivatives in s of the integrand ("fermion derivatives") lead to terms with a different structure than derivatives of the measure ("boson derivatives"); in order that we can treat both types of derivatives in a uniform way we find it convenient to introduce dummy boson fields to replace those that have been differentiated.

(b) We find it more convenient to use s-Wick ordering matched to the measure $d\mu_{C(s)}$.

(c) In order to take advantage of the linear lower bound (I.14) it is important after each step in the cluster expansion to collect together certain terms in such a way as to preserve the form (I.11).

(d) The nonlocality referred to above, while not a serious problem, affects the combinatorics of the cluster expansion. In particular, a set of bonds γ may have from 1 to 6 associated localizations, instead of exactly 2 as in the $P(\phi)_2$ case.

(e) As remarked above, the convergence of the cluster expansion depends on L^p bounds on $(\partial^2/\partial x_i \partial y_j)\partial^\gamma C$. Although $\partial^\gamma C(s; x, y)$ has only mild singularities (i.e. logarithmic) as x or y approaches a bond and hence is any L^p, the singularities of $(\partial^2/\partial x_i \partial y_j)\partial^\gamma C$ are more severe and make it a delicate question as to what L^p space $(\partial^2/\partial x_i \partial y_j)\partial^\gamma C$ belongs to (see Theorem VI.1).

(f) To deduce convergence of the cluster expansion based on the *unit* lattice Z^2, we have to take $|\lambda|$ small as well as m_b and m_f large. The reasons for this are (i) to compensate for positive powers of m_b and m_f introduced by the L^p estimates referred to in (e); and (ii) to bound the partition function for a unit square Δ away from zero as in [9]. The conclusions of Theorem I.1, however, are extended by a scaling argument to all values of λ, m_b, m_f for which the dimensionless ratios λ/m_b and λ/m_f are sufficiently small. It is also possible to develop a convergent cluster expansion for any given value of λ (see the Remark following Theorem V.1) provided one chooses m_b and m_f sufficiently large *and* the basic lattice spacing sufficiently small.

The following miscellany of remarks and conventions is mostly notational:

(i) We denote the twice subtracted determinant (see (I.7)) by $\det_3$ as in [15] and [24] instead of by $\det_{(2)}$ as in Seiler's original paper [22].

(ii) As noted in [25], the combination $(1 - \lambda K)$ and not $(1 + \lambda K)$ as in Seiler [22] corresponds to the Yukawa Hamiltonian with coupling constant $+\lambda$.

(iii) We shall often assert that something is true uniformly in the bare parameters; it is to be understood that the bare parameters lie in the region $\{(\lambda, m_b, m_f) \mid |\lambda| \leqslant 1, m_b \geqslant 1, m_f \geqslant 1\}$.

(iv) For notational convenience we shall usually set $m_b = m_f = m_0$.

(v) λ will be suppressed in most of the formulas until the proof of convergence when it will be required.

(vi) Following standard convention, we shall usually omit spin indices and confuse $\mathcal{H}_\alpha \oplus \mathcal{H}_\alpha$ with $\mathcal{H}_\alpha$.

(vii) Although we have stated the Matthews-Salam-Seiler formula for the scalar theory, all of our results hold for the pseudo-scalar theory obtained by inserting an $i\gamma_5$ in the definition of K: $K = S_0 i\gamma_5 \phi \chi_\Lambda$.

As we were completing this manuscript we received a preprint from J. Magnen and R. Sénéor [14] who have obtained the same results as we do but by a rather different approach. We mention also that D. Brydges [2] has developed a cluster expansion in the Hamiltonian framework for a fermion field ψ moving in an external field ϕ via the interaction $\bar{\psi}\psi\phi$.

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II. Review of the Glimm-Jaffe-Spencer cluster expansion. For the reader's convenience we outline in this section the basic ideas, definitions, and steps in the cluster expansion of Glimm, Jaffe, and Spencer [9]. Our review is no substitute for the presentation in [9] but we hope that it makes this paper reasonably self-contained. Although the following discussion applies strictly only to $P(\phi)_2$, we have tried to phrase matters in a model-independent way.

So consider the situation described in (I.1) and let $\langle A \rangle_\Lambda = \int A(\phi)\, d\nu_\Lambda$ denote the expectation of a function A of the Euclidean fields and let $\mathcal{E}_\Lambda(A) = \langle A \rangle_\Lambda / Z(\Lambda)$ denote the normalized expectation where $Z(\Lambda) = \langle 1 \rangle_\Lambda$ is the partition function. The purpose of the cluster expansion is to prove exponential decoupling or clustering of the form

$$\mathcal{E}_\Lambda(AB) - \mathcal{E}_\Lambda(A)\mathcal{E}_\Lambda(B) = \mathcal{O}(e^{-md(A,B)}). \tag{II.1}$$

Here the constant m is independent of Λ, A, B but depends only on the bare parameters (masses, coupling constants) and is positive for suitable values of these parameters; $d(A, B)$ is the distance between the regions where the functions A and B are localized. It is worth pointing out that clustering such as (II.1) with $m > 0$ leads in a straightforward way to the existence of the infinite volume limit $\lim_{\Lambda\to\infty} \mathcal{E}_\Lambda$ [8, Theorem 2.2.2].

It is no loss of generality in (II.1) to suppose that A and B are each localized in unit squares Δ_A and Δ_B in $\mathbf{R}^2 \setminus \mathfrak{B}$ where $\mathfrak{B} = (\mathbf{Z}^2)^*$. We also take Λ to be a union of unit squares. Furthermore we may suppose that each of A and B satisfies (for the various boundary conditions used below)

$$\mathcal{E}_\Lambda(A) = \mathcal{E}_\Lambda(B) = 0 \tag{II.2}$$

for the general case can always be reduced to this one by a suitable "doubling" of the theory (see [9, Theorem 2.1]). For such A, B the goal (II.1) simplifies to

$$\mathcal{E}_\Lambda(AB) = \mathcal{O}(e^{-md(A,B)}). \tag{II.3}$$

If instead of $\langle \cdot \rangle_\Lambda$ and $\mathcal{E}_\Lambda$ we were to consider $\langle \cdot \rangle_{\Lambda,0}$ and $\mathcal{E}_{\Lambda,0}$ corresponding to the measure $d\nu_{\Lambda,0}$ with D B.C. (Dirichlet boundary conditions) on all the bonds in $\mathfrak{B}$ (see §I); then we would have exact decoupling between lattice squares. Thus, for $\Delta_A \neq \Delta_B$,

$$\mathcal{E}_{\Lambda,0}(AB) = \mathcal{E}_{\Lambda,0}(A)\mathcal{E}_{\Lambda,0}(B) = 0$$

by (II.2) for D B.C.

The cluster expansion proof of (II.3) now consists of relating $\mathcal{E}_\Lambda$ to $\mathcal{E}_{\Lambda,0}$ and showing that

$$\mathcal{E}_\Lambda(AB) - \mathcal{E}_{\Lambda,0}(AB) = \mathcal{O}(e^{-md(A,B)}). \tag{II.4}$$

Considering the unnormalized expectations, one goes from $\langle AB \rangle_{\Lambda,0}$ to $\langle AB \rangle_\Lambda$ by "turning on" the couplings across bonds in such a way as to exhibit the smallness of the differences. This is accomplished as follows. With each bond $b \in \mathfrak{B}$ we associate a parameter $s_b = 0$ or 1 where $s_b = 0$ corresponds to D B.C. on b and $s_b = 1$ corresponds to full coupling across b, i.e. no B.C. on b (later s_b will range throughout the interval $[0, 1]$). For any value of the multiparameter $s = (s_b)_{b\in\mathfrak{B}}$ we have B.C. intermediate between $\langle AB \rangle_{\Lambda,0} = \langle AB \rangle_{\Lambda,s=0}$ and $\langle AB \rangle_\Lambda = \langle AB \rangle_{\Lambda,s=1}$ and we denote the corresponding expectation by $\langle AB \rangle_{\Lambda,s}$. For any function $f(s)$, $b \in \mathfrak{B}$ and set $\Gamma \subset \mathfrak{B}$, let

$$\begin{aligned} (\delta^b f)(s) &= f(s)|_{s_b=1} - f(s)|_{s_b=0}, \\ (\delta^\Gamma f)(s) &= \Big(\Big(\prod_{b\in\Gamma} \delta^b\Big) f\Big)(s). \end{aligned} \tag{II.5}$$

Then we have the (formal) identity

$$f(1) - f(0) = \prod_{\Gamma\subset\mathfrak{B}} (\delta^\Gamma f)(0) \tag{II.6}$$

where the sum is over finite nonempty sets Γ. Note that $\delta^\Gamma f(s)$ is independent of the values of s_b, $b \in \Gamma$; hence setting $s = 0$ in the summand in (II.6) gives D B.C. only on $\Gamma^c = \mathfrak{B} \setminus \Gamma$. The expansion (II.6) corresponds to the Mayer expansion from statistical mechanics. Its usefulness, for suitable functions f, depends on the fact that, as a multiple difference of order $|\Gamma|$,

$$\delta^\Gamma f(0) \approx e^{-c|\Gamma|} \tag{II.7}$$

where the constant $c \to \infty$ as $m_0 \to \infty$ (see (II.20) below). If we had gone from $f(0)$ to $f(1)$ via single differences, we would have obtained a sum with "fewer" terms but would have lost the important decay (II.7).

As remarked after (II.6) $\delta^\Gamma f(0)$ has D B.C. on Γ^c: we regard the line segments in Γ^c as barriers and those in Γ as bonds. With this viewpoint, Γ gives a decomposition of $\mathbf{R}^2$ into connected components: $\mathbf{R}^2 \setminus \Gamma^c = \bigcup X_j$ where the X_j are disjoint. Returning to the "proof" of (II.4) we have from (II.6) that

$$\langle AB \rangle_{\Lambda,1} - \langle AB \rangle_{\Lambda,0} = \sum_{\Gamma \subset \mathfrak{B}} \delta^\Gamma \langle AB \rangle_{\Lambda,0} \tag{II.8}$$

where $\delta^\Gamma \langle AB \rangle_{\Lambda,0} = \delta^\Gamma \langle AB \rangle_{\Lambda|s=0}$. If $\Delta_A \neq \Delta_B$, $\langle AB \rangle_{\Lambda,0} = 0$. In fact, by (II.2) the summand $\delta^\Gamma \langle AB \rangle_{\Lambda,0}$ *will vanish unless* Γ *connects* Δ_A *and* Δ_B *within one connected component* X_j. For this to be the case Γ must be sufficiently large, i.e. $|\Gamma| \gtrsim d(A, B)$. Therefore we expect that by (II.7) each term in (II.8) is $\approx \exp[-cd(A, B)]$.

Now the above intuitive remarks cannot lead to a proof that $\langle AB \rangle_\Lambda$ is $O(e^{-md(A,B)})$ uniformly in Λ since $\langle AB \rangle_\Lambda$ is unnormalized. To fashion a proof from the above ideas it is necessary to divide by $Z(\Lambda)$ and to effect a cancellation. This is accomplished by a partial resummation of the expansion (II.8) as follows. In the sum in (II.8) let $\Gamma = \Gamma_1 \cup \Gamma_2$ where Γ_1 connects Δ_A and Δ_B; i.e., if X is the single component of $\mathbf{R}^2 \setminus \Gamma^c$ containing Δ_A and Δ_B, then $\Gamma_1 = \Gamma \cap \operatorname{int} X$. We obtain

$$\sum_{\Gamma \subset \mathfrak{B}} \delta^\Gamma \langle AB \rangle_{\Lambda,0} = \sum_X \sum_{\Gamma_1 \subset \operatorname{int} X} \sum_{\Gamma_2 \subset X^c} \delta^{\delta^{\Gamma_1 \cup \Gamma_2}} \langle AB \rangle_{\Lambda,0} \tag{II.9}$$

where the X and Γ_1 sums satisfy

(II.10) (i) X is a finite, closed, connected union of lattice squares containing Δ_A and Δ_B,
(ii) $\Gamma \subset \mathfrak{B} \cap \operatorname{int} X$ is such that $X \sim \Gamma_1^c$ is connected.

Now $(\Gamma_1 \cup \Gamma_2) \cap \partial X = \varnothing$ so that $\delta^{\Gamma_1 \cup \Gamma_2} \langle AB \rangle_{\Lambda,0}$ decouples across ∂X:

$$\delta^{\Gamma_1 \cup \Gamma_2} \langle AB \rangle_{\Lambda,0} = \delta^{\Gamma_1} \langle AB \rangle_{\Lambda \cap X,0} \delta^{\Gamma_2} \langle 1 \rangle_{\Lambda \cap X^c,0}. \tag{II.11}$$

By (a slight generalization of) (II.6),

$$\text{(II.12)}\quad \sum_{\Gamma_2 \subset X^c} \delta^{\Gamma_2}\langle 1\rangle_{\Lambda\cap X^c,0} = \langle 1\rangle_{\Lambda\cap X^c}|_{s=1\text{ in }X^c, s=0\text{ in }X} \equiv Z_{\partial X}(\Lambda \sim X),$$

the partition function with interaction in $\Lambda \sim X$ and D B.C. on ∂X. Therefore by (II.8)–(II.12)

$$\langle AB\rangle_\Lambda = \sum_X \sum_{\Gamma_1 \subset \text{int} X} \delta^{\Gamma_1}\langle AB\rangle_{\Lambda\cap X,0} Z_{\partial X}(\Lambda \sim X)$$

so that, dividing by $Z(\Lambda)$,

$$\text{(II.13)}\qquad \mathcal{E}_\Lambda(AB) = \sum_{X,\Gamma_1} \frac{Z_{\partial X}(\Lambda \sim X)}{Z(\Lambda)} \delta^{\Gamma_1}\langle AB\rangle_{\Lambda\cap X,0}$$

where the sums over X, Γ_1 satisfy conditions (II.10) above.

The identity (II.13) is the desired cluster expansion for $\mathcal{E}_\Lambda(AB)$. One deduces (II.4) by proving that the summand in (II.13) is bounded by

$$\text{(II.14)}\qquad \mathcal{O}(e^{-c|\Gamma_1|+d|X|}),$$

the $e^{-c|\Gamma_1|}$ arising from the multiple difference δ^{Γ_1} and the $e^{d|X|}$ from the ratio $Z_{\partial X}|Z$ and the $\exp(O(|\Lambda \cap X|))$ behaviour of an expectation $\langle \cdot \rangle_{\Lambda\cap X}$. Here d is uniformly bounded in the bare parameters whereas $c \to \infty$ as $m_0 \to \infty$. But by (ii) of (II.10), Γ_1 must fill out X; more precisely, (ii) implies that $|\Gamma_1| > |X| - 1$ so that (II.14) is bounded by $O(e^{-(c-d)|X|})$. Since there are at most $O(e^{(\ln 19)|X|})$ terms in (II.13) with a fixed value of $|X|$ [9, Proposition 5.1], we obtain

$$|\mathcal{E}_\Lambda(AB)| \leqslant \mathcal{O} \sum_{|X|>d(A,B)} e^{-(c-d-\ln 19)|X|} \leqslant \mathcal{O}(e^{-md(A,B)})$$

where $m \to \infty$ as $m_0 \to \infty$.

This concludes our "intuitive" description of how the cluster expansion leads to exponential decoupling. We next wish to recall some definitions from [9]. If $\Gamma \subset \mathfrak{B}$, let $C_\Gamma = (-\Delta_\Gamma + m_0^2)^{-1}$ where Δ_Γ is the Laplacian with D B.C. on the bonds in Γ. In order to interpolate between $C_\phi = (-\Delta + m_0^2)^{-1}$ and $C_{\mathfrak{B}} = (-\Delta_{\mathfrak{B}} + m_0^2)^{-1}$, we let the parameters s_b range through [0, 1] and we define the covariance

$$\text{(II.15)}\qquad C(s) = \sum_{\Gamma\subset\mathfrak{B}} \left[\prod_{b\in\Gamma} s_b \prod_{b\in\Gamma^c} (1-s_b)\right] C_{\Gamma^c}.$$

In particular, $C(0) = C_{\mathfrak{B}}$ and $C(1) = C_\phi$.

We denote the Gaussian measure corresponding to $C(s)$ by $d\mu_{C(s)}$ or $d\mu_s$, and the interacting measure in volume Λ by $d\nu_{\Lambda,s} = \rho_{\Lambda,s}\, d\mu_s$. For $P(\phi)_2$ the factor $\rho_{\Lambda,s}$ may be chosen to be independent of s, i.e. $\rho_\Lambda = e^{-U_\Lambda} = \exp(-\int_\Lambda :P(\phi): d^2x)$ and for this reason we assume that we are dealing with $P(\phi)_2$ for the rest of this section. (For the definition of $\rho_{\Lambda,s}$ in Y_2, see the next section.) The s-dependent partition function and (unnormalized) Schwinger functions are

$$Z(\Lambda, s) = \int d\nu_{\Lambda,s}, \qquad ZS(\Lambda, S) = \int \phi(f_1)\ldots\phi(f_n)\, d\nu_{\Lambda,s},$$

where each test function f_j is assumed to be smooth and localized in a lattice square.

In order to expand $F = Z$ or ZS as in (II.6) we must verify that $F(s)$ is *regular at* ∞, i.e. if $\Gamma \nearrow \mathfrak{B}$ through finite subsets, then $F(s(\Gamma)) \to F(s)$ for each s, where

$$s(\Gamma) = \Gamma(s) = \begin{cases} s_b, & b \in \Gamma, \\ 0, & b \in \Gamma^c. \end{cases} \tag{II.16}$$

In order to carry out the factorization as in (II.11) we must next verify that f *decouples at* $s = 0$, i.e., if the D barriers in Γ^c decompose $\mathbf{R}^2$ into a disjoint union

$$\mathbf{R}^2 \sim \Gamma^c = \cup X_j \tag{II.17}$$

then

$$F(\Lambda, s(\Gamma)) = \prod_j F(\Lambda \cap X_j, s(\Gamma \cap X_j)). \tag{II.18}$$

Given that F is regular at ∞ and decouples at $s = 0$ we can obtain a convergent expansion as in (II.13). For $S(\Lambda) = S(\Lambda, s = 1)$ we resum over the components in (II.17) that do not meet the set $X_0 = \cup_j \operatorname{supp} f_j$ and the resulting expansion is, just as in (II.13),

$$S(\Lambda) = \sum_{(X,\Gamma)\in \mathbb{S}} \frac{Z_{\partial X}(\Lambda \sim X)}{Z(\Lambda)} \delta^\Gamma ZS(\Lambda \cap X, 0) \tag{II.19}$$

where the set $\mathbb{S} = \mathbb{S}(X_0)$ is defined by

$$\mathbb{S} = \{(X, \Gamma) | X \text{ a finite union of closed lattice squares with } X_0 \subset X;\ \Gamma \subset \mathfrak{B} \cap \operatorname{int} X \text{ is such that each component of } X \sim \Gamma^c \text{ meets } X_0\}.$$

There are three estimates involved in analyzing (II.19) as indicated in the discussion of (II.13) above. The main one is Proposition 5.3 of [9]:

I. There is a constant $c_1 < \infty$ (uniformly in the bare parameters, X, and $|\Gamma|$) and a norm $|f|$ on test functions such that

$$|\delta^\Gamma ZS(X, 0)| \leqslant e^{-c|\Gamma| + c_1|X|}|f| \tag{II.20}$$

where $c \to \infty$ as $m_0 \to \infty$.

The second estimate is on the ratio of partition functions: for sufficiently large m_0 there is a constant $c_2 < \infty$ (uniformly in Λ, X and the bare parameters) such that

$$|Z_{\partial X}(\Lambda \sim X)/Z(\Lambda)| \leqslant e^{c_2|X|}. \tag{II.21}$$

As shown in §6 of [9], (II.21) follows from an appropriate cluster expansion for Z ("Kirkwood-Salsburg equations"), a bound on Z analogous to (II.20), and the following estimate:

II. Let $Z_{\partial\Delta}(\Delta)$ be the partition function for a unit square Δ with D B.C. on $\partial\Delta$. For sufficiently small $|\lambda|$ and sufficiently large m_0,

$$\tfrac{1}{2} \leqslant |Z_{\partial\Delta}(\Delta)|. \tag{II.22}$$

The third estimate is a combinatorial one:

III. The number of terms in (II.19) with a fixed value of $|X|$ is bounded by $e^{c_3|X|}$.

Obviously, this last estimate is model independent. To summarize: *for any model the convergence of the cluster expansion reduces to proving* (*the analogue of*) *estimates* I *and* II.

For the remainder of this section we discuss the method of proof of I. The first step is to apply the fundamental theorem of calculus to δ^Γ; i.e.

$$\delta^\Gamma ZS(0) = \int_{0<\sigma<\Gamma(1)} \partial^\Gamma ZS(\sigma)\, d^{|\Gamma|}\sigma \tag{II.23}$$

where $\partial^\Gamma = \Pi_{b\in\Gamma}\partial/\partial\sigma_b$ and where, according to (II.16), the integration in (II.23) extends over the region $\sigma_b \in [0, 1]$, $b \in \Gamma$. But one has a simple formula for $\partial d\mu_{c(s)}/\partial s_b$, namely [4]:

$$\frac{\partial}{\partial s_b}\int G(\phi)\, d\mu_{C(s)} = \frac{1}{2}\int\left(\frac{\partial C}{\partial s_b}\cdot\Delta_\phi\right) G\, d\mu_{C(s)} \tag{II.24}$$

where

$$\left(\frac{\partial C}{\partial s_b}\cdot\Delta_\phi\right) G = \int\int \frac{\partial C}{\partial s_b}(s; x, y)\, \frac{\delta^2 G}{\delta\phi(x)\delta\phi(y)}\, dx\, dy. \tag{II.25}$$

The formula for higher derivatives follows from the product rule:

$$\partial^\Gamma \int G(\phi)\, d\mu_{C(s)} = \sum_{\pi\in\mathcal{P}(\Gamma)} \int \prod_{\gamma\in\pi} \frac{1}{2}(\partial^\gamma C\cdot\Delta_\phi) G\, d\mu_{C(s)} \tag{II.26}$$

where $\mathcal{P}(\Gamma)$ is the set of all partitions of Γ. From (II.23) and (II.26) we obtain

$$\delta^\Gamma ZS(X, 0) = \sum_{\pi\in\mathcal{P}(\Gamma)} \int d^{|\Gamma|}s \int d\mu_s \prod_{\gamma\in\pi} \frac{1}{2}\,\partial^\gamma C\cdot\Delta_\phi \Phi(X) e^{-U(X)} \tag{II.27}$$

where $\Phi(X) = \Pi_{j:\operatorname{supp} f_j\subset X}\phi(f_j)$.

Now the derivatives Δ_ϕ in (II.27) produce a sum of products of local polynomials $:Q(\phi(X)):$ brought down from the exponential $e^{-U(X)}$. There are thus a large number of terms in (II.27), for two distinct reasons:

(B_1) The number of terms $|\mathcal{P}(\Gamma)|$ in the sum over partitions is large, of the order $|\Gamma|^{|\Gamma|}$.

(B_2) The number of terms resulting from differentiations Δ_ϕ is large, of the order $|\Gamma|^{|\Gamma|}$; moreover, the total degree of the polynomial brought down is of the order $|\Gamma|$, which results in a number singularity which could be of the order $|\Gamma|^{|\Gamma|}$.

The only hope for controlling these singularities lies with the $\partial^\gamma C$'s. In fact, $\partial^\gamma C(x,y)$ is small for two (essentially) distinct reasons:

(G_1) $\partial^\gamma C(x,y)$ becomes exponentially small as γ becomes large.

(G_2) $\partial^\gamma C(x,y)$ becomes exponentially small as x or y gets far from any bond in γ.

Fortunately, (G_1) controls (B_1) and (G_2) controls (B_2). The intuitive explanation of the latter is this: given $x \in \mathbf{R}^2$ there cannot be too many γ's near x and consequently by (G_2) only a small number of the $\delta/\delta\phi(x)$'s are significant at x. Hence, *locally* the number of terms and the degree of the polynomial ("local number singularities") are effectively bounded independently of $|\Gamma|$.

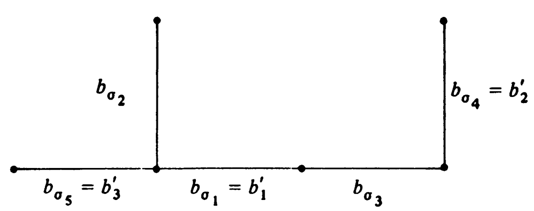
For precise statements of (G_1) and (G_2) we quote Propositions 8.1 and 8.2 from [9], but first some more notation: Let Δ_j be the lattice square whose lower left corner is at $j \in \mathbf{Z}^2$. We let $\chi_j = \chi_{\Delta_j}$ denote the characteristic function of Δ_j, and we define the distance

$$d(\gamma, j) = \max_{b \in \gamma}\left[\operatorname{dist}(b, \Delta_{j_1}) + \operatorname{dist}(b, \Delta_{j_2})\right]. \tag{II.28}$$

If $\gamma = \{b_1, \ldots, b_n\}$ consists of n bonds, and σ is a permutation in S_n, let $l_\sigma(\gamma)$ be the ordered set $(b_{\sigma_1}, \ldots, b_{\sigma_n})$. We define a size $|l_\sigma(\gamma)|$ as follows: Let $b_1' = b_{\sigma_1}$; let b_2' be the first of the b_{σ_j}'s not touching b_1'; let b_3' be the first of the b_{σ_j}'s after b_2' not touching b_2', and so on. Define

$$|l_\sigma| = \sum_i \operatorname{dist}(b_i', b_{i+1}') \tag{II.29}$$

with $|l_\sigma| = 0$ if there is no such b_2'. As an example we have $|l_\sigma| = 3$ for γ and σ as shown:



LEMMA II.1 (PROPOSITION 8.1 OF [9]; APPENDIX OF [28]). *Let* $1 < q < \infty$ *and let* m_0 *be sufficiently large. Then*

(II.30) $\|\chi_{j_1}\partial^{\gamma}C\chi_{j_2}\|_{L^q} < K_4(q)K_8^{|\gamma|}K_6(\gamma)m_0^{-|\gamma|/2q}\exp[-m_0 d(\gamma,j)/3]$

where $K_4(q)$ and K_8 are constants independent of m_0 and

(II.31) $$K_6(\gamma) = \sum_{\sigma\in S_n} e^{-m_0|l_\sigma(\gamma)|/3}.$$

LEMMA II.2 (PROPOSITION 8.2 OF [9]). *For m_0 sufficiently large there is a constant K_7 independent of m_0 such that*

(II.32) $$\sum_{\pi\in\mathcal{P}(\Gamma)}\prod_{\gamma\in\pi} K_6(\gamma) \leq e^{K_7|\Gamma|}.$$

Here, in barest outline, is how the proof goes that G_j controls B_j. The analogous steps for Y_2 require some modifications and will be explained more fully in §IV:

Insertion of localizations. In order to estimate the number of terms in (II.27) as well as the local number singularities $N(\Delta)$, i.e., to control (B_2), and to take advantage of (G_2), we introduce a partition of unity for each local polynomial brought down from the exponential. This amounts to the same thing as writing

(II.33) $$\partial^{\gamma}C = \sum_{j_\gamma}\chi_{j_{\gamma,1}}\partial^{\gamma}C\chi_{j_{\gamma,2}} \equiv \sum_{j_\gamma}\partial^{\gamma}C(j_\gamma)$$

where $j_\gamma = (j_{\gamma,1}, j_{\gamma,2})$ runs through $\mathbf{Z}^4$. The sum over $j = (j_\gamma)$ is pulled to the outside and for a fixed value of j, one estimates the other sums.

Estimates on Q-space integrals. Such estimates lead to a product of factors: the G_j factor $\Pi_{\gamma\in\pi}\|\partial^{\gamma}C(j_\gamma)\|_{L^q}$; the typical boson factor $\Pi_\Delta N(\Delta)!$; and to a linear lower bound factor $e^{O(|X|)}$.

Counting. One introduces $M(\Delta)$ as the number of $\delta/\delta\phi$'s localized in Δ. Both the local number singularity factor $\Pi_\Delta N(\Delta)!$ and the number of terms (B_2) are estimated by $e^{O(|\Gamma|)}(\Pi M(\Delta)!)^{\beta}$ for some β depending only on the degree of the polynomial.

(G_1) *controls* (B_1). This is just an application of Lemma II.2.

(G_2) *controls* (B_2). The (G_2) factor is $\exp[-\Sigma_\gamma m_0 d(\gamma,j_\gamma)/3]$ and the (B_2) singularities have been replaced by

$$\prod_\Delta M(\Delta)!^{\beta} \leq \exp\left[c_\varepsilon\sum_\Delta M(\Delta)^{1+\varepsilon}\right]$$

for any $\varepsilon > 0$. A geometric argument that

(II.34) $$\sum_\Delta M(\Delta)^{3/2} < \text{const}\left(\sum_\gamma d(\gamma,j_\gamma) + |\Gamma|\right)$$

shows that (G_2) controls (B_2) for sufficiently large m_0.

III. Cluster expansion for Y_2: formalism. In this section we formally derive the Y_2 cluster expansion, the proof of convergence being given in §IV. As described in the previous sections, a key point is to define the s-dependent theory so that it decouples at $s = 0$. For Y_2 it is also important to collect terms in such a way as to preserve the fermion structure that enables us to cancel poles with zeroes as in (I.14). For this reason, we shall write down the expansion in a rather explicit way.

Our starting point is the Matthews-Salam-Seiler formula (I.3). We assume without loss of generality that each of the functions in f, g and h is in Schwartz space and has support in a unit square. As we have already stated in (I.17) we shall obtain an s-theory which decouples at $s = 0$ (in the sense of definition (II.18)) by replacing $d\mu$ by $d\mu_{C(s)}$ and, in addition, by replacing the fermion two-point function wherever it occurs by

(III.1) $$S_0(s; x, y) = C(s; x, y)(\not p + m_0) = (-i\not\nabla_y + m_0)C(s; x, y)$$

where $C(s)$ is defined in (II.15). Note that we have set $m_b = m_f = m_0$ and that accordingly we use the same symbol C for both bosons and fermions. The operator inequality $C(s) \leqslant C(1) = C_\phi$ implies that (III.1) defines a bounded operator $S_0(s)$ on $L^2(\mathbf{R}^2)$. Moreover, since

$$\not p = -i(\gamma_0 \partial/\partial x_0 + \gamma_1 \partial/\partial x_1)$$

is a local operator, $S_0(s)$, like $C(s)$, *decomposes across Dirichlet barriers*; i.e., if $s = 0$ on a set of barriers $\mathfrak{B}$ which divides $\mathbf{R}^2$ into a disjoint union, $\mathbf{R}^2 \setminus \mathfrak{B} = \bigcup X_i$, then

(III.2) $$S_0(s) = \bigoplus_i S_0(s) \restriction L^2(X_i).$$

The s-dependent objects corresponding to (I.5), (I.6), (I.10) and (I.12) are defined in the obvious way:

(III.3) $$K(s) = K(\Lambda, s) = S_0(s)\phi\chi_\Lambda,$$

(III.4) $$R(s) = (1 - \lambda K(s))^{-1},$$

(III.5) $$P(s) = (\eta, \cdot\,)_{\Lambda^m \mathfrak{K}} \psi(s),$$

where $\psi(s) = S_0(s) g_1 \wedge \cdots \wedge S_0(s) g_m$, and

(III.6) $$\begin{aligned} \rho(s) = \rho(\Lambda, s) &= \det_{\text{ren}}(1 - \lambda K(\Lambda, s)) \\ &= \det_3(1 - \lambda K(\Lambda, s)) \exp\left[-\lambda^2 B(\Lambda, s)\right] \end{aligned}$$

where

(III.7) $$B = B(s) = B(\Lambda, s) = \tfrac{1}{2}\left[:\mathrm{Tr}\, K(\Lambda, s)^2:_s + \delta m_b^2 :\phi^2(\chi_\Lambda):_s\right],$$

the *Wick subtractions being made with respect to* $C(s)$. Of course, (III.7) is only formal, e.g. the relation $:\mathrm{Tr}\, K^\dagger K: = \delta m_b^2 :\phi^2(\chi_\Lambda):$ involves the infinite constant δm_b^2. What we really mean by (III.7) is the limit as an ultraviolet cutoff is removed,

(III.8) $$B(s) = \lim_{\sigma\to\infty} B_\sigma(s),$$

where we introduce the cutoff σ as follows: Let $h(x) \in C_0^\infty(\{x \mid |x| < \frac{1}{2}\})$ with $h > 0$, $\int h(x)\,dx = 1$ and $h(x) = h(-x)$. Define $h_\sigma(x) = \sigma^2 h(\sigma x)$. Then the ultraviolet cutoff fermion propagator is defined by

(III.9) $$S_{0,\sigma}(s; x, y) = \int S_0(s; x, z) h_\sigma(z - y)\,dz.$$

Set

(III.10a) $$K_\sigma(\Lambda, s) = S_{0,\sigma}(s)\phi\chi_\Lambda,$$
$$\delta m_\sigma^2 = (2\pi^2)^{-1} \int \frac{d^2p}{p^2 + m^2} \tilde{h}_\sigma(p)^2$$

with

$$\tilde{h}_\sigma(p) = \int e^{ipx} h_\sigma(x)\,d^2x$$

and

(III.10b) $$B_\sigma(\Lambda, s) = \tfrac{1}{2}\left[:\mathrm{Tr}\,K_\sigma(\Lambda, s)^2:_s + \delta m_\sigma^2 :\phi^2(\chi_\Lambda):_s\right].$$

In (VII.52) we prove the convergence of (III.8) in $L^2(d\mu_{C(s)})$.

Combining the above definitions we define the partition function

(III.11a) $$Z(s) = \int \rho(s)\,d\mu_{C(s)}$$

and the (unnormalized) Schwinger function, as in (I.3),

(III.11b) $$ZS(s) = \int \Phi T_m(\wedge^m R(s) \cdot P(s))\rho(s)\,d\mu_{C(s)}$$

where we have used (I.11) and introduced the notation

$$T_m(\cdot) = m!\,\mathrm{Tr}_{\wedge^m \mathcal{H}}(\cdot).$$

In §IV we shall show using estimates from §VII that the objects in (III.11) are well defined. However a word is in order here about the meaning of an expression like (III.4) for $R(s)$. As in [22] we let $\mathcal{C}_p(\mathcal{H})$ denote the class of compact operators A on $\mathcal{H}$ with $\|A\|_p^p = \mathrm{Tr}(A^*A)^{p/2} < \infty$; and we let $\mathcal{C}_{p,q;s}$ denote the class of $\mathcal{C}_p(\mathcal{H})$-valued functions $A(\phi)$ with

(III.12) $$\|A\|_{p,q;s} \equiv \left[\int d\mu_s \|A(\phi)\|_p^q\right]^{1/q} < \infty.$$

Since $\|K(s)\|_{4,4;s}$ is finite (see (VII.27)), it follows that $K(s)$ is a well-defined compact operator in the class $\mathcal{C}_4(\mathcal{H})$ for almost all ϕ (w.r.t. $d\mu_{C(s)}$). Thus for each fixed ϕ (except in a set of measure zero) (III.4) is well defined as an operator on $\mathcal{H}$ except for a countable set $\{\lambda|\lambda^{-1} \in \sigma(K(s))\}$. Of course this set of λ will depend on ϕ and so it is quite possible that there is no λ for which $(1-\lambda K)^{-1}$ makes sense a.e. in ϕ. (We mention that Seiler [22] has proved that for every λ, $(1-\lambda K)^{-1}$ is well defined for ϕ in a set of positive measure.) However we note that in (III.11b) the factor $\bigwedge^m R$ can be combined with the $\det_3(1-\lambda K)$ to obtain a cancellation between the poles of $\bigwedge^m R$ and the zeros of $\det_3(1-\lambda K)$. More precisely, by Proposition 5 of the Appendix of [23], if $A \in \mathcal{C}_p$, then

(III.13) $$A \to B = \bigwedge^m (1-A)^{-1} \det_p(1-A)$$

is a continuous map from $\mathcal{C}_p$ to $\mathcal{L}(\bigwedge^m \mathcal{H})$, the bounded operators on $\bigwedge^m \mathcal{H}$. Accordingly we can define $\bigwedge^m R \det_2(1-\lambda K)$ *by continuity* in λ even at the values $\lambda \in \sigma(K)^{-1}$. It will turn out in the course of the cluster expansion that the "operator" $R(s)$ *will always occur in a single factor of the form* $\bigwedge^r R(s)$, and consequently we shall freely use the above remarks to manipulate with $K(s)$ and $R(s) = (1-\lambda K(s))^{-1}$ *as though they were well-defined bounded operators on* $\mathcal{H}$ *for all* ϕ *and all* λ. For notational convenience we shall set $\lambda = 1$ until the end of §IV at which point λ will be resurrected.

With the above definitions and interpretation we now assert that $ZS(s)$ and $Z(s)$ decouple at $s=0$ (see (II.18)). For suppose that $s=0$ on $\mathfrak{B}_0 \subset (\mathbf{Z}^2)^*$ with $\mathbf{R}^2 \sim \mathfrak{B}_0 = \bigcup X_i$ a disjoint union. Let $K = K(\Lambda, s)$ and $K_i = K(\Lambda_i, s)$ where $\Lambda_i = \Lambda \cup X_i$ so that $K = \sum K_i$. Let $\mathcal{H}(X_i)$ be the subspace of $\mathcal{H}$ consisting of functions with support in X_i. Then clearly $K_i = 0$ on $\mathcal{H}(X_j)$ if $i \neq j$, Range $K_i \subset \mathcal{H}(X_i)$ by (III.2), and so

(III.14a) $$K_i K_j = 0 \quad \text{if } i \neq j.$$

Define the operator $R_i = (1-K_i)^{-1}$. As explained above we may assume that $1 \notin \sigma(K)$ and $1 \notin \sigma(K_i)$ for all i so that $R = (1-K)^{-1}$ and R_i are well-defined operators. It then follows from the above properties of K_i that

(III.14b) $$R = R_i \quad \text{on } \mathcal{H}(X_i).$$

In addition we see from (III.14a) that

$$\begin{aligned} \det_n(1-K) &= \det\left[(1-K)\exp\left(\sum_{k=1}^{n-1} \frac{K^k}{k}\right)\right] \\ &= \det\left\{\prod_i \left[(1-K_i)\exp\left(\sum_{k=1}^{n-1} \frac{K_i^k}{k}\right)\right]\right\} \\ &= \prod_i \det_n(1-K_i). \end{aligned}$$

Although it follows formally from (III.8) and (III.10) that B decouples at

$s = 0$,

(III.15)
$$B(\Lambda, s) = \sum_i B(\Lambda_i, s),$$

the proof involves an ultraviolet cutoff argument which we postpone until §VII.5. Using (III.14) and (III.15) we obtain from definition (III.6):

(III.16)
$$\rho(\Lambda, s) = \prod_k \rho(\Lambda_k, s)$$

in contrast to the nonlocality (I.16).

The trace term in (III.11b),

$$T_m\big((\wedge^m R(s))P(s)\big) = \det_{ij}\big((f_i, R(s)S_0(s)g_j)_{L^2}\big),$$

also factors. Since each test function f_i, g_j, h_i is localized in a unit square it must also be localized in one of the Λ_k's.By (III.2), if $\operatorname{supp} g_j \subset \Lambda_k$ then $\operatorname{supp} S_0(s)\, g_j \subset \Lambda_k$ and so, by (III.14b),

$$R(s)S_0(s)\, g_j = R_k(s)S_0(s)\, g_j$$

also has support in Λ_k. Consequently,

$$A_{ij} \equiv (f_i, R(s)S_0(s)\, g_j)_{L^2} = \begin{cases} (f_i, R_k S_0(s)\, g_j) & \text{if } \operatorname{supp} f_i \text{ and } \operatorname{supp} g_j \subset \Lambda_k, \\ 0 & \text{otherwise.} \end{cases}$$

We see that, after a suitable relabeling of the rows, the matrix A_{ij} decomposes into blocks $A_{ij}^{(k)}$ associated with the various regions Λ_k. Obviously $\det A_{ij} = 0$ unless these blocks are square, i.e. unless the same number (m_k) of f_i's as g_j's have support in each Λ_k. We also have the factorization

(III.17)
$$\begin{aligned} T_m\big((\wedge^m R(s))P(s)\big) &= \det A_{ij} = \pm \prod_k \det A_{ij}^{(k)} \\ &= \pm \prod_k T_{m_k}\big((\wedge^{m_k} R_k)P_k\big) \end{aligned}$$

where

$$P_k = \big(D^{-1}f_{i_1} \wedge \cdots \wedge D^{-1}f_{i_{m_k}}, \cdot\big)_{\wedge^{m_k}\mathcal{K}} S_0(s)\, g_{j_1} \wedge \cdots \wedge S_0(s)\, g_{j_{m_k}}$$

and $f_{i_1}, \ldots, f_{i_{m_k}}, g_{j_1}, \ldots, g_{j_{m_k}}$ are the f's and g's with support in Λ_k.

If we insert (III.16) and (III.17) into (III.11) we obtain the desired decoupling since each $\rho(\Lambda_k, s)$ and R_k is a function of the fields in Λ_k:

LEMMA III.1. *$ZS(s)$ and $Z(s)$ decouple at $s = 0$.*

For bounded Λ it follows easily from the results of §VII that ZS and Z are "regular at ∞" (defined before (II.16)). Thus the cluster expansions for ZS and Z are generated exactly as for $P(\phi)_2$ and we obtain for S:

$$\text{(III.18)}\quad S(\Lambda) = \sum_{(X,\Gamma)\in\mathbb{S}} (\pm)\frac{Z_{\partial X}(\Lambda \sim X)}{Z(\Lambda)} \int d^{|\Gamma|}s\,\partial^{\Gamma} ZS(\Lambda\cap X, s(\Gamma)).$$

Here the set $\mathbb{S} = \mathbb{S}(X_0)$ is defined following (II.19) in terms of the union X_0 of the supports of the f_i, g_i, h_i; $ZS(\Lambda \cap X, s(\Gamma))$ is the unnormalized Schwinger function with interaction in $\Lambda \cap X$ and with the f_i, g_i, h_i localized in $\Lambda \cap X$; and the $\pm$ sign arises from the permutations involved in the factorization (III.17) (it is unimportant for the purposes of this paper).

We must now "evaluate" the s-derivatives ∂^{Γ}. Our calculations involve the differentiation formula (II.24) which becomes, in the case of an s-dependent integrand,

$$\text{(III.19)}\qquad \frac{\partial}{\partial s}\int G(s,\phi)\,d\mu_s = \int\left(\frac{\partial G}{\partial s} + \frac{1}{2}\,\frac{\partial C}{\partial s}\cdot\Delta_{\phi} G\right) d\mu_s.$$

Taking higher derivatives and using the fact that $\partial/\partial s$ and $\delta/\delta\phi$ commute, we obtain

$$\text{(III.20)}\quad \partial^{\Gamma}\int G\,d\mu_s = \sum_{\Gamma=\Gamma_b\cup\Gamma_f}\ \sum_{\pi\in\mathcal{P}(\Gamma_b)} \int\left(\prod_{\gamma\in\pi}\frac{1}{2}\,\partial^{\gamma}C\cdot\Delta_{\phi}\right)\partial^{\Gamma_f} G\,d\mu_s$$

where Γ_b and Γ_f are disjoint subsets of Γ which we regard as index sets for boson and fermion derivatives respectively. In §VII.6, we indicate the rigorous justification of the formal relations (III.19) and (III.20) for the class of integrands G considered in this paper.

The boson and fermion derivatives are independent except for the following convention. The integrand G will always contain the factor e^{-B} (see (III.6)) as well as factors brought down by differentiating B. Let $b(s;\,x,y)$ denote the kernel of B, i.e.,

$$\text{(III.21)}\qquad B(s) = \tfrac{1}{2}\int b(s;\,x,y){:}\phi(x)\phi(y){:}_s\,dx\,dy.$$

Since we use matched Wick ordering ($:\ :_s$ denotes $C(s)$ subtractions) we obtain a cancellation between boson and fermion derivatives; e.g.,

$$\left(\frac{\partial}{\partial s} + \frac{1}{2}\,\frac{\partial C}{\partial s}\cdot\Delta_{\phi}\right)B = \tfrac{1}{2}\int\frac{\partial b}{\partial s}(s;\,x,y){:}\phi(x)\phi(y){:}_s\,dx\,dy.$$

The same cancellation occurs for higher s-derivatives. We accordingly adopt the following convention. We drop the cancelled vacuum expectation terms and collect the remaining terms in the form

$$\text{(III.22)}\qquad \frac{1}{2}\int\partial^{\gamma} b(s;\,x,y){:}\phi(x)\phi(y){:}_s\,dx\,dy.$$

Thus no term involving boson derivatives of B occurs in which the same covariance kernel $\partial^{\gamma'}C(x,y)$ is integrated against each variable of b or $\partial^{\gamma}b$.

From definition (III.11b), and formula (III.20),

$$\partial^\Gamma ZS(s) = \sum_{\Gamma=\Gamma_b\cup\Gamma_f}\ \sum_{\pi_b\in\mathcal{P}(\Gamma_b)} \int d\mu_s \prod_{\gamma\in\pi_b}\left(\tfrac{1}{2}\partial^\gamma C\cdot\Delta_\phi\right) \tag{III.23}$$
$$\cdot\, \Phi\partial^{\Gamma_f} T_m(\wedge^m R(s)P(s))\rho(s).$$

We proceed to evaluate a typical fermion derivative (formally). Since $R = (1-K)^{-1}$, we have

$$(\partial/\partial s)R = R(\partial K/\partial s)R. \tag{III.24}$$

Consequently,

$$\frac{\partial}{\partial s}\wedge^m R = \sum_{j=1}^{m}(\wedge^{j-1}R)\wedge\frac{\partial R}{\partial s}\wedge(\wedge^{m-j}R) \tag{III.25}$$
$$= d\wedge^m\left(R\frac{\partial K}{\partial s}\right)\wedge^m R$$

by (III.24), (A.5), and the definition (A.4) of the derivation $d\wedge^m(\cdot)$. Since (formally)

$$\det{}_3(1-K) = \exp\left[\operatorname{Tr}\left(\log(1-K)+K+K^2/2\right)\right]$$

we have

$$\frac{\partial}{\partial s}\det{}_3(1-K) = \operatorname{Tr}\left(-(1-K)^{-1}\frac{\partial K}{\partial s}+\frac{\partial K}{\partial s}+K\frac{\partial K}{\partial s}\right)\det{}_3(1-K)$$
$$= -\operatorname{Tr}\left(RK^2\frac{\partial K}{\partial s}\right)\det{}_3(1-K).$$

Therefore

$$\frac{\partial\rho}{\partial s} = -\left[\operatorname{Tr}\left(RK^2\frac{\partial K}{\partial s}\right)+\frac{\partial B}{\partial s}\right]. \tag{III.26}$$

By (III.25) and (III.26)

$$\frac{\partial}{\partial s}T_m(\wedge^m R\cdot P)\rho = \left\{T_m\left(d\wedge^m\left(R\frac{\partial K}{\partial s}\right)\wedge^m R\cdot P\right)\right. \tag{III.27}$$
$$+\,T_m\left(\wedge^m R\frac{\partial P}{\partial s}\right)-T_m(\wedge^m R\cdot P)$$
$$\left.\cdot\left[\operatorname{Tr}\left(RK^2\frac{\partial K}{\partial s}\right)+\frac{\partial B}{\partial s}\right]\right\}\rho.$$

Recall that in the discussion following (III.13) we asserted that R always occurs in a single factor of the form $\wedge^r R$, this fact being critical for our control of the terms in the cluster expansion. Plainly this is not the case for two terms on the right side of (III.27) and so we must effect a cancellation.

To this end we write $R = RK^2 + 1 + K$ so that

$$R\frac{\partial K}{\partial s} = RK^2\frac{\partial K}{\partial s} + (1+K)\frac{\partial K}{\partial s} = RA_s + E_s$$

where

$$A_s = K^2\frac{\partial K}{\partial s} \quad \text{and} \quad E_s = (1+K)\frac{\partial K}{\partial s}. \tag{III.28}$$

The two offending terms in (III.27) become

$$\begin{aligned} &T_m\left(d\wedge^m\left(R\frac{\partial K}{\partial s}\right)\wedge^m R\cdot P\right) - T_m(\wedge^m R\cdot P)T_1\left(RK^2\frac{\partial K}{\partial s}\right)\\ &\quad = T_m(\wedge^m R\cdot P\, d\wedge^m(RA_s)) - T_m(\wedge^m R\cdot P)T_1(RA_s)\\ &\qquad + T_m(\wedge^m R\cdot P\, d\wedge^m(E_s))\\ &\quad = -T_{m+1}(\wedge^m R\cdot P\wedge RA_s) + T_m(\wedge^m R\cdot P\, d\wedge^m(E_s)) \qquad \text{(III.29)}\\ &\quad = -T_{m+1}(\wedge^{m+1} R\cdot P\wedge A_s) + T_m(\wedge^m R\cdot P\, d\wedge^m(E_s)) \qquad \text{(III.30)} \end{aligned}$$

where in (III.29) we have used the trace formula (A.12) and in (III.30) the elementary identity (A.5). Combining (III.27) and (III.30) we obtain the basic differentiation formula:

$$\begin{aligned} \frac{\partial}{\partial s}T_m(\wedge^m R\cdot P)\rho = \Big\{ &-T_{m+1}(\wedge^{m+1}R\cdot P\wedge A_s)\\ &+ T_m(\wedge^m R\cdot P d\wedge^m(E_s))\\ &+ T_m\left(\wedge^m R\frac{\partial P}{\partial s}\right) - T_m(\wedge^m R\cdot P)\frac{\partial B}{\partial s}\Big\}\rho. \end{aligned} \tag{III.31}$$

For G an operator on $\wedge^m \mathcal{H}$, we introduce the functional

$$\tau_m(G) = T_m(\wedge^m R\cdot G)\rho \tag{III.32}$$

Then (III.31) can be written

$$\begin{aligned} \frac{\partial}{\partial s}\tau_m(P) = \tau_m\left(\frac{\partial P}{\partial s}\right) &- \tau_{m+1}(P\wedge A_s)\\ &+ \tau_m(Pd\wedge^m E_s) - \tau_m(P)\frac{\partial B}{\partial s}. \end{aligned} \tag{III.33a}$$

Each boson derivative gives an analogous formula. That is, if $G(\phi)$ is a function of the fields taking values as an operator on $\wedge^m\mathcal{H}$, then

$$\begin{aligned} \frac{\delta}{\delta\phi(y)}\tau_m(G) = \tau_m\left(\frac{\delta G}{\delta\phi(y)}\right) &- \tau_{m+1}(G\wedge A_y)\\ &+ \tau_m(Gd\wedge^m E_y) - \tau_m(G)\frac{\delta B}{\delta\phi(y)} \end{aligned} \tag{III.33b}$$

where

$$A_y = K^2 \frac{\delta K}{\delta\phi(y)}, \quad E_y = (1+K)\frac{\delta K}{\delta\phi(y)} \quad \text{and} \quad \frac{\delta B}{\delta\phi(y)} = \int b(s;y,y')\phi(y').$$

It is useful to rewrite an operator like $Pd\bigwedge^m E_s$ in (III.33a) as follows. From definition (III.5),

$$P(s) = P_1(s) \wedge P_2(s) \wedge \cdots \wedge P_m(s) \tag{III.34}$$

where $P_j(s) = (D^{-1}f_j, \cdot)_{\mathcal{H}} S_0(s) g_j$ is a projection in $\mathcal{H}$. Therefore by (A.6) we may "spread the $d\bigwedge^m E$ into the P_j's"

$$\begin{aligned} Pd\bigwedge{}^m E = {} & P_1 E \wedge P_2 \wedge \cdots \wedge P_m + P_1 \wedge P_2 E \wedge \cdots \wedge P_m \\ & + \cdots + P_1 \wedge \cdots \wedge P_m E. \end{aligned} \tag{III.35}$$

Before plunging in to evaluate (III.23) by iteration of the derivative formulas (III.33), we pause to comment on the general structure of $\partial^\Gamma ZS$ and the possibility of making estimates. Clearly, fermion derivatives lead to a sum of terms of the form $\tau_r(G)\mathbf{B}$ where $\mathbf{B}$ is a product of quadratic polynomials in ϕ (typically Tr $:K\partial^\gamma K:$) and where G is an r-fold tensor product $G = G_1 \wedge \cdots \wedge G_r$. Here each factor G_i is a product of operators on $\mathcal{H}$ containing at least 3 factors of K (see (III.28)) or a rank one operator P_i (see (III.35)). Since a K factor is in $\mathcal{C}_{3,\frac{1}{3}}$ each G_i will consequently be in $\mathcal{C}_{1,1}$. We may thus estimate $\tau_r(G)$ by (A.14):

$$\begin{aligned} |\tau_r(G)| &\leq r!\,\|\textstyle\bigwedge^r R\|\cdot\|G_1 \wedge \cdots \wedge G_r\|_1 \cdot |\rho| \\ &\leq [\,\|\textstyle\bigwedge^r R\|\;|\rho|\,] \displaystyle\prod_i \|G_i\|_1. \end{aligned}$$

The first factor is in any $L^p(d\mu)$, $p < \infty$ (see (I.14)); so is the second factor and the product $\mathbf{B}$. In fact, the G_i's and $\mathbf{B}$ together give the desired $e^{-c|\Gamma_f|}$ decay because they contain the factors $\|\partial^\gamma K\|_3$.

Now boson derivatives evidently upset this tensor product structure since different factors G_j are linked by $\partial^\gamma C$'s. However if we replace the fields ϕ that have been differentiated by *dummy fields* ϕ_γ then we can restore the tensor product structure so that boson and fermion derivatives lead to terms with the same structure. To this end let r_γ be the replacement operator which acts on polynomials in the field by

$$r_\gamma \phi(x_1)\ldots\phi(x_n) = \sum_{i=1}^{n} \phi(x_1)\ldots\phi(x_{i-1})\phi_\gamma(x_i)\phi(x_{i+1})\ldots\phi(x_n) \tag{III.36}$$

with the convention that r_γ acts only on the original fields ϕ and not on dummy fields. Then we have the formal identity

$$\int \partial^\gamma C(x,y)\frac{\delta^2}{\delta\phi(x)\delta\phi(y)}\phi(x_1)\ldots\phi(x_n) = \mathcal{E}_\gamma r_\gamma^2 \phi(x_1)\ldots\phi(x_n) \tag{III.37}$$

where the "covariance operation" $\mathcal{C}_\gamma$ acts on the pair of ϕ_γ fields by

$$\mathcal{C}_\gamma \phi_\gamma(x)\phi_\gamma(y) = \partial^\gamma C(x,y). \tag{III.38}$$

We shall denote dummy field replacements by subscripts; e.g.,

$$\partial^{\gamma'} K_\gamma = r_\gamma \partial^{\gamma'} K = \partial^{\gamma'} S_0 \phi_\gamma \chi_\Lambda. \tag{III.39}$$

For each γ in the partition π_b of Γ_b there will be a pair of dummy fields ϕ_γ and an $\mathcal{C}_\gamma$. The expression (III.23) will thus involve

$$\mathcal{C}(\pi_b) = \prod_{\gamma \in \pi_b} \mathcal{C}_\gamma. \tag{III.40}$$

In the next section we shall replace $\mathcal{C}(\pi_b)$ by integration over suitable Gaussian measures.

It is now straightforward to iterate the differentiation formulas (III.33) to obtain an explicit formula for (III.23). However this requires a veritable arsenal of notation which we must first introduce. The fermion derivatives, like the boson derivatives, are grouped into sets $\gamma \subset \mathfrak{B}$ belonging to a partition of Γ_f. We adopt the following

Order convention. The bonds b in $\mathfrak{B}$ are ordered in some fixed order with any subset $\Gamma \subset \mathfrak{B}$ inheriting the same order. We first apply the derivatives in Γ_f in this order and then the derivatives in Γ_b in this order. If γ and γ' are two (disjoint) sets of fermion derivatives we say that γ precedes γ' (and we write $\gamma \prec \gamma'$) if the first bond in γ precedes the first bond in γ'. Similarly for boson γ's. Any fermion γ precedes any boson γ.

As in (III.33), a derivative ∂^γ or r_γ may introduce a factor

$$A^\gamma \equiv K^2 \partial^\gamma K \quad \text{or} \quad A_\gamma \equiv K^2 K_\gamma. \tag{III.41}$$

We call such factors, as well as the expressions obtained by further derivatives of (III.41), *type A factors*. For example, $\partial^{\gamma_2} K K \partial^{\gamma_1} K$, $K_{\gamma_3} K_{\gamma_2} K_{\gamma_1}$, or $K K_{\gamma_2} \partial^{\gamma_1} K_{\gamma_3}$ are type A factors. In each of these examples, we say that γ_1 labels the "first" derivative, i.e. the derivative that produces the factor. By our convention of applying derivatives in order, we have $\gamma_1 \prec \gamma_j$ for $j = 2$ or 3; note also that a factor like $K\partial^{\gamma_2} K K_{\gamma_1}$ does not occur since the boson derivatives follow the fermion derivatives. We call factors of the form

$$E^\gamma \equiv (1+K)\partial^\gamma K \quad \text{or} \quad E_\gamma \equiv (1+K) K_\gamma, \tag{III.42}$$

and further derivatives of (III.42), *type E factors*. We call

$$\begin{aligned} B^{\gamma_1,\gamma_2} &\equiv :\mathrm{Tr}\, \partial^{\gamma_2} K \partial^{\gamma_1} K: \equiv \int \partial^{\gamma_1,\gamma_2} b(s;x,y) :\phi(x)\phi(y): \, dx\, dy, \\ B^\gamma &\equiv B^{\gamma,\varnothing} = :\mathrm{Tr}\, K \partial^\gamma K:, \qquad B_\gamma \equiv \int b(s;x,y)\phi(x)\phi_\gamma(y)\, dx\, dy, \end{aligned} \tag{III.43}$$

as well as further r_γ derivatives, *type B factors*. Note that B^{γ_1,γ_2} and B^γ are well defined without the need for a mass term (see Corollary VII.13). We refer to the factors $\phi(h_j)$ in Φ, and their derivatives, as *type Φ factors*, and the factors P_j in (III.34), or their derivatives, as *type P factors*.

We return to (III.23) in which we shall expand the fermion derivatives ∂^{Γ_f} and write the result as a sum over partitions π of Γ. We wish to classify elements γ of π according to what type of derivative they occur in. Accordingly we write each $\pi \in \mathcal{P}(\Gamma)$ as a disjoint union

$$\pi = \pi_b \cup \pi_f \tag{III.44a}$$

where the elements of π_b (resp. π_f) label boson (resp. fermion) derivatives. Then we write

$$\pi_f = \pi_{f,A} \cup \pi_{f,E} \cup \pi_{f,0} \cup \pi_{f,B} \cup \pi_{f,S} \cup \pi_{f,P} \equiv \cup_X \pi_{f,X} \tag{III.44b}$$

where the elements of

$\pi_{f,A}$ label first derivatives in an A factor;
$\pi_{f,E}$ label first derivatives in an E factor;
$\pi_{f,0}$ label all other derivatives in an A or E factor;
$\pi_{f,B}$ label first derivatives in a B factor;
$\pi_{f,S}$ label second derivatives in a B factor;
$\pi_{f,P}$ label derivatives on P.

We perform a similar decomposition on π_b except that each $\gamma \in \pi_b$ has two associated derivatives or ϕ_γ replacements (see (III.36)). Accordingly we let π_b^2 denote the set π_b *doubled* (i.e. for each $\gamma \in \pi_b$ there are two distinguished occurrences of γ in π_b^2) and we write

$$\pi_b^2 = \pi_{b,A} \cup \pi_{b,E} \cup \pi_{b,0} \cup \pi_{b,B} \cup \pi_{b,S} \cup \pi_{b,\Phi} \equiv \cup_X \pi_{b,X} \tag{III.44c}$$

where each of the $\pi_{b,X}$ has a meaning analogous to the corresponding $\pi_{f,X}$, with $\pi_{b,\Phi}$ labelling derivatives of Φ, in analogy to $\pi_{f,P}$. We denote the sum over all possible decompositions of π as in (III.44) by $\Sigma_{\text{decomp}\,\pi}$. Iteration of (III.33) then gives the following explicit sum for (III.23):

$$\partial^\Gamma ZS(s) = \sum_{\pi \in \mathcal{P}(\Gamma)} \sum_{\text{decomp}\,\pi} (-1)^t \int d\mu_s \mathcal{E}(\pi_b) \Phi(\pi_{b,\Phi}) G(\pi_{p,X}) \tag{III.45a}$$

with

$$\begin{aligned} G(\pi_{p,X}) = \partial^{\pi_{f,0S}} r_{\pi_{b,0S}} \tau_r \big(P(\pi_{f,P}) \wedge A(\pi_{f,A}, \pi_{b,A}) \cdot d{\textstyle\bigwedge}^r E(\pi_{f,E}, \pi_{b,E}) \big) \\ \cdot B(\pi_{f,B}, \pi_{b,B}) \end{aligned} \tag{III.45b}$$

where

$$\begin{aligned} t &= |\pi_{f,A}| + |\pi_{b,A}| + |\pi_{f,B}| + |\pi_{b,B}|, \\ r &= m + |\pi_{f,A}| + |\pi_{b,A}|, \end{aligned} \tag{III.46}$$

$$\Phi(\pi_{b,\Phi}) = \Big(\prod_{\gamma \in \pi_{b,\Phi}} r_\gamma \Big) \Phi, \tag{III.47}$$

$$P(\pi_{f,P}) = \Big(\prod_{\gamma\in\pi_{f,P}} \partial^{\gamma}\Big) P_1 \wedge \cdots \wedge P_m, \tag{III.48}$$

where each ∂^{γ} must act on a different factor P_j,

$$A(\pi_{f,A}, \pi_{b,A}) = \Big(\bigwedge_{\gamma\in\pi_{f,A}} A^{\gamma}\Big) \wedge \Big(\bigwedge_{\gamma\in\pi_{b,A}} A_{\gamma}\Big), \tag{III.49}$$

$$d{\bigwedge}^r E(\pi_{f,E}, \pi_{b,E}) = \prod_{\gamma\in\pi_{f,E}} d'{\bigwedge}^r E^{\gamma} \prod_{\gamma\in\pi_{b,E}} d'{\bigwedge}^r E, \tag{III.50}$$

$$B(\pi_{f,B}, \pi_{b,B}) = \prod_{\gamma\in\pi_{f,B}} B^{\gamma} \prod_{\gamma\in\pi_{b,B}} B_{\gamma}, \tag{III.51}$$

$$\partial^{\pi_{f,0S}} = \partial^{\pi_{f,0}}\partial^{\pi_{f,S}} = \prod_{\gamma\in\pi_{f,0}} \partial^{\gamma} \prod_{\gamma\in\pi_{f,S}} \partial^{\gamma}, \tag{III.52}$$

where each ∂^{γ} with $\gamma \in \pi_{f,0}$ (resp. $\pi_{f,S}$) acts on an undifferentiated K occurring in a factor of type A or E (resp. B), and with a similar meaning for

$$r_{\pi_{b,0S}} = r_{\pi_{b,0}} r_{\pi_{b,S}} = \prod_{\gamma\in\pi_{b,0}} r_{\gamma} \prod_{\gamma\in\pi_{b,S}} r_{\gamma}. \tag{III.53}$$

In (III.50) the symbol $d'{\bigwedge}^r$ has the same meaning as $d{\bigwedge}^r$ defined in (A.4) except for the observance of the order convention. By this we mean that when the E factors in (III.45b) are spread into the P and A factors as in (III.35) we omit those terms in which the derivative on the E precedes the derivative on the A.

Also by the order convention, a bosen derivative in $r_{\pi_{b,0S}}$ may be applied to a fermion factor of type A, E or B but a fermion derivative in $\partial^{\pi_{f,0S}}$ may not be applied to a boson factor. Actually we can afford to be a little careless in enforcing these conventions since we shall disregard the order convention, i.e. we shall over-count, when we come to estimating the summand in (III.45).

The formulas (III.18) and (III.45) constitute the cluster expansion whose convergence we prove in §IV. In closing this section we explain how to insert localizations (in analogy to (II.33) for $P(\phi)_2$). It is here that the "nonlocality" of the model enters, for each γ is not necessarily associated with two local monomials in ϕ. For example, the γ giving rise to an A^{γ} (see (III.41)) is associated with a nonlocal cubic in ϕ and therefore requires 3 associated localizations. Indeed, the number of localizations associated with a γ will vary depending on which set $\pi_{p,X}$ γ is in. In Table III.1 we specify, for each type of γ, the number of localizations required and their position. In the following, χ will be the standard symbol for the characteristic function of a unit square; for $j \in \mathbf{Z}^2$ we let $\chi_j = \chi_{\Delta_j}$ denote the characteristic function of the jth lattice square. Insertion of a localization simply means inserting a partition of unity: $1 = \Sigma\chi_j$.

TABLE III.1. INSERTION OF LOCALIZATIONS

Set	Typical term	No. of localizations	Position of localizations
$\pi_{f,A}$	$K^2\partial^\gamma K$	3	$K\chi_3 K\chi_2\partial^\gamma K\chi_1$
$\pi_{f,E}$	$(1+K)\partial^\gamma K$	3	$\chi_3(1+K)\chi_2\partial^\gamma K\chi_1$
$\pi_{f,0}$	$\ldots\partial^\gamma K\ldots$	1	$\partial^\gamma K\chi_1$
$\pi_{f,B}$	$K\partial^\gamma K$	2	$K\chi_2\partial^\gamma K\chi_1$
$\pi_{f,S}$	$\partial^\gamma K\ldots$	1	$\partial^\gamma K\chi_1$
$\pi_{f,P}$	$\partial^\gamma P_j$	0	—
$\pi_{b,A}$	K^2K_γ	3	$K\chi_3 K\chi_2 K_\gamma\chi_1$
$\pi_{b,E}$	$(1+K)K_\gamma$	3	$\chi_3(1+K)\chi_2 K_\gamma\chi_1$
$\pi_{b,0}$	$\ldots K_\gamma\ldots$	1	$K_\gamma\chi_1$
$\pi_{b,S}$	$\int\int\phi_\gamma\partial^{\gamma'}b\phi$	1	$\int\int\phi_\gamma\chi_1\partial^{\gamma'}b\phi$
$\pi_{b,B}$	$\int\int\phi_\gamma b\phi$	2	$\int\int\phi_\gamma\chi_2 b\chi_1\phi$
$\pi_{b,\Phi}$	$\phi_\gamma(f)$	0	—

Note that the localizations for γ in $\pi_{f,0}$, $\pi_{f,S}$, $\pi_{b,0}$, and $\pi_{b,S}$ occur immediately next to localizations already present, but this over-localization produces no difficulties. Let α_0 be the number of γ's in the summand in (III.45) that require localizations, i.e.,

$$\alpha_0 = \sum_{X\neq P}|\pi_{f,X}| + \sum_{X\neq\Phi}|\pi_{b,X}|. \tag{III.54}$$

Then we label the localization indices by $j_{\alpha,\beta}\in\mathbf{Z}^2$ where $\alpha = 1, 2, \ldots, \alpha_0$ and $\beta = 1, \ldots, \beta_0$, with $\beta_0(\alpha) = 1$, 2 or 3 according to the above table. We let $j_\alpha = (j_{\alpha,1}, \ldots, j_{\alpha,\beta_0})$. When we want to emphasize that set γ to which j_α is attached, we write $j_\alpha = j_\gamma$ or $j_{\alpha,\beta} = j_{\gamma,\beta}$.

We insert localizations in (III.45) as specified in the table, indicating the presence of the localization by a subscript j:

$$G(\pi_{p,X}) = \sum_j G_j(\pi_{p,X}) = \sum_{j_1\cdots j_{\alpha_0}} G_j(\pi_{p,X}) \tag{III.55a}$$

where

$$\begin{aligned} G_j(\pi_{p,X}) = {} & r_{\pi_{b,0S},j}\,\partial_j^{\pi_{f,0S}}\tau_r\big(P(\pi_{f,P})\wedge A_j(\pi_{f,A}, \pi_{b,A}) \\ & \cdot d{\textstyle\bigwedge}^r E_j(\pi_{f,E}, \pi_{b,E})\big)B_j(\pi_{f,B}\ \pi_{b,B}) \end{aligned} \tag{III.55b}$$

with $r_{\pi_{b,0S},j}$ and $\partial_j^{\pi_{f,0S}}$ defined as in (III.53) and (III.52) but each K or ϕ to which an r_γ or ∂^γ is applied is also multiplied on the right by $\chi_{j_{\gamma,1}}$,

$$A_j(\pi_{f,A}, \pi_{b,A}) = \Big(\bigwedge_{\gamma \in \pi_{f,A}} K\chi_{j_{\gamma,3}} K\chi_{j_{\gamma,2}} \partial^{\gamma} K\chi_{j_{\gamma,1}} \Big)$$
$$\wedge \Big(\bigwedge_{\gamma \in \pi_{b,A}} K\chi_{j_{\gamma,3}} K\chi_{j_{\gamma,2}} K_{\gamma}\chi_{j_{\gamma,1}} \Big)$$

and so on.

IV. **Convergence of the cluster expansion.** Consider the cluster expansion (III.18):

$$S(\Lambda) = \sum_{(X,\Gamma) \in \mathbb{S}} T(\Lambda, X, \Gamma) \tag{IV.1}$$

where

$$T(\Lambda, X, \Gamma) = \pm \frac{Z_{\partial X}(\Lambda \sim X)}{Z(\Lambda)} \int d^{|\Gamma|}s\, \partial^{\Gamma} ZS(\Lambda \cap X, s(\Gamma)). \tag{IV.2}$$

In this section we prove that (IV.1) converges at a rate governed by $|X|$, uniformly in Λ:

THEOREM IV.1. *Let $\alpha > 0$ be given. For sufficiently small $|\lambda|$ and sufficiently large m_b and m_f,*

$$\sum_{\substack{(X,\Gamma) \in \mathbb{S} \\ |X| > d}} |T(\Lambda, X, \Gamma)| \leqslant \beta e^{-\alpha(d-n-9m)} \tag{IV.3}$$

where β is a constant independent of Λ, d, λ, m_b and m_f. (Here n is the number of bosons and m the number of fermion-antifermion pairs in S.)

The constant β in (IV.3) depends on n and m and on Schwartz space norms of the test functions f, g, h. The form of β is important as input into the Osterwalder-Schrader axioms in §V. However, we find it convenient to appeal to the a priori bound (I.15) of McBryan and Seiler and Simon rather than to keep track of the exact form of β.

The main estimate involved in the proof of Theorem IV.1 is (see §II):

ESTIMATE I. There are constants c_1 and c_2 bounded uniformly in the bare parameters such that

$$\left|\partial^{\Gamma} ZS(\Lambda, S)\right| \leqslant c_1 e^{-\alpha(|\Gamma| - 7m) + c_2|\Lambda|} \tag{IV.4}$$

where $\alpha \to \infty$ as $m_0 \to \infty$ and $\lambda \to 0$.

Most of this section will be devoted to the proof of this estimate based on the explicit formula (III.45):

$$\partial^{\Gamma} ZS = \sum_{\pi \in \mathcal{P}(\Gamma)} \sum_{\text{decomp } \pi} \sum_j (-1)^t \int d\mu_s \mathcal{E}(\pi_b) \Phi(\pi_{b,\Phi}) G_j(\pi_{p,X}) \tag{IV.5}$$

where the localized G_j is defined in (III.55). The logic of the proof is exactly as in the case of $P(\phi)_2$ (see the end of §II). First we interchange the sum over localizations with the sum over partitions (Lemma IV.2). We next count the number of terms in the summand in (IV.5) in terms of $M(\Delta)$'s (Lemma IV.3). We estimate Q-space integrals in Lemmas IV.4, IV.6, IV.7 and Theorem VII.14, in such a way as to extract G_1 and G_2 factors (see §II). These factors are then used to control B_1 and B_2.

Before embarking on this project we remark on notation to be used. In addition to the sharp localizations χ_j introduced in §III we shall also require smooth localizing functions $\zeta_j \in C_0^\infty$ with the properties:

$$\zeta_j = 1 \quad \text{on } \Delta_j: \qquad \operatorname{supp} \zeta_j \subset \left\{x \mid \operatorname{dist}(x, \Delta_j) < \tfrac{1}{4}\right\}. \tag{IV.6}$$

In the following estimates we use the letter c to denote various (not necessarily equal) positive constants which may depend on n, m, f, g, h, . . . but which are independent of s, Λ, Γ and the bare parameters. Similarly we use the letters δ and a to denote various universal positive constants where the constants δ will always satisfy $\delta \leqslant 1$.

We now wish to interchange the sum over partitions with the sum over localizations in (IV.5). The number of localization indices clearly depends on the partition π and its decomposition (III.44). Therefore before interchanging we rewrite the sum over partitions as follows. Let $\alpha_{p,X} = |\pi_{p,X}|$ where $p = b$ or f and $X = A, E, B, 0, S, P$ or Φ. We then write

$$\sum_{\pi \in \mathcal{P}(\Gamma)} \sum_{\text{decomp } \pi} = \sum_{\substack{\alpha_{p,X}=0 \\ \text{all } p,X}}^{2|\Gamma|} \sum_{\pi \in \mathcal{P}_N(\Gamma)} \sum_{\substack{\text{decomp } \pi \\ |\pi_{p,X}| = \alpha_{p,X}}} \tag{IV.7}$$

where $\mathcal{P}_N(\Gamma)$ denotes the set of partitions of Γ into N subsets and $N = \frac{1}{2}\sum_X \alpha_{b,X} + \sum_X \alpha_{f,X}$. There may be no decomposition of π as defined in (III.44) which satisfies the constraints $|\pi_{p,X}| = \alpha_{p,X}$; if so the inner sum in (IV.7) is zero. Since there are 12 $\alpha_{p,X}$'s we may estimate the number of terms in the sum over the $\alpha_{p,X}$'s by $(2|\Gamma| + 1)^{12}$. Assuming the summand is nonnegative we thus have

$$\sum_{\pi \in \mathcal{P}} \sum_{\text{decomp } \pi} \sum_j \leqslant (2|\Gamma| + 1)^{12} \max_{\alpha_{p,X}} \sum_{\pi \in \mathcal{P}_N} \sum_{\substack{\text{decomp } \pi \\ |\pi_{p,X}| = \alpha_{p,X}}} \sum_j .$$

In what follows we fix the $\alpha_{p,X}$ (and hence N) at the maximizing values. Since the number and type of localization indices $j_{\alpha,\beta}$ are now fixed we may pull the sum over the j_α to the outside. In addition, we estimate the number of terms in the sum over decompositions of π by $6^{2|\Gamma|}$, so that

$$(2|\Gamma| + 1)^{12} \sum_{j_\alpha} \sum_{\pi} \sum_{\text{decomp } \pi} \leqslant e^{a|\Gamma|} \sum_{j_\alpha} \sum_{\pi} \max_{\substack{\text{decomp } \pi \\ |\pi_{p,X}| = \alpha_{p,X}}} . \tag{IV.8}$$

Again to save writing we shall not write in the symbol "max" in (IV.8), but shall assume that for each $j = (j_\alpha)$ and $\pi \in \mathcal{P}_N(\Gamma)$ we take the decomposition of π which yields the maximum in (IV.8). From (IV.5) we thus obtain

LEMMA IV.2. *For the maximizing values of* $\alpha_{p,X}$ *and* $\pi_{p,X}$,

$$|\partial^\Gamma ZS(s)| < e^{a|\Gamma|}\sum_j \sum_{\pi\in\mathcal{P}_N(\Gamma)} \int d\mu_s \mathcal{E}(\pi_b)\Phi(\pi_{b,\Phi})G_j(\pi) \tag{IV.9}$$

where $G_j(\pi)$ *is defined in* (III.55).

Suppose now that $G_j(\pi)$ is "computed" by multiplying the $d'\bigwedge' E$'s into the P and A factors by (A.6) and by taking the fermion and boson derivatives in (III.55b). We obtain a sum of "primitive terms" g_i,

$$G_j(\pi) = \sum_{i=1}^{M} g_i \tag{IV.10}$$

where by a primitive term we mean a term of the form

$$g = \Phi\tau_r(F_1 \wedge F_2 \wedge \cdots \wedge F_r)\mathbf{B} \tag{IV.11}$$

where Φ is a product of n ϕ's or ϕ_γ's, $\mathbf{B}$ is a product of factors of type B, and each F_α is a factor of type P, A, PE, AE, PEE etc. We wish to estimate the number of terms $M = M(j, \pi)$ in the sum (IV.10). To facilitate the counting we make the following definitions concerning $G_j(\pi)$:

$$\begin{aligned} M_{AE}(\Delta) &= \text{number of times the localization } \Delta \text{ occurs} \\ &\qquad \text{in an } A \text{ or } E \text{ factor} \\ &= \left|\left\{j_{\gamma,\beta} | \Delta_{j_{\gamma,\beta}} = \Delta;\ \gamma \in \pi_{f,A} \cup \pi_{f,E} \cup \pi_{b,A} \cup \pi_{b,E},\ \beta = 1, 2, 3\right\}\right|. \end{aligned} \tag{IV.12}$$

$$\begin{aligned} M(\Delta) &= \text{total number of times the localization } \Delta \text{ occurs} \\ &= \left|\left\{j_{\alpha,\beta} | \Delta_{j_{\alpha,\beta}} = \Delta;\ \alpha = 1, \ldots, \alpha_0;\ \beta = 1, \ldots, \beta_0(\alpha)\right\}\right|. \end{aligned} \tag{IV.13}$$

LEMMA IV.3. *The number of primitive terms in* (IV.10) *is bounded by*

$$M < e^{c|\Gamma|}\prod_\Delta M(\Delta)!. \tag{IV.14}$$

PROOF. We ignore the order convention in the estimate of M. We first estimate how many primitive terms arise from spreading the $d\bigwedge' E$'s into the P's and A's. Given a square Δ, consider all the $d\bigwedge' E$'s with the leftmost localization in Δ, i.e. $\Delta_{j_{\gamma,3}} = \Delta$. Suppose there are $M_E(\Delta)$ such $d\bigwedge' E$'s. Then there are at most $(M_{AE}(\Delta) - M_E(\Delta) + m)$ factors of type P, A or E which may receive them (there may be previous E factors already multiplied in); hence there are at most

$$\frac{(M_{AE}(\Delta) - M_E(\Delta) + m)!}{(M_{AE}(\Delta) - 2M_E(\Delta) + m)!} \leqslant 2^{M_{AE}(\Delta) - M_E(\Delta) + m} M_E(\Delta)!$$

ways of distributing these E's. Thus we conclude that all together there are at most

$$\prod_{\substack{\Delta \\ M_E(\Delta) > 0}} 2^{M(\Delta) + m} M_E(\Delta)! \tag{IV.15}$$

primitive terms arising from multiplying in the E's.

Next, let $M_d(\Delta)$ be the number of derivatives in $\partial_j^{\pi_{f,os}}$ and $r_{\pi_{b,os},j}$ localized in Δ. Since there are at most $(M(\Delta) - M_d(\Delta))$ places for them to be applied, we estimate the number of ways of applying these derivatives as in (IV.15) by

$$\prod_{\substack{\Delta \\ M_d(\Delta) > 0}} 2^{M(\Delta)} M_d(\Delta)!. \tag{IV.16}$$

The derivatives in the Φ and P factors (III.47) and (III.48) contribute a factor to M but this factor is bounded by $n!m!$. Hence from (IV.15) and (IV.16) we conclude that

$$M \leqslant \prod_{\substack{\Delta \\ M(\Delta) > 0}} c^{M(\Delta)} M_E(\Delta)! M_d(\Delta)!.$$

Now

$$M_E(\Delta)! M_d(\Delta)! \leqslant (M_E(\Delta) + M_d(\Delta))! \leqslant M(\Delta)!,$$

and since each γ has at most 6 associated localizations,

$$\sum_{\Delta} M(\Delta) \leqslant 6|\Gamma|. \tag{IV.17}$$

The last three inequalities yield the lemma. □

The next step in estimating the summand in (IV.9) is to represent the expectation $\mathcal{E}(\pi_b)$, as defined in (III.38) and (III.40), by positive Gaussian measures. There is one simple case in which there is no need for this step or even for $\mathcal{E}_\gamma$; namely, if for some decomposition of π we have both occurrences of some γ in $\pi_{b,\Phi}$, then we obtain a factor $(h_i, \gamma^\gamma C h_j)$ which can be estimated by

$$|(h_i, \partial^\gamma C h_j)| \leqslant c K_6(\gamma) m_0^{-\delta_1 |\gamma|} \tag{IV.18}$$

by (II.30). It is to be understood in the sequel that we always apply the estimate (IV.18) and do not introduce dummy fields for such a γ.

Now, although $\partial^\gamma C(x, y)$ is pointwise positive, it is not positive as an operator except in the case $|\gamma| = 1$. Thus there will not, in general, be a corresponding positive measure. However we can write $\partial^\gamma C$ as a linear combination of suitable positive operators. We do so in two ways depending on whether we wish to exhibit the G_1 or G_2 decay (see §II). For G_2 we note that if b is, say, the first bond in γ, then by (VI.19),

$$\partial^{\gamma} C(s) = \partial^{b} \partial^{\gamma\sim b} C(s) = \sum_{\nu \subset \gamma \sim b} (-1)^{|\gamma|-|\nu|-1} C_{\gamma,\nu}(s) \tag{IV.19a}$$

where

$$C_{\gamma,\nu}(s) = \partial^{b} C(s)\Big|_{\substack{s=1 \text{ on } \nu \\ s=0 \text{ on } \gamma\sim(\nu\cup b)}} . \tag{IV.19b}$$

Since only a single bond derivative is involved in (IV.19b), $\partial^{b} C(s) = \delta^{b} C(s)$ *is* positive definite (see e.g. [12]). In other words, (IV.19) represents $\partial^{\gamma} C$ as a linear combination of $2^{|\gamma|-1}$ positive definite operators, each of which is continuous on $\mathcal{S}(\mathbf{R}^2)$ and each of which has appropriate G_2 decay: as we prove in Corollary VI.2, for any $\varepsilon < \frac{3}{4}$,

$$\| D^{\varepsilon} \zeta_i C_{\gamma,\nu} \zeta_k D^{\varepsilon} \|_{L^2(\mathbf{R}^4)} \leqslant c(\varepsilon) m_0^{a} e^{-\delta m_0 [d(\gamma,i)+d(\gamma,k)]} \tag{IV.20}$$

where the constant $c(\varepsilon)$ is independent of i, γ, ν, k, where

$$D = \left(-\Delta + m_0^2\right)^{1/2},$$

and where (b is the first bond in γ)

$$d(\gamma, i) = \operatorname{dist}(b, \Delta_i). \tag{IV.21}$$

Let $d\mu_{\gamma,\nu}$ be the Gaussian measure on a copy Q_γ of $\mathcal{S}'(\mathbf{R}^2)$ with covariance (IV.19b), and define

$$d\mu_2(\pi_b) = \bigotimes_{\gamma \in \pi_b} d\mu_{\gamma,\nu(\gamma)} \quad \text{on } Q(\pi_b) = \mathop{\times}_{\gamma \in \pi_b} Q_\gamma \tag{IV.22}$$

where $\nu(\gamma)$ is some subset of $\gamma \sim b$. Then we can represent the expectation $\mathcal{E}(\pi_b)$ in (IV.5) as a linear combination of $\Pi_{\gamma \in \pi_b} 2^{|\gamma|-1} = e^{O(|\Gamma|)}$ integrals of the form $\int_{Q(\pi_b)} d\mu_2(\pi_b)$ with ϕ_γ the coordinate function on Q_γ.

For the G_1 decay we restrict our attention to "large" γ's. By definition a *large* γ is one for which $|l_\sigma(\gamma)| \geqslant 1$ for all permutations σ (see (II.29) for the definition of $|l_\sigma(\gamma)|$). It is easy to see that

$$|\gamma| > 7 \text{ implies } \gamma \text{ is large,} \tag{IV.23}$$

and that if γ is large then for all permutations σ,

$$|l_\sigma(\gamma)| \geqslant |\gamma|/14. \tag{IV.24}$$

If γ is not large we say it is *small*. The point of these definitions is that for small γ, $K_6(\gamma) \geqslant 1$ (see (II.31)) and so there is no G_1 decay; whereas for large γ there is G_1 decay and, moreover, we can extract a convergence factor. Explicitly, let

$$G_1(\gamma, \delta) = \sum_{\sigma} e^{-\delta m_0 |l_\sigma(\gamma)|}. \tag{IV.25}$$

Then

$$K_6(\gamma)^{\delta} \leqslant G_1(\gamma, \delta/3) \leqslant e^{-\delta_1 m_0|\gamma|} G_1(\gamma, \delta_2) \tag{IV.26}$$

where $\delta/3 = 14\delta_1 + \delta_2$ with $\delta_i > 0$ (we have used (IV.24)).

For the G_1 decay for large γ we appeal to Lemma VI.11 which asserts that $\partial^{\gamma}C$ may be written as a difference,

$$\partial^{\gamma}C = C_{\gamma,+} - C_{\gamma,-}, \tag{IV.27}$$

where $C_{\gamma,\pm} \geqslant 0$ are not necessarily the positive and negative parts of $\partial^{\gamma}C$, but for any $p < \infty$ can be chosen to satisfy

$$\|C_{\gamma,\pm}\|_{L^p} \leqslant c^{|\gamma|} m_0^a G_1(\gamma, \delta) \tag{IV.28}$$

where $\delta > 0$, ζ_1 and ζ_2 are any two localizations as defined in (IV.6), $c(p)$ is a constant independent of γ, ζ_1, ζ_2, and G_1 is defined in (IV.25). Moreover, for any $\varepsilon < 1$, there is a constant $c(\varepsilon) < \infty$ such that

$$\|D^{\varepsilon} C_{\gamma,\pm} D^{\varepsilon}\|_{L^2} \leqslant c^{|\gamma|} m_0^a G_1(\gamma, \delta). \tag{IV.29}$$

As in the G_2 case, we introduce Gaussian measures $d\mu_{\gamma,\pm}$ on Q_γ with covariance $C_{\gamma,\pm}$ for each large γ. For the small γ's we use a G_2 measure. Then we can write the expectation $\int d\mu_{C(s)} \mathfrak{S}(\pi_b)$ as a linear combination of $e^{O(|\Gamma|)}$ integrals of the form $\int_{Q(\pi_b)} d\mu_1(\pi_b)$ where

$$d\mu_1(\pi_b) = \bigotimes_{\gamma \in \pi_b^L} d\mu_{\gamma,\pm} \bigotimes_{\gamma \in \pi_b^S} d\mu_{\gamma,\nu(\gamma)},$$

where π_b^L (resp. π_b^S) are the large (resp. small) γ's in π_b. When we insert $d\mu_1$ or $d\mu_2$ into (IV.9) it is to be understood that the ν's and $\pm$ signs are fixed to maximize the integral and that the $e^{O(|\Gamma|)}$ bound on the number of terms in the sum over ν and $\pm$ has been absorbed into the $e^{a|\Gamma|}$. Thus from (IV.9) we have

$$|\partial^{\Gamma} Z S(s)| \leqslant e^{a|\Gamma|} \sum_j \sum_{\pi \in \mathcal{P}_N(\Gamma)} \min_{i=1,2} \left| \int_{Q \times Q(\pi_B)} d\mu_{C(s)} \otimes d\mu_i G_j(\pi) \right|. \tag{IV.30}$$

We are now ready to estimate the contribution to (IV.30) of each primitive term in (IV.10), i.e. to estimate an expression of the form

$$\min_i \left| \int d\mu_{C(s)} \otimes d\mu_i \Phi \tau_r(F_1 \wedge \cdots \wedge F_r) \mathbf{B} \right|. \tag{IV.31}$$

From definition (III.32) we find by the "weak linear lower bound" of Theorem VII.14′ that for any $p < \infty$ and $q > p$ there is a constant c independent of m_0, r and Λ such that

$$\|\tau_r(F_1 \wedge \cdots \wedge F_r)\|_{L^p} \leqslant e^{c(r+|\Lambda|)} \left\| \prod_\alpha \|F_\alpha\|_1 \right\|_{L^q} \tag{IV.32}$$

where $\|\cdot\|_1$ denotes the trace norm on $\mathcal{H}$. By Hölder's inequality we then bound (IV.31) by

$$e^{c(r+|\Lambda|)} \min_i \|\Phi\|_{L_i^4} \left\| \prod_{\alpha=1}^{r} \|F_\alpha\|_1 \right\|_{L_i^4} \|B\|_{L_i^4} \tag{IV.33}$$

where L_i^4 denotes the norm in $L^4(Q \times Q(\pi_b), d\mu_{C(s)} \otimes d\mu_i)$.

The last three factors in (IV.33) are estimated in Lemmas IV.4, IV.6, IV.7 below. We shall always estimate the minimum over i in (IV.33) by the geometric mean.

Lemma IV.4.

$$\prod_{i=1}^{2} \|\Phi\|_{L_i^4}^{1/2} \leq c \prod_{\gamma \in \pi_{b,\Phi}} m_0^a G_1(\gamma, \delta) \tag{IV.34}$$

where the factor m_0^a can be replaced by $m_0^{-\delta_1|\gamma|/2}$ if γ occurs twice in $\pi_{b,\Phi}$.

Proof. If γ occurs twice in $\pi_{b,\Phi}$, we simply extract the factor (IV.18) without introducing dummy fields ϕ_γ into Φ or measures $d\mu_{\gamma,\nu}$ or $d\mu_{\gamma,\pm}$. Otherwise, the evaluation of $\|\Phi\|_{L_i^4}$ is a standard calculation involving (IV.20) and (IV.29). The constant c in (IV.34) depends on n and Schwartz norms of the form $\|D^{-\epsilon}h_i\|_{L^2}$. □

Consider next the second factor (involving F_α's) in (IV.33). In order to understand the choices we make in estimating the F_α's, the reader should bear in mind these facts: K is in any $\mathcal{C}_{p,p;s}$ (defined in (III.12)) with $p > 2$; $\partial^\gamma K$ is in any $\mathcal{C}_{p,p;s}$ with $p > \frac{3}{2}$; we can improve the $\mathcal{C}_{p,p;s}$ property of K, in the sense of decreasing p, by applying operators $D^{-\epsilon}$ with $\epsilon > 0$; operators D^ϵ worsen $\mathcal{C}_{p,p;s}$ properties. Precise statements of the above may be found in §VII.2. Consider now a typical factor F_α occurring in (IV.33); e.g.,

$$F_\alpha = AE = \left[K\chi_{j_{\gamma,3}} K\chi_{j_{\gamma,2}} \partial^\gamma K\chi_{j_{\gamma,1}}\right]\left[\chi_{j_{\gamma',3}}(1+K)\chi_{j_{\gamma',2}} \partial^{\gamma'} K\chi_{j_{\gamma',1}}\right] \tag{IV.35}$$

where, of course, $j_{\gamma,1} = j_{\gamma',3}$. We insert smooth localizations ζ_j by $\chi_j = \chi_j\zeta_j$ to obtain

$$F_\alpha = (K\chi_{j_{\gamma,3}})(\zeta_{j_{\gamma,3}} K\chi_{j_{\gamma,2}})(\zeta_{j_{\gamma,2}} \partial^\gamma K\chi_{j_{\gamma,1}}) \cdots (\zeta_{j_{\gamma',2}} \partial^{\gamma'} K\chi_{j_{\gamma',1}}).$$

We indicate schematically (i.e. without writing in the localizations) the manner in which we estimate $\|F_\alpha\|_1$. First,

$$\|F_\alpha\|_1 \leq \|A\|_1 \|E\|. \tag{IV.36}$$

Then, using Hölder's inequality on the $\|A\|_1$ factor we get $\mathcal{C}_4$ and $\mathcal{C}_2$ norms on the K and $\partial^\gamma K$ factors (resp.), i.e.,

$$\|A\|_1 \leq \|K\|_4 \|K\|_4 \|\partial^\gamma K\|_2. \tag{IV.37}$$

We estimate the factors K and $\partial^\gamma K$ in E by operator norms and then by $\mathcal{C}_4$

and $\mathcal{C}_2$ norms (resp.):

(IV.38) $$\|E\| \leqslant (1+\|K\|_4)\|\partial^{\gamma'}K\|_2.$$

Thus,

(IV.39) $$\begin{aligned}\|F_\alpha\|_1 \leqslant \|K\chi_{j_{\gamma,3}}\|_4 \cdot \|\zeta_{j_{\gamma,3}}K\chi_{j_{\gamma,2}}\|_4 \cdot \|\zeta_{j_{\gamma,2}}\partial^{\gamma}K\chi_{j_{\gamma,1}}\|_2 \\ \cdot \left(\delta_{j_{\gamma',3}j_{\gamma',2}} + \|\zeta_{j_{\gamma',3}}K\chi_{j_{\gamma',2}}\|_4\right)\|\zeta_{j_{\gamma',2}}\partial^{\gamma'}K\chi_{j_{\gamma',1}}\|_2.\end{aligned}$$

The procedure for the other possibilities for F_α is similar. If a P factor occurs instead of an A factor we estimate its trace norm by

(IV.40) $$\|\partial^{\gamma}P_i\|_1 = \|D^{-1}f_i\|_{\mathcal{K}}\|\partial^{\gamma}Sg_i\|_{\mathcal{K}}.$$

Note that in this case we may still replace the sharp localization $\chi_{j_{\gamma,3}}$ by $\zeta_{j_{\gamma,3}}$ in the first E factor since the f_i occurring in P_i is localized in a unit square. If additional E factors occur we estimate them as in (IV.38). If derivatives in $\partial^{\pi_{f,0}}$ or $r_{\pi_{b,0}}$ are applied to any of the factors, that does not alter our method of estimation except that K_γ factors, unlike $\partial^{\gamma}K$ factors, require a $\mathcal{C}_4$ norm. This causes one slight complication in the case of an A_γ factor, $A_\gamma = K^2K_\gamma$. In this case we insert factors of D^{ε} and $D^{-\varepsilon}$ where $0<\varepsilon<\frac{1}{6}$; i.e.,

(IV.41) $$\|K\chi_3K\chi_2K_\gamma\chi_1\|_1 \leqslant \|K\chi_3\|_4\|\zeta_3K\chi_2D^{\varepsilon}\|_4 \cdot \|D^{-\varepsilon}\zeta_2K_\gamma\chi_1\|_2.$$

Now the $\mathcal{C}_2$ and $\mathcal{C}_4$ norms that occur in the above inequalities are the square and fourth roots of polynomials in the fields. The kernels of these polynomials have certain L^p properties which we list here and prove in §VII. First we define the following distances: if $i, k \in \mathbf{Z}^2$,

(IV.42) $$d(i,k) = \operatorname{dist}(\Delta_i, \Delta_k)$$

if $j_\gamma \in \mathbf{Z}^{2\beta_0}$, let

(IV.43) $$d(\gamma, j) = d(\gamma, j_\gamma) = \sum_{\beta=1}^{\beta_0} d(\gamma, j_{\gamma,\beta})$$

where $d(\gamma, j_{\gamma,\beta})$ is defined in (IV.21). We set $d(\gamma,j)=0$ if γ has no associated localizations. Then:

LEMMA IV.5. (i) *Let $w_{i,k}^{\gamma}$ be the kernel of $\|\zeta_i\partial^{\gamma}K\chi_k\|_2^2$, i.e.*

$$\|\zeta_i\partial^{\gamma}K\chi_k\|_2^2 = \int w_{i,k}^{\gamma}(x_1,x_2)\phi(x_1)\phi(x_2)\,dx_1\,dx_2.$$

Then for any $p<\frac{12}{11}$,

$$\|w_{i,k}^{\gamma}\|_{L^p(\mathbf{R}^4)} \leqslant c_1(p)c^{|\gamma|}m_0^{a}G_1(\gamma,\delta)e^{-\delta m_0[d(\gamma,i)+d(\gamma,k)]}.$$

(ii) *If w_i^{γ} is the kernel of $\|\partial^{\gamma}K\chi_i\|_2^2$, then for any $p<\frac{12}{11}$*

$$\|w_i^{\gamma}\|_{L^p(\mathbf{R}^4)} \leqslant c_1(p)c^{|\gamma|}m_0^{a}G_1(\gamma,\delta)e^{-\delta m_0 d(\gamma,i)}.$$

(iii) *Let $\varepsilon<\frac{1}{6}$ and $p<12/(11+6\varepsilon)$. There is a kernel $w_{i,j,\varepsilon}$ in $L^p(\mathbf{R}^8)$*

satisfying

(IV.44a) $$\|\zeta_i K\chi_k D^{\varepsilon}\|_4^4 \leqslant \int w_{i,k,\varepsilon}(x_1 \dots x_4)\phi(x_1)\dots\phi(x_4)\,dx$$

with

(IV.44b) $$\|w_{i,k,\varepsilon}\|_{L^p(\mathbf{R}^8)} \leqslant c m_0^a e^{-\delta m_0 d(i,k)}.$$

(iv) *Let* $\varepsilon > 0$ *and* $p < \min[(1-\varepsilon)^{-1}, \frac{12}{11}]$. *There is a kernel* $w_{-\varepsilon,i,k}$ *in* $L^p(\mathbf{R}^4)$ *satisfying*

$$\|D^{-\varepsilon}\zeta_i K\chi_k\|_2^2 \leqslant \int w_{-\varepsilon,i,k}(x_1, x_2)\phi(x_1)\phi(x_2)\,dx$$

with

$$\|w_{-\varepsilon,i,k}\|_{L^p(\mathbf{R}^4)} \leqslant c m_0^a e^{-\delta m_0 d(i,k)}.$$

(v) *For* $p < \frac{12}{11}$, *there is a kernel* $w_j \in L^p(\mathbf{R}^8)$ *satisfying*

$$\|K\chi_j\|_4^4 \leqslant \int w_j(x_1 \dots x_4)\phi(x_1)\dots\phi(x_4)\,dx$$

with $\|w_j\|_{L^p(\mathbf{R}^8)} \leqslant c m_0^a$.

In the estimation of the F_α factors, the decay (G_1 and G_2) will occur explicitly in the kernels as in the above lemma and also in the covariance when the dummy fields are involved. For example, consider an expression like $\int \|\zeta_j K_\gamma \chi_k\|_4^4\,d\mu_i$ and apply (IV.44) with $\varepsilon = 0$:

(IV.45)
$$\begin{aligned}\int \|\zeta_j K_\gamma \chi_k\|_4^4\,d\mu_i &\leqslant \int\!\!\int w_{j,k}(x)\phi(x_1)\dots\phi(x_4)\,dx\,d\mu_i \\ &= \int dx\, w_{j,k}(x)\big[(\chi_k C\chi_k)(x_1,x_2)(\chi_k C\chi_k)(x_3,x_4) + \dots\big]\end{aligned}$$

where C is a covariance $C_{\gamma,\nu}$ or $C_{\gamma,\pm}$ depending on whether $i = 1$ or 2 (see (IV.19) and (IV.27)). We apply Hölder's inequality with $1 < p < \frac{12}{11}$ to bound (IV.45) by

(IV.46)
$$\begin{aligned}\prod_{i=1}^{2}\left(\int \|\zeta_j K_\gamma \chi_k\|_4^4\,d\mu_i\right)^{1/2} &\leqslant 3\|w_{j,k}\|_{L^p}\|\chi_k C\chi_k\|_{L^{p'}}^2 \\ &\leqslant c^{|\gamma|} m_0^a e^{-\delta m_0[d(j,k)+d(\gamma,k)]} G_1(\gamma,\delta)\end{aligned}$$

by (IV.44b), (II.30), and (IV.28), where we have taken a geometric mean as explained before Lemma IV.4.

Let π_{AE} consist of the γ's associated with A and E factors, i.e.

(IV.47) $$\pi_{AE} = \bigcup_{\substack{p=b,f \\ X=A,E,0}} \pi_{p,X}.$$

Then we have

LEMMA IV.6.

$$\prod_{i=1}^{2}\left\|\prod_{\alpha=1}^{r}\|F_\alpha\|_1\right\|_{L_i^4}^{1/2} \leqslant \prod_{\Delta} e^{cM_{AE}(\Delta)}(4M_{AE}(\Delta))!^{1/4} \tag{IV.48}$$

$$\cdot \prod_{\gamma\in\pi_{AE}\cup\pi_{f,P}} c^{|\gamma|}G_1(\gamma,\delta)\cdot\prod_{\gamma\in\pi_{AE}} m_0^a e^{-\delta m_0 d(\gamma,j_\gamma)}.$$

PROOF. As indicated in (IV.36)–(IV.39), each $\|F_\alpha\|_1$ is split into a product of factors, each factor being one of the types listed in the previous lemma. The integrals involving dummy fields factor off, each containing a polynomial of degree at most 8. They are estimated as in (IV.45)–(IV.46) by using Lemma IV.5 to estimate the kernel and (II.30) and (IV.28) to estimate the covariances. The remaining integral over $d\mu_{C(s)}$ involving the original fields ϕ is the integral of a product of lcoalized monomials of the type listed in Lemma IV.5. This integral may be estimated by the Checkerboard Theorem of [12] or by Theorem 9.4 of [9] in order to control the number singularities. The required L^p estimate on the kernels is supplied by Lemma IV.5. It is clear that each γ in π_{AE} contributes a $G_1(\gamma,\delta)$ factor and a decay factor $e^{-\delta m_0 d(\gamma,k)}$ where k is the localization index of the field in the K factor (i.e. $\partial^\gamma K$, K_γ, $\partial^\gamma K_{\gamma'}$, $\partial^{\gamma'}K_\gamma$) in which γ occurs. Note, however, that we have written the estimate (IV.48) in terms of the distance $d(\gamma,j_\gamma)$ involving all the localizations associated with γ (see (IV.43)). This use of $d(\gamma,j)$ is justified by the triangle inequality. For instance, in the bound (IV.46) we have

$$\begin{aligned} d(\gamma,k)+d(j,k) &\geqslant d(\gamma,k)+d(j,k)/2+\left[d(\gamma,j)-d(\gamma,k)\right]/2 \\ &\geqslant\left[d(\gamma,k)+d(\gamma,j)\right]/2. \end{aligned} \tag{IV.49}$$

Clearly the distances $d(\gamma,j_\gamma)$ can always be extracted in this way by one or two applications of the triangle inequality with an appropriate change in the constants δ. Note that each $\gamma\in\pi_{f,P}$ also contributes a $G_1(\gamma,\delta)$ factor by (IV.40); for let $\varepsilon>0$ and χ be the characteristic function of the lattice square in which g_i is localized. Then for large γ,

$$\|\partial^\gamma S g_i\|_{\mathcal{K}} \leqslant \|D^{1/2}\partial^\gamma S\chi D^{-\varepsilon}\|_{L^2(\mathbf{R}^4)}\|D^\varepsilon g_i\|_{L^2}$$

which has $G_1(\gamma,\delta)$ decay by Corollary VI.2. □

Finally we turn to the **B** factor in (IV.33). Let π_B be the γ's associated with the **B** factor, i.e.

$$\pi_B=\bigcup_{\substack{p=b,f\\X=B,S}}\pi_{p,X}$$

and let $M_B(\Delta)$ denote the number of times the localization Δ occurs in a B factor. We shall use the following estimate on the kernels (see Lemma VII.11(b) and VII.12): if $\varepsilon>\frac{2}{3}$,

$$\text{(IV.50)}\qquad \|D^{-\varepsilon}\chi_{j_2}\partial^{\gamma_1,\gamma_2}b\chi_{j_1}D^{-\varepsilon}\|_{L^2(\mathbf{R}^4)} \leqslant cm_0^a \prod_{k=1}^{2} G_1(\gamma_k,\delta)e^{-\delta m_0 d(\gamma_k,j)}$$

and

$$\text{(IV.51)}\qquad \|D^{-\varepsilon}\chi_i b\chi_k D^{-\varepsilon}\|_{L^2(\mathbf{R}^4)} \leqslant c(\varepsilon)m_0^a e^{-\delta m_0 d(i,k)}.$$

LEMMA IV.7.

$$\text{(IV.52)}\qquad \prod_{i=1}^{2}\|\mathbf{B}\|_{L_i^4}^{1/2} \leqslant c\prod_{\Delta} e^{cM_B(\Delta)}(4M_B(\Delta))!^{1/4} \cdot \prod_{\gamma\in\pi_B} c^{|\gamma|}m_0^a G_1(\gamma,\delta)e^{-\delta m_0 d(\gamma,j)}.$$

PROOF. $\mathbf{B}^4$ is a product of quadratic expressions of the form

$$\int :\phi_1(x)\chi_{i_1}(x)w(x,y)\chi_{i_2}(y)\phi_2(y):\,dx\,dy$$

where w is a b-kernel with 0, 1 or 2 γ-derivatives and ϕ_1 and ϕ_2 are original or dummy fields. The norm

$$\|\mathbf{B}\|_{L_i^4} = \left\{\int \mathbf{B}^4\,d\mu_{C(s)}\otimes d\mu_i\right\}^{1/4}$$

is then evaluated in the standard way as a sum over vacuum graphs: a vacuum graph is obtained by grouping the ϕ's in pairs (subject to the restriction that no pair may be drawn from the same :quadratic expression:) with each pair $\phi(x)\phi(y)$ giving rise to a covariance $C(s;\,x,\,y)$, and by grouping the ϕ_γ's in pairs, each pair $\phi_\gamma(x)\phi_\gamma(y)$ giving rise to a covariance $C_\gamma(s;\,x,y)$. In each of the resulting integrals we insert a smooth localization and the operators D^ε and $D^{-\varepsilon}$ with $\frac{2}{3} < \varepsilon < \frac{3}{4}$; for example,

$$\int C(x,y)\chi_k(y)w(y,z)\,dy = \int\left[C(x,y)\zeta_k(y)D_y^\varepsilon\right]\left[D_y^{-\varepsilon}\chi_k(y)w(y,z)\right]dy.$$

We estimate each of the resulting operators $D^\varepsilon\zeta_i C\zeta_k D^\varepsilon$ by its operator norm on $L^2(\mathbf{R}^2)$ and we apply the Schwarz inequality to obtain a product of $L^2(\mathbf{R}^4)$ norms of the functions $D^{-\varepsilon}\chi_i w\chi_k D^{-\varepsilon}$. The latter we estimate by (IV.50) and (IV.51). In most cases $D^\varepsilon\zeta_i C\zeta_k D^\varepsilon$ will actually be Hilbert-Schmidt: If a C_γ covariance is involved, then the operator is Hilbert-Schmidt with appropriate G_i-decay on its norm by (IV.20) and (IV.29). If a $C(s)$ covariance is involved with $d(i,k) > 0$ then we write

$$\text{(IV.53)}\qquad D^\varepsilon\zeta_i C\zeta_k D^\varepsilon = D^\varepsilon\zeta_i C_\phi\zeta_k D^\varepsilon + D^\varepsilon\zeta_i\delta C\zeta_k D^\varepsilon$$

where $\delta C = C - C_\phi$. The first term is Hilbert-Schmidt on $L^2(\mathbf{R}^2)$ by the proof of Lemma III.4 of [11] and satisfies

$$\text{(IV.54a)}\qquad \|D^\varepsilon\zeta_i C_\phi\zeta_k D^\varepsilon\|_{L^2(\mathbf{R}^4)} \leqslant ce^{-m_0 d(i,k)},$$

while the second term is Hilbert-Schmidt by Lemma VI.10 and satisfies

$$\|D^\varepsilon \zeta_i \delta C \zeta_k D^\varepsilon\|_{L^2(\mathbf{R}^4)} \leqslant c m_0^a e^{-\delta m_0 d(i,k)}. \tag{IV.54b}$$

If in (IV.53) $d(i, k) = 0$, then the second term on the right still satisfies (IV.54b) while the first term is evidently a bounded operator (to see this, simply omit the ζ's). We conclude that $D^\varepsilon \zeta_i C \zeta_k D^\varepsilon$ is always a bounded operator with $d(i, k)$ decay as in (IV.54). This decay is important for controlling the number singularities, i.e. the sum over vacuum graphs: using (IV.54b) and Theorem 9.4 of [9] we obtain the estimate $\prod_\Delta e^{c4M_B(\Delta)}(4M_B(\Delta))!$ for this sum. (We do not require such an argument for the ϕ_γ pairings since each field ϕ_γ can occur at most 8 times in $\mathbf{B}^4$.)

Collecting all of these estimates, we obtain (IV.52). Note that an application of the triangle inequality may be required again, just as in (IV.46); e.g. to convert the $d(j_{\gamma,1}, j_{\gamma,2})$ and $d(\gamma, j_{\gamma,1})$ decays of $\int \phi \chi_{j_{\gamma,2}} b \chi_{j_{\gamma,1}} \phi_\gamma$ to a $d(\gamma, j_\gamma)$ decay. $\square$

We now collect the above estimates to obtain the desired bound on (IV.31). The numerical factors in (IV.48) and (IV.52) may be simplified somewhat by

$$\begin{aligned}\left[(4M_{AE}(\Delta))!\,(4M_B(\Delta))!\right]^{1/4} &\leqslant 4^{M_{AE}(\Delta)} M_{AE}(\Delta)!\,4^{M_B(\Delta)} M_B(\Delta)! \\ &\leqslant 4^{M(\Delta)} M(\Delta)!.\end{aligned}$$

By (IV.33), (IV.34), (IV.35), (IV.48) and (IV.52) we thus bound (IV.31) by

$$\begin{aligned}&c \exp\left[c\left(r + |\Lambda| + |\Gamma| + \sum_\Delta M(\Delta)\right)\right] \prod_\Delta M(\Delta)! \\ &\quad\cdot \prod_{\gamma \in \pi_{b,\Phi,2}} m_0^{-\delta|\gamma|} \prod_{\gamma \in \pi'} m_0^a \prod_{\gamma \in \pi_b^2 \cup \pi_f} G_1(\gamma, \delta) e^{-\delta m_0 d(\gamma, j)}\end{aligned} \tag{IV.55}$$

where the set $\pi_{b,\Phi,2}$ consists of those γ's which occur twice in $\pi_{b,\Phi}$ (see Lemma IV.4) and the set $\pi' = \pi_b^2 \cup \pi_f \setminus (\pi_{b,\Phi} \cup \pi_{f,P})$. Note that the factor m_0^a in (IV.34), associated with those γ's applied once to Φ, has been absorbed into the m_0^a associated with the other occurrence of γ in π'. We can simplify (IV.55) somewhat by (IV.17) and the estimate $r \leqslant m + 2|\Gamma|$ (see (III.46)); also we identify the two occurrences of a γ in π_b^2 so that $\pi_b^2 \cup \pi_f$ may be replaced by π, $\pi_{b,\Phi,2}$ replaced by π_Φ consisting of γ's exclusively associated with Φ, and π' redefined to consist of those γ's in π which are not associated exclusively with a P or Φ factor. Note that a $\gamma \in \pi$ may now have as many as six localizations $j_{\gamma,\beta}$. Using Lemmas IV.2 and IV.3 we then obtain

COROLLARY IV.8.

$$|\partial^\Gamma ZS(s)| \leq e^{c(|\Gamma|+|\Lambda|)} \sum_j \sum_{\pi \in \mathcal{P}_N(\Gamma)} \prod_\Delta M(\Delta)!^2 \cdot \prod_{\gamma \in \pi_\Phi} m_0^{-\delta|\gamma|} \prod_{\gamma \in \pi'} m_0^a \prod_{\gamma \in \pi} G_1(\gamma, \delta) e^{-\delta m_0 d(\gamma, j)}. \tag{IV.56}$$

We are left with the same expression as in the $P(\phi)_2$ case except for two differences:

(i) the number of localizations $j_{\gamma,\beta}$ associated with γ varies from 1 to 6, instead of being exactly 2;

(ii) there is a large factor $m_0^{a|\pi'|}$ instead of the convergence factor $m_0^{-a|\Gamma|}$.

Difference (i) is minor, for we still have the relation (II.34) and, consequently, the bound, for $0 < \varepsilon \leq \frac{1}{2}$,

$$\prod_\Delta M(\Delta)!^2 \leq \exp\left[c_\varepsilon \sum_\Delta M(\Delta)^{1+\varepsilon}\right] \leq e^{c|\Gamma|} exp\left[c \sum d(\gamma, j_\gamma)\right]. \tag{IV.57}$$

Thus for sufficiently large m_0 we can absorb the factor $\Pi_\Delta M(\Delta)!^2$ into the G_2 decay factor, i.e. G_2 controls B_2.

As for (ii), we now liberate the coupling constant λ so that each K factor acquires an associated λ and each B an associated λ^2. It is then easy to see that with each $\gamma \in \pi'$ we may associate at least a factor of $|\lambda|^{1/2}$. If we choose λ so that

$$m_0^a |\lambda|^{1/4} \leq 1 \tag{IV.58}$$

we deduce from (IV.56) and (IV.57) that for sufficiently large m_0,

$$|\partial^\Gamma ZS(s)| \leq e^{c(|\Gamma|+|\Lambda|)} \sum_j \sum_\pi \prod_{\gamma \in \pi_\Phi} m_0^{-\delta|\gamma|} \cdot \prod_{\gamma \in \pi'} |\lambda|^{1/4} \prod_{\gamma \in \pi} G_1(\gamma, \delta) e^{-\delta m_0 d(\gamma, j)}. \tag{IV.59}$$

It should now be clear how we extract the convergence factor. For large γ's in π we can extract a factor $e^{-\delta m_0 |\gamma|}$ from $G_1(\gamma, \delta)$ by (IV.26). For small γ's in π_Φ and π' we have convergence factors $m_0^{-\delta|\gamma|}$ and $|\lambda|^{|\gamma|/28}$, respectively; here we have used the fact (IV.23) that a small γ satisfies $|\gamma| \leq 7$. The only remaining γ's are small γ's associated with P factors. But there can be at most m such γ's each with at most 7 bonds. We conclude that for any $\alpha > 0$, if we choose m_0 sufficiently large and $|\lambda|$ sufficiently small so that m_0^δ, $|\lambda|^{1/28}$, $e^{-\delta m_0} \leq e^{-\alpha}$, then we have

$$|\partial^\Gamma ZS(s)| \leq e^{c(|\Gamma|+|\Lambda|)} e^{-\alpha(|\Gamma|-7m)} \cdot \sum_j \sum_{\pi \in \mathcal{P}_N} \prod_{\gamma \in \pi} G_1(\gamma, \delta) e^{-\delta m_0 d(\gamma, j)}.$$

We now rejoin the $P(\phi)_2$ proof: we pull out the factor $\max_{\pi \in \mathcal{P}_N} \Pi_{\gamma \in \pi} e^{-\delta m_0 d(\gamma, j)}$, sum over π to get a factor $e^{O(|\Gamma|)}$ by Lemma II.2 (i.e. G_1 controls B_1), and then sum over each $j_{\alpha,\beta}$ to get another factor $e^{O(|\Gamma|)}$.

Therefore

$$\left|\partial^{\Gamma} ZS(s)\right| \leq ce^{-(\alpha-c)(|\Gamma|-7m)+c|\Lambda|}$$

which is Estimate I (i.e. estimate (IV.4)). The proof of Theorem IV.1 now follows easily:

PROOF OF THEOREM IV.1. It remains to check Estimate II (see (II.22)). We use a dominated convergence type argument, namely Lemma 3.5 of [25]: if $\rho_{\partial\Delta}(\lambda)$ denotes the renormalized determinant with $\Lambda = \Delta$ and $s = 0$ on $\partial\Delta$, then we know (see §VII.4) that, for any $p < \infty$, $\|\rho_{\partial\Delta}(\lambda)\|_{L^p(d\mu_{C(s)})} < \infty$, uniformly for λ in a bounded interval. Since $\rho_{\partial\Delta}(\lambda) \to 1$ pointwise and $d\mu_{C(s)}$ is a probability measure, we conclude that $\rho_{\partial\Delta}(\lambda) \to 1$ in $L^1(d\mu_{C(s)})$ as $\lambda \to 0$. Hence for sufficiently small $|\lambda|$, $Z_{\partial\Delta}(\Delta) = \int \rho_{\partial\Delta}(\lambda)\, d\mu_{C(s)}$ satisfies $\frac{1}{2} \leq |Z_{\partial\Delta}(\Delta)| \leq 2$. (Presumably, $\rho(s)$ and $Z(s)$ are nonnegative as in the $s = 1$ case [22], but we do not have a proof of this.) As in [9] we conclude that by Estimates I–III,

$$\sum_{\substack{(X,\Gamma)\in \mathbb{S} \\ |X|>d}} \left|T(\Lambda, X, \Gamma)\right| \leq c \sum_{|X|>d} e^{-\alpha(|\Gamma|-7m)+c|X|}.$$

But by the definition of $\mathbb{S}$, every component of $X \sim \Gamma^c$ meets X_0 so that

$$|\Gamma| \geq |X| - |X_0| \geq |X| - n - 2m.$$

The theorem follows from the last two inequalities. □

V. **Consequences.** In this section we shall establish the major consequences of the convergence of the cluster expansion (in particular, Theorem I.1). Our proofs are brief since, once the appropriate doubling procedure is introduced, the Y_2 proofs follow those of $P(\phi)_2$. Our first result is

THEOREM V.1 (CLUSTER PROPERTY). *Let $\alpha > 0$ be given. Suppose that $f_A = (f_{A,i})$, $g_A = (g_{A,i})$ and $h_A = (h_{A,i})$ are Schwartz space test functions with support in the region $A \subset \mathbf{R}^2$, and similarly f_B, g_B, h_B have support in the region B. Let $f = (f_A, f_B)$, $g = (g_A, g_B)$, $h = (h_A, h_B)$. Then for any given λ we can choose m_b and m_f sufficiently large so that*

$$\text{(V.1)} \quad \left|S_\Lambda(f, g, h) - S_\Lambda(f_A, g_A, h_A) S_\Lambda(f_B, g_B, h_B)\right| \leq \beta e^{-\alpha(d(A,B)-c)}$$

where $\beta = \beta(f, g, h)$ is independent of Λ and depends on f, g, h only through appropriate (translation invariant) Schwartz space norms, the constant c depends only on the order of S_Λ, and $d(A, B)$ is the distance between A and B.

PROOF. We shall assume that the number m_A of $f_{A,i}$'s equals the number of $g_{A,i}$'s (and similarly for B). If not, then the subtracted term in (V.1) vanishes and we simply apply the cluster expansion to $S_\Lambda(f, g, h)$ without the necessity of a doubling procedure. We remark that it is possible to compare directly the cluster expansions for the Schwinger functions on the left side of (V.1) provided we also multiply $S_\Lambda(f, g, h)$ by $1 = S_\Lambda(1)$ and perform a cluster expansion for it as well. However, as in [9], it is simpler to "double" the theory as follows:

(i) (*Boson doubling*). We let $(Q^{(i)}, \phi^{(i)}, d\mu^{(i)})$ for $i = 1, 2$ be two identical copies of the free boson field on $Q^{(i)} = \mathcal{S}'(\mathbf{R}^2)$. Define a new process by the independent sum $\tilde{\phi} = \phi^{(1)} + \phi^{(2)}$ on $\tilde{Q} = Q^{(1)} \times Q^{(2)}$ with measure $d\tilde{\mu} = d\mu^{(1)} \times d\mu^{(2)}$. We may introduce s B.C. for $\tilde{\phi}$ by $d\tilde{\mu}_{\tilde{C}(s)} = d\mu^{(1)}_{C(s)} \times d\mu^{(2)}_{C(s)}$ with $\tilde{C}(s) = C(s) \oplus C(s)$ on $\tilde{\mathcal{S}} = \mathcal{S}(\mathbf{R}^2) \oplus \mathcal{S}(\mathbf{R}^2)$.

(ii) (*Fermion doubling*). We let $\mathcal{H}^{(i)}$, $i = 1, 2$, be two identical copies of $\mathcal{H}$ and we define $\tilde{\mathcal{H}} = \mathcal{H}^{(1)} \oplus \mathcal{H}^{(2)}$. If $F(\phi)$ is a function on $\mathcal{S}'$ taking values as an operator on $\mathcal{H}$, define the operators on $\tilde{\mathcal{H}}$:

$$F^{(1)}(\tilde{\phi}) = F(\phi^{(1)}) \oplus 0, \qquad F^{(2)}(\tilde{\phi}) = 0 \oplus F(\phi^{(2)}) \tag{V.2a}$$

and

$$\tilde{F}(\tilde{\phi}) = F(\phi^{(1)}) \oplus F(\phi^{(2)}) = F^{(1)}(\tilde{\phi}) + F^{(2)}(\tilde{\phi}). \tag{V.2b}$$

With K given by (III.3), we define

$$\tilde{\rho}(\tilde{\phi}) = \det_{\text{ren}}\big(1 - \tilde{K}(\tilde{\phi})\big) = \det_3(1 - \tilde{K})\exp\left[-\tfrac{1}{2}:\text{Tr}(\tilde{K}^2 + \tilde{K}^\dagger\tilde{K}):\right].$$

The Schwinger functions for the doubled theory are defined as in (III.11) in terms of $\tilde{\rho}$, $d\tilde{\mu}$, and $\tilde{T}_m = m!\text{Tr}_{\wedge^m\tilde{\mathcal{H}}}$. Just as in (III.14)–(III.16) we have the factorization

$$\tilde{\rho}(\tilde{\phi}) = \rho(\phi^{(1)})\rho(\phi^{(2)}) \tag{V.3a}$$

and so

$$\tilde{Z} = \int \tilde{\rho}(\tilde{\phi})\, d\tilde{\mu} = \int \rho(\phi^{(1)})\, d\mu^{(1)} \int \rho(\phi^{(2)})\, d\mu^{(2)} = Z^2. \tag{V.3b}$$

Suppose we rewrite formula (III.11b) for (f, g, h) as

$$S_\Lambda(f, g, h) = Z^{-1} \int T_m(\wedge^m R \cdot Q_A \wedge Q_B)\rho\, d\mu$$

where, for $X = A$ or B, $Q_X = \Phi(h_X) \wedge_{j=1}^{m_X} P_{X,j}$ with

$$P_{X,j} = \big(D^{-1}g_{X,j}, \cdot\,\big)_{\mathcal{H}} S_0 f_{X,j},$$

and where $m = m_A + m_B$. Let $Q_X^{(i)}$ be the operators on $\wedge^{m_X}\tilde{\mathcal{H}}$ defined by

$$Q_X^{(i)} = \Phi^{(i)}(h_X) \bigwedge_{j=1}^{m_X} P_{X,j}^{(i)}$$

where $\Phi^{(i)}(h_X) = \Pi_j \phi^{(i)}(h_{X,j})$ and $P_{X,j}^{(i)}$ is as defined in (V.2a). Then we claim that

$$\begin{aligned} \tilde{S} &\equiv \tilde{Z}^{-1} \int \tilde{T}_m\big(\wedge^m\tilde{R}\,(Q_A^{(1)} - Q_A^{(2)}) \wedge (Q_B^{(1)} - Q_B^{(2)})\big)\tilde{\rho}\, d\tilde{\mu} \\ &= 2\big[S_\Lambda(f, g, h) - S_\Lambda(f_A, g_A, h_A)S_\Lambda(f_B, g_B, h_B)\big]. \end{aligned} \tag{V.4}$$

The identity (V.4) follows from the direct sum structure $\tilde{\mathcal{H}} = \mathcal{H}^{(1)} \oplus \mathcal{H}^{(2)}$. For

$$\begin{aligned} &\bigwedge^m \tilde{R}\,(Q_A^{(1)} - Q_A^{(2)}) \wedge (Q_B^{(1)} - Q_B^{(2)}) \\ &\quad = \bigwedge^m R^{(1)} Q_A^{(1)} \wedge Q_B^{(1)} + \bigwedge^m R^{(2)} Q_A^{(2)} \wedge Q_B^{(2)} \\ &\quad\quad - \bigwedge^{m_A} R^{(1)} Q_A^{(1)} \wedge \bigwedge^{m_B} R^{(2)} \cdot Q_B^{(2)} \\ &\quad\quad - \bigwedge^{m_A} R^{(2)} \cdot Q_A^{(2)} \wedge \bigwedge^{m_B} R^{(1)} \cdot Q_B^{(1)}. \end{aligned} \tag{V.5}$$

We apply T_m to (V.5), observing that by Lemma A.1 we obtain a factorization of the off diagonal terms, e.g.,

$$\begin{aligned} &\tilde{T}_m (\bigwedge^{m_A} R^{(1)} \cdot Q_A^{(1)} \wedge \bigwedge^{m_B} R^{(2)} Q_B^{(2)}) \\ &\quad = \tilde{T}_{m_A} (\bigwedge^{m_A} R^{(1)} \cdot Q_A^{(1)}) \tilde{T}_{m_B} (\bigwedge^{m_B} R^{(2)} \cdot Q_B^{(2)}). \end{aligned} \tag{V.6}$$

Noting that $\tilde{T}_m((\bigwedge^m R^{(i)} Q^{(i)})(\tilde{\phi})) = T_m((\bigwedge^m RQ)(\phi^{(i)}))$, integrating with respect to $\tilde{\rho}\, d\tilde{\mu}$, and invoking (V.3) we deduce (V.4).

The left side of (V.4) has the appropriate form for an application of the cluster expansion, i.e., it has the form (III.11) and, furthermore, it decouples at $s = 0$. For suppose that A and B belong to disjoint components Λ_A and Λ_B of $\mathbf{R}^2 \sim \Gamma^2$. Then just as in (V.6) we have by Lemma A.1,

$$\begin{aligned} &\tilde{T}_m\big[\bigwedge^m \tilde{R} \cdot (Q_A^{(1)} - Q_B^{(2)}) \wedge (Q_B^{(1)} - Q_B^{(2)})\big] \\ &\quad = \tilde{T}_{m_A}\big[\bigwedge^{m_A} \tilde{R}_A (Q_A^{(1)} - Q_A^{(2)})\big] \cdot \tilde{T}_{m_B}\big[\bigwedge^{m_B} \tilde{R}_B (Q_B^{(1)} - Q_B^{(2)})\big] \end{aligned} \tag{V.7}$$

where $R_X = (1 - K_{\Lambda \cap \Lambda_X})^{-1}$. (Here it is the decomposition of the doubled theory induced by the direct sum decomposition of $L^2(\mathbf{R}^2)$ that is involved.) The integral of (V.7) now factors by (III.16). If Γ fails to connect A and B, then the corresponding term $\partial^\Gamma \widetilde{ZS}$ in the expansion factors as

$$\partial^\Gamma \widetilde{ZS} = \prod_{X = A,B} \partial^{\Gamma_X} \int \tilde{T}_{m_X}\big[\bigwedge^{m_X} \tilde{R}_X \cdot (Q_X^{(1)} - Q_X^{(2)})\big] \tilde{\rho}_{\Lambda_X}\, d\tilde{\mu}$$

and so vanishes by the $1 \leftrightarrow 2$ symmetry.

Let Σ be the set of values of (λ, m_b, m_f) for which the cluster expansion for (V.4) converges in the sense of Theorem IV.1 (i.e. Σ consists of sufficiently small λ and sufficiently large m_b, m_f); and let Σ' be those (λ, m_b, m_f) for which (V.4) satisfies the desired bound (V.1). As in [9, Theorem 2.1] we deduce that $\Sigma \subset \Sigma'$. It remains to show that Σ' contains points with arbitrary λ.

But the Schwinger functions $S_{\lambda, m_b, m_f, \Lambda}(f, g, h)$, where we have denoted the dependence on the bare parameters by additional subscripts, are invariant under the change of variables:

$$\lambda \to r\lambda, \quad m_b \to rm_b, \quad m_f \to rm_f, \quad \Lambda \to r^{-1}\Lambda,$$

$$h(x) \to r^2 h(rx) \quad f(x) \to r^{3/2} f(rx), \quad g(x) \to r^{3/2} g(rx).$$

This follows by a simple scaling argument based on the fact that $\phi(x; m_b)$ and

$\phi(r^{-1}x; rm_b)$ are isomorphic processes. Consequently, if $(\lambda, m_b, m_f) \in \Sigma$ with decay rate $e^{-\alpha d(A,B)}$ then so is $(r\lambda, rm_b, rm_j)$ with decay rate $e^{-\alpha d(rA,rB)} = e^{-r\alpha d(A,B)}$. $\square$

REMARK. The results for large λ can also be obtained via a convergent cluster expansion provided we replace the unit squares with squares of side length $L = d/m_0$, where, for convenience, we have taken $m_b = m_f = m_0$ and where d is a constant to be determined. We sketch this argument: First we rescale lengths as in the above proof with $r = 1/m_0$ to obtain an equivalent theory with bare parameters $\lambda' = \lambda/m_0$, $m_b' = m_f' = m_0' = 1$, and interaction region $\Lambda' = m_0\Lambda$. We claim that the cluster expansion for this new theory with lattice spacing $L' = r^{-1}L = d$ converges as in (IV.3) provided L' is chosen large enough and λ' small enough (and hence that the cluster expansion for the original theory with lattice spacing L converges for any λ provided d and m_0 are chosen large enough). To see this, consider the bound which is the analogue of (IV.56):

$$|\partial^{\Gamma'} ZS(X')| \leqslant e^{c|\Gamma'|+c_2|X'|} \sum_{j,\pi} \prod_{\Delta} M(\Delta)!^2 \prod_{\gamma\in\pi'} |\lambda'|^{1/2} \cdot \prod_{\gamma\in\pi} L'^a G_1(\gamma,\delta) e^{-\delta d(\gamma,j)}. \tag{IV.56$'$}$$

Note that the factors G_1 and $e^{-\delta d(\gamma,j)}$ involve distances based on a unit of length L' and that instead of factors m_0^a there are now occur factors of L'^a since the local L^p estimates are calculated over squares of side L'. As in (IV.56), we can extract a convergence factor $e^{-\delta L'|\gamma|}$ from $G_1(\gamma, \delta)$ for large γ's. Next we choose L' sufficiently large for the combinatorial arguments of the cluster expansion (i.e. G_1 controls Σ_π and $e^{-\delta d(\gamma,j)}$ controls $\Pi_\Delta M(\Delta)!^2$). We then choose λ' sufficiently small to give a convergence factor for the small γ's (except for the at most $n + m$ γ's in $\pi \sim \pi'$), thereby controlling the factors L'^a. The only additional feature is that the crucial estimate $|\Gamma| \geqslant |X| - n - 2m$ (see the proof of Theorem IV.1) becomes $|\Gamma'| \geqslant |X'|/L'^2 - n - 2m$. But this presents no problem as the linear lower bound constant c_2 in (IV.56$'$) can be made arbitrarily small by choosing λ' small enough. Finally, for any choice of L', the bound (II.22) on $Z_{\partial\Delta}(\Delta)$ will hold for all sufficiently small λ'.

We now turn to the existence of the infinite volume limit.

PROOF OF THEOREM I.1(a). By the scaling argument used in the proof of Theorem V.1 it is sufficient to prove the theorem for values of λ and m_0 for which the cluster expansion applies. We consider the effect of increasing the interaction region Λ by adding a single lattice square Δ. To do this we interpolate between χ_Λ and $\chi_{\Lambda\cup\Delta}$ by $h_\tau = \chi_\Lambda + \tau\chi_\Delta$ and write

$$S_{\Lambda\cup\Delta} - S_\Lambda = \int_0^1 d\tau \, \frac{d}{d\tau} S(\tau)$$

where $S(\tau)$ is defined as in (I.3) with χ_Λ in (I.5) replaced by h_τ. As in [8, Theorem 2.2.2] it suffices to bound $dS(\tau)/d\tau$ by

$$|dS(\tau)/d\tau| \leqslant ce^{-\alpha d(\Delta, X_0)} \tag{V.8}$$

where α and c are positive constants (independent of Λ and Δ) and X_0 is a fixed set which contains the supports of all the test functions in S.

As in (III.26) and (III.31) we find that

$$\frac{d}{d\tau} Z(\tau) = -\int \left[\mathrm{Tr}(RA_\tau) + \frac{dB}{d\tau} \right] \rho \, d\mu$$

and

$$\frac{d}{d\tau} ZS(\tau)$$
$$= \int \Big\{ -T_{m+1}\big(\wedge^{m+1}R \cdot P \wedge A_\tau\big) + T_m(\wedge^m R \cdot Pd\wedge^m E_\tau) - T_m(\wedge^m R \cdot P) \frac{dB}{d\tau} \Big\} \rho \, d\mu$$

where

$$A_\tau = K(\tau)^2 \frac{dK}{d\tau} = K(\tau)^2 K_\Delta, \quad E_\tau = (1 + K(\tau))K_\Delta,$$

$$\frac{dB}{d\tau} = :\mathrm{Tr}\big[K_\Lambda K_\Delta + \tau\big(K_\Delta^2 + K_\Delta^\dagger K_\Delta\big)\big]: = :\mathrm{Tr}\, K_\Lambda K_\Delta: + \tau B_\Delta.$$

Thus we obtain

$$\frac{d}{d\tau} S(\tau) = Z^{-1} \frac{d}{d\tau} ZS - SZ^{-1} \frac{d}{d\tau} Z = T_E - T_A - T_B$$

where

$$T_E = Z^{-1} \int T_m(\wedge^m R \cdot \ Pd\wedge^m E_\tau)\rho \, d\mu,$$

$$T_A = Z^{-1} \int T_{m+1}\big(\wedge^{m+1}R \cdot P \wedge A_\tau\big)\rho \, d\mu$$
$$\quad - Z^{-2} \int T_m(\wedge^m R \cdot P)\rho \, d\mu \int T_1(RA_\tau)\rho \, d\mu,$$

$$T_B = Z^{-1} \int T_m(\wedge^m R \cdot P) \frac{dB}{d\tau} \rho \, d\mu - Z^{-2} \int T_m(\wedge^m R \cdot P)\rho \, d\mu \int \frac{dB}{d\tau} \rho \, d\mu.$$

Consider first the term T_E. To obtain a bound of the form (V.8), we apply the cluster expansion to T_E and argue that each term in this expansion will vanish unless Γ connects Δ and X_0. Suppose that X_0 and Δ are in two disconnected components, say X_1 and X_2 of $\mathbf{R}^2 \sim \Gamma^c = \bigcup X_i$. Then by (III.14a)

(V.9) $$E_\tau = (1 + K_{\Lambda_2} + \tau K_\Delta)K_\Delta$$

where $\Lambda_j = \Lambda \cap X_j$. Consider a typical product $P_i E_\tau$ obtained from spreading $d \bigwedge^m E_\tau$ into the P's as in (III.35). We have $P_i E_\tau = P_i \chi_{X_0} E_\tau = 0$ since, by (III.2), S_0 has vanishing matrix elements between $X_1 \supset X_0$ and $X_2 \supset \Lambda_2 \cup \Delta$. We conclude by Theorem IV.1 that T_E satisfies a bound of the form (V.8).

The bounds on T_A and T_B are essentially a corollary of Theorem V.1. Consider T_A and write it in terms of the doubled theory:

(V.10) $$T_A = \frac{1}{2}\tilde{Z}^{-1}\int \tilde{T}_{m+1}\left(\bigwedge^{m+1}\tilde{R}\cdot(P^{(1)} - P^{(2)}) \wedge (A_\tau^{(1)} - A_\tau^{(2)})\right)\tilde{\rho}\, d\tilde{\mu}.$$

We apply the cluster expansion to (V.10) and argue as in the proof of Theorem V.1 that if Γ does not connect X_0 and Δ, then $\partial^\Gamma \tilde{Z} T_A$ factors and, hence, vanishes by the $1 \leftrightarrow 2$ symmetry. The nonlocality of A_τ is no problem, for as in (V.9) A_τ will be completely localized in the same component of $\mathbf{R}^2 \sim \Gamma^c$ as Δ, i.e. $A_\tau = (K_{\Lambda_2} + \tau K_\Delta)^2 K_\Delta$.

The case of T_B is similar. The only point to be checked is that $dB/d\tau$ is in every $L^p(d\mu_{C(s)})$. In particular, the term $:\mathrm{Tr}\, K_\Lambda K_\Delta:$ is in L^p without the necessity of a mass counterterm since Λ and Δ do not overlap but at worst touch (see Corollary VII.13). □

Finally we come to the axioms:

Proof of Theorem I.1b. Consider the Osterwalder-Schrader axioms [19]:

(E0′) Distribution Property. We simply appeal to the a priori bounds (I.15) of McBryan and Seiler and Simon.

(E1) Covariance. Translational invariance is a consequence of the convergence of the $\Lambda \to \infty$ limit. Rotational covariance may be deduced by introducing instead a spatial cutoff h which is spherically symmetric (the transformation of spinor indices never changes from that of the free theory).

(E2) Positivity. As remarked in [18], positivity will hold after the removal of cutoffs if it holds with cutoffs. In the (appropriate) ultraviolet cutoff case, positivity follows from the Feynman-Kac formula of [18]. See also [25] for a proof of the relation between the Matthews-Salam-Seiler formula and the Hamiltonian formalism which does not use Euclidean fermi fields.

(E3) Symmetry. This is independent of the spatial cutoff and follows from the determinant structure of (I.3).

(E4) Cluster Property. This is just Theorem V.1.

Note that by scaling the infinite volume theory depends on the bare parameters only through the dimensionless ratios λ/m_b and λ/m_f. □

VI. Derivatives of Green's functions.

VI.1. *Introduction*. In this section we obtain local L^p estimates on the mixed partial derivatives $(\partial^2/\partial x_{i_1}\partial y_{i_2})\partial^\gamma C(s; x, y)$ where the s-derivative $\partial^\gamma C$ is defined in §II. These estimates have two important features: (i) as $|\gamma|$

increases, the L^p properties improve; (ii) the estimates provide the G_1 and G_2 decay needed for convergence of the cluster expansion. As in §IV, we use the letters δ and c to denote various positive constants where δ will always be a universal constant in (0, 1) and c will, in general, depend on various parameters (e.g., p, δ) but not on m_0, γ or j. Our main result is

THEOREM VI.1. *For $\alpha_1, \alpha_2 = 0$ or 1 and $\gamma \subset \mathfrak{B}$ a finite set of bonds, let $p < p_{\alpha_1+\alpha_2}(|\gamma|)$, where $p_i(n)$ is defined in the table below. Then for all $i_1, i_2 = 1$ or 2 and j_1, j_2 in $\mathbf{Z}^2$ we have*

$$\left\| \chi_{j_1}(x)\chi_{j_2}(y) \frac{\partial^{\alpha_1+\alpha_2}}{\partial x_{i_1}^{\alpha_1}\partial y_{i_2}^{\alpha_2}} \partial^\gamma C(s; x, y) \right\|_{L^p(\mathbf{R}^4)} \leqslant m_0 c^{|\gamma|} G_1(\gamma, \delta) e^{-\delta m_0 d(\gamma, j)} \tag{VI.1}$$

where χ_i is the characteristic function of the square Δ_i, and $G_1(\gamma, \delta)$ is defined in (IV.25) *and $d(\gamma, j)$ in* (II.28).

DEFINITION OF $p_i(n)$

	$n=0$	$n=1$	$n>1$	Proof of (VI.1) in
$p_0(n)$	∞	∞	∞	§VI.4
$p_1(n)$	2	3	4	§VI.6
$p_2(n)$	1	3/2	2	§VI.7

REMARKS. 1. By tracing through our proof, the reader can check that any $\delta < \frac{1}{2}$ is allowed in (VI.1) and that for $\alpha_1 = \alpha_2 = 0$ the factor m_0 may be replaced by $m_0^{-2/p}$. A scaling argument indicates that for large m_0 the best possible power of m_0 in (VI.1) is $m_0^{\alpha_1+\alpha_2-2/p}$ (each spatial derivative introduces a factor of m_0).

2. If $\partial^\gamma C$ is replaced by $\delta C \equiv C(s) - C(1)$ or even by $C(s)$ provided that $j_1 \neq j_2$, we obtain L^p estimates on the spatial derivatives similar to the case of $\partial^\gamma C$ with $|\gamma| = 1$ (see Lemma VI.10 in §VI.7).

3. By the equivalence of Sobolev and potential space norms [29, p. 135] together with standard interpolation techniques [20], Theorem VI.1 implies

COROLLARY VI.2. *For any α_1, α_2 in* [0, 1] *and*

$$p < (2 + \min(2, |\gamma|))/(\alpha_1 + \alpha_2),$$

$$\|D_x^{\alpha_1}\zeta_{j_1}(x)\partial^\gamma C(s; x, y)\zeta_{j_2}(y)D_y^{\alpha_2}\|_{L^p} \leqslant m_0^3 c^{|\gamma|} G_1(\gamma, \delta) e^{-\delta m_0 d(\gamma, j)} \tag{VI.2}$$

where $D_x = (-\Delta_x + m_0^2)^{1/2}$ *and where the smooth localization* ζ_j *is defined in* (IV.6).

Our proof of Theorem VI.1 is based on the fact that $(C_\phi - C_\Gamma)(x, y)$ is the solution of a Dirichlet boundary value problem for $-\Delta_x + m_0^2$ with boundary values $C_\phi(x, y)$ on Γ. This enables us to express $\partial^\gamma C$ as a sum involving iterated sequences of Poisson kernels which can be estimated by standard techniques from the theory of partial differential equations. This approach appears to give more direct information about the spatial derivatives of $\partial^\gamma C$ than does the Wiener integral representation (for this method see [6, §V]). At the same time, it provides a completely elementary derivation of the known [9], [28] estimate when $\alpha_1 = \alpha_2 = 0$.

VI. 2 *Iterated Poisson Formula*. From definition (II.15) of $C(s)$ we find by a straightforward calculation that

$$\partial^\gamma C(s) = \sum_{\Gamma \subset \mathfrak{B} \sim \gamma} \prod_{b \in \Gamma} s_b \prod_{b \in \Gamma^c \sim \gamma} (1 - s_b) \delta^\gamma C_{\Gamma^c} \tag{VI.3}$$

where, in the notation of (II.5),

$$\delta^\gamma C_{\Gamma^c} = \delta^\gamma C(s)|_{s=\Gamma(1)} = \delta^\gamma C(s)|_{\substack{s=1 \text{ on } \Gamma \\ s=0 \text{ on } \Gamma^c}}. \tag{VI.4}$$

For example, if $b \in \Gamma^c$, then $\delta^b C_{\Gamma^c} = C_{\Gamma^c \sim b} - C_{\Gamma^c}$. More generally we have by induction that

$$\delta^\gamma C_{\Gamma^c} = \sum_{\nu \subset \gamma} (-1)^{|\gamma| - |\nu|} C_{\Gamma^c \sim \nu}. \tag{VI.5}$$

According to (VI.3), $\partial^\gamma C(s)$ is a convex combination of $\delta^\gamma C_{\Gamma^c}$'s so that *it will be sufficient to prove Theorem* VI.1 *for* $\delta^\gamma C_{\Gamma^c}$ *with bounds independent of* Γ.

We thus study $\delta^\gamma C_{\Gamma^c}$. If $x, y \notin \Gamma^c$ and $\gamma \subset \Gamma^c$ then by (VI.5),

$$(-\Delta_x + m_0^2) \delta^\gamma C_{\Gamma^c}(x, y) = \sum_{\nu \subset \gamma} (-1)^{|\gamma| - |\nu|} \delta(x - y) = 0.$$

Hence $\delta^\gamma C_{\Gamma^c}(\cdot, y)$ is the solution to the Dirichlet problem for $(-\Delta + m_0^2)$ with boundary values $\delta^\gamma C_{\Gamma^c}(z, y)$ on Γ^c. Given a piecewise smooth curve $B \subset \mathbf{R}^2$ we let $u = P_B f$ denote the solution of the boundary value problem

$$(-\Delta + m_0^2) u(x) = 0, \qquad x \notin B; \qquad u = f \quad \text{on } B$$

where f is a continuous function on B (if B is bounded we require that u be regular at ∞, i.e. $u(x) \to 0$ as $x \to \infty$). The kernel $P_B(x, z)$ of P_B (the "Poisson kernel") is given in terms of the normal derivative of the Dirichlet Green's function by (see e.g. [3])

$$P_B(x, z) = \partial C_B(x, z) / \partial n_z. \tag{VI.6}$$

We shall be particularly interested in the case where $B = b$, a single bond, in which case we shall replace the line integral $P_b f(x) = \oint_b P_b(x, z) f(z)\, dz$ around b by an integral along b with P_b being the jump across b of the normal derivative of C_b. According to this notation we have $\delta^\gamma C_{\Gamma^c} = P_{\Gamma^c}\delta^\gamma C_{\Gamma^c}$ where on the right side we regard $\delta^\gamma C_{\Gamma^c}(z, y)$ as being restricted to $z \in \Gamma^c$ in order to provide boundary values for P_{Γ^c} which acts on the first variable z, the variable $y \not\in \Gamma^c$ being regarded as a parameter.

We now claim that if $\gamma \in \Gamma^c$ then

$$\delta^\gamma C_{\Gamma^c} = \sum_{b \in \gamma} P_{\Gamma^c} \chi_b \delta^{\gamma \sim b} C_{\Gamma^c \sim b} \tag{VI.7}$$

where χ_ν is the characteristic function of $\nu \subset \mathfrak{B}$ as a subset of $\mathbf{R}^2$. For $\delta^\gamma C_{\Gamma^c}(z, y) = 0$ if $z \in \Gamma^c \sim \gamma$ so that

$$\delta^\gamma C_{\Gamma^c} = P_{\Gamma^c} \chi_\gamma \delta^\gamma C_{\Gamma^c} = \sum_{b \in \gamma} P_{\Gamma^c} \chi_b \delta^b C_{\Gamma^c}. \tag{VI.8}$$

But by (VI.5) and the fact that $\chi_b C_\gamma = 0$ if $b \in \gamma$,

$$\begin{aligned} \chi_b \delta^\gamma C_{\Gamma^c} &= \sum_{\nu \subset \gamma} (-1)^{|\gamma| - |\nu|} \chi_b C_{\Gamma^c \sim \nu} \\ &= \sum_{b \in \nu \subset \gamma} (-1)^{|\gamma| - |\nu|} \chi_b C_{\Gamma^c \sim \nu} = \chi_b \delta^{\gamma \sim b} C_{\Gamma^c \sim b}. \end{aligned} \tag{VI.9}$$

Combining (VI.8) and (VI.9) we obtain (VI.7). By iterating (VI.7) we deduce

LEMMA VI.3. *Let $\gamma \subset \Gamma^c \subset \mathfrak{B}$ with $\gamma = \{b_1, \ldots, b_n\}$ and let $B = \Gamma^c \sim \gamma$. Then*

$$\begin{aligned} \delta^\gamma C_{\Gamma^c} = \sum_{\sigma \in S_n} & P_{B \cup b_{\sigma_1} \cup \cdots \cup b_{\sigma_n}} \chi_{b_{\sigma_n}} P_{B \cup b_{\sigma_1} \cup \cdots \cup b_{\sigma_{n-1}}} \\ & \cdot \chi_{b_{\sigma_{n-1}}} \cdots \chi_{b_{\sigma_2}} P_{B \cup b_{\sigma_1}} \chi_{b_{\sigma_1}} C_B. \end{aligned} \tag{VI.10}$$

REMARK. We may regard (VI.10) as an integrated form of the representation of $\delta^\gamma C_{\Gamma^c}$ as a Wiener integral over paths which cross every b_j in γ [9]. The sum over permutations σ is a sum over orders of crossing.

VI.3. *Maximum principle.* We shall make frequent use of the following maximum principle:

PROPOSITION VI.4. *Suppose that $(-\Delta + m_0^2)u = 0$ in an open region Λ, and that u is continuous in $\bar{\Lambda}$. Then if* $\sup u > 0$ *it must be attained either on $\partial \Lambda$ or at ∞, and if* $\inf u < 0$ *it must be attained on $\partial \Lambda$ or at ∞.*

PROOF. In Λ, $\Delta u = m_0^2 u$ so if $u(x) > 0$, u cannot have a maximum at x. $\square$

It follows immediately from this principle that if u satisfies $(-\Delta + m_0^2)u = 0$ in Λ, $u \geqslant 0$ on $\partial\Lambda$, and $u(x) \to 0$ as $x \to \infty$, then $u(x) \geqslant 0$ everywhere in Λ. In particular we have

COROLLARY VI.5. *P_Γ is positivity preserving.*

A positivity preserving operator A obviously satisfies $|Af| \leqslant A|f|$. If A and B are two linear operators, by $A \leqslant B$ (pointwise) we mean that $B - A$ is positivity preserving. When A and B have measurable kernels, this is equivalent to saying that $A(x,y) \leqslant B(x,y)$. If $0 \leqslant A \leqslant B$ (pointwise), then

$$|Af| \leqslant A|f| \leqslant B|f|, \tag{VI.11}$$

a relation we shall use below.

A well-known consequence of the maximum principle (see e.g. [11]) is that $0 \leqslant C_\Gamma \leqslant C_\phi$ (pointwise) and, hence,

$$0 \leqslant C_\Gamma(x,y) \leqslant ce^{-m_0|x-y|}\left(1 + \left|\ln(m_0|x-y|)\right|\right). \tag{VI.12}$$

Now clearly if $\gamma \subset \Gamma \subset \mathfrak{B}$, we have $P_\Gamma P_\gamma = P_\gamma P_\Gamma = P_\gamma$ and $\chi_\gamma P_\Gamma = \chi_\gamma$. It follows from Corollary VI.5 that for any $X \subset \mathbf{R}^2$,

$$P_\Gamma \chi_X P_\gamma \leqslant P_\Gamma P_\gamma = P_\gamma \tag{VI.13}$$

and

$$P_\Gamma \chi_\gamma = P_\Gamma \chi_\gamma P_\gamma \leqslant P_\Gamma P_\gamma = P_\gamma. \tag{VI.14}$$

These relations allow us to estimate the products occurring in (VI.10) in terms of *single bond* Poisson kernels:

LEMMA VI.6. *Let $\gamma, B \subset \mathfrak{B}$ with $\gamma = b_1 \cup \cdots \cup b_n \subset B^c$. Let $2 \leqslant i_1 < i_2 < \cdots < i_r \leqslant n$. Then*

$$\begin{aligned} &P_{B \cup b_1 \cup \cdots \cup b_n} \chi_{b_n} P_{B \cup b_1 \cup \cdots \cup b_{n-1}} \chi_{b_{n-1}} \cdots \chi_{b_2} P_{B \cup b_1} \chi_{b_1} \\ &\qquad \leqslant P_{b_{i_r}} \cdots P_{b_{i_1}} P_{b_1} \quad (\textit{pointwise}). \end{aligned} \tag{VI.15}$$

PROOF. Since all the operators involved are positivity preserving, we can use (VI.13) to drop any of the factors $Q_r \equiv P_{B \cup b_1 \cup \cdots \cup b_r} \chi_{b_r}$ (except the first) from the product $Q_n \ldots Q_1$ on the left of (VI.15). Then by (VI.14) we can dominate each remaining Q_j by the corresponding P_{b_j}. $\square$

We shall estimate each term in (VI.10) by first applying (VI.15) and then using the fact that $P_\Gamma(x,z)$ decays exponentially: To see this, suppose that x and y are on the same side of an infinite straight line λ in $\mathbf{R}^2$. Then by the method of images $C_\lambda(x,y) = C_\phi(x,y) - C_\phi(x,\bar{y})$ where $\bar{y}$ is the image of y under reflection in λ. Thus for any $\varepsilon > 0$ there is a c_ε such that for $|x - z| > \varepsilon$,

$$P_\lambda(x,z) = 2(\partial/\partial n_z)C_\phi(x,z) \leqslant c_\varepsilon m_0^{1/2} e^{-m_0|x-z|}/|x-z|^{1/2} \tag{VI.16}$$

since $C_\phi(x-z) \sim c(m_0|x-z|)^{-1/2}e^{-m_0|x-z|}$ for large $|x-z|$ and similarly for its derivatives [1].

But if Γ is an arbitrary subset of $\mathbf{R}^2$, and x is outside the convex hull $\operatorname{conv}\Gamma$ of Γ, we let λ be the straight line separating x from $\operatorname{conv}\Gamma$ with $d(x, \operatorname{conv}\Gamma) = d(x, \lambda)$. Then clearly $P_\Gamma(x,z) = (P_\lambda P_\Gamma)(x,z)$, and since $|P_\Gamma f(y)| \leqslant \sup_{z\in\Gamma}|f(z)|$ by the maximum principle, we obtain from (VI.16):

LEMMA VI.7. *For any $\delta < 1$ there is a constant $c = c(\delta)$ such that if $d(x, \operatorname{conv}\Gamma) > \frac{1}{2}$, then for any f,*

$$|P_\Gamma f(x)| \leqslant ce^{-\delta m_0 d(x,\operatorname{conv}\Gamma)} \sup_{z\in\Gamma}|f(z)|. \tag{VI.17}$$

VI.4. *The $\partial^\gamma C$ decay estimate of Glimm-Jaffe-Spencer.* We now prove the result of [9], [28] that (VI.2) holds when $\alpha_1 = \alpha_2 = 0$ (see Lemma II.1 above). As remarked after (VI.5) it is sufficient to estimate $\delta^\gamma C_{\Gamma^c}$. Consider the sum (VI.10). To each term, call it $T_\sigma(B,\gamma)$, we apply (VI.15), dropping bonds if they touch the last retained bond. From Lemma VI.7 we get precisely the decay factor $e^{-\delta m_0|l_\sigma(\gamma)|}$ as defined in (II.29) for any $\delta < 1$:

$$T_\sigma(B,\gamma) \leqslant c^{|\gamma|}e^{-\delta m_0|l_\sigma(\gamma)|} \sup_{z\in b_{\sigma_1}} C_B(z,y).$$

Summing over σ and using (VI.12) we obtain

$$\delta^\gamma C_{\Gamma^c} \leqslant c^{|\gamma|}G_1(\gamma,\delta)\log\left(1 + d(\gamma,y)^{-1}\right) \tag{VI.18}$$

where G_1 is defined in (IV.25).

To extract the G_2 factor we may reason as follows. Let b be any bond in γ, e.g. the one giving the maximum in (II.28). From (VI.5)

$$\partial^\gamma C_{\Gamma^c} = \delta^b\delta^{\gamma\sim b}C_{\Gamma^c} = \sum_{\nu\subset\gamma\sim b}(-1)^{|\nu|-|\gamma|}\delta^b C_{\Gamma^c\sim\nu}. \tag{VI.19}$$

This represents $\delta^\gamma C_{\Gamma^c}$ as a sum of $2^{|\gamma|-1}$ terms each of which may be bounded by

$$\begin{aligned}\chi_{j_1}\delta^b C_{\Gamma^c\sim\nu}\chi_{j_2} &\leqslant \chi_{j_1}P_b C_{(\Gamma^c\sim\nu)\sim b}\chi_{j_2}\\ &\leqslant ce^{-m_0 d(b,\Delta_{j_1})}\sup_{z\in b}C_\phi(z,y)\delta_{j_2}(y)\end{aligned} \tag{VI.20}$$

by (VI.17) and the bound $C_B \leqslant C_\phi$. If b touches Δ_{j_2} then $\sup_{z\in b}C_\phi(z,y)$ gives a log factor as in (VI.18); otherwise it gives a decay factor $\exp[-m_0 d(b,\Delta_{j_2})]$. Taking the geometric mean of (VI.18) and (VI.20) yields

$$\chi_{j_1}\delta^\gamma C_{\Gamma^c}\chi_{j_2} \leqslant c^{|\gamma|}G_1(\gamma,\delta)e^{-\delta m_0 d(\gamma,j)}\chi_{j_1}\log\left(1 + d(\gamma,y)^{-1}\right)\chi_{j_2} \tag{VI.21}$$

for any $\delta < \frac{1}{2}$. (VI.21) obviously implies (VI.1) in the case $\alpha_1 = \alpha_2 = 0$ with $a = 0$.

VI.5. *Estimates on $\partial C_B(x,y)/\partial y_i$.* To estimate $(\partial/\partial y_i)\delta^\gamma C_{\Gamma^c}(x,y)$ we take the y_i derivative of C_B in (VI.10) and apply the maximum principle method as in the previous subsection (using (VI.11)). However the log factor in (VI.21) becomes a $d(\gamma, y)^{-1}$ singularity which is not in any $L^p, p > 1$. It is therefore not sufficient to use the fact that P_b is a contraction from L^∞ to L^∞ and we must make a more careful analysis of the singularities of $P_b(x, z)$ and $\partial C_B/\partial y_i$: In addition to the $|z - y|^{-1}$ singularity at $z = y$, $\partial C_B(z, y)/\partial y_i$ will have (at worst) $|y - j|^{-1/2}$ singularities for j in the set $\varepsilon(B)$ of endpoints of the individual bonds in B. More precisely:

LEMMA VI.8. *Given y, z in $\mathbf{R}^2$, and $B \subset \mathfrak{B}$, let j be the point in $\varepsilon(B)$ closest to y. Then if $|z - y| \leq \frac{1}{2}$,*

$$\left|\frac{\partial C_B(z,y)}{\partial y_i}\right| \leq c\left\{\frac{1}{|z-y|} + m_0 \right.$$

(VI.22a)

$$\left. + |y - j|^{-1/2}\left[1 + m_0^{1/2} + \left(|y-j| + |z-j|\right)^{-1/2}\right]\right\}$$

where the constant c is independent of m_0, B, z, y. If $|z - y| > \frac{1}{2}$, then for any $\delta \in [0, 1)$,

(VI.22b) $$|\partial C_B(z,y)/\partial y_i| \leq c_\delta e^{-\delta m_0 |z-y|}\left(1 + |y-j|^{-1/2}\right)$$

where the constant c_δ depends only on δ.

REMARK. By (VI.6), the bonds (VI.22) apply to $P_B(z, y)$.

PROOF. We first consider a special case of (VI.22a) where the Green's function can be explicitly evaluated by conformal mapping, namely $m_0 = 0$ and $B = \mathbf{R}^+$. In this case the conformal mapping $\zeta = z^{1/2}$ maps B onto $\mathbf{R}$ and $\mathbf{C} \sim B$ onto the upper half-plane. By the method of images, the corresponding Green's function is

$$C^{(0)}_{\mathbf{R}^+}(z, y) = C^{(0)}_{\mathbf{R}}(\zeta, \eta) = \ln\left(|\bar{\zeta} - \eta| / |\zeta - \eta|\right)$$

where $\zeta = z^{1/2}$, $\eta = y^{1/2}$. Since $|d\eta/dy| = \frac{1}{2}|y|^{-1/2}$ and $|\partial C^{(0)}_{\mathbf{R}}(\zeta, \eta)/\partial \eta_i| \leq c|\zeta - \eta|^{-1}$, we thus have

(VI.23) $$|\partial C^{(0)}_{\mathbf{R}^+}(z,y)/\partial y_i| \leq c|y|^{-1/2}|\zeta - \eta|^{-1}.$$

The proof of (VI.22a) will then be completed by the inequality

(VI.24) $$|y|^{-1/2}|\zeta - \eta|^{-1} \leq c\left[|z-y|^{-1} + |y|^{-1/2}\left(|y| + |z|\right)^{-1/2}\right].$$

To prove (VI.24) we note first that if $|\arg \zeta - \arg \eta| \geq \pi/2$ then $|\zeta - \eta|^2 \geq |\zeta|^2 + |\eta|^2 = |z| + |y|$, so that

(VI.25) $$|y|^{-1/2}|\zeta - \eta|^{-1} \leq |y|^{-1/2}\left(|y| + |z|\right)^{-1/2}.$$

On the other hand, if $|\arg \zeta - \arg \eta| < \pi/2$, we have

$$\text{(VI.26)} \qquad \frac{1}{|y|^{1/2}|\zeta-\eta|}-\frac{2}{|z-y|}=\frac{1}{|y|^{1/2}|\zeta+\eta|}\,\frac{|\zeta+\eta|-2|\eta|}{|\zeta+\eta-2\eta|}.$$

The second factor in (VI.26) has magnitude less than 1 and now $|\zeta+\eta| \geqslant (|z|+|y|)^{1/2}$; hence (VI.25) and (VI.26) imply (VI.24).

By similar conformal mapping techniques we can establish (VI.22a) for the special case where $m_0 = 0$ and where $B = B_0$, a union of 0 to 4 of the semiaxes. In this case there will actually be no $|y|^{-1/2}$ singularity if y is enclosed in an angle of $\pi/2$ or π between two axes, and only a $|y|^{-1/3}$ singularity if the enclosing angle is $3\pi/2$. Also if $B = B_1$, a union of 0 to 4 unit bonds emanating from the origin, and if the origin is the closest lattice point to y, conformal mapping shows that

$$\text{(VI.27)} \qquad \left|\partial C^{(0)}_{B_1}(z,y)/\partial y_i\right| \leqslant c|y|^{-1/2}$$

provided $|z| \geqslant 1$ so that z is bounded away from 0 and y.

Consider now the case of general $B \subset \mathfrak{B}$. If j is the lattice point closest to y, we let B_1 be the set of unit bonds in B touching j and B_0 the union of the rays in B_1 extended from j to ∞. Then since $B_1 \subset B$ we have $C^{(0)}_B = C^{(0)}_{B_1} - P^{(0)}_B C^{(0)}_{B_1}$ as in §VI.2. Similarly $C^{(0)}_{B_0} = C^{(0)}_{B_1} - P^{(0)}_{B_0} C^{(0)}_{B_1}$ so that

$$\text{(VI.28)} \qquad \begin{aligned} C^{(0)}_B(z,y) - C^{(0)}_{B_0}(z,y) &= \int_{B_0\sim B_1} dx\, P^{(0)}_{B_0}(z,x)C^{(0)}_{B_1}(x,y) \\ &\quad - \int_{B\sim B_1} dx\, P^{(0)}_B(z,x)C^{(0)}_{B_1}(x,y). \end{aligned}$$

On the right side of (VI.28), the variable x satisfies $|x-j| \geqslant 1$ so that differentiating (VI.28) with respect to y_i and using (VI.27), we can dominate the right side of (VI.28) by $c|y-j|^{-1/2}$ by the maximum principle. It follows from (VI.28) that $C^{(0)}_B(z,y)$ satisfies the bound (VI.22a):

(VI.29)

$$\left|\frac{\partial C^{(0)}_B(z,y)}{\partial y_i}\right| \leqslant c\left\{\frac{1}{|z-y|}+|y-j|^{-1/2}\left[1+(|y-j|+|z-j|)^{-1/2}\right]\right\}.$$

Next we extend this result to $m_0 > 0$ by means of the resolvent formula

$$\text{(VI.30)} \qquad \frac{\partial C_B(z,y)}{\partial y_i} - \frac{\partial C^{(0)}_B(z,y)}{\partial y_i} = -m_0^2\int C_B(z,x)\frac{\partial C^{(0)}_B(x,y)}{\partial y_i}\,d^2x.$$

Using the bounds (VI.29) and $C_B \leqslant C_\phi$, we find that the right side of (VI.30) can be dominated by

$$\text{(VI.31)} \qquad cm_0^2\int C_\phi(z,x)\left\{|x-y|^{-1}+|y|^{-1/2}\left[1+(|x|+|y|)^{-1/2}\right]\right\}d^2x$$

where we have set $j = 0$. By making the change of variable $\xi = m_0 x$, we can estimate (VI.31) by $c(m_0 + m_0^{1/2}/|y|^{1/2})$ where the constant c is independent of m_0. Thus from (VI.29) and (VI.30) we obtain

$$\left|\frac{\partial C_B(y,z)}{\partial y_i}\right| \leqslant c\left\{\frac{1}{|z-y|} + m_0 + |y|^{-1/2}\left[1 + m_0^{1/2} + (|y| + |z|)^{-1/2}\right]\right\}$$

as desired.

Finally, (VI.22b) follows from (VI.22a). For let D be the disk of radius ε and centre y. Then in $\mathbf{R}^2 \sim D \sim B$, $u(z) = \partial C_B(z,y)/\partial y_i$ satisfies

$$u(z) = \int_{\partial D} P_{B \cup \partial D}(z,x) \frac{\partial C_B(x,y)}{\partial y_i}\, dx. \tag{VI.32}$$

Consequently by (VI.11) and (VI.14)

$$\begin{aligned} |u(z)| &\leqslant \int_{\partial D} P_{\partial D}(z,x) \left|\frac{\partial C_B(x,y)}{\partial y_i}\right| dx \\ &\leqslant ce^{-\alpha m_0(|z-y|-\varepsilon)}\left[\varepsilon^{-1} + m_0 + |y-j|^{-1/2}\left(1 + m_0^{1/2} + \varepsilon^{-1/2}\right)\right] \end{aligned} \tag{VI.33}$$

for any $\alpha < 1$, where we have applied Lemma VI.7 and (VI.22a). Clearly (VI.33) implies (VI.22b) (the powers of m_0 can be absorbed in the exponential decay factor). □

VI.6. *Estimates on* $\partial \delta^\gamma C_{\Gamma^c}(x,y)/\partial y_i$. When γ is empty, the estimate (VI.1) for $\partial C_{\Gamma^c}/\partial y_i$ follows immediately from (VI.22). We move to the case where $|\gamma| = 1$, i.e. γ consists of a single bond b. Now by (VI.10) and (VI.15)

$$\left|\frac{\partial}{\partial y_i} \delta^b C_{\Gamma^c}(x,y)\right| \leqslant \int_b dz\, P_b(x,z) \left|\frac{\partial}{\partial y_i} C_B(z,y)\right| \equiv I(x,y) \tag{VI.34}$$

where $B = \Gamma^c \sim b$. If b does not touch Δ_{j_2}, then $|z - y| \geqslant 1$ in (VI.34) and we may apply (VI.22b) to bound $\partial C_B/\partial y_i$ and (VI.17) to bound P_{Γ^c}. In this way estimate (VI.34) by (inserting $\chi_{j_1}(x)$ and $\chi_{j_2}(y)$)

$$I(x,y)\chi_{j_1}(x)\chi_{j_2}(y) \leqslant c_\delta \chi_{j_1}(x)\chi_{j_2}(y) e^{-\delta m_0 d(b,j)}\left(1 + |y - j_0|^{-1/2}\right) \tag{VI.35}$$

where $\delta < 1$ and j_0 is the point in $\varepsilon(B)$ closest to y. Clearly, (VI.35) is in L^{4-} (i.e. in any L^p, $p < 4$) with the bound (VI.1) on its L^p norm. Since $p_1(1) = 3 < 4$ we have established Theorem VI.1 for this case.

Suppose next that b does touch Δ_{j_2}. We may assume that the origin is at an end of b touching Δ_{j_2} and that the origin is the point of Z^2 closest to y (if not we have the bound (VI.35) as above). Suppose that x_1 labels the coordinate parallel to b ($x_1 > 0$ on b) and x_2 the perpendicular coordinate. Then by (VI.22a) we have

$$|\partial C_B(z,y)/\partial y_i| \leqslant c\left[f(z,y) + g(y; m_0)\right], \tag{VI.36}$$

where

$$f(z,y) = |z-y|^{-1} + |y|^{-1/2}(|y| + |z|)^{-1/2} \tag{VI.37a}$$

and

(VI.37b) $$g(y; m_0) = m_0 + |y|^{-1/2}(1 + m_0^{1/2}).$$

As in (VI.35); the g in (VI.36) contributes a term to $I(x,y)$ which is in L^{4-} with appropriate G_2 decay (and whose m_0 dependence is given at worst by a factor of m_0). To estimate the contribution of $f(z,y)$ to $I(x,y)$ we apply (VI.22a) to $P_b(x, z)$ (note that there is also a singularity at the other endpoint $j = (1, 0)$ of b):

(VI.38) $$P_b(x, z) \le c\big[f(x, z) + g(z; m_0) + f(x - j, z - j) + g(z - j; m_0)\big]$$

where $z = (z_1, 0)$. We insert the bounds (VI.36) and (VI.38) into the definition (VI.34) of $I(x,y)$. Since we have assumed that y is closer to 0 than to j, the most singular term in $I(x,y)$ is

(VI.39) $$J(x,y) = \int_b f(x, z) f(z,y)\, dz_1$$

i.e. the contribution from the first term in (VI.38).

In order to estimate $J(x,y)$ we use the following inequalities:

(VI.40) $$1/|x| \le (1/|x_1|^{a_1})(1/|x_2|^{a_2}),$$

(VI.41) $$1/(|x| + |y|) \le (1/|x_1|^{a_1})(1/|x_2|^{a_2}),$$

for any $a_i \ge 0$ with $a_1 + a_2 = 1$, and

(VI.42) $$\int_{-\infty}^{\infty} \frac{du}{|u - v|^p |u|^q} \le \text{const}|v|^{1-p-q}$$

for any $p, q \in (0, 1)$ with $p + q > 1$. For example, we have by (VI.40) and (VI.42),

(VI.43) $$\int_0^1 dz_1 \frac{1}{|x - z|\,|z - y|} \le c|x_2|^{-r}|x_1 - y_1|^{-s}|y_2|^{-t}$$

where c depends on r, s, t which must satisfy

(VI.44a) $$r, s, t > 0, \qquad r + s + t = 1.$$

We note that in (VI.43) we may take $s = 0$ provided we choose $r + t > 1$, and we may take $r = 0$ if x_1 is bounded away from [0, 1]. As another example, we have, by (VI.40) and (VI.41),

$$\int_0^1 dz_1\, z_1^{-1/2}(|x| + z_1)^{-1/2}|y|^{-1/2}(|y| + z_1)^{-1/2}$$
$$\le c|x_1|^{-a_1}|x_2|^{-a_2}|y_1|^{-b_1}|y_2|^{-b_2}$$

where c depends on a_i, b_i, and

(VI.44b) $$a_i \ge 0, \quad b_i \ge 0, \quad a_1 + a_2 > 0,$$
$$b_1 + b_2 > \tfrac{1}{2}, \quad a_1 + a_2 + b_1 + b_2 > 1.$$

Altogether we find that

$$\text{(VI.45)} \qquad J(x,y) \leqslant c\left[|x_2|^{-r}|x_1 - y_1|^{-s}|y_2|^{-t} + |x_1|^{-a_1}|x_2|^{-a_2}|y_1|^{-b_1}|y_2|^{-b_2}\right]$$

where the indices satisfy (VI.44). If we now satisfy these constraints by choosing $r = s = t = \frac{1}{3}$, $a_1 = a_2 = \frac{1}{4}$ and $b_1 = b_2 = \frac{1}{4}+$, we see that the two terms in (VI.45) are in L^{3-}_{loc} and L^{4-}_{loc}, respectively.

By similar reasoning we see that the term $f(x-j, z-j)$ in (VI.38) makes an additional contribution to the bound on $I(x,y)$ of the form

$$c|x_1 - 1|^{-a_1}|x_2|^{-a_2}|y_1|^{-b_1}|y_2|^{-b_2}$$

where the singularity is milder than (VI.45) in the sense that a_i, b_i satisfy (VI.44b) without the constraint $a_1 + a_2 + b_1 + b_2 > 1$. The g's in (VI.38) lead to terms with no x singularities, but with an m_0 factor. We conclude that

$$\text{(VI.46)} \qquad \|\chi_{j_1} I \chi_{j_2}\|_{L^{3-}} \leqslant c m_0 e^{-\delta m_0 d(b, j_1)}$$

for any $\delta < 1$. The G_2 factor in (VI.46) has been extracted just as in (VI.33), i.e. we enclose b in a small convex set D and let the "boundary values" $I(x,y)$ for $x \in \partial D$ determine $I(x,y)$ for all x. These considerations establish Theorem VI.1 in the case $|\gamma| = 1$.

Finally we turn to the case $|\gamma| \geqslant 2$. As in §VI.4 we use (VI.15) to drop bonds from each term in (VI.10) if a bond touches the last retained bond. We consider the case where the first bond b touches Δ_{j_2}, for otherwise we simply appeal to (VI.35). Suppose that b' is the next bond which does not touch b. We wish to estimate

$$\text{(VI.47)} \qquad I'(x,y) = \int_{b'} dz\, P_{b'}(x,z) I(z,y)$$

where I is defined in (VI.34). Since $\text{dist}(z,b) \geqslant 1$ in (VI.47), $I(z,y)$ has no singularities in z but only in y. More precisely,

$$\text{(VI.48)} \qquad I(z,y) \leqslant c m_0 |y_1|^{-b_1}|y_2|^{-b_2}$$

where $b_i > 0$ and $b_1 + b_2 > \frac{1}{2}$. To see this, consider the most singular contribution $J(z,y)$ to $I(z,y)$ as estimated in (VI.45). Since b' does not touch b, $z \in b'$ must satisfy $|z_1 - \frac{1}{2}| \geqslant \frac{3}{2}$ or $|z_2| \geqslant 1$ or both. If $|z_1 - \frac{1}{2}| \geqslant \frac{3}{2}$, then in (VI.45) we choose $r = 0$, $s = 1$, $t = 0+$, $a_1 = \frac{1}{2}$, $a_2 = 0$, $b_1 + b_2 > \frac{1}{2}$; if $|z_2| \geqslant 1$, then we choose $r = 1$, $s = 0$, $t = 0+$, $a_1 = 0$, $a_2 = \frac{1}{2}$, $b_1 + b_2 > \frac{1}{2}$ (see the statement following (VI.44a)). With these choices $J(z,y)$ satisfies the bound (VI.48). Similarly so do the other contributions to I.

Choosing $b_1 = b_2 = \frac{1}{4}+$ in (VI.48) we see that $I(z,y)$ is in L^{4-}_{loc} in y uniformly in z bounded away from b. Extracting the G_1 and G_2 factors as in §VI.4, we obtain the bound (VI.1). There is still one case remaining in the proof of Theorem VI.1 for $\alpha_1 = 0$, $\alpha_2 = 1$, namely the case where every other bond $b' \in \gamma$ touches the first bond b in the ordering $l_\sigma(\gamma)$, i.e. $|l_\sigma(\gamma)| = 0$. In this case we can show that for b' touching b, (VI.47) satisfies

$$I'(x,y) \leqslant c x_1^{-a_1} x_2^{-a_2} y_1^{-b_1} y_2^{-b_2} \tag{VI.49}$$

where $a_i \geqslant 0$, $b_i \geqslant 0$, $b_1 + b_2 > \frac{1}{2}$, $a_1 + a_2 + b_1 + b_2 > 1$, so that the choice $a_1 = a_2 = b_1 = b_2 = \frac{1}{4}+$ gives $I' \in L^{4-}_{\text{loc}}$ as desired. We omit the proof of (VI.49) since the calculations involve nothing new.

VI.7. *Estimates on* $\partial^2 \delta^\gamma C_{\Gamma^c}(x,y)/\partial x_{i_1} \partial y_{i_2}$. Obviously the maximum principle techniques of §§VI.4, VI.6 do not allow us to take an x derivative. Instead we note that $u(x) = (\partial/\partial y_{i_2})\delta^\gamma C_{\Gamma^c}(x,y)$ satisfies $(-\Delta + m_0^2)u = 0$ for $x \notin \Gamma^c$ and we apply the following lemma (proved at the end of this subsection).

LEMMA VI.9. *Suppose that u satisfies $(-\Delta + m_0^2)u = 0$ in the disk $D = \{x \mid |x - z| < R\}$. Then there are constants c_1 and c_2 independent of m_0 such that*

$$|\partial u(z)/\partial x_i| \leqslant (c_1 + c_2 R^{-1}) \max_{x \in D} |u(x)|. \tag{VI.50}$$

By this lemma, $v(x,y) \equiv \chi_{j_1}\chi_{j_2}\partial^2\delta^\gamma C_{\Gamma^c}(x,y)/\partial x_{i_1}\partial y_{i_2}$ will satisfy a bound of the form (for small $R = d(x, \Gamma^c)$)

$$|v(x,y)| \leqslant c^{|\gamma|} m_0 G(\gamma, j, \delta) d(x, \Gamma^c)^{-1} s(x,y) \tag{VI.51}$$

where $G(\gamma, j, \delta) = G_1(\gamma, \delta) e^{-\delta m_0 d(\gamma, j)}$ and $s(x,y)$ contains the singularities of $u(x)$ as studied in the previous section (see e.g. (VI.45), (VI.48), or (VI.49)). Now the x singularity in (VI.51) is worse than the y singularity. However, we note that x and y play symmetric roles in $v(x,y)$ so that we can obtain a second bound on v in which the roles of x and y are interchanged and we can then take the geometric mean of these two bounds. In addition, we choose the exponents in $s(x,y)$ in such a way as to minimize the factor of $d(x, \Gamma^c)^{-1}$ in $s(x,y)$. For example, consider the singularity (VI.49) that arose in the case $|\gamma| \geqslant 2$ and suppose that $d(x, \Gamma^c) = |x_2|$. Then we choose $a_2 = b_2 = 0$, $a_1 = \frac{1}{2}$, $b_1 = \frac{1}{2}+$ so that

$$\begin{aligned} |v(x,y)| &\leqslant c^{|\gamma|} m_0 G \left[\frac{1}{|x_2|\,|x_1|^{1/2}|y_1|^{1/2+}} \; \frac{1}{|y_2|\,|y_1|^{1/2}|x_1|^{1/2+}} \right]^{1/2} \\ &= c^{|\gamma|} m_0 G \frac{1}{|x_1|^{1/2+}|x_2|^{1/2}|y_1|^{1/2+}|y_2|^{1/2}} \end{aligned}$$

which is in L^{2-}_{loc}. Actually the singularity (VI.49) is the worst that occurs for $|\gamma| \geqslant 2$ and so we have verified (VI.1) for $|\gamma| \geqslant 2$. For $|\gamma| = 1$ the worst singularity is given by the first term in (VI.45), and by the above procedure we find that it gives a term in $L^{3/2-}_{\text{loc}}$. As for γ empty, $\partial^2 C/\partial x_{i_1}\partial y_{i_2}$ has a singularity $|x - y|^{-2}$ which is in L^{1-}. These somewhat brief remarks complete our proof of Theorem VI.1.

PROOF OF LEMMA VI.9. Without loss of generality, we take $z = 0$ and $i = 1$.

By calculating the Green's function for $-\Delta + m_0^2$ for a disk of radius $r' < R$ in terms of modified Bessel functions we can express $u(x)$ in terms of its boundary values on the circle of radius r':

$$u(r,\theta) = \frac{1}{2\pi} \sum_{l=-\infty}^{\infty} \int e^{il(\theta-\theta')} \frac{I_l(m_0 r)}{I_l(m_0 r')} u(r',\theta')\, d\theta' \tag{VI.52}$$

where $r < r'$ and I_l is the modified Bessel function which is finite at the origin; in fact $I_l(0) = 0$ for $l \neq 0$ and $I_0(0) = 1$ (see [1]). Putting $r = 0$ in (VI.52) we thus obtain the mean value property

$$u(0) = \frac{1}{2\pi I_0(r')} \int_0^{2\pi} u(r',\theta')\, d\theta'$$

or, integrating over r' from 0 to R,

$$u(0) = \frac{1}{\pi R^2} \int_{|x|<R} \frac{u(x)}{I_0(m_0|x|)}\, d^2x. \tag{VI.53}$$

(Strictly speaking we should integrate over $|x| \leqslant R' < R$ and later take $R' \to R$.) But $\partial u/\partial x_i$ also satisfies the homogeneous equation in D so that by (VI.53),

$$\begin{aligned}
\frac{\partial u}{\partial x_1}(0) &= \frac{1}{\pi R^2} \int_{|x|<R} \frac{\partial u}{\partial x_1}(x) I_0(m_0|x|)^{-1}\, d^2x \\
&= -\frac{1}{\pi R^2} \int_{|x|<R} u(x) \frac{\partial}{\partial x_1} I_0(m_0|x|)^{-1}\, d^2x \\
&\quad + \frac{1}{\pi R^2 I_0(m_0 R)} \int_{-R}^{R} \Big[u\big((R^2 - x_2^2)^{1/2}, x_2\big) \\
&\qquad\qquad - u\big(-(R^2 - x_2^2)^{1/2}, x_2\big)\Big]\, dx_2
\end{aligned}$$

by integration by parts. Now $I_0(m_0 R)^{-1}$ and $(\partial/\partial x_1) I_0(m_0|x|)^{-1}$ are uniformly bounded in x and R and so by straightforward estimates we obtain (VI.50). □

We conclude this subsection with a proof of Remark 2 following Theorem VI.1:

LEMMA VI.10. *For $\alpha_1, \alpha_2 = 0$ or 1 let $p < 3/(\alpha_1 + \alpha_2)$. Then*

$$\left\| \chi_{j_1}\chi_{j_2} \frac{\partial^{\alpha_1+\alpha_2}}{\partial x_{i_1}^{\alpha_1} \partial y_{i_2}^{\alpha_2}} \delta C \right\|_{L^p} \leqslant c m_0 e^{-\delta m_0 d(j_1, j_2)} \tag{VI.54}$$

where $\delta C = C(s) - C_\Phi$. The same estimate holds when δC is replaced by $\delta C(s)$ provided that $j_1 \neq j_2$.

REMARK. The analogue of (VI.2) also holds for δC.

PROOF. By (II.15) δC is a convex combination of differences $C_{\Gamma^c} = C_{\Gamma^c} - C_\phi$ so that it suffices to prove (VI.54) for δC_{Γ^c}. But as in §VI.3,

$$|\delta C_{\Gamma^c}| = |P_{\Gamma^c} C_\phi| = \left| \sum_{b \in \Gamma^c} P_{\Gamma^c} \chi_b C_\phi \right| \leqslant \sum_{b \in \Gamma^c} P_b C_\phi. \tag{VI.55}$$

The sum in (VI.55) converges because of the exponential decay of P_b or C_ϕ. We can estimate y derivatives of (VI.55) as in §VI.6 for $|\gamma| = 1$ and x derivatives as above in order to obtain (VI.54).

If $j_1 \neq j_2$ then it is easy to check using (VI.40) – (VI.41) that (VI.54) holds with δC replaced by C_ϕ: the $|x - y|^{-\alpha_1 - \alpha_2}$ singularity of $(\partial^{\alpha_1+\alpha_2}/\partial x_{i_1}^{\alpha_1} \partial y_{i_2}^{\alpha_2}) C_\phi(x, y)$ is in L^p with $p < 3/(\alpha_1 + \alpha_2)$ since x and y are in different lattice squares. Hence (VI.54) must hold for $C(s) = \delta C + C_\phi$. □

VI.8. *A decomposition lemma*. The major role played by Theorem VI.1 is to control s derivatives $\partial^\gamma S$ of fermi covariances. However, the following consequence of Theorem VI.1 is used to control s derivatives of boson covariances (see (IV.27)):

LEMMA VI.11. *Suppose $|\gamma| \geqslant 2$, $p < \infty$, $\varepsilon < 1$. There exist positive operators $C_{\gamma,\pm}$ on $L^2(\mathbf{R}^2)$ such that $\partial^\gamma C = C_{\gamma,+} - C_{\gamma,-}$ with*

$$\|C_{\gamma,\pm}\|_{L^p} \leqslant c^{|\gamma|} m_0 G_1(\gamma, \delta) \tag{VI.56}$$

and

$$\|D^\varepsilon C_{\gamma,\pm} D^\varepsilon\|_{L^2} \leqslant c^{|\gamma|} m_0 G_1(\gamma, \delta). \tag{VI.57}$$

PROOF. Let $B = D^\varepsilon \partial^\gamma C D^\varepsilon$. If $\{\xi_j\}$ is a smooth partition of unity, we see from Corollary VI.2 that

$$\begin{aligned} \|B\|_{L^2(\mathbf{R}^4)} &\leqslant \sum_{j_1 j_2} \|D^\varepsilon \xi_{j_1} \partial^\gamma C \xi_{j_2} D^\varepsilon\|_{L^2(\mathbf{R}^4)} \\ &\leqslant m_0^3 c^{|\gamma|} G_1(\gamma, \delta). \end{aligned} \tag{VI.58}$$

That is, B is a Hilbert-Schmidt operator on $L^2(\mathbf{R}^2)$. Thus $B_\pm$, the positive and negative parts of B (in the operator sense), also satisfy the bound (VI.58). We define $C_{\gamma,\pm} = D^{-\varepsilon} B_\pm D^{-\varepsilon}$ so that $\partial^\gamma C = C_{\gamma,+} - C_{\gamma,-}$ and $C_{\gamma,\pm}$ satisfy (VI.57). But (VI.57) implies (VI.56) by the following lemma. □

LEMMA VI.12. *Let*

$$q^{-1} < r^{-1} + \varepsilon/2 \quad \textit{where } 0 < \varepsilon < 2.$$

Then $D_x^{-\varepsilon} D_y^{-\varepsilon}$ is a bounded operator from $L^q(\mathbf{R}^4)$ to $L^r(\mathbf{R}^4)$ with norm bounded by $c m_0^{-4(\varepsilon/2 + r^{-1} - q^{-1})}$.

PROOF. $D^{-\varepsilon}$ is given by convolution with a function $m_0^{2-\varepsilon} d_{-\varepsilon}(m_0|x|)$ which is exponentially decreasing at ∞ and has a singularity $|x|^{\varepsilon - 2}$ at $x = 0$. Thus for any $p < 2(2 - \varepsilon)^{-1}$, (VI.59) $$\|d_{-\varepsilon}(m_0|x|)\|_{L^p(\mathbf{R}^2)} \leqslant c m_0^{-2/p}.$$

Therefore, by Young's inequality,

$$\|D_x^{-\varepsilon} D_y^{-\varepsilon} f(x,y)\|_{L^r(\mathbf{R}^4)} \leqslant c m_0^{4-2\varepsilon-4p^{-1}} \|f\|_{L^q(\mathbf{R}^4)}$$

where $p^{-1} + q^{-1} = 1 + r^{-1}$ and $q^{-1} - r^{-1} < 1 + (\varepsilon - 2)/2 = \varepsilon/2$. □

VII. Estimates.

VII.1. *Introduction*. In this section we establish the estimates used in §§III and IV. In particular, in §VII.2 we study the operators $K(s)$, $\partial^\gamma K(s)$, and $\delta K(s) \equiv K(s) - K(1)$ on $\mathcal{H} = \mathcal{H}_{1/2} \oplus \mathcal{H}_{1/2}$ and show that they are in certain $\mathcal{C}_{p,q;s}$ classes (defined in (III.12)). In §VII.3 we obtain estimates on quadratic expressions such as $B(s)$ (defined in (III.8)). In §VII.4 we prove the "linear lower bound" for $\rho_\Lambda(s)$. Our method of proof is basically that of McBryan [16] in that it involves momentum expansions but we obtain a considerable simplification by using the formalism of §III to control the terms in the expansion. Finally we use the estimates of §§VII.2–VII.3 to complete the proof of decoupling at $s = 0$ (in §VII.5) and to justify the formulas for s derivatives (in §VII.6).

There is a basic complication involved in the estimates for s B.C. In Seiler [22] and subsequent work on Y_2 frequent use was made of the elementary identity

(VII.1) $$(-\not{p} + m_0)\, C\, (\not{p} + m_0) = 1$$

for $C = C_\phi$. Since $C(s)$ is not diagonal in momentum space, (VII.1) is not available for s B.C. when $s \neq 1$. As a replacement for (VII.1) we shall systematically write

$$C(s) = C_\phi + \delta C(s).$$

We then use (VII.1) on the C_ϕ term and the local L^p estimates of the previous section on the δC term, exploiting the fact that $\partial^2 \delta C(s)/\partial x_i \partial y_j$ has better L^p properties than $\partial^2 C(s)/\partial x_i \partial y_j$ (see Lemma VI.10).

VII.2. *Properties of K*. In this section we study the operator-valued function of ϕ,

(VII.2) $$K(\Lambda, s) = S_0(s)\phi\chi_\Lambda = C(s)(\not{p} + m_0)\phi\chi_\Lambda,$$

as well as the related operators δK and $\partial^\gamma K$ defined as in (VII.2) but with $C(s)$ replaced by $\delta C(s)$ and $\partial^\gamma C(s)$, respectively. We intend to calculate products, adjoints, and traces of these "operators" using their integral kernels, e.g.,

(VII.3) $$\mathrm{Tr}_{\mathcal{H}}(K\delta K) = \int \mathrm{tr}\, K(x,y)\delta K(y,x)\, dx\, dy$$

where tr denotes the trace over spinor indices. Why are such operations justified? In certain cases, e.g. in the calculation of $\mathrm{Tr}|\delta K|^2$ as in Lemma VII.1, we are justified in the use of a formula like (VII.3) because we can show that δK is Hilbert-Schmidt on $\mathcal{H}$ for almost all ϕ. But this assertion that $\delta K \in \mathcal{C}_2(\mathcal{H})$ a.e. or the identity (VII.3) must ultimately be justified by passing to momentum cutoff objects:

$$K_{\sigma,\kappa}(\Lambda, s) = S_{0,\sigma}(s)\phi_\kappa\chi_\Lambda \tag{VII.4}$$

where σ denotes the fermion and κ the boson cutoff, $S_{0,\sigma}$ is defined in (III.9), and $\phi_\kappa(x) \equiv \int\phi(y)h_\kappa(x-y)\,dy$ in terms of the same smooth function h used in (III.9).

Since $K_{\sigma,\kappa}$, $\delta K_{\sigma,\kappa}$ are Hilbert-Schmidt we are justified in using formulas like (VII.3) with $\sigma, \kappa < \infty$ and we then pass to the limit $\sigma, \kappa = \infty$. In a number of the following calculations where this limiting procedure is fairly clear we shall simply deal directly with the case $\sigma, \kappa = \infty$. However, we wish to emphasize: expressions like $B = \frac{1}{2}{:}\mathrm{Tr}(K^2 + K^\dagger K){:}$ or ${:}\mathrm{Tr}(K + K^*)^2{:}$ have no meaning independent of the limiting procedure $\sigma \to \infty$.

For $j = (j_1, j_2) \in \mathbf{Z}^4$ and $\varepsilon = (\varepsilon_1, \varepsilon_2) \in \mathbf{R}^2$, we introduce

$$T_{\varepsilon,j} = D^{\varepsilon_1}\zeta_{j_1}K\chi_{j_2}D^{\varepsilon_2} \tag{VII.5}$$

as well as the related operators $\partial^\gamma T_{\varepsilon,j}$ and $\delta T_{\varepsilon,j}$ in which K in (VII.5) is replaced by $\partial^\gamma K$ and δK, respectively. Operators such as (VII.4) and (VII.5) are linear in ϕ. If $T_1, \ldots, T_r$ are r such operators on $\mathcal{H}$ then $\mathrm{Tr}_{\mathcal{H}}(T_1 \ldots T_r)$ will have the form $\int w(x_1, \ldots, x_r)\phi(x_1)\ldots\phi(x_r)\,dx$; we refer to w as the *kernel* of $\mathrm{Tr}_{\mathcal{H}}(T_1 \ldots T_r)$. We use the notation L^{p-} to denote the functions in L^q for any $q < p$; when we write $\|w\|_{L^{p-}} \leqslant c$ we mean that $\|w\|_{L^q} \leqslant c(q)$ for $q < p$, where $c(q)$ may diverge as $q \to p$. The following two lemmas are basic to this section.

LEMMA VII.1. *Given* $\varepsilon_1, \varepsilon_2 < \frac{1}{6}$, *let* $p_0 = 12/(11 + 6\max(\varepsilon_1, \varepsilon_2, -\frac{1}{2}))$.

(a) *The kernel* w *of* $\|\partial^\gamma T_{\varepsilon,j}\|_2^2 = \mathrm{Tr}(\partial^\gamma T_{\varepsilon,j}^*\partial^\gamma T_{\varepsilon,j})$ *is in* $L^{p_0-}(\mathbf{R}^4)$ *and*

$$\|w\|_{L^{p_0-}} \leqslant c m_0^a G_1(\gamma, \delta)e^{-\delta m_0 d(\gamma,j)}. \tag{VII.6a}$$

(b) *The kernel* w *of* $\|\delta T_{\varepsilon,j}\|_2^2$ *is also in* L^{p_0-} *with*

$$\|w\|_{L^{p_0-}} \leqslant c m_0^a e^{-\delta m_0 d(j_1,j_2)}. \tag{VII.6b}$$

REMARKS. 1. We are using the same notation as in the previous sections: c, a, and δ will represent various positive constants as explained after (IV.6); G_1 is defined in (IV.25), $d(\gamma, j)$ in (II.28), and $d(j_1, j_2)$ in (IV.42).

2. Assuming that $\Delta_{j_2} \subset \Lambda$ (otherwise $T_{\varepsilon,j} = 0$), we can omit the χ_Λ in $K(\Lambda, s)$.

PROOF. (a) Recalling that the adjoints T^* on $\mathcal{H}_{1/2}$ and $T^\dagger$ on L^2 are related by $T^* = D^{-1}T^\dagger D$, we have

$$\mathrm{Tr}(\partial^\gamma T_{\varepsilon,j}^*\partial^\gamma T_{\varepsilon,j}) = \mathrm{Tr}\left(D^{-1+2\varepsilon_2}\phi\chi_{j_2}\partial^\gamma S^\dagger\zeta_{j_1}D^{1+2\varepsilon_1}\zeta_{j_1}\partial^\gamma S\chi_{j_2}\phi\right). \tag{VII.7}$$

Letting p_2 be multiplication by m_0, we see from (VII.7) that $w(x, y)$ is given by a sum over $i, i' = 0, 1, 2$ of terms of the form

$$c_{i,i'}D^{-1+2\varepsilon_2}(x-y)\left(B_i^{\dagger}D^{1+2\varepsilon_1}B_{i'}\right)(x,y) \tag{VII.8}$$

where $D^{\alpha}(x-y)$ is the kernel of the operator D^{α}, $c_{ii'}$ is a constant arising from the trace over spinor indices, and B_i is the operator with kernel

$$B_i(x,y) = \zeta_{j_1}(x)(\partial^{\gamma}Cp_i)(x,y)\chi_{j_2}(y) \tag{VII.9}$$

where p_i acts on the argument y of $\partial^{\gamma}C$.

If we now write

$$D^{1+2\varepsilon_1} = \sum_{k=0}^{2} p_k D^{-1+2\varepsilon_1}p_k, \tag{VII.10}$$

then (VII.8) becomes a sum of terms of the form

$$b_{i,i'}D^{-1+2\varepsilon_2}(x-y)\int \overline{C_{ki}(x',x)}\, D^{-1+2\varepsilon_1}(x'-y')C_{ki'}(y',y)\, dx'\, dy' \tag{VII.11}$$

where $C_{ki} = p_k\zeta_{j_1}(\partial^{\gamma}Cp_i)\chi_{j_2}$. When $k = 0, 1$, the $p_k = i\partial/\partial x_{\kappa}$ in C_{ki} acts either on $\partial^{\gamma}C(x,y)$ or on $\zeta_{j_1}(x)$, but in either case we know from Theorem VI.1 that for any $r < \frac{3}{2}$,

$$\|C_{k,i}\|_{L^r(\mathbf{R}^4)} \leq m_0^a c^{|\gamma|}G_1(\gamma,\delta)e^{-\delta m_0 d(\gamma,j)}. \tag{VII.12}$$

But the x' and y' integrations in (VII.11) take place over a finite area, so that we can bound the $L^p(\mathbf{R}^4)$ norm of (VII.11) by

$$\begin{aligned} c\Big\{\int |C_{ki}(x',x)|^p\big|D^{-1+2\varepsilon_2}(x-y)D^{-1+2\varepsilon_1}(x'-y')\big|^p \\ \cdot |C_{ki'}(y',y)|^p\, dx\, dx'\, dy\, dy'\Big\}^{1/p} \end{aligned} \tag{VII.13}$$

$$\leq c\|C_{ki}\|_{L^r}\|C_{ki'}\|_{L^r}\|D^{-1+2\varepsilon_2}(\cdot)\|_{L^q(\mathbf{R}^2)}\|D^{-1+2\varepsilon_1}(\cdot)\|_{L^q(\mathbf{R}^2)}$$

if $2/r + 1/q = 2/p$. In (VII.13) we have applied Young's inequality exploiting the fact that $D^{-\alpha}(x-y)$ is a function of the difference $x - y$. By (VI.59)

$$\|D^{-\alpha}\|_{L^q} < \infty \quad \text{if } q < 2/(2-\min(\alpha,2)) \tag{VII.14}$$

so that in (VII.13) we must take $q < 2/(1 + 2\max(\varepsilon_1, \varepsilon_2, -\frac{1}{2}))$ and $r < 3/2$. Thus if we choose $p < 12/(11 + 6\max(\varepsilon_1, \varepsilon_2, -\frac{1}{2}))$, (VII.12) and (VII.13) imply (VII.6).

(b) If $\partial^{\gamma}C$ is replaced by δC, we argue as in (a) using Lemma VI.10 in place of Theorem VI.1. $\square$

LEMMA VII.2. (a) *Let w be the kernel of* $\mathrm{Tr}(K\chi_{j_1}\delta K\chi_{j_2})$, *or* $\mathrm{Tr}(\delta K\chi_{j_1}\delta K\chi_{j_2})$. *Then for $p < \frac{6}{5}$,*

$$\|w\|_{L^p} \leq c m_0^a e^{-\delta m_0 d(j_1,j_2)}. \tag{VII.15}$$

(b) *Let* w *be the kernel of* $\mathrm{Tr}(\partial^{\gamma_2}K\chi_{j_2}\partial^{\gamma_1}K\chi_{j_1})$ *where* γ_1 *is nonempty. Then for* $p < \frac{6}{5}$,

$$\|w\|_{L^p} \leqslant cm_0^a \prod_{k=1}^{2} G_1(\gamma_k, \delta) e^{-\delta m_0 d(\gamma_k, j)}. \tag{VII.16}$$

PROOF. (a) The more singular case is $\mathrm{Tr}(K\chi_{j_1}\delta K\chi_{j_2})$ with kernel

$$w(x,y) = \chi_{j_1}(x)\chi_{j_2}(y)\mathrm{tr}(S(y,x)\delta S(x,y)).$$

By Theorem VI.1 and Lemma VI.10, S and δS are in L^{2-} and L^{3-}, respectively, so that by Hölder's inequality, $w \in L^{6/5-}$ with norm bounded as in (VII.15).

(b) Same proof as (a). □

REMARK. By the inclusion, $\mathcal{C}_{p_1} \subset \mathcal{C}_{p_2}$ if $p_1 \leqslant p_2$, for the trace classes $\mathcal{C}_p(\mathcal{H})$ on $\mathcal{H}$, we can obtain information about $\mathcal{C}_{2n}$ norms from Lemma VII.1. For example, with $\varepsilon_1, \varepsilon_2 < \frac{1}{6}$,

$$\|\delta T_{\varepsilon,j}\|_4^4 \leqslant \|\delta T_{\varepsilon,j}\|_2^4 = \int w(x_1,x_2)w(x_3,x_4)\phi(x_1)\dots\phi(x_4)\,dx \tag{VII.17}$$

where the kernel in (VII.17) is in $L^p(\mathbf{R}^8)$ if $p < p_0$ with norm $O(e^{-\delta m_0 d(j_1,j_2)})$.

The next two lemmas collect some useful criteria for certain operators to be bounded on $L^p(\mathbf{R}^2)$ or to be in $\mathcal{C}_p(L^2)$, the trace ideal on $L^2(\mathbf{R}^2)$ with norm $\|A\|_p = \mathrm{Tr}(A^\dagger A)^{p/2}$.

LEMMA VII.3. *Suppose* $1 < p < \infty$; $\alpha, \beta \in \mathbf{R}$; *and* $\zeta \in C_0^\infty(\mathbf{R}^2)$.

(a) $\partial D^{-1}/\partial x_k$ *is a bounded operator on* $L^p(\mathbf{R}^2)$.

(b) $D^\alpha \zeta D^{-\alpha}$ *is a bounded operator on* $L^p(\mathbf{R}^2)$ *and the norm of* $\chi_i D^\alpha \zeta_j D^{-\alpha}$ *on* $L^p(\mathbf{R}^2)$ *is* $O(e^{-\delta m_0 d(i,j)})$.

(c) *If* $0 \leqslant \alpha < \beta/2$ *and* $\alpha < 1/2$ *then* $D^\alpha\chi_j D^{-\beta}$ *is a bounded operator on* $L^2(\mathbf{R}^2)$ *and the norm of* $\chi_i D^\alpha \chi_j D^{-\beta}$ *is* $O(e^{-\delta m_0 d(i,j)})$.

PROOF. (a) This follows from [29, Theorem V.3].

(b) For $p = 2$ the boundedness of $D^\alpha\zeta D^{-\alpha}$ was established in [21, Lemma 6.1]. The same proof works for general p (using part (a)). As for the decay, suppose $d(\Delta_i, \Delta_j) > 0$. If $\alpha \geqslant 0$ we choose an integer $n > \alpha/2$ and we write

$$\chi_i D^\alpha \zeta_j = \chi_i D^{2n} D^{\alpha-2n}\zeta_j. \tag{VII.18}$$

Now $D^{2n} = (-\Delta + m_0^2)^n$ is a local operator and we know that for $x \in \Delta_i$, $y \in \operatorname{supp}\zeta_j$, the kernel $D^{\alpha-2n}(x-y)$ is $O(e^{-\delta m_0|x-y|})$ together with all of its derivatives. Thus the norm of (VII.18) on L^p is $O(e^{-\delta m_0 d(i,j)})$. If $\alpha < 0$, we write

$$\chi_i D^\alpha \zeta_j D^{-\alpha} = \chi_i D^\alpha \zeta_j D^{2n} D^{-2n-\alpha}$$

and repeat the above argument with D^{2n} acting to the left.

(c) The boundedness of $D^\alpha\chi_j D^{-\beta}$ was established in [21, Lemma 6.2]. We obtain the exponential decay as in part (b). □

LEMMA VII.4. *Let* $2 \leqslant p \leqslant \infty$; $\alpha, \beta \in \mathbf{R}$; $f \in L^p(\mathbf{R}^2)$; *and* $\zeta \in C_0^\infty(\mathbf{R}^2)$.
(a) *If* $\beta > 2p^{-1}$, *then* $fD^{-\beta} \in \mathcal{C}_p(L^2)$.
(b) *If* $\beta - \alpha > 2p^{-1}$ *then* $D^\alpha \zeta D^{-\beta} \in \mathcal{C}_p(L^2)$.
(c) *If* $\beta - 2\alpha > 2p^{-1}$ *then* $D^\alpha \chi_j D^{-\beta} \in \mathcal{C}_p(L^2)$.

PROOF. (a) This is a special case of Lemma 2.1 of [24].
(b) Letting $\{\xi_i\}$ be a partition of unity where each ξ_i is a translate of a fixed $\xi \in C_0^\infty$, we write

$$D^\alpha \zeta D^{-\beta} = \sum_i (D^\alpha \zeta D^{-\alpha})(D^{-(\beta-\alpha)}\xi_i).$$

By Lemma VII.3(b) the first factor is bounded, and by (a) the second factor is in $\mathcal{C}_p$, so that each term in the sum is in $\mathcal{C}_p$. The sum converges in $\mathcal{C}_p$ since if $d_i = \operatorname{dist}(\operatorname{supp} \zeta, \operatorname{supp} \xi_i) \geqslant 1$, $(\zeta D^{-\beta}\xi_i)(x, y)$ is $O(e^{-\delta m_0 d_i})$ along with all of its derivatives so that the Hilbert-Schmidt norm $\|D^\alpha \zeta D^{-\beta}\xi_i\|_{\mathcal{C}_2}$ is $O(e^{-\delta m_0 d_i})$. Finally, $\mathcal{C}_p \supset \mathcal{C}_2$.
(c) We choose β' to satisfy $2\alpha < \beta' < \beta - 2p^{-1}$ and we write $D^\alpha \chi_j D^{-\beta} = (D^\alpha \chi_j D^{-\beta'}) \cdot (D^{\beta'} \zeta_j D^{-\beta})$. The first factor is bounded by Lemma VII.3(c) and the second factor is in $\mathcal{C}_p$ by part (b). □

We return to estimates of the type in Lemmas VII.1 and VII.2 for the case s = 1:

LEMMA VII.5. (a) *For* $\varepsilon_1, \varepsilon_2 > -\frac{1}{2}$, *let*

$$p_1 = \left(1 - \min\left(\varepsilon_1, \tfrac{1}{2}\right) - \min\left(\varepsilon_2, \tfrac{1}{2}\right)\right)^{-1}.$$

There is a function $w \in L^p(\mathbf{R}^4)$ *for any* $p < p_1$ *such that*

$$\|T_{-\varepsilon,j}(1)\|_2^2 \leqslant \int w(x_1, x_2)\phi(x_1)\phi(x_2)\, dx \tag{VII.19}$$

for almost all ϕ *and*

$$\|w\|_{L^p} \leqslant ce^{-\delta m_0 d(j_1, j_2)}. \tag{VII.20}$$

(b) *If* $\varepsilon_1 = 0$ *and* $0 \leqslant \varepsilon_2 < \frac{1}{2}$, *let* $p_2 = \frac{3}{2}(1 + \varepsilon_2)^{-1}$. *There is a function* $w \in L^p(\mathbf{R}^8)$ *satisfying* (VII.20) *for any* $p < p_2$ *such that for almost all* ϕ,

$$\|T_{\varepsilon,j}(1)\|_4^4 \leqslant \int w(x_1 \ldots x_4)\phi(x_1) \ldots \phi(x_4)\, dx. \tag{VII.21}$$

REMARK. With a little more work it is possible to prove (VII.20) for the actual kernels of $\|T_{-\varepsilon,j}(1)\|_2^2$ and $\|T_{\varepsilon,j}(1)\|_4^4$, but the stated result suffices for our purposes.

PROOF. (a) If $d(j_1, j_2) = 0$ we can eliminate the smooth localization ζ_{j_1}:

$$\|T_{-\varepsilon j}(1)\|_2^2 \leq \|D^{-\varepsilon_1}\zeta_{j_1}D^{\varepsilon_1}\|^2 \|D^{-\varepsilon_1}K(1)\chi_{j_2}D^{-\varepsilon_2}\|_2^2$$
$$\leq c\|D^{-\varepsilon_1}K(1)\chi_{j_2}D^{-\varepsilon_2}\|_2^2$$

by Lemma VII.3(b). The kernel of the right side,

$$w(x_1, x_2) = c\chi_{j_2}(x_1)D^{-1-2\varepsilon_1}(x_1 - x_2)\chi_{j_2}(x_2)D^{-1-2\varepsilon_2}(x_2 - x_1)$$

is easily seen to be in $L^p, p < p_1$ by (VII.14).

If $d(j_1, j_2) > 0$ we retain the ζ_{j_1} and take $w = \ker\|T_{-\varepsilon,j}(1)\|_2^2$. The estimate (VII.20) follows from the fact that for $y \in \Delta_{j_2}$, $\zeta_{j_1}(x)C(1; x - y)$ is a C^∞ function which is $O(e^{-\delta m_0 d(j_1, j_2)})$ together with all of its derivatives.

(b) As in part (a), for $d(j_1, j_2) = 0$ we use $\|T_{\varepsilon,j}(1)\|_4^4 \leq \|\zeta_{j_1}\|_{\text{op}}^4\|K(1)\chi_{j_2}D^{\varepsilon_2}\|_4^4$, and for $d(j_1, j_2) > 0$ we appeal to the smoothness and decay of $\zeta_{j_1}C\zeta_{j_2}$. □

If we intergrate (VII.19) with respect to $d\mu_s$ and apply Hölder's inequality, we see that for $p < p_1$,

$$\int \|T_{-\varepsilon j}(1)\|_2^2\, d\mu_s \leq \int w(x_1, x_2)C(s; x_1, x_2)\, dx \tag{VII.22}$$
$$\leq \|w\|_{L^p}\|\chi_{j_2}C\chi_{j_1}\|_{L^{p'}} \leq ce^{-\delta m_0 d(j_1, j_2)}.$$

It follows from (VII.22) that $K(1)D^{-\varepsilon} \in \mathcal{C}_{2,2;s}$ for any $\varepsilon > 0$. Similarly from (VII.21) we see that $K(1)D^{\varepsilon} \in \mathcal{C}_{4,4;s}$ for any $\varepsilon < \frac{1}{2}$. Using complex interpolation, we can obtain the well-known results [23] that

$$K(1) \in C_{p,q;s} \quad \text{if } p > 2, q < \infty. \tag{VII.23}$$

It is clear from Lemma VII.1 that $\partial^\gamma K$ and δK have better $\mathcal{C}_{p,q;s}$ properties than $K(1)$ in the sense that the restriction $p > 2$ can be improved. For example, by the same argument as in (VII.22), if $\varepsilon_1, \varepsilon_2 < \frac{1}{6}$,

$$\|\partial^\gamma T_{\varepsilon,j}\|_{2,2;s} \leq cm_0^a G_1(\gamma, \delta)e^{-\delta m_0 d(\gamma, j)}. \tag{VII.24}$$

This implies

COROLLARY VII.6. *For any $p > \frac{3}{2}$, $q < \infty$,*

$$\|\zeta_{j_1}\partial^\gamma K\chi_{j_2}\|_{p,q;s} \leq cm_0^a G_1(\gamma, \delta)e^{-\delta m_0 d(\gamma, j)}, \tag{VII.25}$$

$$\|\zeta_{j_1}\delta K\chi_{j_2}\|_{p,q;s} \leq cm_0^a e^{-\delta m_0 d(j_1, j_2)}. \tag{VII.26}$$

REMARK. Writing $K = \delta K + K(1)$ we see by (VII.26) that $K(s)$ enjoys the same $\mathcal{C}_{p,q}$ properties as $K(1)$, e.g. if $p > 2$, $q < \infty$,

$$\|\zeta_{j_1}K(s)\chi_{j_2}\|_{p,q;s} \leq cm_0^a e^{-\delta m_0 d(j_1, j_2)}. \tag{VII.27}$$

PROOF. If $\varepsilon_1 = \varepsilon_2 = \varepsilon < \frac{1}{6}$ we write

$$\zeta_{j_1}\partial^\gamma K\chi_{j_2} = (\zeta D^{-\varepsilon})(\partial^\gamma T_{\varepsilon,j})(D^{-\varepsilon}\zeta_{j_2}) \tag{VII.28}$$

where $\zeta \in C_0^\infty$ covers ζ_{j_1}, i.e., $\zeta\zeta_{j_1} = \zeta_{j_1}$. Now the operator $\zeta D^{-\varepsilon}$ on $\mathcal{H}$ is unitarily equivalent to $D^{1/2}\zeta D^{-\varepsilon-1/2}$ on $\mathcal{H}_0$, which by Lemma VII.4(b) is in

$\mathcal{C}_q$ for any $q > 2/\varepsilon$; similarly for $D^{-\varepsilon}\zeta_{j_2}$. Hence by Hölder's inequality

$$\|\zeta_j \partial^\gamma K \chi_{j_2}\|_{p,2;s} \leq c \|\partial^\gamma T_{\varepsilon,j}\|_{2,2;s} \tag{VII.29}$$

where $p^{-1} = 2^{-1} + 2q^{-1}$; i.e. (VII.29) holds for $p > (\frac{1}{2} + \varepsilon)^{-1}$. Since ε may be chosen arbitrarily close to $\frac{1}{6}$, the restriction on p in (VII.29) is $p > \frac{3}{2}$. (VII.24) and (VII.29) imply (VII.25) for $q = 2$ and by a standard hypercontractivity argument [11] for any $q < \infty$ (of course the constant c in (VII.25) diverges as $p \to \frac{3}{2}$ or $q \to \infty$). The proof of (VII.26) is virtually identical. □

We now have the technical tools in hand for the

Proof of Lemma IV.5. (i) is just Lemma VII.1(a). (ii) follows from a slight modification of (i): in (i) we replace ζ_i by ξ_i where $\{\xi_i\}$ is a suitable partition of unity and then we sum over i, the exponential decay giving convergence. (iii)–(v) are all proved by writing $K = \delta K + K(1)$ and using the triangle inequality to estimate the two resulting terms separately. The $K(1)$ term is estimated by Lemma VII.5 and the δK term by Lemma VII.1(b) and (VII.17). □

In §VII.4 we shall require the following information about the limit $K = \lim_{\kappa\to\infty} K_\kappa$ (K_κ is defined by (VII.4) with $\sigma = \infty$):

Lemma VII.7. *Let $p > 2$, $q < \infty$. There are positive constants c and ε such that for $\kappa \geq 1$,*

$$\|K(\Lambda, s) - K_\kappa(\Lambda, s)\|_{p,q;s} \leq c m_0^a \kappa^{-\varepsilon}. \tag{VII.30}$$

Proof. We verify (VII.30) by following through the proof of (VII.27) (based on (VII.23) and (VII.26)) with ϕ everywhere replaced by $\delta\phi_\kappa = \phi - \phi_\kappa$. When we calculate $\|(K - K_\kappa)D^{-\varepsilon}\|_{2,2;s}$ as in (VII.22) we find that $C(s)$ is replaced by differences such as

$$\zeta C \zeta - h_\kappa^* \zeta C \zeta \quad \text{or} \quad \zeta C \zeta * h_\kappa - h_\kappa^* \zeta C \zeta * h_\kappa \tag{VII.31}$$

where we are entitled to insert $\zeta \in C_0^\infty$ provided $\zeta = 1$ on a sufficiently large set. (*Notation.* h_κ^* acts on the first and $*h_\kappa$ on the second variable of $\zeta(x_1)C(x_1, x_2)\zeta(x_2)$.) By part (a) of the following lemma, these differences go to zero in any $L^p(\mathbf{R}^4)$, $p < \infty$, at a rate $O(\kappa^{-\varepsilon})$ and this establishes (VII.30). □

Lemma VII.8. (a) *Let $\zeta \in C_0^\infty(\mathbf{R}^2)$, let h_κ be defined as in* (III.8), *and let d_κ be one of the functions in* (VII.31). *Then for any $p < \infty$, there is an $\varepsilon > 0$ such that $\|d_\kappa\|_{L^p} = O(\kappa^{-\varepsilon})$.*

(b) *If $\alpha_1, \alpha_2 \leq 1$ with $\alpha_1 + \alpha_2 < \frac{3}{2}$, there is an $\varepsilon > 0$ such that*

$$\|D^{\alpha_1}\zeta_{j_1}(C - h_\kappa^* C)\zeta_{j_2} D^{\alpha_2}\| \leq c m_0^a \kappa^{-\varepsilon} e^{-\delta m_0 d(j_1, j_2)} \tag{VII.32}$$

where $\|\cdot\|$ denotes the operator norm on $L^2(\mathbf{R}^2)$. The same estimate holds if $C - h_\kappa^* C$ is replaced by $C * h_\kappa - h_\kappa^* C * h_\kappa$.

PROOF. (a) Since the operator h_κ^* on $L^p(\mathbf{R}^2)$ has norm less than $\|h_\kappa\|_{L^1} = 1$ (by Young's inequality), it is sufficient to consider the first difference in (VII.31). But for any $p < \infty$ we know by (VI.2) that there is an $\alpha > 0$ such that $D^\alpha \zeta C \zeta \in L^p(\mathbf{R}^4)$. We then apply Lemma VII.9(b) below.

(b) Consider first the case $s = 1$. If $d(j_1, j_2) = 0$ we can drop the ζ's since $D^\alpha \zeta D^{-\alpha}$ is a bounded operator on L^2 (Lemma VII.3(b)). The resulting operator is diagonal in momentum space and easily seen to be $O(\kappa^{\alpha_1+\alpha_2-2})$. If $d(j_1, j_2) > 0$ we obtain an overestimate by putting $\alpha_i = 2$ and letting the local operator D^2 act on the ζ's and C. We obtain a sum of terms, e.g.,

$$\text{(VII.33)} \quad \|\zeta_{j_1} D^2 (C - h_\kappa^* C) D^2 \zeta_{j_2}\| = \|\zeta_{j_1} (D^2 C D^2 - h_\kappa^* D^2 C D^2) \zeta_{j_2}\|.$$

Now if $x_i \in \operatorname{supp} \zeta_{j_i}$, $C(x_1, x_2)$ is C^∞ and, together with all of its derivatives, is $O(e^{-\delta m_0 |x_1 - x_2|})$. Thus, estimating the operator norm of (VII.33) by its Hilbert-Schmidt norm, we can apply Lemma VII.9(b) to extract the convergence factor $O(\kappa^{-\epsilon})$. Actually, to do so we must first "commute" ζ_{j_1} and h_κ^*, using the fact that the commutator is a bounded operator on L^2 with bound $O(\kappa^{-1})$. Explicitly,

$$\begin{aligned} \int |\zeta h_\kappa^* f(x) - h_\kappa^* \zeta f(x)|^2 \, dx &= \int dx \left| \int h_\kappa(x - x')(\zeta(x) - \zeta(x')) f(x') \, dx' \right|^2 \\ \text{(VII.34)} \qquad &\leq c \int dx \left| \int h_\kappa(x - x') |x - x'| f(x') \, dx' \right|^2 \\ &\leq c \left[\int h_\kappa(x) |x| \, dx \right]^2 \|f\|^2 = c\kappa^{-2} \|f\|^2 \end{aligned}$$

where we have used Young's inequality in the last inequality.

Having dispensed with $s = 1$, we complete the proof of (VII.32) by establishing it for δC. Again we estimate (VII.32) by its HS ($= L^2(\mathbf{R}^2)$) norm. We interpolate in α_i between 0 and 1 to estimate this L^2 norm: For $\alpha_i = 1$, it follows from Lemma VI.10 that

$$\|D\zeta_{j_1} (C - h_\kappa^* C) \zeta_{j_2} D\|_{L^{3/2-}} \leq c m_0^a e^{-\delta m_0 d(j_1 j_2)}.$$

For $\alpha_i = 0$ we use (VII.34) and part (a) to show that for any $p < \infty$,

$$\|\zeta_{j_1} (C - h_\kappa^* C) \zeta_{j_2}\|_{L^p} \leq c m_0^a \kappa^{-\epsilon}. \quad \square$$

LEMMA VII.9. (a) *If $f \in L^p(\mathbf{R}^2)$, then $\|f - h_\kappa^* f\|_{L^p} \to 0$ as $\kappa \to \infty$.*

(b) *If $D^\alpha f \in L^p(\mathbf{R}^2)$ for some $\alpha > 0$, then*

$$\text{(VII.35)} \qquad \|f - h_\kappa^* f\|_{L^p} \leq c\kappa^{-\alpha/3} \|D^\alpha f\|_{L^p}.$$

REMARK. The same results hold if $f(x_1, \ldots, x_n) \in L^p(\mathbf{R}^{2n})$ and D^α and h_κ^* act on one of the variables x_i.

PROOF. (a) Since h_κ^* is a bounded operator on L^p, it suffices to prove convergence for a dense set in L^p, say $f \in C_0^\infty$. But if $f \in C_0^\infty \subset L^2$ we find that

$$\|f - h_\kappa^* f\|_2^2 = \int |\tilde{f}(k)|^2 |\tilde{h}(k/\kappa) - 1|^2 \, dk \to 0$$

by the Lebesgue dominated convergence theorem. L^p convergence follows for $p > 2$ by the Hausdorff-Young inequality and for $p < 2$ from the fact that $f - h_\kappa^* f$ has compact support.

(b) By Young's inequality

$$\|f - h_\kappa^* f\|_{L^p} \leqslant \|D^{-\alpha} - h_\kappa^* D^{-\alpha}\|_{L^1} \|D^\alpha f\|_{L^p}. \tag{VII.36}$$

Now since $h_\kappa(x)$ has support in $\{x \mid |x| \leqslant (2\kappa)^{-1}\}$,

$$\begin{aligned} &\left| D^{-\alpha}(x) - \int h_\kappa(x-y) D^{-\alpha}(y)\, dy \right| \\ &\quad = \int |x-y| h_\kappa(x-y) \frac{|D^{-\alpha}(x) - D^{-\alpha}(y)|}{|x-y|}\, dy \\ &\quad \leqslant \sup_{|y-x| \leqslant (2\kappa)^{-1}} |\nabla D^{-\alpha}(y)| \int |x-y| h_\kappa(x-y)\, dy \\ &\quad \leqslant c e^{-\delta m_0 |x|} \sup_{|y-x| \leqslant (2\kappa)^{-1}} |y|^{-3+\alpha} \kappa^{-1} \end{aligned} \tag{VII.37}$$

since $\partial D^{-\alpha}(y)/\partial y_i$ has a singularity $|y|^{-3+\alpha}$ at $y = 0$. Thus for $|x| \geqslant \kappa^{-1/3}$, (VII.37) is $O(e^{-\delta m_0 |x|} \kappa^{-\alpha/3})$ and so the integral of (VII.37) in this region is $O(\kappa^{-\alpha/3})$. In the region $|x| \leqslant \kappa^{-1/3}$ we estimate the two terms on the left of (VII.37) separately, e.g.,

$$\int_{|x| \leqslant \kappa^{-1/3}} |D^{-\alpha}(x)|\, dx \leqslant c \int_{|x| \leqslant \kappa^{-1/3}} |x|^{-2+\alpha}\, dx = c\kappa^{-\alpha/3}.$$

Thus the L^1 norm in (VII.36) is $O(\kappa^{-\alpha/3})$. □

VII.3. *Properties of B.* We turn now to quadratic polynomials such as B_σ, defined in (III.10b), as well as the related objects

$$\delta B_\sigma(s) = B_\sigma(s) - B_\sigma(1), \tag{VII.38}$$

$$B_\sigma^{\gamma_1, \gamma_2}(s) = :\mathrm{Tr}\, \partial^{\gamma_2} K_\sigma \partial^{\gamma_1} K_\sigma:. \tag{VII.39}$$

We also consider a polynomial which enters in the proof of the linear lower bound in §VII.4:

$$W_{1,\sigma} = \tfrac{1}{2} :\mathrm{Tr}\big(K_\sigma^*(1) K_\sigma(1) - K_\sigma(1)^\dagger K_\sigma(1)\big):. \tag{VII.40}$$

Note that the counterterm $:\mathrm{Tr}\, K_\sigma(1)^+ K_\sigma(1):$ in the definition (III.10b) of $B_\sigma(s)$ is s-independent: once we cancel it against the divergences in $B_\sigma(1)$, the remaining term $\delta B_\sigma(s)$ is well defined, as we show below.

We let b_σ, δb_σ, $\partial^{\gamma_1, \gamma_2} b_\sigma$ and $w_{1,\sigma}$ denote the kernels of the above polynomials without the characteristic function χ_Λ. For example,

$$B_\sigma =: \int_{\Lambda\times\Lambda} b_\sigma(s;\, x, y)\phi(x)\phi(y)\, dx\, dy:$$

where

$$\text{(VII.41)} \qquad \begin{aligned} b_\sigma(s;\, x, y) = \tfrac{1}{2}\operatorname{tr}\Big[& S_{0,\sigma}(s;\, x, y) S_{0,\sigma}(s;\, y, x) \\ & + \big(S_{0,\sigma}(1)^\dagger S_{0,\sigma}(1)\big)(x, y)\delta(x - y)\Big]. \end{aligned}$$

When $s = 1$, the kernel $b_\sigma(1;\, x, y)$ is a function of $x - y$. We let

$$w_{0,\sigma}(x - y) = b_\sigma(1;\, x, y)$$

and study

$$\tilde{w}_{i,\sigma}(k) = (2\pi)^{-2}\int e^{ikx} w_{i,\sigma}(x)\, dx:$$

LEMMA VII.10. (a) *Let $\varepsilon > 0$. Then $\tilde{w}_{i,\sigma}(k) \leqslant c(\varepsilon)D(k)^\varepsilon$ uniformly in σ where $D(k) = (k^2 + m^2)^{1/2}$. The limit $\tilde{w}_i(k) = \lim_{\sigma\to\infty}\tilde{w}_{i,\sigma}(k)$ exists in the sense that* $\sup_k(\tilde{w}_i(k) - \tilde{w}_{i,\sigma}(k))D(k)^{-\varepsilon} \to 0$ *and also satisfies $\tilde{w}_i(k) \leqslant c(\varepsilon)D(k)^\varepsilon$.*

(b) *Suppose $\beta > \frac{1}{2}$. Let*

$$v_{i,\sigma}(x, y) = D_x^{-\beta} D_y^{-\beta} \chi_{j_1}(x) w_{i,\sigma}(x - y)\chi_{j_2}(y).$$

As $\sigma \to \infty$, $v_{i,\sigma}$ converges in $L^2(\mathbf{R}^4)$ to

$$v_i(x, y) = D_x^{-\beta} D_y^{-\beta}\chi_{j_1}(x) w_i(x - y)\chi_{j_2}(y)$$

where (the tempered distribution) w_i is the inverse Fourier transform of $\tilde{w}_i$. v_i satisfies the bound

$$\text{(VII.42)} \qquad \|v_i\|_{L^2(\mathbf{R})} \leqslant c m_0^a e^{-\delta m_0 d(j_1, j_2)}.$$

REMARK. In [22], Seiler verified by explicit integration that $\tilde{w}_i(k) = O(\ln|k|)$.

PROOF. (a) We consider only $i = 0$, the case $i = 1$ being similar. From definition (VII.41) we calculate that

$$\text{(VII.43)} \quad \tilde{w}_{0,\sigma}(k) = \frac{2}{(2\pi)^4}\int\left[\frac{m_0^2 - p_+\cdot p_-}{D(p_+)^2 D(p_-)^2}\tilde{h}_\sigma(p_+)\tilde{h}_\sigma(p_-) + \frac{\tilde{h}_\sigma(p)^2}{D(p)^2}\right] dp$$

where $p_\pm = p \pm k/2$. If we write $m_0^2 - p_+\cdot p_- = p_+\cdot k - D(p_+)^2 + 2m_0^2$ and make the change of variable $p \to p + k/2$ in the first part of the integral in (VII.43), we obtain

$$\text{(VII.44)} \quad \begin{aligned} \tilde{w}_{0,\sigma}(k) = \frac{2}{(2\pi)^4}\int \frac{\tilde{h}_\sigma(p)}{D(p)^2}\Bigg[& \frac{2m_0^2 + k\cdot(p + k)}{D(p + k)^2}\tilde{h}_\sigma(p + k) \\ & + \tilde{h}_\sigma(p) - \tilde{h}_\sigma(p + k)\Bigg] dp. \end{aligned}$$

Since $\tilde{h}_\sigma(p) = \tilde{h}(p/\sigma)$, we have, for any α in [0, 1],

(VII.45) $$|\tilde{h}_\sigma(p) - \tilde{h}_\sigma(p+k)| \leqslant c|k|^\alpha/\sigma^\alpha.$$

Using (VII.45) it is fairly easy to show that $\tilde{w}_{0,\sigma}(k) \leqslant c(\varepsilon)D(k)^\varepsilon$, and so we shall spell out only the convergence of $\tilde{w}_{0,\sigma}$ to

$$\tilde{w}_0(k) = \frac{2}{(2\pi)^4}\int \frac{2m_0^2 + k\cdot(p+k)}{D(p)^2 D(p+k)^2}\,dp \equiv \frac{2}{(2\pi)^4}\int f(p,k)\,dp.$$

We write

$$\tilde{w}_o(k) - \tilde{w}_{0,\sigma}(k) = 2\left[A_\sigma(k) + B_\sigma(k)\right]/(2\pi)^4$$

where

$$\begin{aligned} A_\sigma(k) &= \int f(p,k)\left(1 - \tilde{h}_\sigma(p)\tilde{h}_\sigma(p+k)\right)dp \\ &= \int f(p,k)\left[\left(1 - \tilde{h}_\sigma(p)^2\right) + \tilde{h}_\sigma(p)\left(\tilde{h}_\sigma(p) - \tilde{h}_\sigma(p+k)\right)\right]dp \end{aligned}$$

and

$$B_\sigma(k) = \int \frac{\tilde{h}_\sigma(p)}{D(p)^2}\left(\tilde{h}_\sigma(p) - \tilde{h}_\sigma(p+k)\right)dp.$$

Since $|f(p,k)| \leqslant cD(k)/D(p)^2D(p+k)$, we find, using (VII.45), that

$$\begin{aligned} |A_\sigma(k)| &\leqslant cD(k)\int \frac{dp}{D(p)^2 D(p+k)}\,\frac{|p|^\alpha + |k|^\alpha}{\sigma^\alpha} \\ &\leqslant cD(k)^\alpha \log D(k)/\sigma^\alpha \end{aligned}$$

and similarly for B_σ. This establishes the desired convergence of $\tilde{w}_{0,\sigma}$ to $\tilde{w}_0$.

(b) Regarding $v_{i,\sigma}(x,y)$ and $w_{i,\sigma}(x-y)$ as the kernels of operators $V_{i,\sigma}^{\mathrm{op}}$ and $W_{i,\sigma}^{\mathrm{op}}$, we write

(VII.46) $$V_{i,\sigma}^{\mathrm{op}} = \left(D^{-\beta}\chi_{j_1}\right)\cdot\left(W_{i,\sigma}^{\mathrm{op}}D^{-\varepsilon}\right)\cdot\left(D^\varepsilon\chi_{j_2}D^{-\beta}\right)$$

for $2\varepsilon < \beta - \frac{1}{2}$. By part (a) and Lemma VII.4(c) we see that the 3 factors in (VII.46) are in $\mathcal{C}_4$, $\mathcal{C}_\infty$, and $\mathcal{C}_4$ of L^2 so that the product is in $\mathcal{C}_2$. By part (a), this factorization also yields the $L^2(\mathbf{R}^4)$ convergence of $v_{i,\sigma}$ to v_i.

It remains only to establish the exponential decay in (VII.42) when $d(j_1, j_2) > 0$. But this is immediate from (VII.41) (without the need for the $D^{-\beta}$ factors). For if $x \in \Delta_{j_1}$, $y \in \Delta_{j_2}$ we have $\delta(x-y) = 0$ and $S(1;x,y) = O(e^{-\delta m_0|x-y|})$. Since $h_\sigma(x)$ is supported in $\{x|\,|x| < 1/2\sigma\}$, convolution with h_σ preserves this decay and so $w_{0,\sigma}(x-y) = O(e^{-\delta m_0|x-y|})$. The same conclusion holds for $w_{1,\sigma}$ since

$$\ker :\mathrm{Tr}\,K_\sigma(1)^* K_\sigma(1): = \mathrm{tr}\left(S_{0,\sigma}(1)^\dagger D S_{0,\sigma}(1)\right)(x,y)D^{-1}(x-y)$$

and

$$\operatorname{tr} S(1)^{\dagger} D S(1)(x,y) = 2D^{-1}(x-y) = O(e^{-\delta m_0|x-y|}). \quad \square$$

In the case of general s we have

LEMMA VII.11. (a) $\delta b_\sigma \in L^{6/5-}(\mathbf{R}^4)$ *and converges in* $L^{6/5-}$ *to a limit* δb *satisfying*

$$\|\chi_{j_1}\delta b \chi_{j_2}\|_{L^{6/5-}} \leqslant c m_0^a e^{-\delta m_0 d(j_1,j_2)}. \tag{VII.47}$$

(b) $\partial^{\gamma_1,\gamma_2} b_\sigma \in L^{6/5-}(\mathbf{R}^4)$ *and converges in* $L^{6/5-}$ *to a limit* $\partial^{\gamma_1,\gamma_2} b$ *satisfying*

$$\|\chi_{j_1}\partial^{\gamma_1,\gamma_2} b \chi_{j_2}\|_{L^{6/5-}} \leqslant c m_0^a \prod_{k=1}^{2} G_1(\gamma_k,\delta) e^{-\delta m_0 d(\gamma_k, j)} \tag{VII.48}$$

where the $k = 2$ *factor is not present if* γ_2 *is empty.*

REMARK. By (VII.48) and Lemma VI.12 we see that if $\varepsilon > \frac{2}{3}$, $D^{-\varepsilon}\chi_{j_1}\partial^{\gamma_1,\gamma_2} b \chi_{j_2} D^{-\varepsilon}$ is in $L^2(\mathbf{R}^4)$ and satisfies the bound (IV.50).

PROOF. (a) From definition (VII.38),

$$\chi_{j_1}\delta b \chi_{j_2} = \tfrac{1}{2}\ker \operatorname{Tr}\big(\delta K \chi_{j_1}\delta K \chi_{j_2} + K(1)\chi_{j_1}\delta K \chi_{j_2} + \delta K \chi_{j_1} K(1)\chi_{j_2}\big)$$

and similarly for the σ cutoff objects. The bound (VII.47) is a corollary of Lemma VII.2, while the convergence as $\sigma \to \infty$ is a corollary of Lemma VII.9(a).

(b) This follows from Lemmas VII.2(b) and VII.9(a). $\square$

By Lemmas VII.10(b) and VII.11(a) we see that the (distribution) kernel b of B introduced in (III.21) is rigorously defined by $b \equiv w_0 + \delta b$ and has the following properties:

LEMMA VII.12. *Suppose* $\beta > \frac{2}{3}$. *As* $\sigma \to \infty$,

$$D_x^{-\beta} D_y^{-\beta} \chi_{j_1}(x)\big[b_\sigma(x,y) - b(x,y)\big]\chi_{j_2}(y) \to 0 \tag{VII.49}$$

in $L^2(\mathbf{R}^4)$ *and the limit satisfies* (IV.51), *i.e.*

$$\|D^{-\beta}\chi_{j_1} b \chi_{j_2} D^{-\beta}\|_{L^2(\mathbf{R}^4)} \leqslant c m_0^a e^{-\delta m_0 d(j_1,j_2)}. \tag{VII.50}$$

PROOF. We write $b_\sigma - b = (w_{0,\sigma} - w_0) + (\delta b_\sigma - \delta b)$. The contribution of the first term to (VII.49) converges to 0 by Lemma VII.10(b) while the second term contribution converges in L^{r-} where $r^{-1} = 5/6 - \beta/2 < 1/2$ by Lemmas VII.11(a) and VI.12. (VII.50) is obvious from the previous two lemmas. $\square$

From (VII.49) we deduce the convergence (III.8) of B_σ to B in $L^2(d\mu_{C(s)})$. For

$$\begin{aligned}\int (B - B_\sigma)^2\, d\mu_{C(s)} = \int_{\Lambda^4} &\big[b_\sigma(x,y) - b(x,y)\big]\cdot\big[b_\sigma(x',y') - b(x',y')\big] \\ &\cdot\big[C(x,x')C(y,y') + C(x,y')C(y,x')\big]\end{aligned} \tag{VII.51}$$

By a standard procedure we multiply the C's by smooth cutoffs $\zeta \in C_0^\infty$ which cover χ_Λ and introduce factors $D^{-\beta}$ and D^β, $\frac{2}{3} < \beta < \frac{3}{4}$, for each variable in (VII.51). As in Lemma IV.7, $D^\beta \zeta C \zeta D^\beta$ is a bounded operator on $L^2(\mathbf{R}^2)$ and so we obtain from (VII.51),

$$\text{(VII.52)} \quad \int (B - B_\sigma)^2 \, d\mu_{C(s)} \leqslant c \| D^{-\beta} \chi_\Lambda (b_\sigma - b) \chi_\Lambda D^{-\beta} \|^2_{L^2(\mathbf{R}^4)} \to 0.$$

The above estimates also show that:

COROLLARY VII.13. *If $\gamma_1 \neq \varnothing$ and $\Lambda \cap \Delta = \varnothing$ then the quadratic polynomials $\partial^{\gamma_1, \gamma_2} B$ and $:\mathrm{Tr}\, K_\Lambda K_\Delta:$ are well defined in $L^p(d\mu_s)$, $p < \infty$ (without the need for counterterms).*

PROOF. We consider only $:\mathrm{Tr}\, K_\Lambda K_\Delta:$. As in Lemma VII.2(a) the kernel is

$$w(x, y) = \chi_\Lambda(x) \chi_\Delta(y) \mathrm{tr}(S(y, x) S(x, y)).$$

But by Lemma VI.10, $\chi_\Lambda \chi_\Delta S \in L^{3-}(\mathbf{R}^4)$ so that $w \in L^{3/2-}$. It follows easily that $:\mathrm{Tr}\, K_\Lambda K_\Delta: \in L^p$. $\square$

VII.4. *Linear lower bound.* We now come to the basic estimate on

$$\text{(VII.53a)} \qquad w_{m,\Lambda}(s) = \| \textstyle\bigwedge^m R \| \det_3(1 - K) e^{-B}.$$

THEOREM VII.14. *For any $p < \infty$ there are constants c_1 and c_2 independent of m, Λ and s such that*

$$\text{(VII.53b)} \qquad \| w_{m,\Lambda}(s) \|_{L^p(\mu_s)} \leqslant c_1^m e^{c_2 |\Lambda|}.$$

For $s = 1$ this result is due to McBryan [16] and Seiler and Simon [24]. Although it seems to us that either of their proofs can be generalized to the case $s \neq 1$, we shall not actually prove Theorem VII.14 here. Instead we establish the following weaker result which is sufficient for our purposes. Our proof is close in structure to McBryan's while our control of certain additional terms when $s \neq 1$ relies on techniques of Seiler and Simon.

THEOREM VII.14′. *Let $P_1, \ldots, P_m \in \mathcal{C}_{1,q;s}$ for any $q < \infty$. Suppose that each P_j is localized in the sense that $P_j \chi_\Delta = P_j$ for some unit square Δ. Define $P = P_1 \wedge \cdots \wedge P_m$ and*

$$\text{(VII.54)} \qquad w_{m,\Lambda}(s) = T_m(P \textstyle\bigwedge^m R) \det_3(1 - K) e^{-B}.$$

Then for any p and q with $p < q < \infty$ there are constants c_1 and c_2 independent of m, Λ, s and P such that

$$\| w_{m,\Lambda}(s) \|_{L^p(\mu_s)} \leqslant c_1^m e^{c_2 |\Lambda|} \Big\| \prod_j \| P_j \|_1 \Big\|_{L^q(\mu_s)}.$$

REMARKS. 1. By (A.14) and Hölder's inequality, Theorem VII.14 implies Theorem VII.14′. Were it not for the localization hypothesis on the P_j's the converse would be true, for $w_{m,\Lambda}$ given by (VII.53a) can always be expressed in the form (VII.54) in terms of appropriate rank one operators P_j satisfying $\|P_j\|_1 = 1$.

2. In the course of proving Theorem VII.14′ we shall not be explicit about the dependence of c_1 and c_2 on the bare parameters λ and m_0. However it is easy to see that c_1 is uniformly bounded and that $c_2 = O(\lambda m_0^a)$ for some a. Thus our assumption in §IV that c_2 is uniformly bounded is justified once we impose the constraint (IV.58).

3. Our method allows us to include an arbitrary finite mass renormalization $\frac{1}{2}M^2\int_\Lambda :\phi^2:$ in B (see [15], [23]).

4. If one wishes to recover Seiler's result [22] that (VII.53a) with $s = 1$ satisfies $\|w_{m,\Lambda}\|_{L^p} \leqslant c(\Lambda)^m$, then the following method simplifies considerably and gives an elementary alternate proof.

PROOF OF THEOREM VII.14′. The basic idea of the proof is the same as the corresponding proof for $P(\phi)_2$ [12, §VII.4 and Appendix] in which a Duhamel expansion is performed in each unit square in Λ and a Checkerboard Estimate is used to decouple the unit squares. (VII.54) has the same form as the integrand in (III.11b) on which we based our cluster expansion. We now perform a momentum expansion. Let $\Lambda = \bigcup_{i\in I}\Delta_i$ be the decomposition of Λ into unit squares, where we assign some fixed order to the index set $I \subset \mathbf{Z}^2$; if i precedes i' in this ordering we shall write $i < i'$. We consider the sequence $\kappa_j = e^j, j = 1, 2, \ldots,$ of momentum cutoffs, and we define

$$K^{(i)} \equiv S_0\phi\chi_{\Delta_i}, \qquad K_j^{(i)} = S_0\phi_{e^j}\chi_{\Delta_i}$$

where the momentum cutoff field ϕ_κ is defined after (VII.4). The first step in the momentum expansion of (VII.54) is to consider the first i in I and to replace $K = \Sigma K^{(i)}$ by $K_1 = K_1^{(i)} + \Sigma_{i'>i}K^{(i')}$ by interpolating via

$$\tilde{K}_1(t) = (1-t)K_1^{(i)} + tK^{(i)} + \sum_{i'>i} K^{(i')}. \tag{VII.55}$$

That is, with $R_1 = (1 - K_1)^{-1}$, $\tilde{R}_1(t) = (1 - \tilde{K}_1(t))^{-1}$, and

$$B(K) = \tfrac{1}{2} :\mathrm{Tr}\big(K^2 + [K^\dagger K]_{s=1}\big):, \tag{VII.56}$$

$$\begin{aligned} w_{m,\Lambda} - T_m(P \textstyle\bigwedge^m R_1)\det_3(1 - K_1)e^{-B(K_1)} &= \int_0^1 dt\, \frac{d}{dt}\, T_m(P \textstyle\bigwedge^m \tilde{R}_1)\det_3(1 - \tilde{K}_1)e^{-B(\tilde{K}_1)} \\ &= \int_0^1 dt\, \Big\{ -T_{m+1}\big(\textstyle\bigwedge^{m+1} R_1 \cdot P \wedge A_1\big) \\ &\quad + T_m\big(\textstyle\bigwedge^m \tilde{R}_1 P d \bigwedge^m E_1\big) - T_m\big(\textstyle\bigwedge^m \tilde{R}_1 \cdot P\big)B_1 \Big\}\det_3(1 - \tilde{K}_1)e^{-B(\tilde{K}_1)} \end{aligned} \tag{VII.57}$$

as in (III.31). Here

$$A_1(t) = \tilde{K}_1(t)^2 \delta K_1 \quad \text{with } \delta K_1 = K^{(i)} - K_1^{(i)},$$
$$E_1(t) = (1 + \tilde{K}_1(t))\delta K_1 \quad \text{and}$$
$$B_1(t) = \tfrac{1}{2} :\mathrm{Tr}\big(2\tilde{K}_1 \delta K_1^{(i)} + \big[\delta K_1^{(i)\dagger}\tilde{K}_1 + \tilde{K}_1^\dagger \delta K_1^{(i)}\big]_{s=1}\big):.$$

We continue to iterate this procedure replacing the $\tilde{K}_1(t)$ and $\tilde{R}_1(t)$ in (VII.57) by

$$K_2(t) = (1-t)K_1^{(i)} + tK_2^{(i)} + \sum_{i'>i} K^{(i')} \quad \text{and}$$

$$R_2(t) = (1 - K_2(t))^{-1},$$

and so on. Having carried out this (infinite) expansion in Δ_i, we move on (in order) to each of the other $\Delta_{i'}$'s and perform such an expansion in each lattice square. Clearly the resulting terms will have a structure similar to the terms of the cluster expansion (see (III.45)).

We now introduce the requisite notation to describe this structure. For each $i \in I$ and positive integer n_i, let

$$\tilde{K}_{n_i}^{(i)}(t) = \sum_{j=1}^{n_i} \delta t_j^{(i)} K_j^{(i)} + t_{n_i}^{(i)} K^{(i)} \tag{VII.58}$$

where $\delta t_j^{(i)} = t_{j-1}^{(i)} - t_j^{(i)}$ and

$$1 = t_0^{(i)} \geqslant t_1^{(i)} \geqslant \cdots \geqslant t_n^{(i)} \geqslant 0. \tag{VII.59}$$

We also define

$$K_{n_i}^{(i)}(t) = \tilde{K}_{n_i}^{(i)}(t)\big|_{t_{n_i}^{(i)}=0} = \sum_{j=1}^{n_i} \delta t_j^{(i)} K_j^{(i)}.$$

For $n = (n_i)_{i\in I}$ given, we let

$$K_n(t) = \sum_i K_{n_i}^{(i)}(t), \qquad R_n(t) = (1 - K_n(t))^{-1}, \tag{VII.60}$$

and as in (VII.57),

$$\tilde{K}_{n,i,j}(t) = \sum_{i'<i} K_{n_{i'}}^{(i')}(t) + \tilde{K}_j^{(i)}(t) + \sum_{i'>i} K^{(i')}. \tag{VII.61}$$

Then iteration of (VII.57) as described in the previous paragraph yields

$$\begin{aligned} w_{m,\Lambda} = \sum_{n=0}^{\infty} \int dt \sum_{\text{decomp}\,\pi_n} &(-1)^l T_r\big(\textstyle\bigwedge^r R_{n+1} \cdot P \wedge A(\pi_A) d \bigwedge^r E(\pi_E)\big) \\ &\cdot B(\pi_B)\det_3(1 - K_{n+1})e^{-B(K_{n+1})}, \end{aligned} \tag{VII.62}$$

where the sum over n is over n_i from 0 to ∞, $i \in I$; $\int dt$ represents integration with respect to the $|n| = \sum n_i$ variables $t_j^{(i)}$, $i \in I$, $j = 1, 2, \ldots, n_i$, in the region (VII.59);

$$\pi_n = \{\alpha | \alpha = (i,j);\ i \in I, j = 1, \ldots, n_i\}$$

ordered by $(i,j) < (i',j')$ if $i < i'$ or if $i = i'$ and $j < j'$; $\Sigma_{\text{decomp}\,\pi_n}$ is the sum over decompositions of π_n into 3 disjoint subsets $\pi_n = \pi_A \cup \pi_E \cup \pi_B$; $l = |\pi_A| + |\pi_B|$ and $r = m + |\pi_A|$; R_{n+1} and K_{n+1} are defined in (VII.60) where the notation $n+1$ means $(n_i+1)_{i \in I}$; $A(\pi_A) = \bigwedge_{\alpha \in \pi_A} A_\alpha(n,t)$, where for $\alpha = (i,j)$,

(VII.63) $$A_\alpha(n,t) = \tilde{K}_{n,i,j}(t)^2 \delta K_j^{(i)},$$

where $\tilde{K}_{n,i,j}$ is defined in (VII.61) and

(VII.64) $$\delta K_j^{(i)} = K^{(i)} - K_j^{(i)};$$

$d\bigwedge' E(\pi_E) = \prod_{\alpha \in \pi_E} d'\bigwedge' E_\alpha(n,t)$, with the factors taken in increasing order from left to right, where

(VII.65) $$E_\alpha(n,t) = \big(1 + \tilde{K}_{n,i,j}(t)\big)\delta K_j^{(i)};$$

$B(\pi_B) = \prod_{\alpha \in \pi_B} B_\alpha(n,t)$, where

(VII.66) $$B_\alpha(n,t) = :\mathrm{Tr}\Big(\tilde{K}_{n,i,j}\delta K_j^{(i)} + \big[\tilde{K}^\dagger_{n,i,j}\delta K_j^{(i)}\big]_{s=1}\Big):.$$

As in the cluster expansion, $d'\bigwedge^r$ is defined just as $d\bigwedge^r$ except that the order convention is observed: when the E factors are spread into the P and A factors as in (III.35), a product AE is omitted if the E factor precedes the A factor (relative to the ordering of π_n).

The analysis of (VII.62) is similar to the analysis of (III.45) but is somewhat simpler: In each of the A, E, and B factors in (VII.62) we replace $\tilde{K}_{n,i,j}(t)$ by the sum (VII.61) of localized K's according to definitions (VII.63), (VII.65), (VII.66). In addition, we insert an additional partition of unity at the left of each E factor. This expresses the A, E and B factors as a sum of "localized" A, E and B factors. In the sum over localizations each element α of π_A or π_E has 2 associated localization indices, which we denote by $k_\alpha = (k_{\alpha,1}, k_{\alpha,2}) \in \mathbf{Z}^4$, and each element α of π_B has 1 localization index $k_{\alpha,1}$. Since there are $3^{|n|}$ possible decompositions of π_n, we may replace the sum over decompositions by $3^{|n|}$ provided we choose the maximizing decomposition $\pi_n = \pi_A \cup \pi_E \cup \pi_B$.

For each value $k = (k_\alpha)_{\alpha \in \pi_n}$ of the localization indices, we let

(VII.67) $$M(\Delta_i) = n_i + |\{k_{\alpha,\beta} | \Delta_{k_{\alpha,\beta}} = \Delta_i\}|,$$

i.e., $M(\Delta_i)$ is the total number of times the localization Δ_j occurs in a $\delta K_j^{(i)}$ factor or in k. We also let $M_P(\Delta)$ be the number of P_j's with localization Δ. As in (IV.15) the number M of primitive terms obtained by spreading the E factors into the P and A factors is bounded by

(VII.68) $$M < \prod_\Delta 2^{M_P(\Delta) + M(\Delta)} M(\Delta)! = 2^m \prod_\Delta 2^{M(\Delta)} M(\Delta)!.$$

Choosing the primitive term with the largest $L^p(d\mu)$ norm we thus obtain from (VII.62)

$$\|w_{m,\Lambda}\|_{L^p} \leq \sum_{n,k} 3^{|n|} M \Big\| T_r\big(\wedge^r R_{n+1} \cdot \wedge_{\rho=1}^{r} G_\rho\big) \cdot B(\pi_B)\det_3(1-K_{n+1})e^{B(K_{n+1})}\Big\|_{L^p} \tag{VII.69}$$

where each G_ρ is a product of type $P, A, PE, AE, PEE \ldots$ where the A's and E's, as well as the B's in $F(\pi_B)$, are localized.

As in (IV.32) we apply the bound

$$\big|T_r\big(\wedge^r R_{n+1} \cdot \wedge_\rho G_\rho\big)\big| \leq \|\wedge^r R_{n+1}\| \cdot \prod_\rho \|G_\rho\|_1. \tag{VII.70}$$

Substituting (VII.70) into (VII.69) and applying Hölder's inequality, we find that (with $p^{-1} = \sum_{i=1}^{3} q_i^{-1}$)

$$\|w_{m,\Lambda}\|_{L^p} \leq \sum_{n,k} 3^{|n|} M \Big\| \prod_\rho \|G_\rho\|_1 \Big\|_{L^{q_1}} \|B(\pi_B)\|_{L^{q_2}} \|w_{r,\Lambda}(n+1)\|_{L^{q_3}} \tag{VII.71}$$

where

$$w_{r,\Lambda}(n) = \|\wedge^r R_n\| \det_3(1-K_n)e^{-B(K_n)}.$$

We control the growth of the last factor in (VII.71) by the following lemma which we prove below:

LEMMA VII.15. *For any $q < \infty$ there are constants c_i independent of r, n, Λ such that*

$$\|w_{r,\Lambda}(n)\|_{L^q(\mu_s)} \leq e^{c_1 r + c_2|n| + c_3|\Lambda|}. \tag{VII.72}$$

The estimates on the other factors in (VII.71) are so similar to those of §IV that we shall be rather brief. After inserting smooth localizations ζ_k as in §IV we estimate P_j factors by $\|P_j\|_1 = 1$, A factors as in (IV.41), and E factors using $\mathcal{C}_4$ norms on the K's and δK's. This bounds $\Pi\|G_\rho\|_1$ in terms of products of norms of individual localized K's or δK's. These norms are analyzed in Corollary VII.6 and Lemma VII.7 and we may estimate such products by the Checkerboard Theorem of [12] or Theorem 9.4 of [9] obtaining the usual local number singularity factors $\Pi_i M(\Delta_i)!$. Because of the momentum cutoff each K or δK will be a function of the field in a slightly enlarged unit square, but this presents no real problem; e.g., before applying the Checkerboard Estimate we first use Hölder in order to separate overlapping squares.

Although the estimate of the $B(\pi_B)$ term in (VII.71) is not much different from the estimates of Lemma IV.7, we indicate some of the details: When the sum (VII.61) is inserted into definition (VII.66) for B_α, we get a sum over localized B's, $B_\alpha = \sum_{k\in I} B_{\alpha,k}$. Explicitly we have

$$B_\alpha = B_{(i,j)} = :\mathrm{Tr}\Big(\tilde{K}_j^{(i)}\delta K_j^{(i)} + \big[\tilde{K}_j^{(i)\dagger}\delta K_j^{(i)}\big]_{s=1}\Big): + \sum_{k\neq i} :\mathrm{Tr}\, K_{n_k}^{(k)}\delta K_j^{(i)}:$$

(where $K_{n_k}^{(k)} = K^{(k)}$ if $k > i$) so that the localized $B_{\alpha,k}$'s are given by the following formulas: if $k = i$,

$$B_{\alpha,i} = :\mathrm{Tr}\Big(\tilde{K}_j^{(i)}\delta K_j^{(i)} + \big[\tilde{K}_j^{(i)\dagger}\delta K_j^{(i)}\big]_{s=1}\Big):$$

$$= 2\int \chi_i(x)b(x,y)\chi_i(y){:}\phi_f(x)\big(\phi(y) - \phi_{h_j}(y)\big){:}\; dx\, dy$$

where b is the kernel of B (see Lemma VII.12), $\phi_h = h*\phi$, $h_j \equiv h_{\kappa_j}$ and

(VII.73a) $$f(x) = \sum_{j'=1}^{j} \delta t_{j'}^{(i)} h_{j'}(x) + t_j^{(i)}\delta(x);$$

if $k \neq i$,

$$B_{\alpha,k} = :\mathrm{Tr}\, K_{n_k}^{(k)}\delta K_j^{(i)}:$$

$$= \int \chi_k(x)b'(x,y)\chi_i(y){:}\phi_g(x)\big(\phi(y) - \phi_{h_j}(y)\big){:}\; dx\, dy$$

where $g(x) = g^{(k)}(x) = \delta(x)$ if $k > i$, and

(VII.73b) $$g^{(k)}(x) = \sum_{j=1}^{n_k} \delta t_j^{(k)} h_j(x) \quad \text{if} \quad k < i,$$

and where, by Corollary VII.13 and Theorem VI.1,

(VII.74) $$\|\chi_k b' \chi_i\|_{L^{3/2-}} \leqslant c m_0^a e^{-\delta m_0 d(i,k)}.$$

We now estimate $\|B(\pi_B)\|_{L^{q_2}}$ exactly as in Lemma IV.7. Assuming without loss of generality that $q_2 = 2$, we evaluate $\|B(\pi_B)\|_{L^2}^2$ as a sum over contributions of vacuum graphs. The contribution of each vacuum graph is an integral of a product of $\chi b \chi$'s (or $\chi b' \chi$'s) linked by covariances of the form $f_1^* C^* f_2$ where $f_i = f, g$ or $(\delta - h_j)$ as defined in (VII.73). As in Lemma IV.7, we insert smooth localizations ζ_j covering the χ_j's and factors $D^{\pm\alpha}$ where $\frac{2}{3} < \alpha < \frac{3}{4}$. By Lemma VII.8, the operator norms on $L^2(\mathbf{R}^2)$

(VII.75) $$\|D^\alpha \zeta_i f_1^* C^* f_2 \zeta_k D^\alpha\| = O(e^{-\delta m_0 d(i,k)})$$

or $O(e^{-\varepsilon j} e^{-\delta m_0 d(i,k)})$ if one of the f_i's is $(\delta - h_j)$. The factors $D^{-\alpha}\chi b \chi D^{-\alpha}$ are in $L^2(\mathbf{R}^4)$ by Lemma VII.12, the factors $D^{-\alpha}\chi_i b' \chi_k D^{-\alpha}$ are in $L^2(\mathbf{R}^4)$ by (VII.74) and Lemma VII.12 with norm $O(e^{-\delta m_0 d(i,k)})$. Thus the estimation proceeds as in Lemma VII.8 with the sum over vacuum graphs controlled by the exponential decay of the covariances. Note that each $\alpha \in \pi_B$ gives rise to a factor $e^{-\varepsilon j}$ (as does each $\alpha \in \pi_n$).

In this way we obtain for some $\delta > 0$ and $\varepsilon > 0$,

$$\begin{aligned}&\Big\|\prod_\rho \|G_\rho\|_1\Big\|_{L^{q_1}}\cdot\|B(\pi_B)\|_{L^{q_2}}\\ &\leqslant c(\varepsilon)^{|n|}\prod_i\left[c^{M(\Delta_i)}M(\Delta_i)!\prod_{j=1}^{n_i}e^{-\varepsilon j}\right]\cdot\prod_{\alpha\in\pi_n}e^{-\delta m_0 d(\Delta_\alpha,k_\alpha)},\end{aligned}\tag{VII.76}$$

where $\Delta_\alpha = \Delta_i$ if $\alpha = (i,j)$ and

$$d(\Delta_\alpha, k_\alpha) = \sum_\beta d(\Delta_\alpha, k_{\alpha,\beta}).\tag{VII.77}$$

Collecting the estimates (VII.71), (VII.68), (VII.72) and (VII.76) we get

$$\begin{aligned}\|w_{m,\Lambda}\|_{L^p}&\leqslant c_1^{m+|\Lambda|}\sum_n c^{|n|}\prod_i\exp\left(\frac{-\varepsilon n_i^2}{2}\right)\\ &\quad\cdot\sum_k\prod_i c^{M(\Delta_i)}M(\Delta_i)!^2\prod_{\alpha\in\pi_n}e^{-\delta m_0 d(\Delta_\alpha,k_\alpha)},\end{aligned}\tag{VII.78}$$

where we have used the estimates $r = m + |\pi_A| \leqslant m + |n|$ and $\sum_{j=1}^{n_i} j \geqslant n_i^2/2$. By a combinatorial lemma due to McBryan [16, Lemma A.1] the sum over k in (VII.78) is bounded by $c(b)^{|n|}\prod_i(3n_i)!^b$ for any $b > 4$. Thus,

$$\begin{aligned}\|w_{m,\Lambda}\|_{L^p}&\leqslant c_1^{m+|\Lambda|}\prod_i\sum_{n_i}\exp\left[cn_i - \varepsilon n_i^2/2 + 3bn_i\log 3n_i\right]\\ &\leqslant c_1^{m+|\Lambda|}\prod_i c_3 \leqslant c_1^m e^{c_2|\Lambda|},\end{aligned}$$

establishing the theorem. □

PROOF OF LEMMA VII.15 IN THE CASE $s = 1$. We appeal to results of McBryan [15], [16]. As pointed out in [15, Lemma 3.3], the estimate

$$\|\wedge^r R_n\|\cdot|\det_3(1 - K_n)| \leqslant e^{r/2+\operatorname{Tr}(K_n+K_n^*)^2/4}$$

is an elementary generalization of Carleman's Inequality [5]. Hence, since

$$B(K) = \tfrac14 :\operatorname{Tr}(K + K^*)^2: + \tfrac12 :\operatorname{Tr}(K^*K - K^\dagger K):,$$

$$\begin{aligned}|w_{r,\Lambda}(n)| &\leqslant \exp\left[r/2 + \tfrac14\langle\operatorname{Tr}(K_n + K_n^*)^2\rangle\right]\\ &\quad\cdot\exp\left[\tfrac12 :\operatorname{Tr}(K_n^*K_n - K_n^\dagger K_n):\right],\end{aligned}\tag{VII.79}$$

where $\langle\cdot\rangle$ denotes $\int(\cdot)\,d\mu_s$ (here $s = 1$). The first factor on the right is less than $e^{r/2+O(|n+1|)}$ since, by Lemma 2.4 of [16],

$$\left\langle\operatorname{Tr}(K_n(1) + K_n^*(1))^2\right\rangle \leqslant O(|n + 1|).\tag{VII.80}$$

The second factor is in all $L^q(d\mu)$, $q < \infty$, with norm less than $e^{O(|\Lambda|)}$ since, by Lemma 2.1 of [16],

$$\left\|\exp\left[-\tfrac12 :\operatorname{Tr}\left(K_n(1)^*K_n(1) - K_n(1)^\dagger K_n(1)\right):\right]\right\|_{L^p}\leqslant e^{c|\Lambda|}\tag{VII.81}$$

uniformly in Λ. This completes the proof of Lemma VII.15 and, hence, of Theorem VII.14 in the case $s = 1$. $\square$

PROOF OF LEMMA VII.15 IN THE CASE $s \neq 1$. We split K_n into a low and high (fermion) momentum part:

$$K_n(s) = L_{\sigma,n} + H_{\sigma,n}(s)$$

with $L_{\sigma,n} = \sum_{i \in I} S_{0,\sigma}(1)\phi_{\kappa_{n_i}}\chi_{\Delta_i}$ for some appropriate choice of σ (to be determined). It is convenient, but not essential, in the sequel to assume that the σ cutoff is sharp (i.e. that $\tilde{h}_\sigma$ is the characteristic function of $\{k \mid |k| < \sigma\}$). This eliminates high-low cross-terms $\operatorname{Tr}(L^* H)$ or $\operatorname{Tr}(L^\dagger H)$. We now write (with $H_n = H_{\sigma,n}(S)$)

$$\begin{aligned} w_{r,\Lambda}(n) = \|\wedge^r R_n\| &\det\left[(1 - K_n)\exp\left(K_n + \tfrac{1}{2}H_n^2\right)\right] \\ &\cdot \exp\left[-\tfrac{1}{2} :\operatorname{Tr}\left(H_n^2 + H_n^* H_n\right): + \tfrac{1}{2}\langle \operatorname{Tr}\left(K_n^2 - H_n^2\right)\rangle \right. \\ &\qquad \left. + \tfrac{1}{2} :\operatorname{Tr}\left(H_n^* H_n - K_n(1)^\dagger K_n(1)\right): \right]. \end{aligned} \tag{VII.82}$$

In order to bound (VII.82) we note that

$$\begin{aligned} &\left| \|\wedge^r R_n\| \det\left[(1 - K_n)\exp\left(K_n + \tfrac{1}{2}H_n^2\right)\right]\right| \\ &\qquad \leqslant \exp\left[c_1 r + c_2\|L_n\|_1 + \tfrac{1}{2}\operatorname{Tr}\left(H_n^2 + H_n^* H_n\right)\right]. \end{aligned} \tag{VII.83}$$

(VII.83) follows by combining Lemma 2.3 of [23] with the fact that if A is symmetric with positive part A_+, then

$$\det{}_3(1 + A_+) < \exp\tfrac{1}{2}\operatorname{Tr} A_+^2 \leqslant \exp\tfrac{1}{2}\operatorname{Tr} A^2.$$

We thus obtain, as a generalization of (VII.79),

$$\begin{aligned} |w_{r,\Lambda}(n)| < \exp&\left[c_1 r + c_2\|L\|_1\right]\exp\left[\tfrac{1}{2}\langle \operatorname{Tr}\left(K_n^2 + H_n^* H_n\right)\rangle\right] \\ &\cdot \exp\left[\tfrac{1}{2} :\operatorname{Tr}\left(H_n^* H_n - K_n(1)^\dagger K_n(1)\right): \right]. \end{aligned} \tag{VII.84}$$

Since $\|L\|_1 < \sum_{i \in I}\|L\chi_i\|_1$, we find, by the Checkerboard Estimate and the argument of Lemma 2.5 of [23], that the factor $e^{c_2\|L\|_1}$ is in all L^p with norm bounded by $e^{O(|\Lambda|)}$ uniformly in s. The proof of the lemma will then be completed by the following two lemmas corresponding to (VII.80) and (VII.81). $\square$

LEMMA VII.16. *There is a constant $c(\sigma)$ independent of n, s, Λ such that*

$$\left\langle \operatorname{Tr}\left(K_n^2 + H_{\sigma,n}^* H_{\sigma,n}\right)\right\rangle \leqslant c|n + 1|. \tag{VII.85}$$

LEMMA VII.17. *For every $p < \infty$, there are a σ and $c(\sigma)$ independent of s and n such that*

$$\left\|\exp\left[\tfrac{1}{2} :\operatorname{Tr}\left(H_{\sigma,n}^* H_{\sigma,n} - K_n(1)^\dagger K_n(1)\right): \right]\right\|_{L^p(\mu_s)} < e^{c|\Lambda|}. \tag{VII.86}$$

PROOF OF LEMMA VII.16. We first isolate the s-independent high momentum cancellation, leaving a number of terms involving expectations of quadratic monomials in ϕ which can be estimated uniformly in n. By elementary algebra,

$$\begin{aligned}\left\langle \operatorname{Tr}\left(K_n^2 + H_n^* H_n\right)\right\rangle &= \left\langle \operatorname{Tr}\left(K_n(1)^2 + K_n(1)^* K_n(1)\right)\right\rangle + 2\langle \operatorname{Tr} K_n(1)\delta K\rangle \\ &\quad + \langle \operatorname{Tr} \delta K^2\rangle - \langle \operatorname{Tr} L^* L\rangle + \langle \operatorname{Tr} \delta K^* H_n(1)\rangle \\ &\quad + \langle \operatorname{Tr} H_n(1)^* \delta K\rangle + \langle \operatorname{Tr} \delta K^* \delta K\rangle,\end{aligned} \tag{VII.87}$$

where $\delta K = K_n(s) - K_n(1)$. Now by (VII.80) the first term on the right of (VII.87) is $O(|n+1|)$. We claim that the remaining terms are $O(|\Lambda|)$ uniformly in s and n, and are thus $O(|n+1|)$. Bounds uniform in n will follow by Lemma VII.8 if we can establish bounds in the absence of a boson momentum cutoff. For the terms $\langle \operatorname{Tr} K(1)\delta K\rangle$ and $\langle \operatorname{Tr} \delta K^2\rangle$ we insert localizations and then apply Lemma VII.2(a) and Hölder's inequality using the $e^{-\delta m_0 d(j_1,j_2)}$ decay factor to obtain an $O(|\Lambda|)$ bound.

We illustrate the bounds on the "adjoint" terms in (VII.87) by the case of $\operatorname{Tr}(\delta K^* H(1))$. We insert localizations and apply Lemma VII.18 below (with $\alpha = \frac{1}{2}$, $A = H(1)$, $B = \delta K$, and $\frac{3}{2} < q < 2$) to write

$$\begin{aligned}\left|\operatorname{Tr}(\delta K^* H(1))\right| &= \left|\sum_{j_1,j_2} \operatorname{Tr}\left((\delta K \chi_{j_1})^* H(1)\chi_{j_2}\right)\right| \\ &\leqslant c \sum_{j_1,j_2} \|\delta K \chi_{j_1}\|_q \|H(1)\chi_{j_2}\|_{q'} e^{-\delta m_0 d(j_1,j_2)}.\end{aligned}$$

Now $d(j_1, j_2)$ is a function of the difference $|j_1 - j_2|$ so we can apply Young's inequality on l^2:

$$\begin{aligned}\left|\operatorname{Tr}(\delta K^* H(1))\right| &\leqslant c\left(\sum_j \|\delta K \chi_j\|_q^2\right)^{1/2}\left(\sum_j \|H(1)\chi_j\|_{q'}^2\right)^{1/2} \\ &\leqslant \tfrac{1}{2} c \sum_j \left(\|\delta K \chi_j\|_q^2 + \|H(1)\chi_j\|_{q'}^2\right) \\ &\leqslant c \sum_{j\in I} \left(\|\delta K \chi_j\|_q^2 + \|K(1)\chi_j\|_{q'}^2 + \|L\chi_j\|_2^2\right),\end{aligned} \tag{VII.88}$$

where in the last line we have used the triangle inequality and the inclusion $\mathcal{C}_2 \subset \mathcal{C}_{q'}$ if $q' > 2$. By (VII.23) and (VII.26), $\langle \|\delta K \chi_j\|_q^2\rangle$ and $\langle \|K(1)\chi_j\|_{q'}^2\rangle$ are finite, as is $\langle \|L\chi_j\|_2^2\rangle$ because of the momentum cutoff. Thus $\langle \operatorname{Tr}(\delta K^* H(1))\rangle = O(|\Lambda|)$ and similarly for the other terms in (VII.87). $\square$

In the proof of the above lemma we have used

LEMMA VII.18. *For any α suppose that $A\chi_1 \in \mathcal{C}_q(\mathcal{H}_\alpha)$ and $B\chi_2 \in \mathcal{C}_{q'}(\mathcal{H}_\alpha)$*

with $1/q + 1/q' = 1$. Then there exist positive numbers c, δ independent of A and B such that

$$|\mathrm{Tr}(A\chi_1(B\chi_2)^*)| \leqslant ce^{-\delta m_0 d(\Delta_1,\Delta_2)}\|A\chi_1\|_{\mathcal{C}_q(\mathcal{H}_\alpha)}\|B\chi_2\|_{\mathcal{C}_{q'}(\mathcal{H}_\alpha)}.$$

PROOF. The case $d(\Delta_1, \Delta_2) = 0$ is just Hölder's inequality. If $d(\Delta_1, \Delta_2) > 0$ we write

$$\begin{aligned} |\mathrm{Tr}(A\chi_1(B\chi_2)^*)| &= \mathrm{Tr}(A\chi_1 D^{-\alpha}\chi_2 B^\dagger D^\alpha) \\ &= \mathrm{Tr}\big((D^{\alpha/2}A\chi_1 D^{-\alpha/2})(D^{\alpha/2}\zeta_1 D^{-\alpha}\zeta_2 D^{\alpha/2})(D^{-\alpha/2}\chi_2 B^\dagger D^{\alpha/2})\big) \\ &\leqslant \|A\chi_1\|_{\mathcal{C}_q(\mathcal{H}_\alpha)}\|B\chi_2\|_{\mathcal{C}_{q'}(\mathcal{H}_\alpha)}\|D^{\alpha/2}\zeta_1 D^{-\alpha}\zeta_2 D^{\alpha/2}\|_{\mathrm{op}(L^2)} \end{aligned}$$

and then apply Lemma VII.3(b). □

PROOF OF LEMMA VII.17. Since $C(s) \leqslant C(1)$ as operators we can invoke conditioning [11] to put $s = 1$ in the measure $d\mu_s$ and Wick ordering (the s-dependence of $H_{\sigma,n}(s)$ remains). We then write

$$:\mathrm{Tr}\big(H^*_{\sigma,n}H_{\sigma,n} - K_n(1)^\dagger K_n(1)\big): = A_1 + A_2 + A_3$$

with

$$\begin{aligned} A_1 &= :\mathrm{Tr}\big(H_{\sigma,n}(1)^* H_{\sigma,n}(1) - K_n(1)^\dagger K_n(1)\big): + M^2\int_\Lambda :\phi_n^2(x):\, dx, \\ A_2 &= -2\,\mathrm{Re}\langle \mathrm{Tr}\, H_{\sigma,n}(1)^*\delta K\rangle - \langle \mathrm{Tr}\,\delta K^*\delta K\rangle, \\ A_3 &= 2\,\mathrm{Re}\,\mathrm{Tr}(\delta K^* H_{\sigma,n}(1)) + \mathrm{Tr}(\delta K^*\delta K) - M^2\int_\Lambda :\phi_n^2(x):\, dx, \end{aligned}$$

where the constant M^2 is to be determined below and

$$\phi_n(x) = \sum_{k\in I}\chi_k(x)\phi_{g^{(k)}}(x)$$

where $g^{(k)}$ is defined in (VII.73b). By Hölder's inequality it is sufficient to bound each of $\|e^{1/2A_j}\|_{L^p}$ separately.

Now $A_1 = \int a_{\sigma,M}(x,y):\phi_n(x)\phi_n(y):\, dx\, dy$, where for any M we can choose σ sufficiently large so that $a_{\sigma,M}(x,y)$ is the kernel of a negative operator (see Lemma 2.8 of [23]). Hence by explicit Gaussian integration (see e.g. Lemma 3.3 of [22])

$$\|e^{A_1/2}\|_{L^p(\mu)} \leqslant \exp\big(\|A_1\|^2_{L^2(\mu)}\big).$$

From definition (VII.40),

$$A_1 = 2\int w_1(x,y):\phi_n(x)\phi_n(y): - :\mathrm{Tr}\, L^*_{\sigma,n}L_{\sigma,n}: + M^2\int_\Lambda :\phi_n^2:,$$

and so we have the bound

$$\text{(VII.89)}\quad \|A_1\|_{L^2}^2 \le c\left\{\left\|\int w_1{:}\phi_n\phi_n{:}\right\|_{L^2}^2 + \|{:}\mathrm{Tr}\, L^*L{:}\|_{L^2}^2 + \left\|M^2\int {:}\phi_n^2{:}\right\|_{L^2}^2\right\}.$$

It is easy to see that each of the terms in (VII.89) is $O(|\Lambda|)$. For example,

$$\left\|\int w_1{:}\phi_n\phi_n{:}\right\|_{L^2}^2 = \sum_{i,j,i',j'\in I}\int \chi_i(x)w_1(x,y)\chi_j(y)\chi_{i'}(x')w_1(x',y')\chi_{j'}(y')$$

$$\cdot\left[C_{ii'}(x,x')\,C_{jj'}(y,y') + C_{ij'}(x,y')C_{i'j}(x',y)\right]\cdot dx\,dx'\,dy\,dy'$$

where $C_{ii'} = g^{(i)*}C^*g^{(i')}$. As usual we insert smooth localizations and factors of $D^{\pm\alpha}$ for some α in $(\frac{1}{2},\frac{3}{4})$ and apply estimates (VII.42) and (VII.75). The exponential decay provided by these estimates controls the sum over i, j, i', j' and gives an $O(|\Lambda|)$ bound. As for the second term in (VII.89), the $O(|\Lambda|)$ bound follows from the fact that ${:}\mathrm{Tr}\, L^*L{:} = \int l_\sigma{:}\phi_n\phi_n{:}$, where $l_\sigma(x,y) = 2D^{-1}(x-y)h_\sigma^* D^{-1}(x-y) \in L^{2-}_{\mathrm{loc}}$ (because of the momentum cutoff) and $\|\chi_i l_\sigma \chi_j\|_{L^{2-}} = O(e^{-\delta m_0 d(i,j)})$. The last term in (VII.89) is obviously $O(|\Lambda|)$.

Now $A_2 = O(|\Lambda|)$ by the previous lemma. It thus remains to show that there is an $M < \infty$ independent of s such that $\|e^{A_3}\|_{L^p} = e^{O(|\Lambda|)}$. To do so, we appeal to the analysis of such problems by Seiler and Simon [24, §III]. In (VII.88) we use complex interpolation to replace the $\mathcal{C}_{q'}$ norm by $\mathcal{C}_2$ and $\mathcal{C}_4$ norms (see [23, Lemma 2.7]) and (VII.29) to replace the $\mathcal{C}_q$ norm (provided $q > \frac{12}{7}$) by a $\mathcal{C}_2$ norm:

$$\begin{aligned}|\mathrm{Tr}(\delta K^*H)| &\le c\sum_{j\in I}\Big(\|\delta K\chi_j D^{\varepsilon_1}\|_2^2 + \|K(1)\chi_j D^{-\varepsilon_2}\|_2^2 \\ &\qquad + \|K(1)\chi_j D^{\varepsilon_3}\|_4^2 + \|L\chi_j\|_2^2\Big) \qquad \text{(VII.90)}\\ &\equiv c\sum_{j\in I}\sum_{i=1}^4 \chi_j^{(i)},\end{aligned}$$

where $\varepsilon_1 = 2/q - 1 < 1/6$, $\varepsilon_2 > 0$, $\varepsilon_3 < 1/2$. By Hölder's inequality it is sufficient to show that there is an M such that for each i,

$$\prod_{j\in I}\exp\left(cX_j^{(i)} - \tfrac{1}{2}M^2\int_{\Delta_j}{:}\phi_{g^{(j)}}^2{:}\right)$$

has L^p norm $e^{O(|\Lambda|)}$. By the Checkerboard Theorem (after use of Hölder's inequality to remove the slight overlap caused by the cutoff functions $g^{(j)}$), we need only show that for some M,

$$\left\| \exp\left(X_j^{(i)} - \tfrac{1}{2} M^2 \int_{\Delta_j} :\phi_{g^{(j)}}: \right) \right\|_{L^p} < \infty.$$

According to the method of [24] this will be the case if for some M

$$\int \left(X_j^{(i)} \right)^2 d\nu_{Mj} < e^{-2}, \tag{VII.91}$$

where $d\nu_{Mj}$ is the Gaussian measure on $\mathcal{S}'$ with mean zero and covariance $C(1 + M^2 \chi_j C)^{-1}$. (We may drop the ultraviolet cutoff $g^{(j)}$ in the proof of (VII.91).) The cases $i = 2, 3, 4$ of (VII.91) are dealt with in [24] and the case $i = 1$ is similar since we know that the kernel of $X_j^{(1)} = \|\delta K \chi_j D^{\epsilon_1}\|_2^2$ is pointwise bounded by an s-independent function in $L^{p_0 -}$ (see Lemma VII.1). $\square$

VII.5. *Decoupling at $s = 0$*. We now complete the proof of Lemma III.1 by verifying (III.15); i.e., if $s = 0$ on $\mathcal{B}_0 \subset (\mathbf{Z}^2)^*$ with $\mathbf{R}^2 \sim \mathcal{B}_0 = \bigcup X_i$ a disjoint union, then in $L^2(d\mu_{C(s)})$,

$$B(\Lambda, s) = \sum_i B(\Lambda_i, s) \tag{VII.92}$$

where $\Lambda_i = \Lambda \cap X_i$. To prove (VII.92) we must show that the cross-terms between different Λ_i's vanish:

$$\int :\phi(x) \chi_{\Lambda_i}(x) b(s; x, y) \chi_{\Lambda_j}(y) \phi(y): \, dx \, dy = 0 \tag{VII.93}$$

in $L^2(d\mu_{C(s)})$ if $i \neq j$. If we estimate the $L^2(d\mu_{C(s)})$ norm of (VII.93) as in (VII.52), we see that (VII.93) follows from the assertion that $D^{-\beta} \chi_1 b \chi_2 D^{-\beta} = 0$ in $L^2(\mathbf{R}^4)$ for some $\beta < \frac{3}{4}$ where $\chi_1 = \chi_{\Lambda_i}$ and $\chi_2 = \chi_{\Lambda_j}$. This in turn follows if for any $f_1, f_2 \in \mathcal{S}(\mathbf{R}^2)$,

$$\int f_1(x) \chi_1(x) b(s; x, y) \chi_2(y) f_2(y) \, dx \, dy = 0. \tag{VII.94}$$

Suppose first that $d \equiv \operatorname{dist}(\operatorname{supp} f_1 \cup \operatorname{sup} f_2, \mathcal{B}_0) > 0$. Then if $\sigma > (2d)^{-1}$ we claim that $f_1(x) \chi_1(x) b_\sigma(s; x, y) \chi_2(y) f_2(y) = 0$. For consider formula (VII.41a) for $b_\sigma = b_{0,\sigma}$. Certainly $\delta(x - y)$ vanishes, as does $S_{0,\sigma}(s; x, y)$, since $S_0(s; x, y) = 0$ if $x \in \Lambda_i, y \in \Lambda_j$ and convolution with h_σ (see (III.9)) "blurs" the support of S_0 by a distance of only $1/2\sigma$. By the convergence (VII.49) of b_σ to b as $\sigma \to \infty$ we thus deduce (VII.94) in the case $d > 0$.

When $d = 0$ we simply approximate f_i by $f_i^{(n)}$ with $d > 0$ such that, as $n \to \infty$, $f_i^{(n)} \to f_i$ in both L^p, any $p < \infty$, and in $\mathcal{H}_\beta$ for some $\beta > 0$. Then $\chi_i f_i^{(n)} \to \chi_i f_i$ in L^p and in $\mathcal{H}_\alpha$ for $\alpha < \operatorname{Min}(1, \beta)/2$ (by Lemma VII.3(c)). Since b can be written as a sum $b = w_0 + \delta b_0$ where $w_0 D^{-\alpha}$ defines a bounded operator on $L^2(\mathbf{R}^2)$ and $\delta b_0 \in L^{12/11-}_{\mathrm{loc}}$ (see Lemma VII.12), we conclude that

$$\int f_1^{(n)} \chi_1 b \chi_2 f_2^{(n)} \to \int f_1 \chi_1 b \chi_2 f_2$$

and this establishes (VII.94) and the decoupling (VII.92).

VII.6. *Boson functional derivatives.* In [4] Dimock and Glimm proved the "integration by parts" formula

$$\text{(VII.95)} \qquad \frac{\partial}{\partial s_b} \int F(s, \phi)\, d\mu_{C(s)} = \int \left(\frac{\partial}{\partial s_b} + \frac{1}{2} \frac{\partial C}{\partial s_b} \cdot \Delta_\phi \right) F\, d\mu_{C(s)}$$

for a large class of functions F of the field ϕ (actually their F's did not have an explicit s dependence). Their method consists essentially of two parts. First they show that (VII.95) holds if F is a cylinder function of the form $F(\phi) = \hat{F}(\phi(f_1), \ldots, \phi(f_n))$ for some fixed $f_1, \ldots, f_n \in \mathcal{S}$ and polynomially bounded function $\hat{F}$ in $C^2(\mathbf{R}^n)$. This is proved by explicit Gaussian integration and integration by parts. In this case the functional derivative $\delta/\delta\phi$ is defined by

$$\text{(VII.96)} \qquad \left(\frac{\delta}{\delta\phi(x)} F \right)(\phi) = \sum_{l=1}^{n} f_l(x) \partial_l \hat{F}(\phi(f_1) \ldots \phi(f_n))$$

where ∂_l denotes the partial derivative with respect to the lth variable. Then for the family of functions F of interest they construct an approximating sequence of cylinder functions $\{F_N\}$ such that as $N \to \infty$ the integrals with respect to $d\mu_s$ of F_N, $\partial F_N/\partial s_b$, and $(\partial C/\partial s_b) \cdot \Delta_\phi F_N$ converge to the integrals of F, $\partial F/\partial s_b$ and $(\partial C/\partial s_b) \cdot \Delta_\phi F$, respectively. The validity of (VII.95) for F then follows by the elementary

LEMMA VII.19. *If $f_n \in C^1([0, 1])$, $f_n \to f$ and $f_n' \to g$ uniformly as $n \to \infty$, then $f \in C^1([0, 1])$ and $f' = g$.*

We have applied formula (VII.95) to the functions $F(s, \phi) = \tau_m(G)$ (see (III.32)), where the s-derivatives are computed by the rules (III.24)–(III.26) and the ϕ-derivatives are computed similarly just as if ϕ were a single variable provided that we define

$$\text{(VII.97)} \qquad \delta H(\phi)/\delta\phi(y) = H(\delta_y)$$

for linear functions H of ϕ (e.g. $H = \partial^\gamma K$). Of course, the functions $F = \tau_m(G)$ are not specifically covered by the results of Dimock and Glimm and in this subsection we verify (VII.95) for such F by following their procedure. However, we wish to emphasize that we do not claim to be rigorously justifying partial calculations such as (III.25). Rather, it is always necessary to collect terms as in (III.31) and (III.33) so that $R = (1 - K)^{-1}$ occurs in a single factor of the form $\bigwedge^r R$. For formulas like (III.24)–(III.27) do not even make sense if $1 \in \sigma(K)$. However after collecting terms we can assert:

LEMMA VII.20. *Let K be a $\mathcal{C}_3$-valued continuously differentiable function of s in an interval $I \subset \mathbf{R}$, i.e. $K \in C^1(I, \mathcal{C}_3(\mathcal{H}))$, $P \in C^1(I, \mathcal{C}_1(\bigwedge^n \mathcal{H}))$, $B \in$*

$C^1(I)$, and $\tau_n(P)$ be as in (III.32). *Then $\tau_n(P)$ is differentiable in I and* (III.33a) *holds.*

PROOF. The formulae (III.25), (III.26), (III.27) and so (III.33a) certainly hold for all s such that $1 \notin \sigma(K(s))$ (see for example [5]). If $1 \in \sigma(K(s))$ then there is an $\varepsilon > 0$ such that $1 \notin \sigma(\lambda K(s))$ for $0 < |\lambda - 1| < \varepsilon$. So (III.33a) holds for K replaced by λK in the definition of $\tau_n(P)$. But the right-hand side of (III.33a) is continuous in λ by results of the Appendix of [23]. Thus by Lemma VII.19, (III.33a) does indeed hold at $\lambda = 1$ even if $1 \in \sigma(K(s))$. $\square$

We define cylinder function approximants to $K(\phi)$ so as to preserve the algebraic structure of the expressions involved. First we apply boson and fermion momentum cutoffs, approximating $K(\phi)$ by $K_{\sigma,\kappa}(\phi)$ (see (VII.4)) which is in $\mathcal{C}_1(\mathcal{H})$ for all $\phi \in \mathcal{S}'$. Let $\tau_{m,\sigma,\kappa}(G)$ denote the corresponding cutoff version of $\tau_m(G)$. Then for the arguments G of τ_m which occur in the derivative formula (III.33), we know that $\int \tau_{m,\sigma,\kappa}(G)\, d\mu_s \to \int \tau_m(G)\, d\mu_s$. This follows by a dominated convergence type argument as in the proof of Theorem IV.1, since by the estimates of the preceding subsections $\tau_{m,\sigma,\kappa}(G) \in L^p(d\mu_s)$ for any $p < \infty$, uniformly in σ and κ. By Lemma VII.19 it thus suffices to prove (VII.95) for $K_{\sigma,\kappa}$. We henceforth fix σ and κ which will be suppressed in the notation.

Next, we choose an orthonormal basis $\{e_i\}$ in $\mathcal{H}_{1/2}$ and approximate $K^{\#} = K$ or $K^{\dagger}$ by the finite rank operators

(VII.98) $$K_N^{\#}(\phi) = E_N K^{\#}(\phi) E_N = \sum_{i,j \leq N} e_i\big(e_i, K^{\#}(\phi) e_j\big)(e_j, \cdot\,)$$

with matrix elements

(VII.99) $$K_N^{\#}(\phi)_{ij} = \begin{cases} \big(e_i, K^{\#}(\phi)e_j\big) = \phi\big(k_{ij}^{\#}\big) & \text{if } i,j \leq N, \\ 0 & \text{if } i \text{ or } j > N, \end{cases}$$

where $k_{ij}^{\dagger} = \overline{k_{ji}}$ and

(VII.100) $$\begin{aligned} k_{ij}(y) &= \iint dx\, e_i(x) DS_{0,\sigma}(x,z)\chi_\Lambda(z) e_j(z) h_\kappa(z-y)\, dz \\ &= K(\delta_y)_{ij}. \end{aligned}$$

Letting F_N be the approximant to F obtained by replacing K by K_N we verify (III.33) by means of Lemma VII.20. In this connection, note that the definition (VII.96) of $\delta F_N/\delta\phi(x)$ does indeed coincide with the prescription for calculating a $\delta/\delta\phi$ derivative given in (VII.97). We illustrate this in the simple case $F = \det_3(1 - K)$:

$$\frac{\delta}{\delta\phi(x)}\det_3(1-K(\phi)) = \sum_{i,j} k_{ij}(x)\frac{\partial}{\partial K_{ij}}\det_3(1-K(\phi)) \quad \text{(by (VII.96))}$$

$$= \sum_{i,j} K(\delta_x)_{ij} K(\phi)^2 \left[\frac{\partial}{\partial K_{ij}} K\right](\phi)\det_3(1-K(\phi)) \quad \text{(by (VII.100))}$$

$$= K(\phi)^2\Big(\sum_{i,j} K(\delta_x)_{ij} J_{ij}\Big)\det_3(1-K(\phi))$$

$$= K(\phi)^2 K(\delta_x)\det_3(1-K(\phi))$$

since $J_{ij} = \partial K/\partial K_{ij}$ has lm matrix element $(J_{ij})_{lm} = \delta_{il}\delta_{jm}$.

Since $K_{\sigma,\kappa} \in \mathcal{C}_1$ for all $\phi \in \mathcal{S}'$, $K_N \to K$ in $\mathcal{C}_1$, as $N \to \infty$, pointwise in ϕ. So if $F(\phi)$ is of the form $F(\phi) = \tau_m(G)$ as defined in (III.32), then $F_N \to F$, $\partial F_N/\partial s_b \to \partial F/\partial s_b$, $(\partial C/\partial s_b)\cdot\Delta_\phi F_N \to (\partial C/\partial s_b)\cdot\Delta_\phi F$ pointwise in ϕ. Since $\|K_N\|_1 < \|K\|_1$, the F_N, $\partial F_N/\partial s_b$, and $(\partial C/\partial s_b)\cdot\Delta_\phi F_n$ are dominated uniformly by $c_1 e^{c_2\|K\|_1 + m}$ which is in $L^1(d\mu_s)$ so that by the dominated convergence theorem, their integrals converge. This establishes (VII.95) for $K = K_{\sigma,\kappa}$.

Appendix. *Antisymmetric tensor products.* In this Appendix we collect and prove those properties of fermi statistics that we have used throughout the paper. We begin with some definitions. Given a separable Hilbert space $\mathcal{H}$, let $\bigwedge^n\mathcal{H}$ be its n-fold antisymmetric tensor product. It is convenient to regard $\bigwedge^n\mathcal{H}$ as a subspace of the ordinary tensor product $\otimes^n\mathcal{H}$, and to identify an operator A on $\bigwedge^n\mathcal{H}$ with the operator on $\otimes^n\mathcal{H}$ which equals A on $\bigwedge^n\mathcal{H}$ and is zero on $(\bigwedge^n\mathcal{H})^\perp$. We let π_n be the orthogonal projection in $\otimes^n\mathcal{H}$ onto $\bigwedge^n\mathcal{H}$. The antisymmetric tensor product of vectors $e_1, \ldots, e_n \in \mathcal{H}$ is

$$\text{(A.1)} \qquad e_1 \wedge \cdots \wedge e_n = \pi_n e_1 \otimes \cdots \otimes e_n = \frac{1}{n!}\sum_{\sigma\in S_n} \operatorname{sgn}\sigma\, e_{\sigma_1} \otimes \cdots \otimes e_{\sigma_n}.$$

If $\{e_i\}$ is an orthonormal basis for $\mathcal{H}$, then $\{(n!)^{1/2} e_{i_1} \wedge \cdots \wedge e_{i_n}\}$ with $i_1 < i_2 < \cdots < i_n$ is an orthonormal basis for $\bigwedge^n\mathcal{H}$. Note that

$$\text{(A.2)} \qquad (f_1 \wedge \cdots \wedge f_n, g_1 \wedge \cdots \wedge g_n) = (n!)^{-1}\det(f_i, g_j)$$

from which we obtain the basic identity (I.11). Given n (bounded) operators $A_1, \ldots, A_n$ on $\mathcal{H}$, their antisymmetric tensor product is defined on $\bigwedge^n\mathcal{H}$ by

$$\text{(A.3a)} \qquad A_1 \wedge A_2 \wedge \cdots \wedge A_n = \pi_n A_1 \otimes \cdots \otimes A_n \pi_n.$$

Alternatively, $A_1 \wedge \cdots \wedge A_n$ may be defined by its action on a basis vector

$$\text{(A.3b)} \qquad (A_1 \wedge \cdots \wedge A_n) e_{i_1} \wedge \cdots \wedge e_{i_n} = \frac{1}{n!}\sum_{\sigma\in S_n} A_{\sigma_1} e_{i_1} \wedge \cdots \wedge A_{\sigma_n} e_{i_n}.$$

We write $\bigwedge^n A = A \wedge A \wedge \cdots \wedge A$ (n factors). If A and B are operators on $\bigwedge^n \mathcal{H}$ and $\bigwedge^m \mathcal{H}$, respectively, we define $A \wedge B$ by $\pi_{n+m} A \otimes B \pi_{n+m}$. Finally, given an operator E on $\mathcal{H}$ we define its derivation $d \bigwedge^n E$ on $\bigwedge^n \mathcal{H}$ by $d \bigwedge^n E = nE \wedge I \wedge \cdots \wedge I$, or alternatively by its action on a basis vector:

$$\text{(A.4)} \qquad d\bigwedge^n E e_{i_1} \wedge \cdots \wedge e_{i_n} = \sum_{j=1}^{n} e_{i_1} \wedge \cdots \wedge E e_{i_j} \wedge \cdots \wedge e_{i_n}.$$

It is easy to see that the operator product $\wedge$ is commutative, associative, and distributive with respect to addition. However, it is not true in general that $(\bigwedge_{i=1}^n A_i) \cdot (\bigwedge_{i=1}^n B_i) = \bigwedge_{i=1}^n A_i B_i$. What is true is that

$$\text{(A.5)} \qquad (\bigwedge_{i=1}^n A_i) \bigwedge^n B = \bigwedge_{i=1}^n (A_i B)$$

as is obvious from (A.3). We also find that

$$\text{(A.6)} \qquad (\bigwedge_{i=1}^n A_i)\, d\bigwedge^n E = \sum_{i=1}^{n} A_1 \wedge \cdots \wedge A_i E \wedge \cdots \wedge A_n.$$

For put $B = (1 + sE)$ in (A.5). Since $(d/ds) \bigwedge^n B|_{s=0} = d \bigwedge^n E$, we deduce (A.6) by differentiating (A.5).

If $\mathrm{Tr}_{\wedge^n \mathcal{H}}(\cdot)$ denotes the trace on $\bigwedge^n \mathcal{H}$, we introduce $T_n(\cdot) = n! \mathrm{Tr}_{\wedge^n \mathcal{H}}(\cdot)$. Let $A \in \mathcal{C}_1(\mathcal{H})$, the trace class on $\mathcal{H}$. As emphasized by Simon [27],

$$\text{(A.7)} \qquad \det(1 + \lambda A) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} T_n(\bigwedge^n A)$$

where the series converges for all λ by virtue of the bound [27, Lemma 3.1]

$$\text{(A.8)} \qquad |T_n(\bigwedge^n A)| \leq n! \|\bigwedge^n A\|_1 \leq \|A\|_1^n.$$

(The subscript 1 denotes trace norm on $\bigwedge^n \mathcal{H}$ and $\mathcal{H}$, respectively.) Now suppose $A_1, \ldots, A_n \in \mathcal{C}_1(\mathcal{H})$. Then it is obvious from (A.7) that

$$\text{(A.9)} \quad T_n(A_1 \wedge A_2 \wedge \cdots \wedge A_n) = \frac{\partial^n}{\partial \lambda_1 \cdots \partial \lambda_n} \det\left(1 + \vec{\lambda} \cdot \vec{A}\right)\Big|_{\lambda_1 = \cdots = \lambda_n = 0}$$

where $\vec{\lambda} \cdot \vec{A} = \sum_{j=1}^n \lambda_j A_j$. The relevant properties of $T_n(A_1 \wedge \cdots \wedge A_n)$ can now be read off from (A.9). For consider the trace formula that we used to collect terms in the cluster expansion:

$$\text{(A.10)} \qquad \begin{aligned} T_n(A_1 \wedge \cdots \wedge A_n) &= T_{n-1}(A_1 \wedge \cdots \wedge A_{n-1}) \cdot T_1(A_n) \\ &\quad - T_{n-1}\left(A_1 \wedge \cdots \wedge A_{n-1} \cdot d\bigwedge^{n-1} A_n\right). \end{aligned}$$

It is proved as follows. Letting $R = (1 + \lambda_n A_n)^{-1}$ we have

$$\det(1+\vec{\lambda}\cdot\vec{A}) = \det\left[(1+\lambda_n A_n)\left(1+R\sum_{j=1}^{n-1}\lambda_j A_j\right)\right]$$

$$= \det(1+\lambda_n A_n)\det\left(1+\sum_{j=1}^{n-1}\lambda_j R A_j\right).$$

Therefore by (A.9),

$$\text{(A.11)}\quad \begin{aligned}&\frac{\partial^{n-1}}{\partial\lambda_1\ldots\partial\lambda_{n-1}}\det(1+\vec{\lambda}\cdot\vec{A})|_{\lambda_1=\cdots=\lambda_{n-1}=0}\\ &\quad = \det(1+\lambda_n A_n)T_{n-1}\left(\wedge^{n-1}R\cdot A_1\wedge\cdots\wedge A_{n-1}\right).\end{aligned}$$

Since $\partial\wedge^{n-1}R/\partial\lambda_n|_{\lambda_n=0} = -d\wedge^{n-1}A_n$, we deduce (A.10) by differentiating (A.11) with respect to λ_n. By taking linear combinations we can generalize (A.10) to

$$\text{(A.12)}\qquad T_n(B\wedge A) = T_{n-1}(B)T_1(A) - T_{n-1}(Bd\wedge^{n-1}A)$$

for A and B trace class operators on $\mathcal{H}$ and $\wedge^{n-1}\mathcal{H}$, respectively.

We also have the factorization used in (V.6) and (V.7):

LEMMA A.1. *Suppose that the operators $Q_{A,i}$, $i = 1,\ldots,m_A$, and $Q_{B,j}$, $j = 1,\ldots,m_B$, satisfy $Q_{A,i}Q_{B,j} = 0$ for all i and j. Then*

$$\text{(A.13)}\quad T_{m_A+m_B}\left(\left(\wedge Q_{A,i}\right)\wedge\left(\wedge Q_{B,j}\right)\right) = T_{m_A}\left(\wedge Q_{A,i}\right)T_{m_B}\left(\wedge Q_{B,j}\right).$$

PROOF. The left side of (A.13) is given by

$$\frac{\partial^{m_A+m_B}}{\partial\lambda_{A,1}\ldots\partial\lambda_{B,m_B}}\det(1+\vec{\lambda}_A\cdot\vec{Q}_A+\vec{\lambda}_B\cdot\vec{Q}_B)|_{\vec{\lambda}_A=\vec{\lambda}_B=0}.$$

But $\det(1+\lambda_A\cdot Q_A+\lambda_B\cdot Q_B) = \det(1+\lambda_A\cdot Q_A)\det(1+\lambda_B\cdot Q_B)$. □

Finally we note this generalization of (A.8):

LEMMA A.2. *Given $A_1,\ldots,A_n \in \mathcal{C}_1(\mathcal{H})$,*

$$\text{(A.14)}\qquad \|\wedge_i A_i\|_1 \leqslant \frac{1}{n!}\prod_i \|A_i\|_1.$$

PROOF. We first prove (A.14) in the special case where each A_i is a rank one operator $P_i = (f_i,\cdot)g_i$ with $\|f_i\| = \|g_i\| = 1$.

Letting $f = f_1\wedge\cdots\wedge f_n$ and $g = g_1\wedge\cdots\wedge g_n$, we have $P \equiv \wedge_i P_i = (f,\cdot)g$, $P^* = (g,\cdot)f$, $|P|^2 = \|g\|^2(f,\cdot)f$ and $|P| = \|f\|\cdot\|g\|P_f$, where P_f is the orthogonal projection onto $\{f\}$. Hence,

$$\|\wedge_i P_i\|_1 = \mathrm{Tr}|P| = \|f\|\cdot\|g\| \leqslant \frac{1}{n!},$$

where in the last inequality we have used (A.2) and Hadamard's determinant inequality [5]. But each A_i has a canonical representation $A_i = \Sigma_j a_i^{(j)} P_i^{(j)}$, with $P_i^{(j)}$ a rank one operator of the above type, $a_i^{(j)} > 0$, and $\|A_i\|_1 = \Sigma_j a_i^{(j)}$. Therefore by the triangle inequality,

$$\begin{aligned}\|\wedge_i A_i\|_1 &\leqslant \sum_{j_1 \cdots j_n} a_1^{(j_1)} \cdots a_n^{(j_n)} \|P_1^{(j_1)} \wedge \cdots \wedge P_n^{(j_n)}\|_1 \\ &\leqslant \frac{1}{n!} \Sigma a_1^{(j_1)} \cdots a_n^{(j_n)} = \frac{1}{n!} \prod_i \|A_i\|_1. \quad \square\end{aligned}$$

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