TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 235, January 1978

LATTICE-VALUED BOREL MEASURES. II

BY

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ABSTRACT. Let T be a completely regular Hausdorff space, $C_b(T)$ the set of all bounded real-valued continuous functions on T, E a boundedly monotone complete ordered vector space, and $\varphi: C_b(T) \to E$ a positive linear map. It is proved that under certain conditions there exist σ-additive, τ-smooth or tight E-valued measures on T which represent φ.

Let T be a completely regular Hausdorff space, $C_b(T)$ the vector-lattice of all bounded, real-valued functions on T. Let E be a boundedly complete partially ordered vector space and $\varphi: C_b(T) \to E$ a positive linear map, i.e., $f \in C_b(T), f \geq 0$ implies $\varphi(f) \geq 0$. In case T is compact, it is known ([2], [5]) that a quasi-regular Borel measure μ on T which represents φ, i.e., $\varphi(f) = \int f d\mu$, $\forall f \in C(T)$, all continuous real-valued functions on T (see [4], [5] for details). The more general case of a Hausdorff completely regular space is the aim of study in this paper.

For a topological space Y let $\mathfrak{B}(Y)$ be the σ-algebra of all Borel subsets of Y and $\mathfrak{B}_0(Y)$ the σ-algebra of all Baire subsets of Y (that is, the smallest σ-algebra which makes each bounded continuous function on Y measurable). Let $B(Y)$ ($B_0(Y)$) be the space of all bounded Borel (Baire) measurable functions on Y. For basic facts about vector lattices we refer to [1] (see also [2], [4]–[9]). We shall make use of the result proved in [2], that if a boundedly σ-complete vector lattice E contains a vector subspace F which is monotone order σ-closed and if F contains a vector sublattice G of E, then F contains the order σ-closure of G. If S is a Stonian (σ-Stonian) compact Hausdorff space we define a mapping $\psi: B(S) \to C(S)$ ($\psi_1: B_0(S) \to C(S)$), $\psi(f) = f$ ($\psi_1(f) = f$) except on a meagre subset of S. It is easy to verify that ψ and ψ_1 are positive order σ-continuous linear maps and for any increasing net $\{f_\alpha\} \subset C(S)$, with $\sup f_\alpha = f \in B(S)$, $\psi(f) = \sup \psi(f_\alpha)$, sup being taken in the boundedly complete vector lattice $C(S)$ [2] (these maps are called Loomis-Sikorski maps [5]).

Throughout the paper any E-valued measure μ on any σ-algebra is required

Received by the editors July 7, 1975 and, in revised form, April 28, 1976.

AMS (MOS) subject classifications (1970). Primary 28A55, 46G10, 28A45; Secondary 28A25, 28A30.

Key words and phrases. τ-smooth measures, tight measures, weakly σ-distributive lattices, monotone order σ-continuous, order σ-continuous, monotone order σ-closed, order σ-closed.

to be nonnegative and σ-additive with respect to the order of E, i.e., whenever $\{F_n\}$, $1 \leqslant n < \infty$, is a monotonic sequence in the σ-algebra then $\mu(\bigcup_{n=1}^{\infty} F_n) = \bigvee_{n=1}^{\infty} \mu(F_n)$. Integration with respect to these measures is taken in the sense of [4], [5]. Throughout this paper $\varphi: C_b(T) \to E$ is assumed to be a given positive linear map, T being a Hausdorff completely regular space, and E a boundedly monotone complete vector lattice assumed to be over the field of real numbers. Denoting by X the Stone-Čech compactification of T, we get a positive linear map $\bar{\varphi}: C(X) \to E$, $\bar{\varphi}(f) = \varphi(f|_T)$, which is represented by a quasi-regular Borel measure $\bar{\mu}$ on X, by the

PROPOSITION 1 (WRIGHT [4], [5]). *Given a positive linear map φ: $C_b(T) \to E$, there exists a unique quasi-regular E-valued Borel measure $\bar{\mu}$ on X such that*

$$\varphi(f|_X) = \int f d\bar{\mu}, \quad \text{for all } f \in C(X).$$

PROOF. This theorem is proved in [5]. By quasi-regularity we mean that for any open subset V of X, $\bar{\mu}(V) = \sup\{\bar{\mu}(C): C \text{ compact}, C \subset V\}$.

To get a measure on T we first note the following result.

LEMMA 2. (i) $\mathfrak{B}_0(T) = \{E \cap T: E \in \mathfrak{B}_0(X)\}$.
(ii) $\mathfrak{B}(T) = \{E \cap T: E \in \mathfrak{B}(X)\}$.

Proof is similar to Lemma C [6].

COROLLARY 3. *There is a unique E-valued Baire measure μ on X which represents φ, i.e.,*

$$\varphi(f) = \int f d\mu, \quad \text{for all } f \in C_b(T),$$

if and only if $\bar{\mu}(A) = 0$ for any $A = \mathfrak{B}_0(X)$ with $A \cap T = \varnothing$.

PROOF. If $\bar{\mu}(A) = 0, \forall A \in \mathfrak{B}_0(X)$, with $A \cap T = \varnothing$, then defining $\mu(P) = \bar{\mu}(P_0)$, where $P_0 \cap T = P$ for a $P_0 \in \mathfrak{B}_0(X)$, for any $P \in \mathfrak{B}_0(T)$, it is easy to see that μ is well defined, is countably additive, and $\int f d\bar{\mu} = \int f|_X d\mu$, $\forall f \in C(X)$. Conversely, if there is such a μ, then $\int f d\bar{\mu} = \int f|_X d\mu$, $\forall f \in C(X)$. This means $\{f \in B_0(X): \int f d\bar{\mu} = \{f|_X d\mu\}\}$ contains $C(X)$ and is monotone order σ-closed. Thus $\int f d\bar{\mu} = \int f|_X d\mu$, $\forall f \in \mathfrak{B}_0(X)$, and so the result follows (cf. [6, Theorem E]).

For each positive e in E, let

$$E[e] = \{a \in E: \exists\lambda > 0 \text{ such that } -\lambda e \leqslant a \leqslant \lambda e\}.$$

Thus $E[e]$ is an order-unit space and can be equipped with the order-unit norm. Since φ and $\bar{\mu}$ take their values in E ($\phi(1)$) there is no loss of generality in supposing E is an order-unit space with order unit $e = \phi(1)$. Let E be equipped with order-unit norm.

PROPOSITION 4. *A sufficient condition for the existence of a unique E-valued Baire measure μ on T which represents φ is that there exists a weakly σ-distributive* [6], *boundedly σ-complete vector lattice W such that E can be embedded (without alternation of suprema) in W and, whenever $\{f_n\}$ is a monotonic decreasing sequence in $C_b(T)$ with pointwise infimum* 0, *then* $\bigwedge_{n=1}^{\infty}\phi(f_n) = 0$.

PROOF. In this case, $\bar{\mu}: B_0(X) \to E$ is regular [6]. The given condition gives $\bar{\mu}(Z) = 0$ for any zero set Z of X, $Z \cap T = \varnothing$ (by zero-set we mean $f^{-1}\{0\}$, for some $f \in C(X)$). By regularity $\mu(P) = 0$ for any $P \in \mathfrak{B}_0(X)$, $P \cap T = \varnothing$. Corollary 3 now gives the result.

PROPOSITION 5. *A sufficient condition for the existence of a unique E-valued Baire measure μ on T which represents φ is that whenever $\{f_n\}$ is a monotone decreasing sequence in $C_b(T)$ which pointwise converges to* 0 *then* $\|\varphi(f_n)\| \to 0$.

PROOF. Proceeding as in [2] we see that $\bar{\varphi}(C(X))$ is embedded, as an ordered vector space, in $C(S)$ for some Stonian compact Hausdorff space S, preserving arbitrary suprema and infima and $\bar{\varphi}(1)$ being the constant function 1 in $C(S)$ (this can also be done by taking MacNeille-Dedekind completion of E [4], [9]). This gives us a positive linear map φ: $C_b(T) \to C(S) \subset B_0(S)$, with pointwise order in $B_0(S)$. Since $B_0(S)$ is trivially weakly σ-distributive, it follows from [8, Theorem 3.4] that φ extends to a linear, positive, monotone order σ-continuous map $\varphi: B_0(T) \to B_0(S)$. Since $(\psi_1 \circ \varphi)^{-1}(E)$ is monotone order σ-closed and contains the lattice $C_b(T)$, we get $(\psi_1 \circ \varphi)^{-1}(E) = B_0(T)$. Defining $\mu = \psi_1 \circ \varphi|_{\mathfrak{B}_0(T)}$ we get the desired Baire measure. The uniqueness is easy to verify.

REMARK. It is enough to assume in Proposition 4 and 5 that E is boundedly monotone order σ-complete.

DEFINITION. (a) An E-valued measure $\mu: \mathfrak{B}(T) \to E$ is said to be τ-smooth if, whenever $\{U_\alpha\}$ is an increasing net of open sets, $\mu(\bigcup U_\alpha) = \bigvee_\alpha \mu(U_\alpha)$.

(b) An E-valued measure μ: $\mathfrak{B}(T) \to E$ is said to be tight if for any open set U, $\mu(U) = \bigvee\{\mu(C)$: C compact, $C \subset U\}$.

We list some properties of these measures.

PROPOSITION 6. (i) *If μ is τ-smooth then for any decreasing net $\{g_\alpha\}$ of bounded, upper semicontinuous, real-valued functions with pointwise* $\inf g_\alpha = g \in B(X_0)$, $\int g d\mu = \bigwedge_\alpha \int g_\alpha d\mu$; *also for an open set V, in T,* $\mu(V) = \bigvee\{\mu(P): P \subset V, P$ *closed in* $T\}$.

(ii) *If μ is tight, then the following statements hold*:

(a) *μ is τ-smooth.*

(b) *If a net $\{f_\alpha\}$, in $B(T)$, converges to $f \in B(T)$, uniformly on compact* *subsets of T, $\|f_\alpha\| \leqslant 1$, $\forall\alpha$, and E is a boundedly complete vector lattice, then $\int f_\alpha d\mu \to 0$ (order convergence).*

(c) *If V_1 and V_2 are open subsets of T, then $\mu(V_1 \setminus V_2) = \bigvee\{\mu(C): C$ compact, $C \subset V_1 \setminus V_2\}$.*

(d) *For a bounded, nonnegative, lower semicontinuous function f on T, $\int f d\mu = \bigvee\{\int g d\mu : 0 \leqslant g \leqslant f$, g simple and a combination of characteristic functions of disjoint compact subsets of $T\}$.*

PROOF. (i) This follows by using the inequality

$$\frac{1}{n}\sum_{i=1}^{n}\mu\left(\left\{x \in T: f(x) \geqslant \frac{i}{n}\right\}\right) \leqslant \int f d\mu \leqslant \frac{1}{n}\mu(T) + \frac{1}{n}\sum_{i=1}^{n}\mu\left(\left\{x \in T: f(x) \geqslant \frac{i}{n}\right\}\right),$$

valid for any $n \geqslant 1$ and for any measurable f, $0 < f < 1$. The regularity property follows from the fact that every point of T has a nbd. base consisting of closed sets.

(ii) (a) is trivially obvious. To prove (b), we have, for any compact subset C of T,

$$\begin{aligned}\int f d\mu = \int_C f d\mu + \int_{T\setminus C} f d\mu &\leqslant \lim \int_C f_\alpha d\mu + \mu(T \setminus C)\\ &= \underline{\lim}\left(\int f_\alpha d\mu - \int_{T\setminus C} f_\alpha d\mu\right) + \mu(T \setminus C)\\ &\leqslant \underline{\lim} \int f_\alpha d\mu + 2\mu(T \setminus C).\end{aligned}$$

In a similar way we get $\int f d\mu \geqslant \overline{\lim} \int f_\alpha d\mu - 2\mu(T \setminus C)$, from which the result follows. Proof of (c) is straightforward. To prove (d), assume $0 < f < 1$, fix a positive integer n and let $V_i = \{x \in T: f(x) > i/n\}$, $0 \leqslant i \leqslant n$. V_i's are open and

$$\sum_{i=1}^{n} \frac{i-1}{n}\chi_{V_{i-1}\setminus V_i} \leqslant f \leqslant \sum_{i=1}^{n} \frac{i}{n}\chi_{V_{i-1}\setminus V_i}.$$

Using (c) and the fact

$$\int\left(\sum_{i=1}^{n}\frac{i}{n}\chi_{V_{i-1}\setminus V_i} - \sum_{i=1}^{n}\frac{i-1}{n}\chi_{V_{i-1}\setminus V_i}\right)d\mu = \frac{1}{n}\mu(V_0) = \frac{1}{n}\mu(T)$$

we get the result.

COROLLARY 7. *There exists an E-valued τ-smooth measure μ on T which represents φ, if and only if $\mu(P) = 0$, for each Borel set $P \in \mathfrak{B}(X)$, $P \cap T = \varnothing$.*

PROOF. If $\bar{\mu}(P) = 0$ for any $P \in \mathfrak{B}(X)$, $P \cap T = \varnothing$, we define $\mu(Q) = \bar{\mu}(\hat{Q})$ for any $Q \in \mathfrak{B}(T)$, $\hat{Q}$ being in $\mathfrak{B}(X)$ such that $\hat{Q} \cap T = Q$. It is easily verified that μ is well defined and is countably additive. Also, since $\bar{\mu}$ is τ-smooth and $\bar{\mu}(P) = 0$, $\forall P \in \mathfrak{B}(X)$, $P \cap T = \varnothing$, it easily follows that μ is τ-smooth. Further, $\int f d\bar{\mu} = \int f|_X d\mu$, $\forall f \in C(X)$. Conversely, if there is such a μ, then

$$\int f d\bar{\mu} = \int f|_X d\mu, \forall f \in C(X).$$

This means $H = \{f \in B(X): \int f d\bar{\mu} = \int f|_X d\mu\}$ is a monotone order σ-closed subspace of $B(X)$ and contains all bounded upper semicontinuous on X (using Proposition 6). Since the subspace, of $B(X)$, generated by upper semicontinuous functions on X is a vector sublattice of $B(X)$ (simple verification) we have $H = B(X)$ and so the result follows (cf. Corollary 3).

PROPOSITION 8. *A sufficient condition for the existence of a unique τ-smooth E-valued Borel measure μ on T which represents φ is that*

(i) *whenever $\{f_\alpha\}$ is a decreasing net in $C_b(T)$ with* $\inf f_\alpha = 0$ (*pointwise order*) *than* $\bigwedge \varphi(f_\alpha) = 0$, *and*

(ii) *E is embedded, as an ordered vector space, in a weakly* (σ, ∞)-*distributive vector lattice* [7], *preserving arbitrary suprema and infima.*

PROOF. Idea of proof is same as Proposition 4. The measure $\bar{\mu}: \mathfrak{B}(X) \to E$ is regular in this case. Proceeding as in Proposition 4 and using Corollary 7, we get the result.

PROPOSITION 9. *A sufficient condition for the existence of a unique τ-smooth E-valued Borel measure μ on T which represents φ is that whenever $\{f_\alpha\}$ is a decreasing net in $C_b(T)$ with* $\inf f_\alpha = 0$ (*pointwise order*), *then* $\|\varphi(f_\alpha)\| \to 0$.

PROOF. As in Proposition 5, $\bar{\phi}(C(X))$ can be considered embedded, as an ordered vector space, in $C(S)$ for a Stonian compact Hausdorff space S, preserving arbitrary suprema and infima. This gives us a positive linear map $\varphi: C_b(T) \to C(S) \subset B_1(S)$, $B_1(S)$ being all bounded real-valued functions on S with pointwise order. Since $B_1(S)$ is boundedly complete and weakly (σ, ∞)-distributive, using Proposition 8, we get a τ-smooth $B_1(S)$-valued measure μ_0 on T representing φ. Now $H = \{f \in B(T): \int f d\mu_0 \in B(S)\}$ is a monotone order σ-closed subspace on T and so $H = B(T)$ (same argument as in Corollary 7). The required measure is $\mu = \psi \circ \mu_0|_{\mathfrak{B}(T)}$. To prove it is E-valued let $H_1 = \{f \in B(T): \int f d\mu \in E\}$. Then H_1 is a monotonic order σ-closed subspace of $B(T)$ and contains upper semicontinuous bounded functions on T. Arguing as in Corollary 7, we prove $H_1 = B(T)$. This proves μ is E-valued. Uniqueness is easily verified.

PROPOSITION 10. *A necessary and sufficient condition for φ being representable by a unique tight Borel measure μ on T is that $\varphi(1) = \bar{\mu}(X) = \bigvee\{\bar{\mu}(C)$: C compact, $C \subset T\}$. If E is a boundedly complete vector lattice then this will happen if and only if for any uniformly bounded net $\{f_\alpha\} \subset C_b(T)$ such that $f_\alpha \to 0$ uniformly on compact subsets of X_0, $\varphi(f_\alpha) \to 0$ in E (order convergence).*

PROOF. First suppose that the condition is satisfied. Let B be any Borel subset of X disjoint from T and let C be any compact subset of T. Then $\bar{\mu}(X) \geqslant \bar{\mu}(B \cup C) = \bar{\mu}(B) + \bar{\mu}(C)$. Thus $\bar{\mu}(B) = 0$ and so by Corollary 7 there exists a well-defined τ-smooth E-valued Borel measure μ on T with $\bar{\mu}(B) = \mu(B \cap T)$, $\forall B \in \mathfrak{B}(X)$. In particular, $\bar{\mu}(C) = \mu(C)$ for any compact $C \subset T$. If P is a closed subset of X, then for any compact $C \subset T$, we have

$$\mu(P \setminus P \cap C) = \mu(P \setminus C) \leqslant \mu(T \setminus C) \leqslant \bar{\mu}(X \setminus C) = \bar{\mu}(X) - \bar{\mu}(C),$$

and so $\mu(P) = \bigvee\{\mu(P \cap C): C$ compact in $T\}$. Now for any open set U, in T, $\mu(U) = \bigvee\{\mu(P): P \subset U,\ P$ closed in $X\}$ (Proposition 6). Hence μ is tight. Converse and uniqueness are easy to verify. Let E be a boundedly complete vector lattice and suppose that φ satisfies the hypothesis. We define a partial order on $I = \{(C, \alpha): C$ a compact subset of T and α a finite subset of $T \setminus C\}$, $(C_2, \alpha_2) \geqslant (C_1, \alpha_1)$ if $C_2 \supset C_1$ and $\alpha_2 \supset \alpha_1 \setminus C_2$. I becomes a directed set. Define, $\forall (C, \alpha) \in I, f_{C,\alpha} \in C(X), 0 \leqslant f_{C,\alpha} \leqslant 1$,

$$f_{C,\alpha} = \begin{cases} 0, & \text{on } \alpha, \\ 1, & \text{on } C. \end{cases}$$

Evidently $f_{C,\alpha}|_T \to 1$ uniformly on compact subsets of T and so $\bar{\mu}(f_{C,\alpha}) \to \bar{\mu}(1)$ (order convergence) in E. For a $(C_0, \alpha_0) \in I$, $\inf\{\bar{\mu}(f_{C,\alpha}):(C, \alpha) \geqslant (C_0, \alpha_0)\} \leqslant \inf\{\bar{\mu}(f_{C_0,\alpha}):(C_0, \alpha) \geqslant (C_0, \alpha_0)\} = \bar{\mu}(C_0)$ and so $\varphi(1) = \bar{\mu}(1) = \bigvee\{\bar{\mu}(C)$: C compact, $C \subset T\}$. The converse is straightforward.

REMARK. The second characterization of Proposition 10 is the definition of the tight functional given in [3].

Using similar methods we have the following sufficient condition for the measure extension to hold in any boundedly σ-complete vector lattice.

PROPOSITION 11. *Let $\mathfrak{A}$ be an algebra of subsets of a set Y, $\mathfrak{A}^\sigma$ be the σ-algebra generated by $\mathfrak{A}$, and E a boundedly monotone σ-complete partially ordered vector space. Let $q:\mathfrak{A} \to E$ be a positive, finitely additive set function such that whenever $\{A_n\}$ is a monotone decreasing sequence in $\mathfrak{A}$ with $\bigcap_{n=1}^{\infty} A_n = \varnothing$, then $\|q(A_n)\| \to 0$. Then there exists a countably additive E-valued measure q^σ on $\mathfrak{A}^\sigma$ which extends q.*

PROOF. As in Proposition 5, we can consider $q(\mathfrak{A}) \subset C(S) \subset B_0(S)$, for some Stonian compact Hausdorff space S. With pointwise order on $B_0(S)$, it is weakly σ-distributive and so we have a countably additive measure $\mu:\mathfrak{A}^\sigma \to$

$B_0(S)$ [6]. The desired measure is $q^\sigma = \psi_1 \circ \mu$. It is easy to verify that q^σ is E-valued.

I am very grateful to the referee for making many useful suggestions which simplified some proofs.

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