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LINEAR ISOTOPIES IN E^2

BY

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ABSTRACT. This paper deals with continuous families of linear embeddings (called linear isotopies) of finite complexes in the Euclidean plane E^2. Suppose f and g are two linear embeddings of a finite complex P with triangulation T into a simply connected open subset U of E^2 so that there is an orientation preserving homeomorphism H of E^2 to itself with $H \circ f = g$. It is shown that there is a continuous family of embeddings h_t: $P \to U$ ($t \in [0, 1]$) so that $h_0 = f$, $h_1 = g$, and for each t, h_t is linear with respect to T.

It is also shown that if P is a PL star-like disk in E^2 with a triangulation T which has no spanning edges and f is a homeomorphism of P which is the identity on Bd P and is linear with respect to T, then there is a continuous family of homeomorphisms h_t: $P \to P$ ($t \in [0, 1]$) such that $h_0 = \mathrm{id}$, $h_1 = f$, and for each t, h_t is linear with respect to T. An example shows the necessity of the "star-like" requirement. A consequence of this last theorem is a linear isotopy version of the Alexander isotopy theorem–namely, if f and g are two PL embeddings of a disk P into E^2 so that $f|\mathrm{Bd}\, P = g|\mathrm{Bd}\, P$, then there is a linear isotopy with respect to some triangulation of P which starts at f, ends at g, and leaves the boundary fixed throughout.

1. Introduction. This paper deals with finite complexes linearly embedded in the Euclidean plane E^2 and continuous families of such linear embeddings which are called linear isotopies.

Suppose (P, T) is a compact triangulated PL complex in a simply connected open subset U of E^2 and g is a linear homeomorphism of P into U that can be extended to an orientation preserving homeomorphism of E^2 onto itself. It is shown in Theorem 7.2 that there is a linear isotopy h_t ($t \in [0, 1]$) of (P, T) into U such that $h_0 = \mathrm{id}$ and $h_1 = g$. Each h_t is linear on each simplex of T.

Theorem 7.2 follows from the result (Theorem 7.1) that if (D, T) is a triangulated disk in E^2, then there is a linear isotopy h_t ($t \in [0, 1]$) of (D, T) into E^2 such that $h_0 = \mathrm{id}$, $h_1(D)$ is a convex disk, and h_t is decreasing in the sense that if $0 \leqslant s \leqslant t \leqslant 1$, $h_t(D) \subset h_s(D)$.

Suppose (P, T) is a complex linearly embedded in E^2 and g is a linear

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homeomorphism of (P, T) into E^2 that is fixed on a certain subcomplex P' of P. Conditions are investigated under which there is a linear isotopy h_t ($t \in [0, 1]$) of (P, T) into E^2 such that

$h_0 = \mathrm{id}$, each h_t is fixed on P', and $h_1 = g$.

It is shown in Theorem 4.1 that if (P, T) is a star-like disk without spanning edges and $P' = \mathrm{Bd}\, P$, then for each g there is such an h_t ($t \in [0, 1]$). However, Example 4.2 shows that the result does not always hold if some 1-simplex of T is permitted to span P.

§2 treats the question of extending the linear embedding into E^2 of a subcomplex C^1 of a finite complex C to a linear embedding of all of C into E^2. Although we do not get necessary and sufficient conditions that such an embedding can be extended, we get some partial solutions (mostly in the case where C is a triangulated disk and $C^1 = \mathrm{Bd}\, D$). Our methods give a new proof of Fary's theorem which states that the extension is possible if C is a 1-complex which can be topologically embedded in E^2 and $C^1 = \varnothing$.

A linear isotopy version of the Alexander Isotopy Theorem for PL disks in E^2 appears in §5, namely, if P is a PL disk in E^2 and f is a PL homeomorphism of P fixed on $\mathrm{Bd}\, P$, then there is a triangulation T of P and a linear isotopy $h_t\colon (P, T) \to E^2$ ($t \in [0, 1]$) with $h_0 = \mathrm{id}$, $h_1 = f$, and for each t, $h_t|\mathrm{Bd}\, P = \mathrm{id}$.

§3 contains the definitions of a linear isotopy and the related notion of a push. It is shown that a linear isotopy of a finite complex in E^n can be extended to a linear isotopy of all of E^n.

§§4 and 5 contain the theorems about linear isotopies of disks with fixed boundaries.

Linear degeneration is defined in §6. This concept is a variation of a collapse and is used to prove the theorems in §7 mentioned in the second paragraph of the introduction.

In future papers Starbird will make use of some variations of the theorems in this paper to investigate questions about linear isotopies in E^3.

2. Linear embeddings. In this section we investigate the question of extending the linear embedding of a part of a complex into E^2 to a linear embedding of the entire complex.

DEFINITION. Let C be a complex with a triangulation T. Then $h\colon C \to E^n$ is a *linear embedding* of C into E^n if h is an embedding which is linear on each simplex in T.

Notation. When we wish to emphasize a particular triangulation on a complex, we may write (C, T) to stand for a complex C with a triangulation T. We use $\mathrm{St}(v)$ and $\mathrm{Lk}(v)$ to denote the star and link respectively of a vertex v in a complex C.

DEFINITION. A disk D in E^2 is *star-like* if and only if there exists a point s

in E^2 (necessarily in Int D) such that for every point x in Bd D, the straight segment xs meets Bd D only at x. We say that such a point s *can see* every point of Bd D.

Note that if D is a star-like polygonal disk in E^2, the set of points from which all of Bd D can be seen is a convex, open subset of Int D.

THEOREM 2.1. *Let P be a polygonal disk with a triangulation T which contains no spanning edge, i.e., if v and w are nonadjacent vertices on* Bd P, *then vw is not a simplex in the triangulation T. Then if h:* Bd $P \to E^2$ *is a linear embedding of* Bd P *such that h*(Bd P) *bounds a star-like disk in E^2, then there exists a linear embedding H: $P \to E^2$ shich extends h.*

PROOF. The proof is by induction on n, the number of interior vertices in the triangulation T.

If $n = 0$, P is a 2-simplex and there is nothing to do.

Assume that the theorem is true for triangulations with k interior vertices where $k < n$. Suppose T has n interior vertices. Let v be an interior vertex of T so that St(v) contains an edge on Bd P. Let s be a point in E^2 which can see every point of h(Bd P). We will have H map v to s. For every vertex $w \in$ (Bd $P \cap$ St(v)), $H(vw)$ will be the straight segment from $h(w)$ to s. These segments divide the interior of h(Bd P) into star-like subdisks each of which meets the requirements of the theorem. Complete the proof by applying induction in each subdisk.

The following theorem is a corollary of Theorem 2.1.

THEOREM 2.2. *Let P be a polygonal disk with a triangulation T and let h:* Bd $P \to E^2$ *be a linear embedding such that h*(Bd P) *bounds a convex disk D and for each spanning* 1-*simplex vw of P, $h(v)$ and $h(w)$ are the ends of a straight spanning segment of D. Then there exists a linear embedding H: $P \to E^2$ which extends h.*

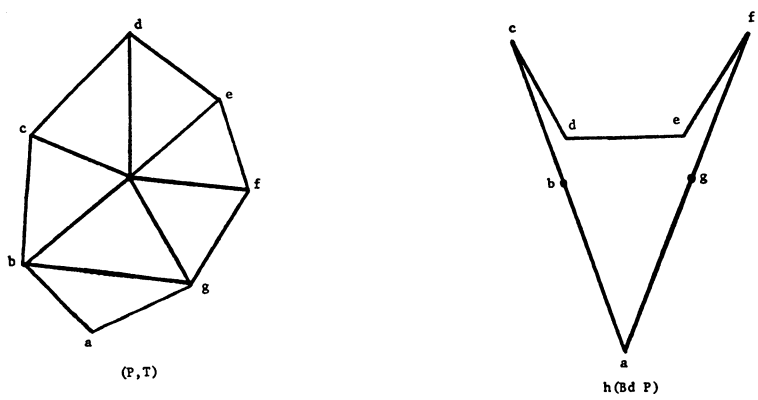


FIGURE 2.1

EXAMPLE 2.1. It is to be noted that Theorem 2.2 is false if instead of supposing that D is convex we suppose instead that it is merely star-like as can be seen by Figure 2.1.

THEOREM 2.3 (FARY [4]). *Let Γ be a finite graph with no loops or multiple edges. If Γ can be topologically embedded in E^2, then it can be linearly embedded in E^2.*

PROOF. Incorporate the embedded Γ into the 1-skeleton of a topological triangulation of a disk in E^2 with no spanning edges. Re-embed the boundary of the disk linearly so that it bounds a star-like disk in E^2 and apply Theorem 2.1. Then Γ will be embedded in the 1-skeleton of a linearly embedded disk and its edges will therefore be straight.

Question 2.1. If C^1 is a subcomplex of a complex C, under what conditions can a linear embedding of C^1 into E^2 be extended to a linear embedding of C? Theorem 2.3 provides an answer if C is a finite 1-complex and $C^1 = \emptyset$. Theorem 2.1 only gives a sufficient condition even in the case where C is a triangulated disk without spanning edges and $C^1 = \text{Bd } C$. In the case where C is a triangulated disk and $C^1 = \text{Bd } C$, might a necessary and sufficient condition be that the embedding could be extended to send each acyclic 1-dimensional subcomplex of C into the disk in E^2 bounded by image of C^1.

Question 2.2. There are 2-complexes which can be topologically embedded in E^3, but not linearly embedded. Suppose a 2-complex can be topologically embedded in E^4, can it then be linearly embedded in E^4? What higher dimensional analogs of the Fary Theorem are true?

3. Linear isotopies and the Fundamental Extension Theorem. In this section we define linear isotopies and pushes of complexes in E^n and prove that a linear isotopy of a finite complex in E^n can be extended to a linear isotopy of all of E^n. This is the only section in which theorems about E^n for n greater than two are proved. Whitehead proved that any finite complex linearly embedded in E^n can be incorporated into a triangulation of E^n without subdividing the complex. This result is given as Theorem 3.1 below and we include a proof for completeness.

DEFINITION. Let C be a complex with a triangulation T. A *linear isotopy* of C is a continuous family of embeddings h_t: $C \to E^n$ ($t \in [0, 1]$) so that for every $t \in [0, 1]$, h_t is a linear embedding of C in E^n. If we wish to emphasize the triangulation involved, we will write that h_t ($t \in [0, 1]$) is a linear isotopy of (C, T). A linear isotopy h_t: $C \to E^n$ ($t \in [0, 1]$) is called a *simple push* if it is fixed on each vertex except one. A finite sequence of such simple pushes, performed one after the preceding, is called a *push*. It may be noted that if there is a linear isotopy taking one embedding h of a finite complex C into E^n onto another embedding g, then $h(C)$ *can be pushed to* $g(C)$.

THEOREM 3.1 (WHITEHEAD [8, THEOREM 5]). *Suppose (C, T_c) is a triangulated finite complex linearly embedded in E^n. Then there is a triangulation T of E^n such that each simplex of T_c is a simplex of T.*

PROOF. The proof is by induction on the number of simplexes in T_c. The theorem is true if T_c has only one simplex. Assume T_c has n simplexes and the theorem is true for complexes with fewer. Let σ be a simplex of T_c that is not a face of any other simplex of T_c and let T' be a triangulation of E^n such that

(1) each simplex of T_c other than σ is an element of T'.

We also suppose that if t' is a simplex of T' that contains an element of T_c that is not a face of σ, then

(2) $t' \cap \operatorname{Int} \sigma = \varnothing$.

If condition (2) above is not satisfied by the first T' selected, some subdivision of this T' leads to one satisfying both conditions (1) and (2).

Let T'' be a triangulation of E^n such that

σ is a simplex of T'' and $t' \cap t''$ is either empty or a face of σ where $t'' \in T''$ containing σ as a face and $t' \in T'$ containing as a face an element of T_c which is not a face of σ.

Let S' be the convex cellular subdivision of E^n whose elements are the intersections of elements of T' and T''. Note that each proper face of σ is an element of S'. By subdividing S' we get a triangulation S of E^n such that

each proper face of σ is an element of S and S is a subdivision of both T' and T''.

Let

$$U = (C - \sigma) \cup (\cup \operatorname{Int} t_i') \quad \text{for } t_i' \in T' \text{ with } t_i' \cap (C - \sigma) \neq \varnothing,$$

$$V = \cup \operatorname{Int} t_i'' \quad \text{for } t_i'' \in T'' \text{ having } \sigma \text{ as a face.}$$

Then U, V are disjoint open sets and each of $\bar{U}$, $\bar{V}$, $E^n - (U \cup V)$ is the union of elements of S.

We start defining T by putting each simplex of S which is in $E^n - (U \cup V)$ in T. The proper faces of σ are now in T.

Now consider $\bar{U}$. It is the union of simplexes of T' and $\operatorname{Bd} U$ is already triangulated by T. To get other simplexes of T in $\bar{U}$ we subdivide the simplexes of T' in $\bar{U}$ one at a time proceeding upward through dimensions. If $t' \in T'$ in $\bar{U}$ and triangulation T on $\operatorname{Bd} t'$ has already been defined, we let t' be an element of T if T did not properly divide $\operatorname{Bd} t'$, but if T properly divided $\operatorname{Bd} t'$ we obtain simplexes of T in t' by coning over $\operatorname{Bd} t'$ from the baricenter of t'. Note that each simplex of T_c other than σ has now been placed in T.

The simplexes of T in $\bar{V}$ are obtained by subdividing the elements of T'' in V in a fashion similar to that in which the elements of T' in $\bar{U}$ were subdivided as described in the preceding paragraph.

In the proof of Theorem 3.1 we regarded (C, T_c) as a finite complex so that we could use induction. However, once the result is proved for the finite case, it can be extended to the infinite case as follows.

THEOREM 3.2. *Suppose (C, T_c) is a triangulated complex (perhaps infinite) linearly embedded as a closed subset of E^n. Then there is a triangulation T of E^n such that each simplex of T_c is a simplex of T.*

PROOF. The theorem is proved by enlarging C until the enlargement contains all of E^n.

Let B_1 be the closed n-ball with center at the origin and radius 1 while U_1 is a bounded open set containing B_1 such that each element of T_c that intersects B_1 lies in U_1. By Theorem 3.1 there is a triangulation T_1 of E^n such that each simplex of T_c that intersects $\overline{U}_1$ is a simplex in T_1. We suppose that T_1 is adjusted so that each simplex of T_1 that intersects B_1 lies in U_1. Let C_1 be the union of C and the simplexes of T_1 that intersect B_1. A simplex in the triangulation T_{c_1} of C_1 is either a simplex of T_c or the face of a simplex of T_1 that intersects B_1.

In a similar fashion we get a sequence of triangulated complexes (C_1, T_{c_1}), $(C_2, T_{c_2}), \ldots$ such that $T_{c_i} \subset T_{c_{i+1}}$ and C_i contains the ball with center at the origin and radius i. A simplex is an element of the required triangulation T if and only if it belongs to some T_{c_i}.

THEOREM 3.3 (THE FUNDAMENTAL EXTENSION THEOREM). *Let (C, T_c) be a triangulated finite complex linearly embedded in E^n and let $h_t\colon C \to E^n$ ($t \in [0, 1]$) be a linear isotopy of C such that $h_0 =$ identity. Then there is a triangulation T of E^n with $T_c \subset T$ and a linear isotopy $H_t\colon E^n \to E^n$ ($t \in [0, 1]$) with compact support such that $H_0 = \mathrm{id}$ and $H_t|C = h_t$.*

PROOF. For every $s \in [0, 1]$, let T_s be a triangulation of E^n which contains $h_s(C, T_c)$ as a subcomplex. Theorem 3.1 asserts that there is such a T_s. For each T_s there is an $\varepsilon_s > 0$ and a linear isotopy

$$H_t^s\colon E^n \to E^n \qquad (t \in [s - \varepsilon_s, s + \varepsilon_s] \cap [0, 1])$$

that moves the vertices of T_s which are in $h_s(C)$ according to h_t, leaves all other vertices of T_s fixed, and extends linearly to the simplexes of T_s.

We can find two finite collections of points in $[0, 1]$, $\{t_i\}$ $(i = 0, 1, \ldots, m)$ with $0 = t_0 < t_1 < \cdots < t_m = 1$ and $\{s_i\}$ $(i = 1, 2, \ldots, m)$ so that $H_t^{s_i}$ is defined for $t \in [t_{i-1}, t_i]$, that is, $[t_{i-1}, t_i] \subset [s_i - \varepsilon_{s_i}, s_i + \varepsilon_{s_i}]$.

Now $H_t\colon E^n \to E^n$ ($t \in [0, 1]$) is defined to be

$$H_t^{s_1}(H_{t_0}^{s_1})^{-1} \quad \text{if } t \in [t_0, t_1],$$
$$H_t^{s_2}(H_{t_1}^{s_2})^{-1}H_{t_1}^{s_1}(H_{t_0}^{s_1})^{-1} = H_t^{s_2}(H_{t_1}^{s_2})^{-1}H_{t_1} \quad \text{if } t \in [t_1, t_2],$$
$$\cdots$$
$$H_t^{s_{i+1}}(H_{t_i}^{s_{i+1}})^{-1}H_{t_i} \quad \text{if } t \in [t_i, t_{i+1}],$$

One can obtain T as follows. Let $T_0 = H_{t_0}^{s_1}(T_{s_1})$. Then T_0 is a triangulation of E^n with $T_c \subset T_0$ and H_t is linear on T_0 for t in $[0, t_1]$. Also $H_{t_1}^{-1}$ is linear on $H_{t_1}(T_0)$. Let T_1' be a triangulation of E^n which is a refinement of both $H_{t_1}(T_0)$ and $H_{t_1}^{s_2}(T_{s_2})$ and has $H_{t_1}(T_c)$ as subcomplex and let $T_1 = H_{t_1}^{-1}(T_1')$. Also let T_2' be a triangulation of E^n which is a refinement of each of $H_{t_2}(T_1)$ and $H_{t_2}^{s_3}(T_{s_3})$ and has $H_{t_2}(T_c)$ as a subcomplex and let $T_2 = H_{t_2}^{-1}(T_2')$. Continuing in this fashion one gets $T_3, T_4, \ldots, T_{m-1}$. The required triangulation T is T_{m-1}.

Question 3.1. Is Theorem 3.3 true if we drop the finite condition, permit (C, T) to be an infinite complex linearly embedded as a closed subset of E^n, and do not require compact support?

4. Linear isotopies of disks with fixed boundaries. Let f and g be two linear embeddings of a triangulated disk (P, T) into E^2 so that $f|\text{Bd } P = g|\text{Bd } P$. In this section we investigate conditions under which there is a linear isotopy or push from f to g which leaves the boundary fixed. Cairns proved the following theorem in this vein.

CAIRNS' THEOREM [2]. *Let (P, T) be a triangulated disk linearly embedded in E^2 where T has only* 3 1-*simplexes in* Bd P *and let f be a linear homeomorphism of (P, T) which is the identity on the boundary. Then there is a push of (P, T) from the identity to f which keeps* Bd P *fixed.*

In this section we prove several variations of Cairns' Theorem. The first variation (Theorem 4.1) removes the hypothesis that T has only 3 1-simplexes in the boundary and replaces it by the hypotheses that P is star-like and T has no spanning 1-simplex. Examples show that neither the "star-like" hypothesis nor the "no spanning 1-simplex" hypothesis can be removed.

THEOREM 4.1. *Let (P, T) be a triangulated, star-like disk linearly embedded in E^2 where T contains no spanning* 1-*simplex. Let g: $P \to E^2$ be a linear homeomorphism of (P, T) which is the identity on* Bd P. *Then there is a push taking (P, T) to $(P, g(T))$ which leaves* Bd P *fixed throughout.*

PROOF. The proof is by induction on n, the number of interior vertices in the triangulation T. The proof bears a resemblance to the proof of Theorem 2.1. The scheme of the proof is to take an interior vertex v in T whose star contains an edge on Bd P and push P, leaving Bd P fixed, by h_t^1 ($t \in [0, 1]$)

so that $h_1^1(v)$ can see all of Bd P. Then we push $(P, g(T))$, leaving Bd P fixed, by h_t^2 $(t \in [0, 1])$ so that $h_1^1(v) = h_1^2(g(v))$. For every edge vw in T where $w \in$ Bd P, $h_1^1(vw) = h_1^2 g(vw)$. As in Theorem 2.1, we then apply induction in each of the subdisks into which the $h_1^1(vw)$'s divide P to finish the proof. What we need to show is that there is a push of (P, T) taking v to a point which can see all of Bd P and similarly a push of $(P, g(T))$ taking $g(v)$ to the same point. The following two lemmas are helpful in constructing these pushes.

LEMMA 4.2. *Let (P, T) be a triangulated star-like disk linearly embedded in E^2. Let $v_0v_1v_2$ be a simplex of T such that v_0 is an interior vertex and there is a point $s \in$ Int $v_0v_1v_2$ which can see every point of* Bd P. *Then for each neighborhood N of s there is a push of (P, T) that moves v_0 into N, leaves* Bd P *fixed, and is such that any vertex that is moved at all is moved in a straight line toward s.*

PROOF. We first describe a continuous family of linear maps f_t: $(P, T) \to P$ $(t \in [0, 1])$ which (except for not being homeomorphisms) satisfy the conclusion of the lemma. Then we pick a t_0 near 1 and approximate f_t $(t \in [0, t_0])$ by a suitable linear isotopy.

The family f_t $(t \in [0, 1])$ will be described by specifying each map f_t on the vertices of T and then extending linearly to P.

The map f_t $(t \in [0, 1])$ is described as follows. It begins by moving v_0 directly toward s leaving all other vertices fixed. We suppose the motion is linear on v_0. We will extend this map by coning from v_0 to Lk(v_0). Either this extension is a homeomorphism for every position of v_0 as it moves to s or it is not. If it is, the lemma is proved. If not, let t_1 be the first moment at which the cone map is not a homeomorphism.

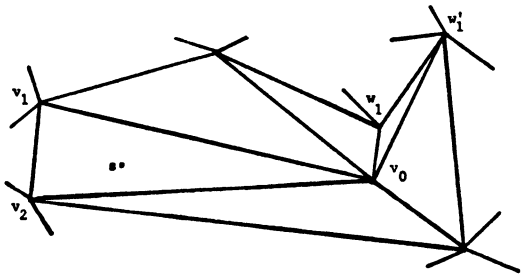


FIGURE 4.1

If $t_1 < 1$, let w_1 be a vertex which is in the way. See Figure 4.1. (If there is more than one such vertex, move each of them as described for w_1 in what follows.) That is, w_1 lies on the segment between $f_{t_1}(v_0)$ and w_1', a vertex in Lk(v_0). (Note that w_1 is also a vertex in Lk(v_0).) The map f_{t_1} smashes the simplex $v_0w_1w_1'$ and we say that w_1 is killed. From now on f_t on w_1 will be

defined as follows: $f_t(w_1)$ ($t \in [t_1, 1]$) will be the point on the intersection of the segment $w_1 s$ and the segment $f_t(v_0) f_t(w_1')$.

We continue defining f_t by moving v_0 and w_1 toward s as described and leaving all other vertices fixed and extending linearly. Let $t_2 \in [t_1, 1]$ be the first point in $[t_1, 1]$ at which the map $f_{t_2} f_{t_1}^{-1}$ is not a homeomorphism of $f_{t_1}(P)$. If $t_2 < 1$, we find a vertex w_2 analogous to w_1 above and a 1-simplex $w_2' w_2''$ so that $w_2 \in f_{t_2}(w_2' w_2'')$. We say that w_2 is killed at time t_2. We define the map f_t ($t \in [t_2, 1]$) on w_2, as before, to be the point of intersection of the segment $w_2 s$ and $f_t(w_2') f_t(w_2'')$. Continue until v_0 has been moved to s.

Let $w_1, w_2, \ldots, w_n$ be the vertices that were killed and t_i ($i = 1, 2, \ldots, n$) be the time at which w_i was killed. For convenience we suppose $0 < t_1 < t_2 < \cdots < t_n < 1$. Let t_0 be an element of $(t_n, 1)$ such that $f_{t_0}(v_0) \in N$. We show how to adjust f_t ($t \in [0, t_0]$) so that the adjustment is a linear isotopy.

We note that w_n was killed when $w_n' w_n''$ was moved up to it from the non-s side. If w_n had been moved toward s slightly before $w_n' w_n''$ arrived, the killing would have been delayed. In fact, if w_n had been continuously moved toward s and slightly ahead of $f_t(w_n)$, it would not have been killed in the time interval $[0, t_0]$. (The interval $[0, t_0]$ was used instead of $[0, 1]$ because difficulty would have been encountered at $t = 1$.) Similarly w_{n-1} could have been moved continuously ahead of $f_t(w_{n-1})$ to prevent its being killed in the time interval $[0, t_0]$. Moving backward through the w_i's we find that all of the w_i's could have been moved slightly ahead of their images under f_t so as to have a linear isotopy rather than a 1-parameter family of maps.

Note that every vertex that moved, moved directly toward s, a point which could see all of Bd P. Any vertex w_i which was killed was on $f_{t_i}(w_i' w_i'')$. Therefore, w_i originally was in the triangle $s\, w_i' w_i''$. Hence w_i was an interior vertex since the triangle $s\, w_i' w_i''$ is contained in P. Note that Bd P remained fixed in f_t ($t \in [0, t_0]$) and also remained fixed by the linear isotopy approximation. Replacing the linear isotopy by a push completes the proof of this lemma.

LEMMA 4.3. *Let (P, T) be a star-like disk with no spanning* 1-*simplex linearly embedded in E^2. Let $s \in$* Int P *be a point which can see* Bd P *and let N be a neighborhood of s. Then there is a push of P such that* Bd P *is left fixed and each interior vertex is taken into N.*

PROOF. We suppose N is the interior of a circle each of whose points can see all of Bd P. Let v be an interior vertex belonging to a 2-simplex of T whose interior intersects N. Use Lemma 4.2 to move v into N. Iterated use of Lemma 4.2 carries the other interior vertices into N.

We use Lemmas 4.2 and 4.3 to finish the proof of Theorem 4.1. Let s be a point that can see Bd P. Let v be an interior vertex of T which contains an edge uw of Bd P in its star. Using Lemma 4.3 on (P, T) move v to a point v'

very close to point s and using it on $(P, g(T))$ move $g(v)$ to a point v'' very close to s. They should be so close that there is a point $s' \in uwv' \cap uwv''$ so that s' can see Bd P. Now apply Lemma 4.2 to move v' to s' (by aiming for a point beyond s' on ray $v's'$) and v'' to s'. Apply induction as described in the beginning of the proof of Theorem 4.1 and the theorem is proved.

We list two corollaries of Theorem 4.1. The first is a variation of a theorem of Ho [6].

COROLLARY 4.4. *Let (P, T) be a triangulated, convex disk linearly embedded in E^2. Let $g\colon P \to P$ be a linear homeomorphism of (P, T) which is the identity on* Bd P. *Then there is a push of (P, T) taking (P, T) to $(P, g(T))$ which leaves* Bd P *fixed throughout.*

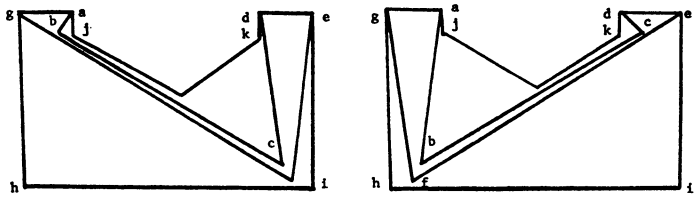
COROLLARY 4.5 (CAIRNS). *Let (B, T) be a triangulated complex. Let g_0 and g_1 be two linear embeddings of (B, T) into E^2 so that there is an orientation-preserving homeomorphism f of E^2 with $f \circ g_0 = g_1$. Then there is a push h_t $(t \in [0, 1])$ of (B, T) so that $h_0 = g_0$ and $h_1 = g_1$.*

PROOF. Incorporate $g_0(B)$ and $g_1(B)$ into equivalent triangulations of a convex PL disk and apply Corollary 4.4.

In a recent paper [7], an example is presented to show that the analog in E^3 of Corollary 4.5 is false.

We now give an example to show that the hypothesis of star-likeness in Theorem 4.1 is necessary.

EXAMPLE 4.1. This is an example of a polygonal disk (P, T) linearly embedded in E^2 and a linear homeomorphism $g\colon P \to P$ with $g|\text{Bd } P$ the identity such that there is no linear isotopy of (P, T) taking the identity map to g while leaving Bd P fixed.



(i) (ii)

FIGURE 4.2

The (i) and (ii) parts of Figure 4.2 show two embeddings of two spanning arcs inside a polygonal disk P. Although it is possible to isotope one embedding of these arcs to the other keeping Bd P fixed, it is impossible to linearly isotope one to the other keeping Bd P fixed since at the moment when the image of the segment bc is horizontal it is impossible to linearly

embed efg. Since these spanning arcs in the two embeddings can be incorporated into equivalent triangulations of P, the example shows that Theorem 4.1 is false without the star-like requirements.

EXAMPLE 4.2. A slight modification of Example 4.1 shows the necessity of the "no spanning 1-simplex" hypothesis in Theorem 4.1. Just add a rectangular disk to the bottom of the disk P of Example 4.1 to make the resulting disk P' star-like but keep the bottom of P as a spanning 1-simplex of P'.

5. The Alexander Isotopy Theorem. The standard proof [1] of the Alexander Isotopy Theorem shows that for every PL homeomorphism f of a PL planar disk P to itself which fixes the boundary there is a PL isotopy h_t: $P \to P$ ($t \in [0, 1]$) so that $h_0 = \mathrm{id}$, $h_1 = f$, and for each $t \in [0, 1]$, $h_t|\mathrm{Bd}\, P = \mathrm{id}$. It is not necessarily true that for each t, h_t is a linear homeomorphism of P with respect to the same triangulation of P. In fact, the standard proof of Alexander's theorem makes use of a process which squeezes badness out of existence in a limit. This type of argument cannot yield a linear isotopy. Here we show that the PL isotopy h_t *can* be chosen so that there is one triangulation T with respect to which each h_t is a linear homeomorphism. In a future paper Starbird will prove the analogous theorem for 3-cells in E^3.

LEMMA 5.1. *Let P be a PL planar disk in E^2 and r and s be two PL embeddings of an arc A into P so that $r(A) \cap \mathrm{Bd}\, P = s(A) \cap \mathrm{Bd}\, P = r(\mathrm{Bd}\, A) = s(\mathrm{Bd}\, A) = \{p, q\}$. Then there is a triangulation T_1 of A and a linear isotopy h_t: $(A, T_1) \to E^2$ ($t \in [0, 1]$) so that $h_0 = r$, $h_1 = s$, and for each t, $h_t(\mathrm{Bd}\, A) = h_t(A) \cap \mathrm{Bd}\, P = \{p, q\}$.*

PROOF. Find a spanning PL arc B in P so that $B \cap \mathrm{Bd}\, P = \mathrm{Bd}\, B \cap \mathrm{Bd}\, P = B \cap r(A) = B \cap s(A) = \{p, q\}$. One can use the disks bounded by $B \cup r(A)$ and $B \cup s(A)$ respectively to guide a push of $r(A)$ to B and a push of B to $s(A)$.

THEOREM 5.2. *Let P be a PL disk PL embedded in E^2 and f be a PL homeomorphism of P onto itself which leaves* Bd P *fixed. Then there is a triangulation T and a linear isotopy h_t: $(P, T) \to E^2$ ($t \in [0, 1]$) so that $h_0 = \mathrm{id}$, $h_1 = f$, and, for each $t \in [0, 1]$, $h_t|\mathrm{Bd}\, P = \mathrm{id}$.*

PROOF. Let T' be a triangulation of P with respect to which f is linear. We first find a subdivision T of T' and a linear isotopy g_t: $(P, T) \to E^2$ ($t \in [0, 1]$) with the properties that $g_0 = \mathrm{id}$, for each t in $[0, 1]$, $g_t|\mathrm{Bd}\, P = \mathrm{id}$, and $g_1|(\text{1-skeleton of } T') = f|(\text{1-skeleton of } T')$. The linear isotopy g_t is obtained as follows. Let $\{\sigma_1, \sigma_2, \ldots, \sigma_n\}$ be a shelling of the triangulation T'. Use Lemma 5.1 to push Bd σ_1 to $f(\mathrm{Bd}\, \sigma_1)$ leaving Bd P fixed and extend this push to a push of all of P using Theorem 3.3. Now Bd σ_2 is partly where it should be under f and partly subdivided and in the wrong place. Now push Bd σ_2 to

where it should be under f using Lemma 5.1 and Theorem 3.3 as above leaving Bd P and all of the moved σ_1 fixed. Continue in this manner until the linear isotopy g_t is produced.

Next note that for each 2-simplex σ_i of T', $g_1|\sigma_i$ and $f|\sigma_i$ are two linear homeomorphisms of $(\sigma_i, T|\sigma_i)$ such that $g_1|\text{Bd } \sigma_i = f|\text{Bd } \sigma_i$. By Corollary 4.4 there is a linear isotopy taking g_1 to f which leaves the boundary of σ_1 fixed. Performing g_t ($t \in [0, 1]$) followed by these linear isotopies on each σ_i yields a desired linear isotopy of P from id to f.

Question 5.1. If P is a polygonal disk in E^2, is there an $\varepsilon > 0$ such that for any triangulation T of P of mesh less than ε and any linear homeomorphism h of (P, T) into E^2 fixed on Bd P, it is possible to push $(h(P), h(T))$ back to (h, T) while leaving Bd P fixed?

6. Linear degeneration. In §7 we improve Corollary 4.5 by restricting the linear isotopy in the conclusion to take place in a restricted part of the plane. (See Theorems 7.1, 7.2, and 7.3.) The proofs of those theorems make use of the concept of linear degeneration which is a method of transforming a triangulated complex (C, T) similar to a collapse. However, we are interested in the linear structure of C, so each transformation will be linear with respect to the given triangulation T. Theorem 6.1 will be used in the next section.

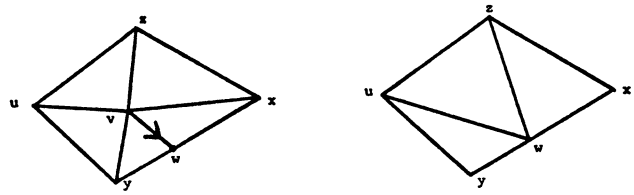


ILLUSTRATION 6.1

A linear degeneration is shown in Illustration 6.1 and its formal definition is given below. It consists of moving the vertex v to a vertex w in its link so that at each moment before the end, the map is a linear homeomorphism whereas at the last moment v was mapped to w and the 2-simplexes vxw and vyw were mapped simplicially to wx and wy respectively. Note that if the vertices u and z did not exist, this movement would have resulted in a collapse. Moving v to y would not be a linear degeneration because the final map would not be a linear embedding of the simplex vxw.

DEFINITION. Let (C, T) be a triangulated complex linearly embedded in E^2 and let vw be a 1-simplex of C. A continuous family of maps f_t: $C \to E^2$ ($t \in [0, 1]$) obtained by moving v along vw to w and coning out to $\text{Lk}(v)$ while leaving $C - \text{St}(v)$ fixed is a *simple linear degeneration* provided

(1) f_t is monotone nonincreasing in the sense that if $s \leqslant t, f_t(C) \subseteq f_s(C)$,

(2) for t in $[0, 1)$, f_t is a linear embedding with respect to T, and

(3) f_1 is a linear embedding of each 2-simplex of T which does not have vw as an edge.

Then $f_1(C)$ is regarded as a new complex linearly embedded in E^2 and having one fewer vertex than C. If a finite sequence of simple linear degenerations transforms a complex C to a complex C^1, then C *linearly degenerates* to C^1 and the composition of the simple linear degenerations is a *linear degeneration*.

The main tools used in this section are lemmas about the existence of low order vertices in triangulations of disks. Ho [5, Lemma 2.17] noted that the Euler characteristic formula $V - E + F = 2$ can be used to verify the following.

6-LEMMA. *Let T be a triangulation of a disk D, V_I be the number of interior vertices of T, V_B be the number of boundary vertices, and for each vertex v, $o(v)$ be the order of v (number of 1-simplexes ending at v). Then*

$$\left(6V_I - \sum_{v_i \in \operatorname{Int} D} o(v_i)\right) + \left(4V_B - \sum_{v_j \in \operatorname{Bd} D} o(v_j)\right) = 6.$$

The 6-Lemma implies the following.

5-3 LEMMA. *If T is a triangulation of a disk, then either some interior vertex of T has order less than or equal to 5 or there are three boundary vertices each of which has order less than or equal to 3.*

We find the 5-3 Lemma useful when it is used in conjunction with the following fact about 3, 4, and 5 sided polygons.

STAR-LIKE LEMMA. *Let J be a polygonal simple closed curve in E^2 with 3, 4, or 5 sides. Then J bounds a star-like disk D and has a vertex v such that for each point w of J not on an edge of J containing v, vw spans D. Also, if a point s of* Int *D can see every point of J, then every point on the open segment (sv) can see all of J.*

We use the 5-3 Lemma and the Star-like Lemma to prove a theorem similar to a shelling theorem.

THEOREM 6.1. *Let P be a polygonal disk in E^2 having a triangulation T. Then (P, T) linearly degenerates to a 2-simplex in such a way that at each stage the image of P is a disk. In fact, if vw is a boundary 1-simplex of T, we can pick the degeneration so that vw remains fixed during the degeneration and is an edge of the final 2-simplex.*

PROOF. The proof is by induction on the number n of 2-simplexes in T. If $n = 1$, there is nothing to do. Assume the theorem for k less than n and let T have n 2-simplexes.

If T has a spanning 1-simplex st, then st divides P into two disks D_1 and D_2 where $vw \subset D_2$. Induction implies that D_1 degenerates to a 2-simplex having st as a side and a move of the opposite vertex of this 2-simplex produces a degeneration to D_2. Induction implies that D_2 in turn degenerates to a 2-simplex having vw as an edge. Hence, we suppose that T has no spanning 1-simplex. Hence no boundary vertex is of order 2 (unless P is a 2-simplex).

Case 1. Suppose there is an interior vertex s of order less than 6. The Star-like Lemma implies that there is a vertex t in $\mathrm{Lk}(s)$ so that s can be moved to t to produce a simple linear degeneration. The image of Bd P is left fixed under this degeneration and induction implies that the linear degeneration of P to a 2-simplex can be completed.

Case 2. Suppose there is no interior vertex of order less than 6. It follows from the 5-3 Lemma that there is a boundary vertex s other than v or w which is of order 3. One can obtain a decreasing linear degeneration by moving s. Note that vw remained fixed during this move.

EXAMPLE 6.1. While we could strengthen Theorem 6.1 so as to leave any given 1-simplex of T fixed or any 2-simplex that intersects Bd P fixed, the example illustrated by Figure 6.1 shows that we could not hope to keep a given interior 2-simplex fixed. No linear degeneration of P to a 2-simplex leaves the center 2-simplex fixed.

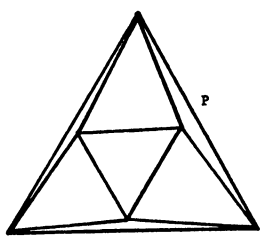


FIGURE 6.1

We next investigate the possibilities of degenerating a triangulated disk P to complexes other than a 2-simplex. A necessary condition for linearly degenerating a complex K to a complex L is that there exist a linear map of K onto L. In the next theorem we investigate the possibility of linearly degenerating a disk to an arc. We make use of the Shortest Arc Property defined below.

DEFINITION. Let P be a disk with triangulation T and A be an arc which is the union of 1-simplexes of T. Then A has the *Shortest Arc Property* if and only if no edge path in T that joins the ends of A contains fewer 1-simplexes of T than A does.

Note that if there is a linear retraction from a triangulated disk (P, T) to a boundary arc A, then A must have the Shortest Arc Property.

THEOREM 6.2. *Suppose (P, T) is a triangulated disk and A is an arc in the 1-skeleton of T with its ends on* Bd P. *Then the following are equivalent*:

(1) *The disk (P, T) linearly degenerates to A leaving A fixed.*

(2) *There is a linear retraction of T to A.*

(3) *The arc A has the Shortest Arc Property.*

PROOF. Our plan is to show that

$$\mathrm{LR} \Rightarrow \mathrm{SAP} \Rightarrow \mathrm{LD} \Rightarrow \mathrm{LR}$$

where LR means T linearly retracts to A, SAP means that A has the Shortest Arc Property in P, and LD means that T linearly degenerates to A. Since the first and third of these implications are trivial, we concentrate on the second. This implication is demonstrated by using induction on the number n of 2-simplexes in T. In the cases where T has only one or two 2-simplexes, SAP $\Rightarrow$ LD. We suppose T has n 2-simplexes and that the implication follows for triangulations with fewer.

Our plan is to linearly degenerate the closure of each component of $P - A$ onto the part of A on its boundary so we suppose with no loss of generality that $A \subset \mathrm{Bd}\, P$.

Case 1. Suppose there is an edge path E joining two vertices v_0' and v_1' of A such that $E \not\subset \mathrm{Bd}\, P$ and E has the same number of 1-simplexes as the number in A between v_0' and v_1'. Then there is an edge path B between vertices v_0 and v_1 of A with the above properties and such that $B \cap A = \{v_0, v_1\}$. Then B separates P. Let C be the closure of a component of $P - B$ missing the part of A between v_0 and v_1. We show in the next paragraph that $(B \cup A) \cap C$ has the Shortest Arc Property with respect to C.

If $(B \cup A) \cap C$ did not have the Shortest Arc Property with respect to C, there would be a too-short arc B' in C from w_0 to w_1 such that $B' \cap [(B \cup A) \cap C] = \{w_0, w_1\}$. This is impossible because there is no place for w_0 and w_1 to lie. They cannot both be on A or else A does not have the Shortest Arc Property. They cannot be both on B or else we could replace a part of B by B' and get a shorter arc than B between v_0 and v_1. One cannot belong to A and the other to B for if w_0 were in A, w_1 in B, and v_0 between w_0 and w_1 on $(B \cup A) \cap C$, the union of B' and the part of B from w_1 to v_1 would be shorter than the arc on A from w_0 to v_1. Hence there is no place for w_0 and w_1 and $(B \cup A) \cap C$ has the Shortest Arc Property in C.

Since C has fewer than n 2-simplexes, it follows by induction that C linearly degenerates to $(B \cup A) \cap C$. Note that $B \cap C$ is a spanning arc of P and the closure of $P - C$ is a disk D which has fewer than n 2-simplexes. Induction implies that there is a linear degeneration of D onto $D \cap A$. The composition of these two linear degenerations is the required one for Case 1.

Case 2. Suppose there is no edge path E as supposed in Case 1. The 5-3

Lemma implies that there is a vertex v_0 of T such that v_0 is an interior vertex of order less than 6 or v_0 is a boundary point of order less than 4 which is not an end of A.

We note that v_0 does not lie on A. If it did, it could not be of order 2 since A has the Shortest Arc Property and it could not be of order 3 or else Case 1 would apply.

If $v_0 \in \operatorname{Bd} P - A$, the linear degeneration can be started by moving v_0. Induction can be used to show that it can be completed.

If v_0 is an interior vertex, it follows from the Star-like Lemma that we can start the linear degeneration by moving v_0 to a vertex w_0 of $\operatorname{Lk}(v_0)$. This degeneration reduced the number of 2-simplexes and induction implies that the linear degeneration can be completed if A has the Shortest Arc Property with respect to the new triangulation. However, it does have this property since if there were a too-short arc B' in the new triangulation, it would pass through w_0 and the inverse of B' would contain an arc B through v_0 showing that Case 1 applied.

One can easily construct an example showing the necessity in Theorem 6.2 of requiring that the ends of A be on Bd P. See Figure 6.2.

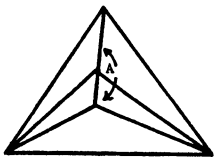


FIGURE 6.2

7. Linear isotopies in open subsets of E^2. In this section we show that only a part of E^2 need be used in pushing a complex from one linear embedding to another. The following theorems are variations of Cairn's result listed above as Corollary 4.5.

THEOREM 7.1. *Let (P, T) be a triangulated disk linearly embedded in E^2. Then there is a linear isotopy h_t: $P \to P$ $(t \in [0, 1])$ of (P, T) such that $h_0 =$ identity, $h_1(P)$ is convex, and $h_t(P)$ $(t \in [0, 1])$ is monotonically decreasing.*

PROOF. By Theorem 6.1, P can be linearly degenerated to a 2-simplex of T. This degeneration can be approximated by a monotonically nonincreasing push of (P, T) which pushes P to a convex set. This fact can be proved by induction on n where the inductive statement is that if (P, T) is a triangulated disk in E^2 with no three collinear vertices, then an n step linear degeneration of P followed by a nonincreasing push taking P to a convex disk can be replaced by a nonincreasing push of P to a convex disk.

The push so obtained can be replaced by a monotonically decreasing linear isotopy to prove Theorem 7.1.

THEOREM 7.2. *Let (P, T) be a triangulated disk, U be a connected open subset of E^2, and f_0 and f_1 be linear embeddings of P in U such that there is an orientation preserving homeomorphism G of E^2 so that $G \circ f_0 = f_1$. Then there is a push h_t: $P \to U$ $(t \in [0, 1])$ of (P, T) into U so that $h_0 = f_0$ and $h_1 = f_1$.*

PROOF. Using Theorem 7.1, $f_0(P)$ can be linearly isotoped to a convex set which can in turn be linearly isotoped to a small convex set. Likewise $f_1(P)$ can be linearly isotoped to a small set. These linear isotopies can be replaced by pushes. We can then push the shrunk $f_0(P)$ into a convex open set which also contains the shrunk $f_1(P)$. Now Corollary 4.4 can be applied to finish the proof as was done in Corollary 4.5.

THEOREM 7.3. *Let (C, T) be a triangulated complex, D be an open disk in E^2, and f_0 and f_1 be two linear embeddings of C into D such that there is an orientation preserving homeomorphism G of E^2 so that $G \circ f_0 = f_1$. Then there is a push h_t: $C \to D$ $(t \in [0, 1])$ of (C, T) into D so that $h_0 = f_0$ and $h_1 = f_1$.*

PROOF. Incorporate $f_0(C)$ and $f_1(C)$ into equivalently triangulated disks contained in D and apply Theorem 7.2.

EXAMPLE 7.1. It should be noted that the hypothesis that D be a disk in Theorem 7.3 is necessary as shown by Figure 7.1.

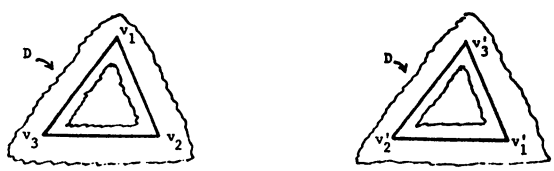


FIGURE 7.1

Question 7.1. In §4 and this section, Cairns' Theorems were modified by keeping part of a complex fixed during a push and by restricting the part of the plane in which the push takes place. Another possible avenue of inquiry is suggested in the following question: Let $\{A_i\}_{i=1}^n$ be a collection of arcs. Let f_0, f_1: $\bigcup A_i \to E^2$ be maps of the disjoint union of the A_i's into E^2 so that $f_0, f_1|A_i$ is linear for each i and so that there is an orientation preserving homeomorphism G of E^2 so that $G \circ f_0 = f_1$. Then is there an isotopy h_t: $E^2 \to E^2$ $(t \in [0, 1])$ so that $h_0 = \text{id}$, for each i, $h_1(f_0(A_i)) = f_1(A_i)$, and for each $t \in [0, 1]$ and each i, $h_t|f_0(A_i)$ is a linear homeomorphism?

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