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THE RIEMANN HYPOTHESIS FOR SELBERG'S ZETA-FUNCTION AND THE ASYMPTOTIC BEHAVIOR OF EIGENVALUES OF THE LAPLACE OPERATOR

BY

BURTON RANDOL

ABSTRACT. Much of that part of the theory of the Riemann zeta-function based on the Riemann hypothesis carries over to zeta-functions of Selberg's type, and in this way one can get asymptotic information about various eigenvalue problems. The methods are illustrated in the case of a compact Riemann surface.

0. Let $\mathfrak{S}$ be a compact Riemann surface, regarded as the quotient of the upper half-plane H^+ by a discontinuous group Γ. We assume that H^+ is endowed with the metric $y^{-2}((dx)^2 + (dy)^2)$, and we denote the volume of $\mathfrak{S}$ by A. Denoting by $\triangle$ the Laplace operator on $\mathfrak{S}$, it is known [3] that the eigenvalues $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots$ for the problem $\triangle f + \lambda f = 0$ on $\mathfrak{S}$ satisfy the asymptotic law

$$\sum_{\lambda_n < T} 1 = \frac{A}{4\pi} T + o(T^{1/2}).$$

In §2 of this paper, we will show that this estimate for the remainder term can be improved to $O(T^{1/2}/\log T)$. In §3, we will obtain an estimate in the other direction, by showing that the remainder term satisfies $\Omega(\log^{1/2-\varepsilon} T)$, for any $\varepsilon > 0$. The truth accordingly lies somewhere between the laws

$$O(T^{1/2}/\log T) \quad \text{and} \quad \Omega(\log^{1/2-\varepsilon} T).$$

The methods depend on the fact that the Riemann hypothesis for the Selberg zeta-function is almost true, in the sense that any possible exceptional zeros are all located in the real segment (0, 1).

I have recently learned that Pierre Bérard of the University of Paris 7 has succeeded in showing that the $O(T^{1/2}/\log T)$ half of the estimate is also valid for surfaces of variable negative curvature [1]. His methods, which are based on [3], are radically different from those of this paper, and the two somewhat complement each other. Both techniques extend to some higher-dimensional cases, in the case of this paper, via the zeta-functions of Selberg's

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type recently introduced by Gangolli [4]. In addition, the methods of this paper can also be applied, at least in two dimensions, to the case in which Γ has torsion, and the admissible functions are defined on H^+ and may transform nontrivially under Γ, e.g., by a character.

The idea for treating the eigenvalue remainder problem via the Selberg zeta-function came from reading lecture notes of Langlands [6], and most of §1 of this paper is adapted from these notes. In [6], Langlands uses a counterpart of the Riemann-von Mangoldt theorem to obtain an estimate which in this context would be $O(T^{1/2})$. The technique for reducing this by a logarithmic factor is adapted from Littlewood's work on consequences of the Riemann hypothesis ([11, Chapter 14]).

I would like to thank Dennis Hejhal for spotting an error in an earlier version of this paper. Hejhal also informs me that Selberg has obtained results similar to those of this paper, and that these will be appearing in Hejhal's forthcoming book on the trace formula.

1. We will illustrate the method in the case of a compact Riemann surface, and begin by recalling a few facts about the Selberg zeta-function $Z(s)$ associated with the surface [4], [7], [10]. To begin with, $Z(s)$, for Re $s > 1$, is defined by the product

$$\prod_{\gamma} \prod_{n=0}^{\infty} (1 - e^{-l_\gamma(s+n)}),$$

where l_γ runs through the set of lengths of primitive closed geodesics on $\mathcal{S}$. $Z(s)$ is uniformly bounded and bounded away from zero in any half-plane of the form Re $s > 1 + \varepsilon$ $(\varepsilon > 0)$, and setting $s = \sigma + it$, the sum $\Sigma_\gamma \Sigma_{n=0}^{\infty} e^{l_\gamma(\sigma+n)}$ is convergent for $\sigma > 1$, so $Z(s)$ tends to 1 exponentially as $\sigma \to \infty$. Furthermore, $Z(s)$ can be continued into an entire function of order 2, having a sequence of zeros at $1, 0, -1, -2, \ldots,$ with the zero at $s = 1$ simple, and having all additional zeros in the critical strip $0 <$ Re $s < 1$. The zeros in the critical strip are located at points which are solutions of the equations $s(1 - s) = \lambda_n$ $(n \geqslant 1)$. The multiplicity of such a zero is the same as the multiplicity of the corresponding eigenvalue, except that $s = \frac{1}{2}$, if it occurs as a zero, has double the multiplicity of the corresponding eigenvalue. The expression $s(1 - s)$ is symmetric about the point $s = \frac{1}{2}$, so the zeros of $Z(s)$ within the critical strip occur in symmetric pairs about $s = \frac{1}{2}$. These zeros are all located on the union of the real segment $0 < s < 1$ with the critical line Re $s = \frac{1}{2}$. It is possible to exhibit surfaces for which zeros occur in $\frac{1}{2} < s < 1$ [8], and such zeros correspond to eigenvalues in $(0, \frac{1}{4})$.

LEMMA 1. *Suppose $\sigma < \frac{1}{2}$ fixed. Then there exists a bounded function $f(t)$, such that as $|t| \to \infty$, $Z(\sigma + it) = f(t)e^{A(1/2-\sigma)|t|}Z(1 - \sigma - it)$.*

PROOF. $Z(s)$ satisfies the functional equation

$$Z(s) = Z(1-s)e^{A\int_0^{s-1/2} w \tan \pi w \, dw}$$

[10]. Since $A = 4\pi(g-1)$ by the Gauss-Bonnet theorem, where g is the genus of $\mathfrak{S}$, the second factor on the right is well-defined. Now as $|t| \to \infty$,

$$\tan \pi(\sigma + it) = \frac{t}{|t|} i + O(e^{-2\pi|t|}),$$

which implies that at points on a vertical line away from the real axis,

$$A\int_0^{s-1/2} w \tan \pi w \, dw = \frac{t}{|t|}\frac{Ai}{2}\left(s-\tfrac{1}{2}\right)^2 + O(1)$$

$$= A\left(\tfrac{1}{2}-\sigma\right)|t| + \frac{t}{|t|}\frac{Ai}{2}\left(\left(\sigma-\tfrac{1}{2}\right)^2 - t^2\right) + O(1),$$

where we are assuming that the integration is carried out along the line segment joining the origin to $s - \frac{1}{2}$. This proves the lemma.

LEMMA 2. $Z(\sigma + it) = e^{O(|t|)}$, *uniformly in any bounded strip* $b_1 < \sigma < b_2$.

PROOF. Since $Z(s)$ is of order 2, this is implied by the Phragmén-Lindelöf theorem and Lemma 1, if we enlarge the strip if necessary so as to place its right-hand boundary in the half-plane $\operatorname{Re} s > 1$, and make its left-hand boundary the reflection of the right-hand boundary through the line $\operatorname{Re} s = \frac{1}{2}$.

Suppose now a is a positive number satisfying $1 < a < 5/4$. We will estimate, as a function of T, the number of zeros of $(Z(s))^2$ in the rectangle R, defined by the inequalities $1 - a < \operatorname{Re} s < a$, $-T < \operatorname{Im} s < T$, assuming that T is selected so that no zero occurs on the boundary of R. Call this function $2N(T)$. Then the number of zeros of $Z(s)$ itself in this rectangle is $N(T)$.

Define $\xi(s) = (Z(s))^2 e^{\Phi(s-1/2)}$, where $\Phi(s) = -A\int_0^s w \tan \pi w \, dw$. For the moment, $\Phi(s)$ is only defined up to multiples of $2\pi i$, while $e^{\Phi(s)}$ is well defined. We now specify $\Phi(s)$ in the open upper and lower half-planes to be the value obtained by carrying out the integration along the straight line segment joining the origin to s. If s is on the real line, and not one of the points $\pm 1/2$, $\pm 3/2$, $\pm 5/2, \ldots,$ we define $\Phi(s)$ by the requirement of continuity as s is approached from the upper half-plane.

Now since $(Z(1-s))^2 = (Z(s))^2 e^{2\Phi(s-1/2)}$ and $e^{\Phi(s)+\Phi(-s)} \equiv 1$, it follows that $\xi(1-s) = \xi(s)$. Note that $\xi(s)$ is real on the real axis, and so satisfies $\xi(\bar{s}) = \overline{\xi(s)}$. The number of zeros of $(Z(s))^2$ in R is

$$\frac{1}{2\pi i}\int_{\partial R}\frac{\xi'(s)}{\xi(s)}\,ds = \frac{1}{2\pi}\operatorname{Im}\left(\int_{\partial R}\frac{\xi'(s)}{\xi(s)}\,ds\right),$$

and it follows in the standard manner [2] from the functional equation for $\xi(s)$, and from the fact that $\xi(\bar{s}) = \overline{\xi(s)}$, that this integral is equal to

$$\frac{2}{\pi} \operatorname{Im}\left(\int_C \frac{\xi'(s)}{\xi(s)}\, ds \right),$$

where C is the portion of ∂R consisting of the vertical segment from a to iT, plus the horizontal segment from $a + iT$ to $\frac{1}{2} + iT$.

Now $\xi'(s)/\xi(s) = 2Z'(s)/Z(s) + \Phi'(s - \frac{1}{2})$, so

$$\begin{aligned}
\frac{2}{\pi} \operatorname{Im}\left(\int_C \frac{\xi'(s)}{\xi(s)}\, ds \right) &= \frac{4}{\pi} \operatorname{Im}\left(\int_C \frac{Z'(s)}{Z(s)}\, ds \right) + \frac{2}{\pi} \operatorname{Im}\left(\int_C \Phi'(s - \tfrac{1}{2}) ds \right) \\
&= \frac{4}{\pi} \operatorname{Im}\left(\int_C \frac{Z'(s)}{Z(s)}\, ds \right) + \frac{2}{\pi} \operatorname{Im}\left(\Phi(iT) - \Phi\left(a - \tfrac{1}{2}\right) \right) \\
&= \frac{2}{\pi} \operatorname{Im} \Phi(iT) + \frac{4}{\pi} \operatorname{Im}\left(\int_C \frac{Z'(s)}{Z(s)}\, ds \right) + \eta \\
&\qquad\qquad \left(\eta = -\Phi\left(a - \tfrac{1}{2}\right)\right) \\
&= \frac{A}{\pi} T^2 + \frac{4}{\pi} \operatorname{Im}\left(\int_C \frac{Z'(s)}{Z(s)}\, ds \right) + O(1),
\end{aligned}$$

so

$$N(T) = \frac{AT^2}{2\pi} + 4S(T) + O(1), \quad \text{where } S(T) = \frac{1}{\pi} \operatorname{Im}\left(\int_C \frac{Z'(s)}{Z(s)}\, ds \right),$$

and the $O(1)$ term is continuous and tends to a limit as $T \to \infty$. Note that the function $S(T)$, which is $1/\pi$ times the variation of the argument of $Z(s)$ along C, plays the same role in this analysis as does its counterpart from the theory of the Riemann zeta-function. In accordance with custom, we extend the definition of $S(T)$ to T's for which $\frac{1}{2} + iT$ is a zero of $Z(s)$ by defining it to be $\lim_{\varepsilon \downarrow 0} \frac{1}{2}(S(T + \varepsilon) + S(T - \varepsilon))$.

Lemma 3. $S(T) = O(T)$.

Proof. $S(T) = h(T) + O(1)$, where $h(T)$ is the variation of the argument of $S(T)$ along the segment from $a + iT$ to $\frac{1}{2} + iT$. Now $h(T)$ is bounded by a multiple of the number of zeros of $\operatorname{Re} Z(s)$ on this segment, since the point $Z(s)$ cannot move between the right and left half-planes without crossing the imaginary axis. On the segment, the real part of $Z(s)$ coincides with $f(w) = \frac{1}{2}(Z(w + iT) + Z(w - iT))$, where w runs from $\frac{1}{2}$ to a on the real axis. But $f(w)$ is evidently holomorphic in a neighborhood of a closed disk D of, say, radius 1 about the point $s = a$, and this disk contains the real segment from $\frac{1}{2}$ to a well within itself, so we may apply Jensen's theorem to conclude that there exists a constant $C > 0$ such that the number of zeros of $f(w)$ on this segment is less than or equal to $C\int_{\partial D} \log^+|f(w)|\,|dw|$. It then follows from Lemma 2 that this integral is $O(T)$, which proves Lemma 3.

2. The discussion which follows is adapted from Chapter 14 of [11], "Consequences of the Riemann hypothesis". Cf. especially Theorem 14.13 of [11]. We begin by noting that $\log Z(s)$ can be defined in the half-plane $\operatorname{Re} s \geq \frac{1}{2}$, minus the real segment $[\frac{1}{2}, 1]$, and minus the zeros of $Z(s)$ on the critical line. We will always suppose the branch to be the one which is real for s real and greater than 1. We will also henceforth suppose $t \geq 10$.

LEMMA 4. *$\int_{1/2}^{a}|\log Z(\sigma + it)|\,d\sigma = O(t)$, and consequently,*

$$\int_{1/2}^{a} \log|Z(\sigma + it)|\,d\sigma = O(t).$$

PROOF. By Lemma 3 of this paper and Lemma α on page 49 of [11], taking $M = t$, $s_0 = a + it$, $r = 4$, and denoting a typical root of $Z(s)$ by ρ, we find that

$$\frac{Z'(s)}{Z(s)} = \sum_{|s-\rho|<2} (s-\rho)^{-1} + O(t) \qquad \left(\tfrac{1}{2} \leq \sigma \leq a\right).$$

Integrating this along the segment from $a + it$ to $\sigma + it$, we find that

$$\log Z(\sigma + it) - \log Z(a + it)$$
$$= \sum_{|s-\rho|<2} (\log(\sigma + it - \rho) - \log(a + it - \rho)) + O(t).$$

Now $\log Z(a + it)$, $\log(a + it - \rho)$, and $\int_{1/2}^{a}|\log(\sigma + it - \rho)|\,d\sigma$ are uniformly bounded for $|s - \rho| \leq 2$, and since $N(T) = AT^2/2\pi + O(T)$, there are $O(t)$ zeros satisfying $|s - \rho| \leq 2$, so we conclude that

$$\int_{1/2}^{a}|\log Z(\sigma + it)|\,d\sigma = O(t). \quad \text{Q.E.D.}$$

DEFINITION. $S_1(T) = \int_0^T S(y)\,dy$.

LEMMA 5. $S_1(T) = O(T)$.

PROOF. Suppose $\beta > a$, and let Q be the contour consisting of the rectangle with vertices $\frac{1}{2}$, β, $\beta + Ti$ and $\frac{1}{2} + Ti$, indented infinitesimally at the zeros of $Z(s)$. Then by Cauchy's theorem, $\int_Q \log Z(s)\,ds = 0$, and hence $\operatorname{Re} \int_Q \log Z(s)\,ds = 0$. This is the same as to say that

$$\int_{1/2}^{\beta}\log|Z(\sigma)|\,d\sigma-\int_0^T\arg Z(\beta+it)\,dt$$
$$-\int_{1/2}^{\beta}\log|Z(\sigma+iT)|\,d\sigma+\pi\int_0^T S(t)\,dt=0.$$

Now as $\sigma\to\infty$, $Z(\sigma+it)=1+O(e^{-c\sigma})$, for some $c>0$, so $\log Z(\sigma+it)=O(e^{-c\sigma})$, which implies that as $\beta\to\infty$, the first integral tends to a constant, and the second integral tends to zero. Also

$$\int_a^{\beta}\log|Z(\sigma+iT)|\,d\sigma=O(1),$$

uniformly in β and T. We conclude that

$$S_1(T)=\frac{1}{\pi}\int_{1/2}^{a}\log|Z(\sigma+iT)|\,d\sigma+O(1), \tag{1}$$

so Lemma 5 follows from Lemma 4.

LEMMA 6. *Suppose* $\frac{1}{2}<\sigma\leqslant a$. *Then*

$$\log Z(s)=i\int_{t/2}^{2t}\left(s-\tfrac{1}{2}-iy\right)^{-1}S(y)\,dy+O(1).$$

PROOF. Suppose $\beta>a$, and let L be the contour consisting of the rectangle with vertices at $\frac{1}{2}+\frac{1}{2}ti$, $\beta+\frac{1}{2}ti$, $\beta+2ti$, and $\frac{1}{2}+2ti$, indented infinitesimally at the zeros of $Z(s)$. Then by Cauchy's theorem,

$$\log Z(s)=\frac{1}{2\pi i}\int_L(w-s)^{-1}\log Z(w)\,dw.$$

The contribution to this integral from the two horizontal segments is $O(1)$, since on the portions of these segments having real part greater than a, $|\log Z(w)|$ is uniformly L^1, while the contribution from the remaining parts of the segments is

$$O\left(t^{-1}\int_{1/2}^{a}|\log Z(\sigma+Bi)|\,d\sigma\right),\quad \text{where } B=\frac{t}{2}\text{ or }2t,$$

and this is $O(1)$ by Lemma 4. The contribution to the rightmost segment can be eliminated by letting $\beta\to\infty$, so we conclude that

$$\log Z(s)=\frac{1}{2\pi}\int_{t/2}^{2t}\left(s-\tfrac{1}{2}-iy\right)^{-1}\log Z\left(\tfrac{1}{2}+iy\right)dy+O(1). \tag{2}$$

Repeating this argument with the point $1-\bar{s}$, which lies outside the region bounded by L, we note that

$$\frac{1}{2\pi i}\int_L\left(w-(1-\bar{s})\right)^{-1}\log Z(w)\,dw=0,$$

which implies as before that

$$\frac{1}{2\pi}\int_{t/2}^{2t}\left(\tfrac{1}{2}-\bar{s}-iy\right)^{-1}\log Z\left(\tfrac{1}{2}+iy\right)dy + O(1) = 0.$$

Taking conjugates, and multiplying by -1,

$$\frac{1}{2\pi}\int_{t/2}^{2t}\left(s-\tfrac{1}{2}-iy\right)^{-1}\overline{\log Z\left(\tfrac{1}{2}+iy\right)}\,dy + O(1) = 0. \tag{3}$$

Subtracting (3) from (2), we find that $\log Z(s) = i\int_{t/2}^{2t}(s-\frac{1}{2}-iy)^{-1}S(y)dy + O(1)$.

DEFINITION. $\phi(t) = \max_{1<u<t}|S_1(u)|$. (Note that $\phi(t) \not\equiv 0$.)

LEMMA 7. *For* $\sigma > \frac{1}{2}$, $0 < \xi < t/2$,

$$\log Z(s) = i\int_{t-\xi}^{t+\xi}\left(s-\tfrac{1}{2}-iy\right)^{-1}S(y)\,dy + O\left(\xi^{-1}\phi(2t)\right) + O(1). \tag{4}$$

PROOF. In view of Lemma 6, it suffices to show that $\int_{t+\xi}^{2t}(s-\frac{1}{2}-iy)^{-1}S(y)\,dy$ and $\int_{t/2}^{t-\xi}(s-\frac{1}{2}-iy)^{-1}S(y)\,dy$ are both $O(\xi^{-1}\phi(2t))$.

We will deal with the first of these, the treatment of the second being almost identical. Now

$$\begin{aligned}\int_{t+\xi}^{2t}\left(s-\tfrac{1}{2}-iy\right)^{-1}S(y)\,dy &= \left(s-\tfrac{1}{2}-iy\right)^{-1}S_1(y)\Big]_{t+\xi}^{2t}\\ &\quad - i\int_{t+\xi}^{2t}\left(s-\tfrac{1}{2}-iy\right)^{-2}S_1(y)\,dy\\ &= O\left(\xi^{-1}\phi(2t)\right) + O\left(\phi(2t)\int_{t+\xi}^{2t}\left(\left(\sigma-\tfrac{1}{2}\right)^2+(y-t)^2\right)^{-1}dy\right)\\ &= O\left(\xi^{-1}\phi(2t)\right).\end{aligned}$$

LEMMA 8. $S(T) = O((T\phi(2T))^{1/2})$.

PROOF. $N(T) = AT^2/2\pi + R(T)$, where $R(T) = 4S(T) + O(1)$, so it clearly suffices to derive the estimate for $R(T)$.

Suppose $x \in (0, T)$. Now $N(T+x) - N(T) \geqslant 0$, so $R(T+x) - R(T) \geqslant -cTx$, for some $c > 0$. Also,

$$\begin{aligned}\int_T^{T+x}R(y)dy &= xR(T) + \int_0^x(R(T+y)-R(T))\,dy\\ &\geqslant xR(T) - cT\int_0^x y\,dy\\ &= xR(T) - cTx^2/2.\end{aligned}$$

I.e.,

$$\begin{aligned}R(T) &< x^{-1}\int_T^{T+x} R(y)dy + cTx/2 \\ &= 4x^{-1}(S_1(T+x) - S_1(T)) + O(1) + cTx/2 \\ &= O\left(x^{-1}\phi(2T)\right) + O(Tx).\end{aligned}$$

Setting $x = (T^{-1}\phi(2T))^{1/2}$, one side of the bound follows. The other side is obtained in the same way, taking the integral over $[T - x, T]$.

LEMMA 9.

$$\int_{1/2}^{1/2+\log^{-1}t} \log|Z(\sigma + it)|d\sigma = O\left(t^{1/4}\phi^{3/4}(4t)\log^{-1/2}t\right).$$

PROOF.

$$\int_{1/2}^{1/2+\log^{-1}t} \log|Z(\sigma + it)|d\sigma = \int_{1/2}^{1/2+\log^{-1}t} \mathrm{Re}(\log Z(\sigma + it))d\sigma.$$

Now the real part of the right side of (4) can be written as

$$\int_0^{\xi} y\left(\left(\sigma - \tfrac{1}{2}\right)^2 + y^2\right)^{-1}(S(t-y) - S(t+y))dy + O\left(\xi^{-1}\phi(2t)\right) + O(1),$$

so

$$\begin{aligned}\int_{1/2}^{1/2+\log^{-1}t} &\log|Z(\sigma + it)|d\sigma \\ &= \int_0^{\xi}\left(\arctan(y^{-1}\log^{-1}t)\right)(S(t-y) - S(t+y))dy \\ &\quad + O\left(\xi^{-1}\phi(2t)\log^{-1}t\right) + O(\log^{-1}t) \\ &= O\left(\xi(t\phi(4t))^{1/2}\right) + O\left(\xi^{-1}\phi(4t)\log^{-1}t\right) + O(\log^{-1}t)\end{aligned}$$

by Lemma 8. If we now set $\xi = t^{-1/4}\phi^{1/4}(4t)\log^{-1/2}t$, the result follows.

3. We will now derive the upper estimate for the eigenvalue remainder term stated at the beginning of this paper. It is clearly implied by the following result:

THEOREM 1. $S(T) = O(T/\log T)$.

PROOF. Suppose $\sigma > \frac{1}{2} + \log^{-1}t$, $0 < \xi < t/2$. Then by Lemmas 7 and 8,

$$\begin{aligned}|\log Z(\sigma + it)| &< O\left((t\phi(4t))^{1/2}\right)\int_{t-\xi}^{t+\xi}\left(\left(\sigma - \tfrac{1}{2}\right)^2 + (y-t)^2\right)^{-1/2} dy \\ &\quad + O\left(\xi^{-1}\phi(2t)\right) + O(1) \\ &= O\left(\xi(t\phi(4t))^{1/2}\left(\sigma - \tfrac{1}{2}\right)^{-1}\right) + O\left(\xi^{-1}\phi(4t)\right) + O(1) \\ &= O\left(\xi(t\phi(4t))^{1/2}\log t\right) + O\left(\xi^{-1}\phi(4t)\right) + O(1).\end{aligned}$$

Setting $\xi = t^{-1/4}\phi^{1/4}(4t)\log^{-1/2}t$, the last expression becomes $O(t^{1/4}\phi^{3/4}(4t)\log^{1/2}t)$. Let $G(t) = t^{1/4}\phi^{3/4}(4t)\log^{1/2}t$. Then there exists $c_0 > 0$ such that $G(t) > c_0 t^{1/4}$, and as we have just shown, there exists $M > 0$ such that $|\log Z(\sigma + it)| < MG(t)$ for $\frac{1}{2} + \log^{-1}t < \sigma < a$.

We will now improve this estimate by an application of the Hadamard three circles theorem.

For fixed t, let the circles C_1, C_2, and C_3 be all centered at $(\frac{3}{2} + \log^{-1}t) + it$, and suppose they pass through the points $\frac{5}{4} + it$, $\sigma + it$, and $(\frac{1}{2} + \log^{-1}t) + it$, respectively. Denoting their radii by r_1, r_2, and r_3, we then have $r_1 = \frac{1}{4} + \log^{-1}t$, $r_2 = \frac{3}{2} + \log^{-1}t - \sigma$, and $r_3 = 1$. Let $M_j = \max_{s\in C_j}|\log Z(s)|$. Then M_1 is uniformly bounded, and $M_3 = O(G(t+1)) = O(H(t))$, where $H(t) = t^{1/4}\phi^{3/4}(5t)\log^{1/2}t$. By the three circles theorem, it therefore follows that $M_2 < c(H(t))^{\alpha}$ for some $c > 0$, where $\alpha = (\log(r_2/r_1))(\log^{-1}(r_3/r_1))$.

I.e., $M_2 < c(H(t))(H(t))^{\alpha-1}$. Now

$$\begin{aligned}\alpha - 1 &= (\log(r_2/r_1) - \log(r_3/r_1))(\log^{-1}(r_3/r_1))\\ &= c_1\log(r_2/r_3) \qquad (c_1 = \log^{-1}(r_3/r_1))\\ &= c_1\log r_2 = c_1\log\left(\frac{3}{2} + \log^{-1}t - \sigma\right)\\ &< c_2\left(\left(\tfrac{1}{2} - \sigma\right) + \log^{-1}t\right) \quad \text{for some } c_2 > 0.\end{aligned}$$

But $H(t) > c_0 t^{1/4}$, so we conclude that

$$M_2 < c_3(H(t))t^{c_4((1/2-\sigma)+\log^{-1}t)}, \quad \text{with } c_3, c_4 > 0.$$

The last quantity is $O(H(t)t^{c_4(1/2-\sigma)})$, since $t^{\log^{-1}t} = O(1)$. We conclude that

$$\begin{aligned}\int_{1/2+\log^{-1}t}^{a}\log|Z(\sigma+it)|d\sigma &= O\left(H(t)\int_{1/2+\log^{-1}t}^{a}t^{c_4(1/2-\sigma)}d\sigma\right)\\ &= O(H(t)\log^{-1}t)\\ &= O(t^{1/4}\phi^{3/4}(5t)\log^{-1/2}t).\end{aligned}$$

But by Lemma 9, $\int_{1/2}^{1/2+\log^{-1}t}\log|Z(\sigma+it)|\,d\sigma$ satisfies the same estimate, since $\phi(4t) < \phi(5t)$. It follows that

$$\int_{1/2}^{a}\log|Z(\sigma+it)|d\sigma = O(t^{1/4}\phi^{3/4}(5t)\log^{-1/2}t),$$

which implies, by (1), that $S_1(t) = O(t^{1/4}\phi^{3/4}(5t)\log^{-1/2}t)$. S Since $t^{1/4}\phi^{3/4}(5t)\log^{-1/2}t$ is an increasing function for large t, we conclude that for $t < T$, $S_1(t) = O(T^{1/4}\phi^{3/4}(5T)\log^{-1/2}T)$, and therefore

$$\phi(T) = O(T^{1/4}\phi^{3/4}(5T)\log^{-1/2}T). \tag{5}$$

Set $\Psi(T) = \max_{10 < t < T}\phi(t)t^{-1}\log^2 t$. Then $\Psi(T)$ is nondecreasing, and $\phi(T)$

$< \Psi(T)T\log^{-2}T$, so (5) implies that

$$\phi(T) = O\left(\Psi^{3/4}(5T)T\log^{-2}T\right) \quad \text{or} \quad \phi(T)T^{-1}\log^2 T = O\left(\Psi^{3/4}(5T)\right),$$

which implies that $\Psi(T) = O(\Psi^{3/4}(5T))$.

Now there exists a sequence of T_n's tending to infinity such that $\Psi(5T_n) < 5\Psi(T_n)$, for if not, $\Psi(5^n T) \geq 5^n\Psi(T)$ for large T, or $\Psi(T) > cT$, for arbitrarily large T, which is impossible, since $\phi(T) = O(T)$, and hence $\Psi(T) = O(\log^2 T)$. We conclude that $\Psi(T_n) = O(\Psi^{3/4}(T_n))$, and hence that $\Psi(T_n) = O(1)$. Since Ψ is nondecreasing, this implies that $\Psi(T) = O(1)$, and hence $\phi(T) = O(T\log^{-2}T)$. By Lemma 8, this proves the theorem.

COROLLARY. *There exists $c > 0$ such that $Z(\frac{1}{2} + it)$ always has a zero in* $[T, T + c\log^{-1}T]$ $(T > 1)$.

4. In this section, we will show that for any $\varepsilon > 0$, $S(T) = \Omega(\log^{1/2-\varepsilon}T)$, which implies that the eigenvalue remainder term is always $\Omega(\log^{1/2-\varepsilon}T)$. As before, the methods are adapted from Littlewood's work on consequences of the Riemann hypothesis [11, Chapter 14].

DEFINITIONS. Suppose γ is a closed geodesic on $\mathcal{S}$, not necessarily primitive. Let $l_\gamma =$ the length of γ, $N_\gamma = e^{l_\gamma}$, and $\Lambda(\gamma) = l_{\gamma_0}$, where $\gamma = \gamma_0^n$, with γ_0 primitive.

LEMMA 10. *There is a positive $\theta < 1$, such that*

$$\Psi(x) = \sum_{N_\gamma < x} \Lambda(\gamma) = x + O(x^\theta)$$

and

$$\pi(x) = \sum_{N_\gamma < x} 1 = \mathrm{li}(x) + O(x^\theta\log^{-1}x).$$

PROOF. This is known [5], [9].

LEMMA 11. *For* $\mathrm{Re}\, s > 1$, $Z'(s)/Z(s) = \sum_\gamma \Lambda(\gamma)N_\gamma^{-s} + Z'(s+1)/Z(s+1)$, *where the summation is carried out over all closed geodesics on $\mathcal{S}$.*

PROOF. For $\mathrm{Re}\, s > 1$,

$$\begin{aligned} \frac{Z'(s)}{Z(s)} &= \sum_\gamma \Lambda(\gamma)\left(1 - N_\gamma^{-1}\right)^{-1} N_\gamma^{-s} \qquad (\text{cf. } [7]) \\ &= \sum_\gamma \Lambda(\gamma)N_\gamma^{-s} + \sum_\gamma \Lambda(\gamma)(N_\gamma - 1)^{-1}N_\gamma^{-s} \\ &= \sum_\gamma \Lambda(\gamma)N_\gamma^{-s} + \sum_\gamma \Lambda(\gamma)\left(1 - N_\gamma^{-1}\right)^{-1}N_\gamma^{-(s+1)}. \quad \text{Q.E.D.} \end{aligned}$$

LEMMA 12. *For an integer $N > 2$, suppose there is a t_0 such that the distance* *of* $(2\pi)^{-1}t_0 l_\gamma$ *from the nearest integer is* $< 1/8$, *provided* $N_\gamma < N$. *Then for* $\sigma_0 = 1 + \log^{-1/2}N$, $|Z'(\sigma_0 + it_0)/Z(\sigma_0 + it_0)| > \frac{1}{4}\log^{1/2}N$, *if* N *is sufficiently large.*

PROOF. By Lemma 11, $Z'(\sigma + it)/Z(\sigma + it) = \sum_\gamma \Lambda(\gamma)N_\gamma^{-\sigma-it} + O(1)$ for $\sigma > 1$. It follows that

$$\left|\frac{Z'(\sigma + it_0)}{Z(\sigma + it_0)}\right| > \sum_{N_\gamma < N} \Lambda(\gamma)N_\gamma^{-\sigma}\cos(t_0 l_\gamma) - \sum_{N_\gamma > N} \Lambda(\gamma)N_\gamma^{-\sigma} + O(1)$$

$$> \frac{1}{2}\sum_{N_\gamma < N} \Lambda(\gamma)N_\gamma^{-\sigma} - \sum_{N_\gamma > N} \Lambda(\gamma)N_\gamma^{-\sigma} + O(1).$$

Now it follows from Lemma 10, writing the last two sums as Stieltjes integrals, that $\sum_{N_\gamma<N}\Lambda(\gamma)N_\gamma^{-\sigma} = \sigma(\sigma - 1)^{-1}(1 - N^{1-\sigma}) + O(1)$, and $\sum_{N_\gamma>N}\Lambda(\gamma)N_\gamma^{-\sigma} = \sigma(\sigma - 1)^{-1}N^{1-\sigma} + O(1)$, which implies the lemma.

REMARK. In what follows, it will be of interest to have upper bounds for the growth of $Z'(s)/Z(s)$ on vertical lines to the right of $\operatorname{Re} s = \frac{1}{2}$. We note at this point that since by Lemma 2, $\log^+|Z(\sigma + it)| = O(t)$ uniformly for $\sigma > \sigma_0 > \frac{1}{2}$, it follows from the Borel-Carathéodory theorem [12, pp. 174–175] applied to appropriate circles of fixed radius with centers sliding up a vertical line located in $\operatorname{Re} s > 1$, that $\log Z(\sigma + it) = O(t)$ uniformly for $\sigma > \sigma_0 > \frac{1}{2}$, and hence, by Cauchy's formula for the derivative, $Z'(\sigma + it)/Z(\sigma + it) = O(t)$ uniformly for $\sigma > \sigma_0 > \frac{1}{2}$.

LEMMA 13. *Suppose* $1 < M_\sigma(t) < t$ *is, for each* $\sigma > \frac{1}{2}$, *a function of* t *such that* $Z'(\sigma + it)/Z(\sigma + it) = O(M_\sigma(t))$, *and such that for each* $k > 0$, *there exists* $c(k) > 0$ *with*

$$M_\sigma(t) > c(k)\int_{t-k}^{t+k} M_\sigma(x)\,dx.$$

Let α *and* σ *be fixed, with* $\frac{1}{2} < \alpha < \sigma < 2$, *and suppose* t *and* δ, *which we will permit to vary, are subject to the constraint* $e^{-\sqrt{t}} < \delta < 1$. *Then as* $t \to \infty$,

$$\frac{Z'(\sigma + it)}{Z(\sigma + it)} = \sum \Lambda(\gamma)N_\gamma^{-\sigma-it}e^{-\delta N_\gamma} + O(\delta^{\sigma-\alpha}M_\alpha(t)) + O(1). \tag{6}$$

PROOF. Setting as usual $s = \sigma + it$, we have by Lemma 11 and the lemma on page 128 of [11],

$$\sum_\gamma \Lambda(\gamma)N_\gamma^{-s}e^{-\delta N_\gamma} = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\left(\frac{Z'(w)}{Z(w)} - \frac{Z'(w+1)}{Z(w+1)}\right)\Gamma(w - s)\delta^{s-w}dw. \tag{7}$$

Let D be an open disk of radius 1 which is centered on the real axis. Suppose, moreover, that ∂D cuts the real axis in $(0, \frac{1}{2})$, and does not intersect any zero of $Z(s)$. Denote by C_α the contour consisting of the portion of the line $\operatorname{Re} s = \alpha$ exterior to D, plus the intersection of ∂D with the half-plane $\operatorname{Re} s < \alpha$, the whole oriented upward.

Now since $\Gamma(z)$ is of uniformly exponential decrease in vertical strips, we can move the path of integration in (7) leftward to C_α, provided we take account of the residues thus picked up. Bearing in mind that $\Gamma(z)$ has a simple pole at $z = 0$, with residue 1, we find that

$$\begin{aligned} \sum_\gamma \Lambda(\gamma) N_\gamma^{-s} e^{-\delta N_\gamma} &= \frac{Z'(s)}{Z(s)} - \frac{Z'(s+1)}{Z(s+1)} + \sum_n r_n \Gamma(s_n - s)\delta^{s-s_n} \\ &\quad + \frac{1}{2\pi i}\int_{C_\alpha}\left(\frac{Z'(w)}{Z(w)} - \frac{Z'(w+1)}{Z(w+1)}\right)\Gamma(w-s)\delta^{s-w}\,dw, \end{aligned} \tag{8}$$

where the s_n's range over the poles of $Z'(s)/Z(s)$ in D, and r_n is the corresponding residue. The rightmost of these poles is the pole at $s = 1$, which is always present.

Now the integral on the right in (8) is, for some $c > 0$,

$$O\left(\delta^{\sigma-\alpha}\int_{-\infty}^{\infty} e^{-c|y-t|} M_\alpha(y)\,dy\right) = O(\delta^{\sigma-\alpha} M_\alpha(t)),$$

by our assumptions on $M_\alpha(t)$.

On the other hand, the functions of the form $\Gamma(s_n - s)\delta^{s-s_n}$ are clearly $o(1)$ for large t, since $\delta > e^{-\sqrt{t}}$ and the gamma-function is of exponential decrease in vertical strips, so the lemma is established.

LEMMA 14. *$Z'(1+it)/Z(1+it) = O(\log t)$, and hence, by Phragmén-Lindelöf, $Z'(\sigma+it)/Z(\sigma+it) = O(\log t)$ uniformly, for $\sigma \geq 1$.*

PROOF. Take $\sigma = 1$, $\alpha = \frac{3}{4}$, and $M_\sigma(t) = t$ in the last lemma. We then find, subject to the constraint $e^{-\sqrt{t}} \leq \delta$, that for t large,

$$\begin{aligned} \left|\frac{Z'(1+it)}{Z(1+it)}\right| &< \sum_\gamma \Lambda(\gamma) N_\gamma^{-1} e^{-\delta N_\gamma} + O(\delta^{1/4} t) + O(1) \\ &< \sum_{N_\gamma < M} \Lambda(\gamma) N_\gamma^{-1} + \sum_{N_\gamma > M} \Lambda(\gamma) e^{-\delta N_\gamma} + O(\delta^{1/4} t) + O(1). \end{aligned}$$

Now it follows immediately from Lemma 10 that $\sum_{N_\gamma < M} \Lambda(\gamma) N_\gamma^{-1} = O(\log M)$. Furthermore,

$$\sum_{N_\gamma > M} \Lambda(\gamma) e^{-\delta N_\gamma} = \int_M^\infty e^{-\delta x} d\Psi(x)$$

$$= \Psi(x) e^{-\delta x}\Big]_M^\infty + \delta \int_M^\infty \Psi(x) e^{-\delta x} dx$$

$$= O\left(x e^{-\delta x}\Big]_M^\infty + \delta \int_M^\infty x e^{-\delta x} dx \right)$$

$$= O\left(\int_M^\infty e^{-\delta x} dx \right) = O(\delta^{-1} e^{-\delta M}).$$

Accordingly, $|Z'(1 + it)/Z(1 + it)| = O(\log M) + O(\delta^{-1} e^{-\delta M}) + O(\delta^{1/4} t) + O(1)$. If we now take $\delta = t^{-8}$ and $M = t^9$, the result follows.

Lemma 15. *Suppose $\varepsilon > 0$. Then if $\sigma > \frac{1}{2}$ is sufficiently close to $\frac{1}{2}$, $Z'(\sigma + it)/Z(\sigma + it) = \Omega(\log^{1/2-\varepsilon} t)$.*

Remark. It is quite possible that the above estimate is much too low. In any case, as we shall presently see, Ω-questions of this sort are closely connected with Diophantine properties of the numbers l_γ.

Proof of Lemma 15. Assume the lemma is false. Then for σ sufficiently near $\frac{1}{2}$, which we will henceforth suppose it to be, we can, for some $\varepsilon > 0$, take $M_\sigma(t) = \log^{1/2-\varepsilon} t$ in Lemma 13. Suppose now δ and η are small positive numbers. Define the integer N by setting $N = [\delta^{-1-\eta}]$. Then it follows from Lemma 10 and the Dirichlet theorem [11, p. 152] that there exists a t satisfying $1 \leqslant t \leqslant N^N$, such that $(2\pi)^{-1} t l_\gamma$ is at distance $\leqslant 1/N$ from the nearest integer, provided $N_\gamma \leqslant N$. Let us assume for the moment that such a t can be found in the interval $[\log^2 \delta^{-1}, N^N]$. Then $e^{-\sqrt{t}} \leqslant \delta$, so Lemma 13 applies, and taking the real part of (6), we find that for $\frac{1}{2} < \alpha < \sigma$, with σ sufficiently near $\frac{1}{2}$,

$$\operatorname{Re} \frac{Z'(\sigma + it)}{Z(\sigma + it)} = \sum_\gamma \Lambda(\gamma) N_\gamma^{-\sigma} \cos(t l_\gamma) e^{-\delta N_\gamma} + O(\delta^{\sigma-\alpha} \log^{1/2-\varepsilon} t) + O(1)$$

$$= \sum_{N_\gamma \leqslant N} \Lambda(\gamma) N_\gamma^{-\sigma} \cos(t l_\gamma) e^{-\delta N_\gamma} + O\left(\sum_{N_\gamma > N} e^{-\delta N_\gamma} \right)$$

$$+ O(\delta^{\sigma-\alpha} \log^{1/2-\varepsilon} t) + O(1).$$

Now for δ small, or what is the same thing, for N large,

$$\sum_{N_\gamma \leqslant N} \Lambda(\gamma) N_\gamma^{-\sigma} \cos(t l_\gamma) e^{-\delta N_\gamma} \geqslant \frac{1}{2} \sum_{N_\gamma \leqslant N} \Lambda(\gamma) N_\gamma^{-\sigma} e^{-\delta N_\gamma}.$$

But $\sum_{N_\gamma \leqslant N} \Lambda(\gamma) N_\gamma^{-\sigma} e^{-\delta N_\gamma} = \int_0^N e^{-\delta x} x^{-\sigma} d\Psi(x)$, and it follows easily from

Lemma 10 and the fact that $\delta N \to \infty$ as $\delta \to 0$, that there exists $c > 0$ such that

$$\frac{1}{2} \sum_{N_\gamma < N} \Lambda(\gamma) N_\gamma^{-\sigma} e^{-\delta N_\gamma} > c\delta^{\sigma-1} \quad \text{for } \sigma \in \left(\tfrac{1}{2}, 1\right],$$

provided δ is sufficiently small. Since $\sum_{N_\gamma > N} e^{-\delta N_\gamma} = O(\delta^{-1} e^{-\delta N})$, we conclude that for small δ,

$$|Z'(\sigma + it)/Z(\sigma + it)|$$
$$> c\delta^{\sigma-1} + O(\delta^{-1} e^{-\delta N}) + O(\delta^{\sigma-\alpha} \log^{1/2-\varepsilon} t) + O(1).$$

But $t < N^N$, or $\log t < N \log N$, with $N = [\delta^{-1-\eta}]$, so it easily follows that for η small,

$$|Z'(\sigma + it)/Z(\sigma + it)| > c \log^{1-\sigma-\eta'} t + O(\log^{1/2-\varepsilon} t) + O(1),$$

where $\eta' \to 0$ as $\eta \to 0$.

If η is sufficiently small and σ sufficiently near $\frac{1}{2}$, this contradicts our assumption that $Z'(\sigma + it)/Z(\sigma + it) = O(\log^{1/2-\varepsilon} t)$. Lemma 15 will accordingly be proved if we can show that there is always a suitable t in the interval $[\log^2 \delta^{-1}, N^N]$. Suppose not. Then setting $\sigma_0 = 1 + \log^{-1/2} N$, Lemma 12 implies that for large N,

$$|Z'(\sigma_0 + it)/Z(\sigma_0 + it)| > \tfrac{1}{4} \log^{1/2} N > c_2 t^{1/4} \quad \text{since } N = [\delta^{-1-\eta}]$$

and we are assuming that $t < \log^2 \delta^{-1}$. The first inequality shows that such t's can be arbitrarily large, and the second then contradicts Lemma 14.

Corollary. *Suppose $\varepsilon > 0$. Then for $\sigma_0 > \frac{1}{2}$ sufficiently close to $\frac{1}{2}$, it is false that* $\log Z(s) = O(\log^{1/2-\varepsilon} t)$, *uniformly in* $\operatorname{Re} s > \sigma_0$.

Proof. If such an estimate were valid, we would immediately obtain a contradiction, since on a line $\operatorname{Re} s = \sigma > \sigma_0$, $Z'(s)/Z(s)$ can be obtained from $\log Z(s)$ by the Cauchy integral for the derivative, about a small circle of fixed radius $< \sigma - \sigma_0$ surrounding the point s, and elementary estimates on the size of such an integral would than lead to a contradiction of the last lemma.

Corollary. *Suppose $\varepsilon > 0$. Then for $\sigma_0 > \frac{1}{2}$ sufficiently close to $\frac{1}{2}$, it is false that* $\operatorname{Im} \log Z(s) = O(\log^{1/2-\varepsilon} t)$, *uniformly in* $\operatorname{Re} s > \sigma_0$.

Proof. This follows immediately from the previous corollary and the Borel-Carathéodory theorem.

Theorem 2. *For any $\varepsilon > 0$, $S(T) = \Omega(\log^{1/2-\varepsilon} T)$.*

Proof. Suppose $S(T) = O(\log^{1/2-\varepsilon} T)$ for some $\varepsilon > 0$. Now by Lemma 6,

$$\log Z(s) = i\int_{t/2}^{2t}\left(s - \tfrac{1}{2} - iy\right)^{-1} S(y)\, dy + O(1).$$

Taking imaginary parts,

$$\begin{aligned}\operatorname{Im}\log Z(s) &= \int_{t/2}^{2t}\left(\sigma - \tfrac{1}{2}\right)\left(\left(\sigma - \tfrac{1}{2}\right)^2 + (t-y)^2\right)^{-1} S(y)\, dy + O(1)\\ &= O\left(\log^{1/2-\varepsilon} t\right)\int_{t/2}^{2t}\left(\sigma - \tfrac{1}{2}\right)\left(\left(\sigma - \tfrac{1}{2}\right)^2 + (t-y)^2\right)^{-1} dy + O(1),\end{aligned}$$

which contradicts the last corollary.

REMARKS. This method can be used to show somewhat more. Namely, that $S(T) = \Omega_{\pm}(\log^{1/2-\varepsilon} T)$. It is also likely that by pushing the method, the estimate can be improved very slightly.

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DEPARTMENT OF MATHEMATICS, GRADUATE CENTER (CUNY), NEW YORK, NEW YORK 10036