TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 237, March 1978

ON THE GROUP OF AUTOMORPHISMS OF AFFINE ALGEBRAIC GROUPS

BY

DONG HOON LEE

ABSTRACT. We study the conservativeness property of affine algebraic groups over an algebraically closed field of characteristic 0 and of their group of automorphisms. We obtain a certain decomposition of affine algebraic groups, and this, together with the result of Hochschild and Mostow, becomes a major tool in our study of the conservativeness property of the group of automorphisms.

1. **Introduction.** Let G be an affine algebraic group over a field F, with Hopf algebra $\mathcal{Q}(G)$ of polynomial functions on G, in the sense of [2] and let $W(G)$ denote the group of all affine algebraic group automorphisms of G. Then $\mathcal{Q}(G)$ may be viewed as a right $W(G)$-module, with $W(G)$ acting by composition $f \to f \circ \alpha$ on $\mathcal{Q}(G)$.

We recall, from [3], that G is said to be *conservative* if $\mathcal{Q}(G)$ is locally finite as a $W(G)$-module. As is shown in [3], the conservativeness of G characterizes the existence of a suitable affine algebraic group structure on $W(G)$ and the obstruction to the conservativeness of a connected G is realized as the presence of certain central tori in G, when the base field F is algebraically closed and of characteristic 0.

In the present study of $W(G)$, we exploit the above results and technique of [3] and, accordingly, we refer to [2] and [3] for standard facts concerning affine algebraic groups and their automorphism group.

The following are brief descriptions of the contents appearing in each section: In §2, we examine reductive affine algebraic groups and their conservativeness and, in §3, we establish a certain $W(G)$-invariant decomposition of G when G is conservative. Finally, in §4, we use the result obtained in §3 to study the structure of $W(G)$.

The following notation is standard throughout: Let G be an affine algebraic group. Then G_1 denotes the connected component of the identity element of G and $Z(G)$ the center of G. If $x \in G$, we use I_x to denote the inner automorphism of G that is induced by x, and, for a subset S of G,

Received by the editors July 9, 1976.

AMS (MOS) subject classifications (1970). Primary 20G15.

Key words and phrases. Affine algebraic groups, conservative, Hopf algebra, antipodes, comultiplication.

$\mathrm{Int}_G(S)$ denotes $\{I_x\colon x \in S\}$. In the case where $S = G$, we simply write $\mathrm{Int}(G)$ instead of $\mathrm{Int}_G(G)$.

2. **Reductive groups and conservativeness.** For an affine algebraic group G over a field F, let $\mathfrak{L}(G)$ denote the Lie algebra of G, and for a morphism $\rho\colon G \to H$ of affine algebraic groups, $\mathfrak{L}(\rho)$ denotes the Lie algebra homomorphism induced by ρ. Thus $\mathfrak{L}(G)$ consists of all F-linear maps $X\colon \mathfrak{A}(G) \to F$ such that $X(fg) = X(f)g(1) + f(1)X(g)$ for all $f, g \in \mathfrak{L}(G)$, and the map $\mathfrak{L}(\rho)\colon \mathfrak{L}(G) \to \mathfrak{L}(H)$ is given by $\mathfrak{L}(\rho)(X)(f) = X(f \circ \rho)$, $f \in \mathfrak{A}(G)$ and $X \in \mathfrak{L}(G)$. For $x \in G$ and $f \in \mathfrak{A}(G)$, we write $x \cdot f$ for the left translate of f by x, which is given by $(x \cdot f)(y) = f(yx)$ for $y \in G$ and define $x/f\colon W(G) \to F$ by $(x/f)(\alpha) = f(\alpha(x))$.

With this preparation, we prove the following characterization of conservative reductive affine algebraic groups.

THEOREM 2.1. *Let G be a reductive affine algebraic group over an algebraically closed field F of characteristic* 0. *Then G is conservative if and only if* $\mathrm{Int}(G)$ *is of finite index in $W(G)$.*

PROOF. Suppose $\mathrm{Int}(G)$ is of finite index in $W(G)$. Then the Hopf algebra $\mathfrak{A}(G)$ is locally finite as an $\mathrm{Int}(G)$-module. Since $\mathrm{Int}(G)$ is a normal subgroup of $W(G)$, it is then locally finite as a $W(G)$-module, proving that G is conservative.

Suppose, conversely, that G is conservative. Thus, by Theorem 2.1, [4], $W(G)$ is an affine algebraic group and its F-algebra $\mathfrak{A}(W(G))$ of polynomial functions on $W(G)$ is generated by the functions x/f, $x \in G$ and $f \in \mathfrak{A}(G)$, and their antipodes.

We first show that the F-space $\mathfrak{L}(W(G))$ may be identified with an F-subspace of the space $Z^1(G, \mathfrak{L}(G))$ of all nonhomogeneous rational 1-cocycles of G with coefficients in $\mathfrak{L}(G)$ relative to the adjoint action of G on $\mathfrak{L}(G)$. To do this, we let $\sigma \in \mathfrak{L}(W(G))$ and, for each $x \in G$, we define

$$\sigma_x\colon \mathfrak{A}(G) \to F$$

by

$$\sigma_x(f) = \sigma(x/x^{-1} \cdot f), \qquad f \in \mathfrak{A}(G).$$

Then we see easily that $\sigma_X \in \mathfrak{L}(G)$ for all $x \in G$, and we also have

$$\sigma_{xy} = \sigma_x + \mathrm{Ad}(x)(\sigma_y), \qquad x, y \in G. \tag{1}$$

To see this, let $\gamma\colon \mathfrak{A}(G) \to \mathfrak{A}(G) \otimes \mathfrak{A}(G)$ be the comultiplication of the Hopf algebra $\mathfrak{A}(G)$. For each $f \in \mathfrak{A}(G)$, we write

$$\gamma(f) = \sum_{i=1}^{n} f_i \otimes g_i, \qquad f_i, g_i \in \mathfrak{A}(G). \tag{2}$$

Then we have

(3) $$f(xy) = \sum_{i=1}^{n} f_i(x) g_i(y) \quad \text{for } x, y \in G.$$

Now let $\alpha \in W(G)$. Then

$$\begin{aligned}\big(xy/(xy)^{-1}\cdot f\big)(\alpha) &= f\big(\alpha(x)\alpha(y)y^{-1}x^{-1}\big)\\ &= f\big(\alpha(x)x^{-1}\cdot I_x\big(\alpha(y)y^{-1}\big)\big)\\ &= \sum_{i=1}^{n} f_i\big(\alpha(x)x^{-1}\big)g_i\big(I_x\big(\alpha(y)y^{-1}\big)\big) \quad \text{(by (3))}\\ &= \sum_{i=1}^{n} \big(x/x^{-1}\cdot f_i\big)(\alpha)\big(y/y^{-1}\cdot (g_i \circ I_x)\big)(\alpha).\end{aligned}$$

That is, we have

(4) $$xy/(xy)^{-1}\cdot f = \sum_{i=1}^{n} \big(x/x^{-1}\cdot f_i\big)\cdot\big(y/y^{-1}(g_i \circ I_x)\big).$$

Now

$$\begin{aligned}\sigma_{xy}(f) = \sigma\big(xy/(xy)^{-1}\cdot f\big) &= \sigma\left(\sum_{i=1}^{n}(x/x^{-1}\cdot f)\cdot\big(y/y^{-1}(g_i \circ I_x)\big)\right)\\ &= \sum_{i=1}^{n}\sigma\big(x/x^{-1}\cdot f_i\big)g_i(1) + \sum_{i=1}^{n} f_i(1)\sigma\big(y/y^{-1}\cdot(g_i \circ I_x)\big).\end{aligned}$$

However, we have (using (3))

$$x/x^{-1}\cdot f = \sum_{i=1}^{n}\big(x/x^{-1}\cdot f_i\big)g_i(1), \quad \text{and}$$

$$y/y^{-1}\cdot(f \circ I_x) = \sum_{i=1}^{n}\big(y/y^{-1}\cdot(g_i \circ I_x)\big)f_i(1)$$

Hence

$$\begin{aligned}\sigma_{xy}(f) &= \sigma\big(x/x^{-1}\cdot f\big) + \sigma\big(y/y^{-1}\cdot(f \circ I_x)\big) = \sigma_x(f) + \sigma_y(f \circ I_x)\\ &= \big(\sigma_x + \mathrm{Ad}(x)(\sigma_y)\big)(f),\end{aligned}$$

proving (1).

For each $\sigma \in \mathfrak{L}(W(G))$, define $\sigma'\colon G \to \mathfrak{L}(G)$ by $\sigma'(x) = \sigma_x$, $x \in G$. Then we easily see that $\sigma' \in Z^1(G, \mathfrak{L}(G))$. Since the functions x/f, together with their antipodes, generate $\mathcal{R}(W(G))$ as an F-algebra, it follows that the F-linear map $\sigma \to \sigma'$ is injective, under which we identify $\mathfrak{L}(W(G))$ with an F-subspace of $Z^1(G, \mathfrak{L}(G))$.

We next consider the morphism of affine algebraic groups $\nu\colon G \to W(G)$, which is given by $\nu(x) = I_x$, $x \in G$.

We compute the image of $\mathfrak{L}(G)$ under the F-linear map $\mathfrak{L}(\nu)\colon \mathfrak{L}(G) \to$

$\mathfrak{L}(W(G))$, $\mathfrak{L}(W(G))$ being identified with an F-subspace of $Z^1(G, \mathfrak{L}(G))$.

To do this, we first note that $X(f') = -X(f)$ for all $f \in \mathfrak{A}(G)$ and $X \in \mathfrak{L}(G)$. This may be seen as follows: Write $\gamma(f) = \sum_{i=1}^n f_i \otimes g_i$ as in (2). Then, by (3),

$$f(1) = f(xx^{-1}) = \sum_{i=1}^n f_i(x)g_i'(x) = \left(\sum_{i=1}^n f_ig_i'\right)(x),$$

which implies that $\sum_{i=1}^n f_ig_i'$ is constant.

Hence

$$\begin{aligned}0 = X\left(\sum_{i=1}^n f_ig_i'\right) &= \sum_{i=1}^n X(f_i)g_i'(1) + \sum_{i=1}^n f_i(1)X(g_i')\\ &= X\left(\sum_{i=1}^n f_ig_i(1)\right) + X\left(\sum f_i'(1)g_i'\right)\\ &= X(f) + X(f')\end{aligned}$$

and $X(f') = -X(f)$ follows.

For $X \in \mathfrak{L}(G)$, $x \in G$, and $f \in \mathfrak{A}(G)$, we have

$$\mathfrak{L}(\nu)(X)(x)(f) = \mathfrak{L}(\nu)(X)(x/x^{-1}\cdot f) = X\big((x/x^{-1}\ f)\cdot \nu\big).$$

But $(x/x^{-1}\cdot f)\cdot \nu = \sum_{i=1}^n f_i\cdot(g_i\cdot \nu(x))'$.

Hence

$$\begin{aligned}\mathfrak{L}(\nu)(X)(x)(f) &= X\left(\sum_{i=1}^n f_i\cdot(g_i\cdot \nu(x))'\right)\\ &= \sum_{i=1}^n X(f_i)(g_i\cdot\nu(x))'(1) + \sum_{i=1}^n f_i(1)X(g_i\cdot\nu(x)')\\ &= X\left(\sum_{i=1}^n f_ig_i(1)\right) - X\left(\sum_{i=1}^n f_i(1)(g_i\cdot\nu(x))\right)\\ &= X(f) - X(f\cdot\nu(x)) = (X - \mathrm{Ad}(x)(X))(f).\end{aligned}$$

That is, $\mathfrak{L}(\nu)(X)(x) = X - \mathrm{Ad}(x)(X)$, and we see that $\mathrm{Im}(\mathfrak{L}(\nu))$ is equal to the subsapce $B^1(G, \mathfrak{L}(G))$ of $Z^1(G, \mathfrak{L}(G))$ consisting of all 1-coboundaries of G.

Since G is reductive, $H^1(G, \mathfrak{L}(G)) = 0$. Hence $\mathrm{Im}(\mathfrak{L}(\nu)) = B^1(G, \mathfrak{L}(G)) = Z^1(G, \mathfrak{L}(G))$. Since F is algebraically closed, the surjectivity of $\mathfrak{L}(\nu)$ implies that $\mathrm{Im}(\nu) = \mathrm{Int}(G)$ is open in $W(G)$ and hence $\mathrm{Int}(G)$ is of finite index in $W(G)$.

THEOREM 3.2. *Let G be an affine algebraic group over an algebraically closed* *field F of characteristic* 0. *Then G is conservative if a maximal reductive subgroup of G is conservative.*

PROOF. Let G_u denote the unipotent radical of G, and let P be a maximal reductive subgroup of G. Since F is of characteristic 0, a theorem of Mostow (see [2, Theorem 14.2]) assures that we have a semidirect product decomposition $G = G_u \cdot P$. By the conjugacy of maximal reductive subgroups, we may assume that P is conservative, and we have $W(G) = \mathrm{Int}(G) \cdot \mathscr{A}$, where $\mathscr{A}$ is the subgroup of $W(G)$ consisting of all $\alpha \in W(G)$ leaving P invariant.

Let $\mathscr{A}_P$ denote the restriction image of $\mathscr{A}$ in $W(P)$. Then $\mathrm{Int}(P) < \mathscr{A}_p$, and, since P is conservative, $W(P)/\mathrm{Int}(P)$ is finite by Theorem 2.1. It follows that $\mathscr{A}_p/\mathrm{Int}(P)$ is also finite.

From this point on, we can copy the argument used in [3, p. 539] for the proof of conservativeness of G when P is a connected semi-simple algebraic subgroup and conclude that G is conservative. This establishes Theorem 2.2.

3. **$W(G)$-invariant decomposition of** G. For a subset $\mathscr{A}$ of $W(G)$, let $G^{\mathscr{A}}$ denote the set consisting of all $x \in G$ such that $\alpha(x) = x$ for all $\alpha \in \mathscr{A}$.

We prove the following result which will then be used in §4 for out study of $W(G)$.

THEOREM 3.1. *Let G be a connected conservative affine algebraic group over an algebraically closed field F of characteristic* 0, *and let T be the maximal central torus of $W(G)_1$. Then there exists a $W(G)$-invariant algebraic vector subgroup Z of G such that $G = Z \times G^T$.*

PROOF. If T is trivial, then the assertion holds trivially. Thus we assume that T is of dimension $\geqslant 1$.

For each $x \in G$, the inner automorphism I_x induced by x commutes with every element of T. Hence, for $\alpha \in T$ and $x \in G$, we have $x^{-1}\alpha(x) \in Z(G)$.

We define, for each $\alpha \in T$, η_α: $G \to Z(G)$ by $\eta_\alpha(x) = x^{-1}\alpha(x)$, $x \in G$.

Then η_α is a morphism of affine algebraic groups. Since G is connected, it follows that $\eta_\alpha(x) \in Z(G)_1$ for all $x \in G$. Now we choose a maximal reductive subgroup P of G so that $G = G_u \cdot P$ (semidirect). We first show that every element of P is T-fixed. To do this, we choose a maximal torus D of P. Then $P = D \cdot P'$, where P' denotes the commutator subgroup of P, and $P' <$ Ker η_α implies that every element of P' is T-fixed. Hence it is enough to show that every element of D is T-fixed. Let K be the maximal torus of $Z(G)$. Then the torus $\eta_\alpha(D)$ is contained in K, and hence we see that every element α of T leaves D invariant. Consider the polynomial map

$$\phi: T \times D \to D,$$

given by $\phi(\alpha, x) = \alpha(x)$, and define, for each $x \in D$, ϕ_x: $T \to D$ by $\phi_x(\alpha) = \alpha(x)$. Then clearly ϕ_x is a polynomial map. Let $x \in D$ be of order $m < \infty$.

Then $\phi_x(\alpha)$ is also of order m for all $\alpha \in T$. Since D contains only a finite number of elements of order m, it follows from the connectedness of T that $\operatorname{Im} \phi_x = \{x\}$. That is, $\alpha(x) = x$ for all $\alpha \in T$. Since the elements in D of finite order form a dense subset of D, it follows that T leaves every element of D fixed.

Next we show that if U denotes the unipotent radical of $Z(G)$, then $G = U \cdot G^T$. The morphism $\eta_\alpha \colon G \to Z(G)$ for $\alpha \in T$ maps G_u into U. Hence η_α induces a morphism $\mu_\alpha \colon G_u \to U$ of affine algebraic groups. Let μ_α^0 denote $\mathfrak{L}(\mu_\alpha) \colon \mathfrak{L}(G_u) \to \mathfrak{L}(U)$. The natural action of T on U determines a T-module structure on the F-space $\mathfrak{L}(U)$, and this in turn defines a T-module structure on the F-space $\operatorname{Hom}_F(\mathfrak{L}(G_u), \mathfrak{L}(U))$.

We then have

$$\mu_\beta^0 = \mu_\beta^0 + \alpha \cdot \mu_\beta^0, \qquad \alpha, \beta \in T. \tag{1}$$

To prove (1), we note that $\exp_U \cdot \mu_\alpha^0 = \mu_\alpha \cdot \exp_{G_u}$, where $\exp_U$, $\exp_{G_u}$ denote the exponential maps for U, G_u, respectively. Hence for $X \in \mathfrak{L}(G_u)$,

$$\begin{aligned}
\exp \mu_{\alpha\beta}^0(X) &= \mu_{\alpha\beta}(\exp X) = (\exp X)^{-1}\alpha\beta(\exp X) \\
&= (\exp X)^{-1}\alpha(\exp X)\alpha\big((\exp X)^{-1}\beta(\exp X)\big) \\
&= \mu_\alpha(\exp X)\alpha(\mu_\beta(\exp X)) = \exp \mu_\alpha^0(X)\alpha\big(\exp \mu_\beta^0(X)\big) \\
&= \exp \mu_\alpha^0(X)\exp\big(\mathfrak{L}(\alpha)(\mu_\beta^0(X))\big) = \exp\big(\mu_\alpha^0(X) + \alpha \cdot \mu_\beta^0(X)\big).
\end{aligned}$$

Hence it follows that $\mu_{\alpha\beta}^0(X) = \mu_\alpha^0(X) + \alpha \cdot \mu_\beta^0(X)$, proving (1).

The identity (1) defines a rational T-module structure on the F-space $F \oplus \operatorname{Hom}_F(\mathfrak{L}(G_u), U)$, if we define the T-action by $\alpha \cdot (r, \phi) = (r, r\mu_\alpha^0 + \alpha \cdot \phi)$ for $\alpha \in T$, $r \in F$ and $\phi \in \operatorname{Hom}_F(\mathfrak{L}(G_u), \mathfrak{L}(U))$. Since T is reductive, the T-submodule $\operatorname{Hom}_F(\mathfrak{L}(G_u), \mathfrak{L}(U))$ has a 1-dimensional T-invariant complement in $F \oplus \operatorname{Hom}_F(\mathfrak{L}(G_u), \mathfrak{L}(U))$. This complement contains exactly one element of the form $(1, \phi)$.

Hence $(1, \phi) = \alpha \cdot (1, \phi) = (1, \mu_\alpha^0 + \alpha \cdot \phi)$ for all $\alpha \in T$ and this implies that $\mu_\alpha^0 = \phi - \alpha \cdot \phi$, $\alpha \in T$.

For each $X \in \mathfrak{L}(G_u)$, we have

$$\begin{aligned}
\exp \phi(X) &= \exp\big(\mu_\alpha^0(X) + \alpha \cdot \phi(X)\big) \\
&= \exp\big(\mu_\alpha^0(X)\big)\exp\big(\mathfrak{L}(\alpha)(\phi(X))\big) \\
&= (\exp X)^{-1}\alpha(\exp X)\alpha(\exp \phi(X)).
\end{aligned}$$

Hence $\exp_{G_u} X \cdot \exp_U \phi(X) \in G^T$ for all $X \in \mathfrak{L}(G_u)$. Since $\exp_{G_u}(\mathfrak{L}(G_u)) = G_u$, it follows that $G_u < U \cdot G^T$, and $p < G^T$ implies $G = U \cdot G^T$.

Now we consider the rational T-module $\mathfrak{L}(U)$. Since T is a torus over an algebraically closed field, we may decompose the F-space $\mathfrak{L}(U)$ as

$$\mathfrak{L}(U) = \sum_{\chi \neq 1} L_\chi + \mathfrak{L}(U)^T,$$

where L_χ is the weight space $\{X \in \mathfrak{L}(U)\colon \alpha \cdot X = \chi(\alpha)X \text{ for all } \alpha \in T\}$ corresponding to the weight $\chi\colon T \to F^*$, and $\mathfrak{L}(U)^T$ is the T-fixed part of $\mathfrak{L}(U)$.

Since T is a normal subgroup of $W(G)$, $W(G)$ permutes the weights of T in $\mathfrak{L}(U)$. Hence the F-subspace $Z = \sum_{\chi = 1} L_\chi$ is $W(G)$-invariant. Let $Z = \exp_U Z$. Then $U = Z \times U^T$ and this implies that $G = Z \times G^T$ follows. Clearly Z is $W(G)$-invariant and the theorem is proved.

REMARK. Since T is a normal subgroup of $W(G)$, it follows that G^T is also $W(G)$-invariant. As we will see in §4, T is central in $W(G)$ and, in fact, a direct factor of $W(G)$.

4. Decomposition and conservativeness of $W(G)$.

THEOREM 4.1. *Let G be a conservative connected affine algebraic group over an algebraically closed field F of characteristic* 0. *Then the maximal central torus of $W(G)_1$ is of dimension ≤ 1 and is a direct factor of $W(G)$.*

PROOF. Let T be the maximal central torus of $W(G)_1$, and assume that T is nontrivial. Then we have a $W(G)$-invariant decomposition $G = Z \times G^T$ (Theorem 3.1). Hence we have $W(G) \simeq W(Z) \times W(G^T)$ as affine algebraic groups and the restriction map $T \to W(Z)$ is injective.

Let $\mathfrak{z}$ denote the Lie algebra of Z. Then the affine algebraic group $W(Z)$ may be identified with the affine algebraic group $GL(\mathfrak{z})$ of all F-linear automorphisms of $\mathfrak{z}$. Since F is algebraically closed, the center of $W(Z)$ is a 1-dimensional torus and is a direct factor of $W(Z)$. Since every element of $W(Z)$ can be extended to an element of $W(G)$, we see easily that the restriction map sends T isomorphically onto the center of $W(Z)$. Hence our assertion follows.

In [2], Hochschild proved that, if G is a nonabelian unipotent affine algebraic group, then the maximal central torus of $W(G)_1$ is trivial and hence that $W(G)_1$ is conservative. The assertion does not hold for arbitrary solvable affine algebraic groups (see the example in [2, p. 111]).

The following theorem characterizes those nonabelian solvable groups G for which $W(G)_1$ is conservative.

THEOREM 4.2. *Let G be a connected conservative solvable nonabelian affine algebraic group over an algebraically closed field of characteristic* 0. *Then the following are equivalent*:

(i) *$W(G)_1$ is conservative.*

(ii) *The connected component of the center of $W(G)_1$ is unipotent* (*i.e.* $T = 1$).

(iii) *G cannot be a product $G = Z \times H$ of a nontrivial algebraic vector subgroup Z and an algebraic subgroup H, both of which are invariant under $W(G)$.*

PROOF. (iii) $\to$ (ii) follows from Theorem 3.1 and the subsequent remark.

(ii) $\to$ (iii) holds because of the decomposition $W(G) = W(Z) \times W(H)$.

(ii) $\to$ (i) follows from Theorem 3.2 of [4].

It remains to show (i) $\to$ (ii).

Let K be a maximal torus of G so that $G = G_u \cdot K$ (semidirect).

If K is trivial, then G is unipotent and nonabelian, and hence (ii) holds (see [2, p. 110]).

(1) Suppose $\dim K \geqslant 2$. Then the maximal central torus of G is trivial by Theorem 3.2 [4] and this implies that the torus $\mathrm{Int}_G(K) \simeq KZ(G)/Z(G)$ is of dimension $\geqslant 1$. Since $\mathrm{Int}(G)$ is a normal algebraic subgroup of $W(G)$, it follows that the algebraic torus $\mathrm{Int}_G(K)$ is contained in the radical of $W(G)_1$ and hence is central in a maximal reductive group containing it. Since $W(G)_1$ is conservative, (ii) follows from Theorem 3.2 of [3].

(2) Suppose $\dim K = 1$. If K is central in G, then $G = G_u \times K$, and hence $W(G) \simeq W(G_u) \times \mathbf{Z}_2$. Since G_u is nonabelian, (ii) follows immediately.

Therefore we may assume that the identity component of the center of G is unipotent. Then $\mathrm{Int}_G(K)$ is a 1-dimensional torus. Assume that (ii) does not hold, and let T be the maximal central torus of $W(G)_1$. Then $T \cap \mathrm{Int}_G(K) = \{1\}$, for if $\alpha \in T$ is of the form $\alpha = I_x$ for some $x \in K$, then the decomposition $G = Z \times G^T$ in Theorem 3.1 implies that $\alpha = 1$.

Since T centralizes $\mathrm{Int}_G(K)$, it follows that $T' = T \cdot \mathrm{Int}_G(K)$ (direct) is an algebraic torus of dimension 2.

Since T' is contained in the radical of $W(G)_1$, it follows that T' is central in a maximal reductive subgroup containing T'. (See [1, Chapter III].) Hence again by Theorem 3.2 of [3], $W(G)_1$ cannot be conservative, contradicting (i). Therefore $T = \{1\}$ and (ii) is proved.

REFERENCES

1. A. Borel, *Linear algebraic groups*, Benjamin, New York, 1969. MR **40** #4273.

2. G. Hochschild, *Introduction to affine algebraic groups*, Holden-Day, San Francisco, 1971. MR **43** #3268.

3. G. Hochschild and G. D. Mostow, *Automorphisms of affine algebraic groups*, J. Algebra **22** (1972), 535–543.

DEPARTMENT OF MATHEMATICS, CASE WESTERN RESERVE UNIVERSITY, CLEVELAND, OHIO 44106