TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 234, Number 2, 1977

DERIVATIVES OF ENTIRE FUNCTIONS AND A QUESTION OF PÓLYA. II

BY

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ABSTRACT. It is shown that if f is an entire function of infinite order, which is real on the real axis and has, along with f', only real zeros, then f'' has nonreal zeros (in fact, infinitely many). The finite order case was treated by the authors in a preceding paper. The combined results show that the only real entire functions f for which f, f', and f'' have only real zeros are those in the Laguerre-Pólya class, i.e.

$$f(z) = z^m \exp\{-az^2 + bz + c\} \prod_n \left(1 - \frac{z}{z_n}\right) e^{z/z_n},$$

$a \geq 0$, b, c and the z_n real, and $\sum z_n^{-2} < \infty$. This gives a strong affirmative version of an old conjecture of Pólya.

In our previous paper [1], we showed that if f is an entire function of finite order which is real on the real axis and has the property that f, f', and f'' have only real zeros, then f is in the Laguerre-Pólya class, i.e. f has the form

$$f(z) = z^m \exp\{-az^2 + bz + c\} \prod_n \left(1 - \frac{z}{z_n}\right) e^{z/z_n}$$

with $a \geq 0$, b, c and each z_n real, and $\sum z_n^{-2} < \infty$. In this sequel we eliminate the condition of finite order, and we prove what we termed the "Hypothetical Theorem" in [1], thereby fully affirming an old conjecture of Pólya for real entire functions. Background to this problem and a related problem of Wiman was given by us in [1]. We establish the

THEOREM. *Let f be a real entire function of infinite order with only real zeros. If f' has only real zeros, then f'' has an infinite number of nonreal zeros.*

We shall confine ourselves to functions of infinite order satisfying

$$\log \log M(r, f) = O(r \log r) \tag{1.1}$$

since, as we pointed out in [1], Levin and Ostrovskiĭ [4] have obtained this theorem for real entire functions for which (1.1) does not hold. (Actually, Levin's and Ostrovskiĭ's result [4, Theorem 2] does not require that the zeros of f' be real. Without this assumption on the zeros of f', it is probable that our Theorem is still true. We also remark that for entire functions of infinite

Received by the editors May 13, 1976.
AMS (MOS) subject classifications (1970). Primary 30A66; Secondary 30A68, 30A70.

order the assumption of reality for f is necessary, as the example $f(z) = \exp(e^{iz})$ indicates. This observation is due to A. Edrei.)

We shall assume, without loss of generality, that $f(0) \neq 0$, $f'(0) \neq 0$. We denote by $\{a_n\}$ the sequence of distinct zeros of f and we enumerate them as follows:

$$\ldots < a_{k-1} < a_k < a_{k+1} < \ldots \qquad (-\infty \leqslant \alpha \leqslant k \leqslant \omega \leqslant +\infty,\ k \text{ finite}). \tag{1.2}$$

By Rolle's Theorem, f' has at least one zero in each interval (a_k, a_{k+1}); choose exactly one and denote it by b_k so that we have (reindexing if necessary)

$$a_k < b_k < a_{k+1} \quad \text{for all } k \text{ with } b - 1 < 0 < a_1. \tag{1.3}$$

As in [1], we set

$$\psi(z) = \begin{cases} \dfrac{z - b_0}{z - a_0} \displaystyle\prod_{k \neq 0} \dfrac{1 - z/b_k}{1 - z/a_k} & \text{if } \omega = +\infty, \\ \dfrac{z - b_0}{(z - a_0)(a_\omega - z)} \displaystyle\prod_{k \neq 0, \omega} \dfrac{1 - z/b_k}{1 - z/a_k} & \text{if } \omega < +\infty, \end{cases} \tag{1.4}$$

and note, as in [1], that ψ is meromorphic and maps the upper half-plane into the upper half-plane. This follows from the interlacing property (1.3) (cf. [3, pp. 308–309]).

(If f has no zeros we set $\psi(z) \equiv 1$, and we set $\psi(z) = (z - a_0)^{-1}$ if f has only one zero.)

We have, therefore,

$$(f'/f)(z) = \phi(z)\psi(z) \tag{1.5}$$

with ϕ real entire, and having only real zeros. A similar discussion applied to f' which is also real and, by hypothesis, has only real zeros gives

$$(f''/f')(z) = \phi_1(z)\psi_1(z) \tag{1.6}$$

with ψ_1 of the same form as ψ and obtained from the zeros of f' and f'' as ψ was obtained from the zeros of f and f'.

We proceed to show that ϕ_1 must have infinitely many zeros, at most two of which can be real. This latter statement as well as the statements of all the lemmas to follow are predicated on the assumption that f satisfies the hypothesis of the Theorem together with the growth condition (1.1). Coupled with the result of Levin and Ostrovskiĭ mentioned above, a proof of this assertion about the zeros of ϕ_1 establishes the Theorem.

We first prove

LEMMA 1. *The functions ϕ and ϕ_1 belong to the Laguerre-Pólya class and are of order, at most, one.*

PROOF. We first prove our assertion for ϕ.

Since ψ maps the upper half-plane into the upper half-plane (and, hence, the lower half-plane into the lower half-plane), ψ satisfies the inequalities of Carathéodory [3, p. 18].

$$(1.7) \qquad C_1 \frac{|\sin \theta|}{r} < |\psi(re^{i\theta})| < C_2 \frac{r}{|\sin \theta|} \qquad (\theta \neq 0, \pi),$$

where C_1 and C_2 are positive constants (depending on ψ). From (1.7) we see that

$$(1.8) \qquad m\left(r, \frac{1}{\psi}\right) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |\psi(re^{i\theta})| \, d\theta = O(\log r).$$

Applying standard estimates from the Nevanlinna theory, it follows readily from (1.5), (1.8), and (1.1), that

$$(1.9) \qquad m(r, \phi) \leqslant m(r, f'/f) + m(r, 1/\psi) = O(r \log r),$$

where m, as in (1.7), is the usual Nevanlinna proximity functional. Since ϕ is real entire with only real zeros and, in view of (1.9), at most of order one, ϕ is in the Laguerre-Pólya class.

It is sufficient now to observe that since f satisfies the growth property (1.1) so does f'. Then, repeating the above proof step by step while replacing f by f', f' by f'', ϕ by ϕ_1, and ψ by ψ_1, we conclude that ϕ_1 is also in the Laguerre-Pólya class and of order not exceeding one.

We require Lemma 1 for the proof of

LEMMA 2. *If ϕ has an infinite number of zeros, so does ϕ_1.*

PROOF. Suppose ϕ_1 has only a finite number of zeros. Since the order of ϕ_1 does not exceed one by Lemma 1,

$$(1.10) \qquad \phi_1(z) = e^{\alpha_1 z} P_1(z)$$

where α_1 is a real constant and P_1 is a polynomial.

By (1.7) applied to ψ_1,

$$(1.11) \qquad \begin{aligned} |(f''/f')(iy)| &= |\phi_1(iy)| \, |\psi_1(iy)| = |P_1(iy)| \, |\psi_1(iy)| \\ &= O\left(|y|^N\right) \qquad (|y| \to +\infty) \end{aligned}$$

for some integer N.

On the other hand, (1.5) implies that $f'(z) = f(z)\phi(z)\psi(z)$ so that

$$(1.12) \qquad \begin{aligned} (f''/f')(z) &= (f'/f)(z) + (\phi'/\phi)(z) + (\psi'/\psi)(z) \\ &= \phi(z)\psi(z) + (\phi'/\phi)(z) + (\psi'/\psi)(z), \end{aligned}$$

and, therefore,

(1.13) $$|f''/f'(iy)| \geqslant |\phi(iy)|\ |\psi(iy)| - \left|\frac{\phi'}{\phi}(iy)\right| - \left|\frac{\psi'}{\psi}(iy)\right|.$$

Since ϕ is in the Laguerre-Pólya class and has order $\leqslant 1$,

$$\phi(z) = e^{\alpha z} g(z)$$

where α is a real constant and g is a canonical product of genus 0 or 1 with an infinite number of real zeros. A well-known, and easily verified, growth property of such a function is given by

(1.14) $$|\phi(iy)| = |g(iy)| \neq O(|y|^n)$$

for any finite n.

In addition, it is also well known and readily verified that

(1.15) $$|(\phi'/\phi)(iy)| = o(|y|)$$

and that

(1.16) $$|(\psi'/\psi)(iy)| = O(1).$$

From (1.13)–(1.16) we have an immediate contradiction of (1.11). Consequently, ϕ_1 must have infinitely many zeros. We now prove

LEMMA 3. *ϕ_1 has an infinite number of zeros.*

PROOF. Our argument will exploit the Mittag-Leffler expansion of the function ψ in (1.4). It is given by [3, pp. 310–311]

(1.17) $$\psi(z) = \gamma z + \delta + \sum_k A_k\left(\frac{1}{a_k - z} - \frac{1}{a_k}\right),$$

where $\gamma \geqslant 0$, δ is real, each $A_k > 0$, and

(1.18) $$\sum_k \frac{A_k}{a_k^2} < +\infty.$$

In view of Lemma 2, we are done if ϕ has infinitely many zeros. We assume, therefore, that

(1.19) $$\phi(z) = e^{\alpha z} P(z),$$

where α is a real constant and P a polynomial.

First we show that f of infinite order forces $\alpha \neq 0$. Suppose $\alpha = 0$. Then it follows from (1.5)and (1.17) that

(1.20) $$\frac{f'}{f}(z) = P(z)\psi(z) = P(z)\left\{\gamma z + \delta + \sum_k A_k\left(\frac{1}{a_k - z} - \frac{1}{a_k}\right)\right\},$$

with γ, δ and the A_k's as above. Equating residues at $z = a_k$, we get

(1.21) $$P(a_k) = -m_k/A_k < 0,$$

since $A_k > 0$ and m_k denotes the multiplicity of the zero a_k of f.

Let d be the degree of P. We choose $M > 0$ so that for all k,

$$0 < -P(a_k) \leqslant M|a_k|^d. \tag{1.22}$$

Then (1.18), (1.21), and (1.22) imply that

$$\frac{1}{M}\sum_k \frac{m_k}{|a_k|^{d+2}} \leqslant \sum_k \frac{-m_k}{a_k^2 P(a_k)} = \sum_k \frac{A_k}{a_k^2} < +\infty. \tag{1.23}$$

Thus, if $\alpha = 0$ in (1.19), the zeros of f have a finite exponent of convergence and f may be written in the form

$$f(z) = \pi(z)e^{h(z)} \tag{1.24}$$

where π is a canonical product of finite genus having only real zeros and h is a real entire function. Then

$$(f'/f)(z) = (\pi'/\pi)(z) + h'(z) = P(z)\psi(z), \tag{1.25}$$

so that $h'(z) = P(z)\psi(z) - (\pi'/\pi)(z)$ and

$$m(r, h') \leqslant m(r, P) + m(r, \psi) + m(r, \pi'/\pi) + \log 2. \tag{1.26}$$

Each of the first three terms on the right-hand side of (1.26) is $O(\log r)$; the first since P is a polynomial, the second because ψ satisfies (1.7), and the third as a consequence of Nevanlinna's important lemma on the logarithmic derivative [5, pp. 56–63] applied to π which is of finite order.

It follows then that

$$m(r, h') = O(\log r), \tag{1.27}$$

which in turn implies that h' is a polynomial and, therefore, that h is a polynomial.

Thus, if $\alpha = 0$, we find that f is of finite order contradicting our hypothesis.

So we take $\alpha \neq 0$ in (1.19) and we treat this case in two subcases:

Case 1. Suppose f has an infinite number of zeros:

Then, using (1.18) and a residue argument similar to that in (1.20) $-$ (1.23), we get

$$\sum_k \frac{m_k}{e^{\alpha a_k}|a_k|^{d+2}} < +\infty, \tag{1.28}$$

where, as before, d denotes the degree of P.

Since $\alpha \neq 0$, the convergence of the series in (1.28) implies that either f has a finite number of positive zeros or a finite number of negative zeros. Say the former case holds. (The argument is similar in the latter situation.) Then (1.28) implies that

$$\alpha < 0, \tag{1.29}$$

which together with (1.19) and (1.29) gives

$$(f'/f)(z) = e^{\alpha z}P(z)\psi(z), \qquad \alpha < 0. \tag{1.30}$$

The assumption that ϕ has only finitely many zeros together with the fact that f has only finitely many positive zeros implies that f' also has only finitely many positive zeros.

Now, suppose ϕ_1 has only a finite number of zeros; then f'' has only a finite number of positive zeros since f' has this property. Moreover,

$$\phi_1(z) = e^{\alpha_1 z}P_1(z), \tag{1.31}$$

and a residue argument identical to the preceding one shows that $\alpha_1 < 0$, since f (and, therefore, f', by Rolle's Theorem) has an infinite number of zeros.

By (1.16) and (1.31), we have, on the one hand,

$$(f''/f')(z) = e^{\alpha_1 z}P_1(z)\psi_1(z), \tag{1.32}$$

and, by (1.30),

$$\begin{aligned}(f''/f')(z) &= (f'/f)(z) + \{(f'/f)'/(f'/f)\}(z)\\ &= e^{\alpha z}P(z)\psi(z) + \alpha + (P'/P)(z) + (\psi'/\psi)(z),\end{aligned} \tag{1.33}$$

with $\alpha < 0$ and $\alpha_1 < 0$. Letting $z = x \to +\infty$ in (1.32) and (1.33), we see that (1.32) implies that $(f''/f')(x) \to 0$ while (1.33) implies that $(f''/f')(x) \to \alpha$. (That $(\psi'/\psi)(x) \to 0$, as $z = x \to +\infty$, when $\omega < +\infty$ follows readily from (1.4).) Since $\alpha \neq 0$, we again have a contradiction, and we conclude that ϕ_1 has infinitely many zeros.

Case 2. f has finitely many zeros.

Suppose again that ϕ_1 has finitely many zeros. We still have (1.32) and (1.33), but we cannot infer anything about the signs of α and α_1 from the convergence of (1.28) and its analogue for f' when the number of zeros of f and f' is finite. However, in this case, ψ, ψ'/ψ, and ψ_1 are rational functions; hence, by letting $z = x \to +\infty$ or $-\infty$ (depending on the signs of α and α_1), we find once again that (1.32) and (1.33) are incompatible with $\alpha \neq 0$.

This completes the proof of the lemma.

Our final lemma is

LEMMA 4. *With at most two exceptions, the zeros of ϕ_1 are nonreal.*

PROOF. We know from Lemma 1 that ϕ is in the Laguerre-Pólya class. Consequently, as is well known and simple to verify,

$$(\phi'/\phi)'(x) \leqslant 0, \tag{1.34}$$

for $x \neq a_k$, a zero of ϕ.

The proof of Lemma 4 is now identical to the proof of Lemma 7 of [1], except that (1.34) replaces inequality (3.16) of [1].

PROOF OF THE THEOREM. Combining Lemmas 3 and 4, we see that ϕ_1 has infinitely many nonreal zeros and, as a result, so does f''.

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