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BOUNDED POINT EVALUATIONS AND SMOOTHNESS PROPERTIES OF FUNCTIONS IN $R^p(X)$[1]

BY

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ABSTRACT. Let X be a compact subset of the complex plane **C**. We denote by $R_0(X)$ the algebra consisting of the (restrictions to X of) rational functions with poles off X. Let m denote 2-dimensional Lebesgue measure. For $p \geq 1$, let $L^p(X) = L^p(X, dm)$. The closure of $R_0(X)$ in $L^p(X)$ will be denoted by $R^p(X)$. Whenever p and q both appear, we assume that $1/p + 1/q = 1$.

If x is a point in X which admits a bounded point evaluation on $R^p(X)$, then the map which sends f to $f(x)$ for all $f \in R_0(X)$ extends to a continuous linear functional on $R^p(X)$. The value of this linear functional at any $f \in R^p(X)$ is denoted by $f(x)$. We examine the smoothness properties of functions in $R^p(X)$ at those points which admit bounded point evaluations. For $p > 2$ we prove in Part I a theorem that generalizes the "approximate Taylor theorem" that James Wang proved for $R(X)$.

In Part II we generalize a theorem of Hedberg about the convergence of a certain capacity series at a point which admits a bounded point evaluation. Using this result, we study the density of the set X at such a point.

PART I. SMOOTHNESS PROPERTIES OF FUNCTIONS IN $R^p(X)$

Let X be a compact subset of the complex plane **C**. We denote by $R_0(X)$ the algebra consisting of the (restrictions to X) of rational functions with poles off X. Let m denote 2-dimensional Lebesgue measure. For $p \geq 1$, let $L^p(X) = L^p(X, dm)$. The closure of $R_0(X)$ in $L^p(X)$ will be denoted by $R^p(X)$. Whenever p and q both appear, we will assume that $1/p + 1/q = 1$.

1. Bounded point derivations.

DEFINITION (1.1). For $x \in X$ we say that x admits a *bounded point derivation of order s* on $R^p(X)$ if there exists a constant C such that $|f^{(s)}(x)| \leq C\|f\|_p$ for all $f \in R_0(X)$.

When x admits a bounded point derivation of order s on $R^p(X)$, the map $f \mapsto f^{(s)}(x)/s!$ extends from $R_0(X)$ to a bounded linear functional on $R^p(X)$.

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We denote this bounded linear functional by D_x^s.

DEFINITION (1.2). When x admits a bounded point derivation of order 0, we say that x admits a *bounded point evaluation.* For $f \in R^p(X)$ we define $f(x) = D_x^0 f$.

DEFINITION (1.3). For each $p \geqslant 2$ the *inner set* for $R^p(X)$ is the set of points in X which admit bounded point evaluations, and we denote it by $S^p(X)$.

PROPOSITION (1.1). *For each $p \geqslant 2$, $S^p(X)$ is an F_σ set.*

PROOF. Write $S^p(X) = \bigcup_{n=1}^{\infty} S_n^p(X)$ where

$$S_n^p(X) = \{x \in X \mid |f(x)| \leqslant n\|f\|_p \text{ for all } f \in R^p(X)\}.$$

We show that each set $S_n^p(X)$ is closed. Suppose that $\{x_k\} \subset S_n^p(X)$ and that $x_k \to x \in X$. Let $L_{x_k} f = f(x_k)$ and observe that the L_{x_k} are a family of linear functionals bounded in norm by n. Since $L_{x_k} f \to f(x)$ for $f \in R_0(X)$, and $R_0(X)$ is dense in $R^p(X)$, it follows that $x \in S_n^p(X)$. Thus each $S_n^p(X)$ is closed.

2. **Potentials and representing functions.** In this paper z will denote the identity function.

DEFINITION (2.1). Let ψ be a positive nondecreasing function on $(0, \infty)$. For each $g \in L^q(X)$, $q \geqslant 1$, we define the ψ-potential of g, U_g^ψ, by

$$U_g^\psi(y) = \int \frac{|g|}{\psi(|z - y|)}\, dm.$$

If $1/\psi(|z|)$ is locally summable with respect to m, Fubini's theorem implies that U_g^ψ is locally summable; hence $U_g^\psi < \infty$ a.e. (m).

DEFINITION (2.2). When $\psi(r) = r$, we denote U_g^ψ by $\tilde{g}$.

DEFINITION (2.3). When $\psi(r) = r^q$, $1 < q < 2$, we denote U_g^ψ by U_g^q.

DEFINITION (2.4). We define the Cauchy transform of g to be

$$\hat{g}(y) = \int (z - y)^{-1} g\, dm \quad \text{for all } y \text{ where } \hat{g}(y) < \infty.$$

For the proof of the following lemma we refer the reader to Sinanjan [16] or Brennan [1, pp. 10–11]. Brennan's proof uses the Cauchy transform.

LEMMA (2.1). *Let $X \subset \mathbf{C}$ be compact and have no interior. Then $R^p(X) = L^p(X)$ for $1 \leqslant p < 2$.*

It follows from the Riesz representation theorem that if $x \in S^p(X)$, then there is a $g \in L^q(X)$ such that $f(x) = \int fg\, dm$ for all $f \in R^p(X)$. We call such a g a *representing function* for x. If $R^p(X) \neq L^p(X)$, there is a nonzero function $g \in L^q(X)$ such that $\int fg\, dm = 0$ for all $f \in R^p(X)$. We call such a g an *annihilating function.*

The following lemma was proved by Bishop for the sup norm case: We assume that $1 < q < 2$.

LEMMA (2.2). *Let* $g \in L^q(X)$ *be an annihilating function. Suppose that* $\hat{g}(y)$ *is defined and* $\neq 0$, *and that* $(z-y)^{-1}g \in L^q(X)$. *Then* $\hat{g}(y)^{-1}(z-y)^{-1}g$ *is a representing function for* y.

PROOF. If $f \in R_0(X)$, then $f = f(y) + (z-y)h$ for some $h \in R_0(X)$. Hence

$$\int (z-y)^{-1} fg\, dm = f(y)\hat{g}(y) + \int hg\, dm = f(y)\hat{g}(y).$$

COROLLARY (2.1). *Let* $g \in L^q(X)$ *be a representing function for* x. *Let*

$$c(y) = \int (z-x)(z-y)^{-1} g\, dm = 1 + (y-x)\hat{g}(y).$$

Then $c(y)^{-1}(z-x)(z-y)^{-1}g$ *is a representing function for* y *whenever* $c(y)$ *is defined and* $\neq 0$.

PROOF. $(z-x)g$ is an annihilating function.

We now present a lemma of Brennan in [2, p. 288] which will be very useful.

LEMMA (2.3). *If* $p > 2$, *then* $R^p(X) \neq L^p(X)$ *if and only if* $S^p(X)$ *has positive* 2-*dimensional measure.*

PROOF. Suppose that $S^p(X) \neq \varnothing$ and $x \in S^p(X)$ is represented by a nonzero function $g \in L^q(X)$. Then $R^p(X) \neq L^p(X)$ because $(z-x)g \in L^q(X)$, and $\int (z-x) gf\, dm = 0$ for all $f \in R^p(X)$.

Now suppose that $R^p(X) \neq L^p(X)$ and let $g \in L^q(X)$ be a nonzero annihilating function. Then $\hat{g}$ fails to vanish on a set of positive measure in X. Hence there is a set $S \subset X$ of positive measure such that for $y \in S$, $\hat{g}(y) \neq 0$ and $\hat{g}(y)^{-1}(z-y)^{-1}g \in L^q(X)$. It follows from Corollary (2.1) that $S \subset S^p(X)$, and the lemma is proved.

REMARK. If we know that there is an $x \in S^2(X)$, the difficulty in showing that there are other points in $S^2(X)$ by the above method is that $z^{-1} \notin L^2_{\text{loc}}$.

3. **Admissible functions.** Fix $x \in \mathbf{C}$ and let $\Delta_n = \{y \in \mathbf{C}: |y-x| < 1/n\}$. We say that a set $E \subset C$ has *full area density* at x if $\lim_{n\to\infty} m(E \cap \Delta_n)/m(\Delta_n) = 1$. Let F be a function defined on X, $x \in X$. We say that a is the *approximate limit* of F at x, and write app $\lim_{y\to x} F(y) = a$ if there exists a subset E of X having full area density at x, such that $\lim_{y\to x; y\in E} F(y) = a$. We say that F is *approximately* continuous at x if app $\lim_{y\to x} F(y) = F(x)$.

If ϕ is a positive function on $(0, \infty)$ with $\lim_{r\to 0}\phi(r) = 0$, we say that F admits ϕ as a *modulus of approximate continuity* at x if $|F(y) - F(x)| \leq$

$\phi(|y-x|)$ for all y in a set having full area density at x. We say that F satisfies an approximate Hölder condition of order α at x if F admits Cr^{α} as a modulus of approximate continuity at x for some constant C.

DEFINITION (3.1). We say that ϕ is an admissible function if

(a) ϕ is a positive, nondecreasing function defined on $(0, \infty)$, and

(b) the associated function ψ, defined by $\psi(r) = r/\phi(r)$, is nondecreasing, with $\psi(0+) = 0$.

EXAMPLE. For any α, $0 \leq \alpha < 1$, $\phi(r) = r^{\alpha}$ is admissible.

REMARKS. 1. If ϕ is admissible and $0 \leq \beta \leq 1$, then ϕ^{β} is also admissible because $r/\phi^{\beta}(r) = (r/\phi(r)) \cdot \phi^{1-\beta}(r)$.

2. In using an admissible function ϕ we will often refer to the triangle inequality: $\phi(r) \leq \phi(r_1) + \phi(r_2)$ whenever $r \leq r_1 + r_2$. This follows from the definition of an admissible function since

$$\begin{aligned}\phi(r) &\leq \phi(r_1 + r_2) = (r_1 + r_2)/\psi(r_1 + r_2)\\ &\leq r_1/\psi(r_1) + r_2/\psi(r_2) = \phi(r_1) + \phi(r_2).\end{aligned}$$

Wang introduced a special kind of admissible function in [17, p. 349].

DEFINITION (3.2). We say that the admissible function ϕ is *nice* if $\int_0^1 \phi(r)^{-1}\,dr < \infty$.

For each q, $1 \leq q < 2$, we will be interested in a subset of the set of nice admissible functions.

DEFINITION (3.3). We say that the admissible function ϕ is q-nice if $\int_0^1 r^{1-q}\phi(r)^{-q}\,dr < \infty$.

Note that a nice admissible function is 1-nice and that $\phi(r) = r^{\alpha}$ is q-nice for $\alpha < (2-q)/q$. When $p > 2$, the q-nice admissible functions will be the most likely ones to be moduli of approximate continuity for functions in the unit ball of $R^p(X)$ at points in $S^p(X)$.

The following lemma is due to Wang [17]:

LEMMA (3.1). *Let* $g \in L^q(X)$, $q \geq 1$, *and let* $x \in X$. *Then there exists a nice admissible function* ϕ *with* $\phi(0+) = 0$ *such that* $\phi(|z-x|)^{-1}g \in L^q(X)$.

PROOF. See Wang [17].

Our proof of the next lemma is in the spirit of Browder's result [3, p. 157]. It will be useful for studying the density of X at points in $S^p(X)$. Let $E \subset X$ be measurable. Define ρ_n by $\pi\rho_n^2 = m(\Delta_n \setminus E)$. Denote $m|\Delta_n \setminus E$ by m_n.

LEMMA (3.2). *Let* ψ *be associated with an admissible* ϕ. *For* q, $0 < q < 2$, *let* $\tau = \psi^q$. *Then if* $g \in L^1(X)$,

$$\lim_{n\to\infty} \frac{n^q}{\rho_n^{2-q}} \int \tau(|y-x|)\, U_g^{\tau}(y)\, dm_n(y) = 0.$$

PROOF. Define

$$F_n(\zeta) = n^q \rho_n^{q-2} \int \psi(|y - x|)^q \cdot \psi(|\zeta - y|)^{-q}\, dm_n(y).$$

Then $F_n(x) < \infty$ and if $\zeta \neq x$, we have for large n

$$|F_n(\zeta)| \leqslant n^q \rho_n^q \psi(n^{-1})^q \cdot \psi(|x - \zeta| - n^{-1})^q \to 0 \quad \text{as } n \to \infty.$$

Next, we will show that the F_n are bounded independently of n. Let $D_n = \Delta(\zeta, \rho_n)$. Since ψ^q is increasing,

$$\begin{aligned}
|F_n(\zeta)| &\leqslant n^q \rho_n^{q-2} \psi(n^{-1})^q \int \psi(|y - \zeta|)^{-q}\, dm_n(y) \\
&\leqslant n^q \rho_n^{q-2} \psi(n^{-1})^q \int_{D_n} \psi(|y - \zeta|)^{-q}\, dm(y) \\
&= 2\pi n^q \rho_n^{q-2} \psi(n^{-1})^q \int_0^{\rho_n} \psi(r)^{-q} r\, dr \\
&\leqslant 2\pi n^q \rho_n^{q-2} \psi(n^{-1})^q \phi(\rho_n)^q \int_0^{\rho_n} r^{1-q}\, dr \\
&= 2\pi n^q \rho_n^{q-2} \psi(n^{-1})^q \phi(\rho_n)^q \rho_n^{2-q} (2 - q)^{-1} \\
&< 2\pi (2 - q)^{-1}.
\end{aligned}$$

Thus, the F_n converge boundedly a.e. to zero. We apply the dominated convergence theorem and Fubini's theorem to obtain the lemma.

LEMMA (3.3). *Let ψ be associated with an admissible ϕ. For $0 < q < 2$, let $\tau = \psi^q$. Then if $g \in L^1(X)$, and $\delta > 0$, the set $E = \{y \in \mathbf{C}: \tau(|y - x|) U_g^\tau(y) < \delta\}$ has full area density at x.*

PROOF. It is sufficient to prove that $\lim_{n\to\infty} m(\Delta_n \setminus E)/m(\Delta_n) = 0$ where $\Delta_n = \Delta(x, 1/n)$. We observe that since

$$m(\Delta_n \setminus E) \leqslant \delta^{-1} \int_{\Delta_n} \tau(|y - x|) U_g^\tau(y)\, dm(y),$$

it is sufficient to prove that

$$\lim_{n\to\infty} n^2 \int_{\Delta_n} \tau(|y - x|) U_g^\tau(y)\, dm(y) = 0.$$

This follows from Lemma (3.2) if we take E in that lemma to be the empty set.

4. **The main theorem.** The following lemma in the sup norm case is due to Wilken [20]. For $x \in S^p(X)$, $p > 2$, it gives a condition for x to admit a bounded point derivation of order s.

LEMMA (4.1). *Suppose there exist a representing function $g \in L^q(X)$ for*

$x \in S^p(X)$, $p > 2$, and a nonnegative integer s such that $(z-x)^{-s}g \in L^q(X)$. Let $c_j = \int (z-x)^{-j} g\, dm$ $(0 \leqslant j \leqslant s)$ and define $G_0, \ldots, G_s$ by:

$$G_0 = g, \qquad G_j = (z-x)^{-j}g - \sum_{k<j} c_{j-k} G_k.$$

Then D_x^j exists, and $D_x^j f = \int f G_j\, dm$ for all $f \in R^p(X)$, $0 \leqslant j \leqslant s$.

An additional lemma will be needed in proving the theorem.

LEMMA (4.2). *Let s be a nonnegative integer, and $g \in L^q(X)$, $1 \leqslant q < 2$. Suppose that $(z-x)^{-s}g \in L^q(X)$. Set $H_j = (z-x)^{-j}g$ $(0 \leqslant j \leqslant s)$. For any $f \in L^p(X)$ and $y \in \mathbf{C}$*

$$\int (z-y)^{-1} fg\, dm = \sum_{j=1}^{s} (y-x)^{j-1} \int f H_j\, dm + (y-x)^s \int (z-y)^{-1} f H_s\, dm.$$

PROOF. Since $H_j = (z-x)H_{j+1}$ for $0 \leqslant j \leqslant s$,

$$\int (z-y)^{-1} f H_j\, dm = \int f H_{j+1}\, dm + (y-x) \int (z-y)^{-1} f H_{j+1}\, dm$$

which implies the lemma.

Our main theorem generalizes the "approximate Taylor's theorem" which Wang obtained for functions in $R(X)$ [17, p. 352].

THEOREM (4.1). *Let ϕ be an admissible function and s a nonnegative integer. Suppose that $p > 2$ and that there is an $x \in S^p(X)$ represented by a $g \in L^q(X)$ such that $(z-x)^{-s}\phi(|z-x|)^{-1}g \in L^q(X)$. Then for every $\varepsilon > 0$ there is a set E in X having full area density at x such that for every $f \in R^p(X)$*

(i) *$f = \sum_{j=0}^{s} (D_x^j f)(z-x)^j + R$ where $R \in R^p(X)$ satisfies*

(ii) *$|R(y)| \leqslant \varepsilon |y-x|^s \phi(|y-x|) \|f\|_p$ for all $y \in E$, and*

(iii) *$\operatorname{app\,lim}_{y \to x} \{R(y)/|y-x|^s \phi(|y-x|)\} = 0$.*

PROOF. Since $(z-x)^{-s}g \in L^q(X)$, Lemma (4.1) implies that the D_x^j exist for $0 \leqslant j \leqslant s$. To each D_x^j, $0 \leqslant j \leqslant s$, there corresponds a constant C_j such that $|D_x^j f| \leqslant C_j \|f\|_p$ for all $f \in R^p(X)$. By Minkowski's inequality there is another constant C such that if R is defined as in (i), $\|R\|_p \leqslant C\|f\|_p$ for all $f \in R^p(X)$.

Choose $\delta > 0$ so that $0 < C\delta(1-\delta)^{-1} < \varepsilon/2$. If $y \in E_1 = \{y \in \mathbf{C}: |y - x|\tilde{g}(y) < \delta\}$, then $c(y) = 1 + (y-x)\hat{g}(y)$ is well defined, and $|c(y)| \geqslant 1 - \delta$. By Corollary (2.1),

$$R(y) = c(y)^{-1}\int [R(z-x)/(z-y)]\, g\, dm$$
$$= c(y)^{-1}\int R[1 + (y-x)/(z-y)]\, g\, dm$$
$$= c(y)^{-1}(y-x)\int [R/(z-y)]\, g\, dm.$$

Next, we claim that $R(y) = c(y)^{-1}(y-x)^{s+1}\int (z-x)^{-s}(z-y)^{-1}Rg\, dm$. This claim depends on Lemma (4.2). Each of the functions $(z-x)^{-j}g$, $0 \leqslant j \leqslant s$, is a linear combination of functions representing D_x^k, $0 \leqslant k \leqslant j$, which implies that $\int (z-x)^{-j}Rg\, dm = 0$ for $0 \leqslant j \leqslant s$, and the claim is proved.

Factoring $g = \phi(|z-x|)h$ where $h \in L^q(X)$, we obtain by the "triangle inequality" that

$$|g| \leqslant \phi(|z-y|)|h| + \phi(|y-x|)|h|.$$

Consequently,

$$|R(y)| \leqslant |c(y)|^{-1}|y-x|^{s+1}\Big[\int |z-y|^{-1}|z-x|^{-s}\phi(|z-y|)|Rh|\, dm$$
$$+ \int |z-y|^{-1}|z-x|^{-s}\phi(|y-x|)|Rh|\, dm\Big].$$

Denote the first integral by I_1 and the second by I_2. We have

$$I_1 = |c(y)|^{-1}|y-x|^s\phi(|y-x|)\psi(|y-x|)\int \psi(|z-y|)^{-1}|z-x|^{-s}|Rh|\, dm.$$

Let $\tau = \psi^q$, $k = (z-x)^{-sq}h^q$, and

$$E_2 = \{y \in \mathbf{C}: \tau(|y-x|)U_k^\tau(y) < \delta^q\}.$$

For $y \in E_2$ we apply Hölder's inequality to obtain

$$I_1 \leqslant (1-\delta)^{-1}|y-x|^s\phi(|y-x|)\tau(|y-x|)^{1/q}\left\{\int |R|^p\, dm\right\}^{1/p}\{U_k^\tau(y)\}^{1/q}$$
$$\leqslant (1-\delta)^{-1}|y-x|^s\phi(|y-x|)C\|f\|_p\delta$$
$$\leqslant (\varepsilon/2)|y-x|^s\phi(|y-x|)\|f\|_p.$$

To estimate I_2 we define

$$E_3 = \{y \in \mathbf{C}: |y-x|^q U_k^q(y) < \delta^q\} \quad \text{and let } y \in E_2 \cap E_3.$$

By Hölder's inequality,

$$I_2 \leqslant (1-\delta)^{-1}|y-x|^s\phi(|y-x|)|y-x|\int |z-y|^{-1}|z-x|^{-s}|Rh|\,dm$$

$$\leqslant (1-\delta)^{-1}|y-x|^s\phi(|y-x|)|y-x|\left\{\int |R|^p\,dm\right\}^{1/p}\{U_k^q(y)\}^{1/q}$$

$$\leqslant (1-\delta)^{-1}|y-x|^s\phi(|y-x|)C\|f\|_p\delta$$

$$\leqslant (\varepsilon/2)|y-x|^s\phi(|y-x|)\|f\|_p.$$

By Lemma (3.3) the set $E = E_2 \cap E_3$ has full area density at x, and we have proved that for $y \in E$

$$|R(y)| \leqslant I_1 + I_2 \leqslant \varepsilon|y-x|^s\phi(|y-x|)\|f\|_p$$

for any $f \in R^p(X)$. To prove (iii) let $L_y f = R(y)/|y-x|^s\phi(|y-x|)$. The above result implies that $\|L_y\| \leqslant \varepsilon$ for $y \in E$. Let $y \to x$ in such a way that y stays in E. Then $L_y f \to 0$ as $y \to x$ for $f \in R_0(X)$, and since $R_0(X)$ is dense in $R^p(X)$, (iii) follows.

An interesting consequence of the above theorem is that we can take the limit of Newton quotients in the set E to evaluate $D_x^1 f$. For f a function defined on a subset of X, $h \in \mathbf{C}$, we set

$$\Delta_h f = f(z+h) - f$$

so $\Delta_h f$ is a function defined on a subset of X. We define inductively $\Delta_h^0 = \mathrm{id}$, $\Delta_h^j = \Delta_h \circ \Delta^{j-1}$ for $j \geqslant 1$. The sup norm version of the following corollary is proved in [17].

COROLLARY (4.1). *If x admits a bounded point derivation of order s on $R^p(X), p > 2$, then for all $f \in R^p(X)$*

$$D_x^s f = \operatorname*{app\,lim}_{h\to 0} \frac{\Delta_h^s f(x)}{s!\,h^s}.$$

LEMMA (4.3). *Let ϕ be a q-nice admissible function. If $x \in S^p(X)$, $p > 2$, then $\{y \in X$: $\exists$ a function g_y that represents y for $R^p(X)$ and satisfies $\phi(|z-y|)^{-1}g_y \in L^q(X)\}$ has full area density at x.*

PROOF. Let $g \in L^q(X)$ represent x.
Let

$$F = \left\{y \in \mathbf{C}\colon \int |z-y|^{-q}\phi(|z-y|)^{-q}|g|^q\,dm < \infty\right\}.$$

Since $|z|^{-q}\phi(|z|)^{-q}$ is locally summable with respect to m, $m(\mathbf{C}\setminus F) = 0$. Fix δ, $0 < \delta < 1$, and put $E = F \cap E_1$ where $E_1 = \{y \in \mathbf{C}\colon |y-x|\tilde{g}(y) < \delta\}$. By Lemma (3.3) the set E has full area density at x. For each $y \in E$ the function $g_y = c(y)^{-1}[(z-x)/(z-y)]g$ represents y. Moreover,

$$\int \phi(|z-y|)^{-q}|g_y|^q\,dm = |c(y)|^{-q}\int |z-y|^{-q}\phi(|z-y|)^{-q}|z-x|^q|g|^q\,dm$$

$$\leqslant C\int |z-y|^{-q}\phi(|z-y|)^{-q}|g|^q\,dm < \infty.$$

This proves the lemma.

COROLLARY (4.2). *Suppose that ϕ is q-nice. Then at almost every point of $S^p(X)$, $p>2$, the functions in the unit ball of $R^p(X)$ admit ϕ as a modulus of approximate continuity.*

PROOF. Combine Theorem (4.1) with Lemma (4.3).

In particular, it follows that at a.e. $x \in S^p(X)$, $p > 2$, the unit ball of $R^p(X)$ satisfies an approximate uniform Hölder condition of order α for every $\alpha < (2-q)/q$.

LEMMA (4.4). *Let ϕ be admissible and $g \in L^q(X)$, $1 \leqslant q < 2$. Then if $\phi(|z-x|)^{-1}g \in L^q(X)$, $\delta > 0$, and*

$$E = \left\{ y \in \mathbf{C}\colon |y-x|^q \int |y-z|^{-q}|g|^q\,dm < \delta \right\},$$

it follows that $m(\Delta_n \setminus E) = o(\phi(n^{-1})^2/n^2)$.

PROOF. We observe that

$$m(\Delta_n \setminus E) \leqslant \delta^{-1}\int |y-x|^q \int |z-y|^{-q}|g|^q\,dm\,dm_n(y).$$

Factor $g = \phi(|z-x|)h$ where $h \in L^q(X)$. Then

$$|g|^q \leqslant C\left[\phi(|z-y|)^q|h|^q + \phi(|y-x|)^q|h|^q\right]$$

where C is some constant. We have

$$m(\Delta_n \setminus E) \leqslant \delta^{-1}C\left[\int |y-x|^q \int |z-y|^{-q}\phi(|z-y|)^q|h|^q\,dm\,dm_n(y)\right.$$

$$\left. + \int |y-x|^q \int |z-y|^{-q}\phi(|y-x|)^q|h|^q\,dm\,dm_n(y)\right].$$

By substituting $|y-x|^q = \phi(|y-x|)^q\psi(|y-x|)^q$ in the first integral, and using the fact that $\phi(|y-x|)^q \leqslant \phi(n^{-1})^q$ for $y \in \Delta_n$, we obtain

$$m(\Delta_n \setminus E) \leqslant \delta^{-1}C\phi(n^{-1})^q\left[\psi(|y-x|)^q \int \psi(|z-y|)^{-q}|h|^q\,dm\,dm_n(y)\right.$$

$$\left. + \int |y-x|^q \int |z-y|^{-q}|h|\,dm\,dm_n(y)\right].$$

Let A_n denote the sum of the two integrals on the right. Replacing $m(\Delta_n \setminus E)$ by $\pi\rho_n^2$, we obtain

$$\pi\rho_n^2 \leqslant \delta^{-1}C\phi(n^{-1})^q\rho_n^{2-q}n^{-q}(A_n)$$

where $\lim_{n\to\infty} A_n = 0$ by Lemma (3.2). Divide both sides by ρ_n^{2-q} to get

$$\pi\rho_n^q \leqslant \delta^{-1}C\phi(n^{-1})^q n^{-q}(A_n).$$

Now raise both sides to the power $2/q$, and the conclusion of the lemma follows.

In the next corollary we consider functions $f \in R^p(X)$ to be defined on $\mathbf{C}$ by setting $f(x) = 0$ for $x \notin X$.

Corollary (4.3). *Let* $\varepsilon > 0$. *If* $x \in S^p(X)$, $p > 2$, *is represented by* $g \in L^q(X)$, *and* $(z - x)^{-\alpha}g \in L^q(X)$ *for some* $\alpha > q - 1$, *then there is an integer* N_x *depending on* x *such that for* $n > N_x$

$$m(\Delta_n)^{-1}\int_{\Delta_n}|f - f(x)|\, dm \leqslant \varepsilon\|f\|_p \quad \textit{for all } f \in R^p(X).$$

Proof. Let E be the set in the conclusion of Theorem (4.1) when $\varepsilon/2$ and $x \in S^p(X)$ are given and $\phi(r) \equiv 1$.

$$\begin{aligned} m(\Delta_n)^{-1}&\int_{\Delta_n}|f - f(x)|\, dm \\ &\leqslant m(\Delta_n)^{-1}\left[\int_{\Delta_n\cap E}|f - f(x)|\, dm + \int_{\Delta_n\setminus E}|f - f(x)|\, dm\right] \\ &\leqslant (\varepsilon/2)\|f\|_p m(\Delta_n)^{-1}m(\Delta_n \cap E) + \pi^{-1}n^2\int_{\Delta_n\setminus E}|f - f(x)|\, dm \\ &\leqslant (\varepsilon/2)\|f\|_p + \pi^{-1}n^2\int_{\Delta_n\setminus E}|f - f(x)|\, dm. \end{aligned}$$

Let $\chi_{\Delta_n\setminus E}$ be the characteristic function of $\Delta_n \setminus E$. Then by Hölder's inequality,

$$\begin{aligned} \pi^{-1}n^2\int_{\Delta_n\setminus E}|f - f(x)|\, dm &= \pi^{-1}n^2\int \chi_{\Delta_n\setminus E}|f - f(x)|\, dm \\ &\leqslant Cn^2\left[m(\Delta_n \setminus E)\right]^{1/q}\|f\|_{L^p(\Delta_n\setminus E)} \end{aligned}$$

where C is a constant. By Lemma (4.4)

$$\left[m(\Delta_n \setminus E)\right]^{1/q} = o(n^{-(2/q)-(2\alpha/q)}).$$

Thus if $\alpha > q - 1$, we can choose an integer N_x so that $n > N_x$ implies that $Cn^2[m(\Delta_n \setminus E)]^{1/q} \leqslant \varepsilon/2$. Hence,

$$\begin{aligned} m(\Delta_n)^{-1}\int_{\Delta_n}|f - f(x)|\, dm &\leqslant (\varepsilon/2)\|f\|_p + (\varepsilon/2)\|f\|_{L^p(\Delta_n\setminus E)} \\ &\leqslant \varepsilon\|f\|_p. \end{aligned}$$

This completes the proof.

COROLLARY (4.4). *If* $p > 2 + \sqrt{2}$, *then for a.e.* $x \in S^p(X)$,

$$\lim_{n\to\infty} m(\Delta_n)^{-1}\int_{\Delta_n} |f - f(x)|\, dm = 0 \quad \textit{for any } f \in R^p(X).$$

PROOF. This follows from Lemma (4.3) and Corollary (4.3).

Given $f \in L^1(dm)$, the set of points $x \in \mathbf{C}$ such that

$$\lim_{n\to\infty} m(\Delta_n)^{-1}\int_{\Delta_n} |f - f(x)|\, dm = 0$$

is called the Lebesgue set of f. For an arbitrary $f \in L^1(dm)$, a.e. (m) point $x \in \mathbf{C}$ belongs to the Lebesgue set of f (see [5, p. 156]). The above corollary identifies points belonging to the Lebesgue sets of all $f \in R^p(X)$. It would be interesting to know whether the corollary holds for $p > 2$.

PART II. CAPACITY AND BOUNDED POINT EVALUATIONS

1. **Capacity theorems.** Before proving a capacity result about bounded point evaluations, we will need two lemmas of Hedberg [9]. Let Ω denote the complex plane when $p > 2$ and the unit disk when $p = 2$.

DEFINITION (1.1). Let $X \subset \Omega$ be a compact set. Then

$$\Gamma_q(X) = \inf_{\omega} \int |\text{grad }\omega|^q\, dm$$

where the inf is taken over Lipschitz functions ω with compact support contained in Ω such that $\omega(z) \geqslant 1$ on X.

For noncompact sets F, q-capacity is defined by $\Gamma_q(F) = \sup_{K\subset F}\Gamma_q(K)$, K compact.

Let U be an open set (bounded if $p = 2$) in the complex plane and denote by $L_a^p(U)$ the space of analytic functions in $L^p(U)$. If f is analytic in $\Omega \setminus X$ where $X \subset \Omega$ is compact, we write $\alpha(f) = (2\pi i)^{-1}\int_C f(z)\, dz$ where C is any Jordan curve in Ω enclosing X.

LEMMA (1.1). *Let* $X \subset \Omega$ *be compact. Then there are positive constants* C_1 *and* C_2, *depending only on* p, *such that*

$$C_1\Gamma_q(X)^{1/q} \leqslant \sup_f |\alpha(f)| \leqslant C_2\Gamma_q(X)^{1/q}$$

where the sup *is taken over functions* f *in* $L_a^p(\Omega)$, $2 \leqslant p < \infty$, *with* $\int_{\Omega\setminus X}|f(z)|^p\, dm \leqslant 1$.

We denote the annulus $\{z\colon 2^{-n-1} \leqslant |z - x| \leqslant 2^{-n}\}$ by $A_n(X)$. We write $A_n = A_n(0)$.

LEMMA (1.2). *Let* $X \subset \Omega$ *be compact. There is a constant* C, *depending only on* p, *such that for* $z \notin A_{n-1} \cup A_n \cup A_{n+1}$

$$|f(z)| \leqslant \frac{C\Gamma_q(A_n \setminus X)^{1/q}}{||z| - 2^{-n}|} \|f\|_{\Omega\setminus X,p}$$

for f analytic outside $A_n \setminus X$, $f(\infty) = 0$ and $\int_{\Omega\setminus X}|f(z)|^p\, dm < \infty$.

The following theorem was proved in the sup norm case by Wang [18, p. 223]. Wang essentially followed O'Farrell [13], who elaborated on a method of Gamelin [7, p. 206]. We assume that $x = 0$ and that $0 \in \partial X$.

THEOREM (1.1). *Let ϕ be an admissible function and s a nonnegative integer. Suppose that there is a function $v \in L^q(X)$ which represents 0 for $R^p(X)$ such that $|z|^{-s}\phi(|z|)^{-1}v \in L^q(X)$. Then*

$$\sum_1^\infty 2^{q(s+1)n}\phi(2^{-n})^{-q}\Gamma_q(A_n \setminus X) < \infty.$$

PROOF. Suppose that

$$\sum_1^\infty 2^{q(s+1)n}\phi(2^{-n})^{-q}\Gamma_q(A_n \setminus X) = \infty.$$

We will show that this leads to a contradiction. We may assume that for each n

$$2^{q(s+1)n}\pi(2^{-n})^{-q}\Gamma_q(A_n \setminus X) \leqslant 1.$$

If not, choose Y_n compact, $Y_n \subset A_n$ such that

$$\tfrac{1}{2} \leqslant 2^{q(s+1)n}\phi(2^{-n})^{-q}\Gamma_q(A_n \setminus X \cup Y_n) \leqslant 1,$$

and set $Y = \overline{\bigcup Y_n} \cup X$. Then define $v^*(z) = v(z)$ for $z \in X$ and $v^*(z) = 0$ for $z \in Y \setminus X$. Clearly, $|z|^{-s}\phi(|z|)^{-1}v^* \in L^q(Y)$ and v^* represents 0 for $R^p(Y)$.

Now choose integers $M_1 \leqslant N_1 < M_2 \leqslant N_2 < \cdots$ so that

$$1 \leqslant \sum_{n=M_j}^{N_j} 2^{q(s+1)n}\phi(2^{-n})^{-q}\Gamma_q(A_n \setminus X) \leqslant 2.$$

For each n we choose by Lemma (1.1) compact sets $K_n \subset A_n \setminus X$ and functions $f_n \in L_a^p(\Omega \setminus K_n)$ so that:

(i) $$|\alpha(f_n)| \geqslant C_1 2^{-1}\Gamma_q(A_n \setminus X)^{1/q}\left\{\int_{\Omega\setminus K_n}|f_n(z)|^p\, dm\right\}^{1/p}$$

(ii) $$= C_1 2^{-1}\Gamma_q(A_n \setminus X)^{1/q}\|f_n\|_{\Omega\setminus K_n,p},$$

$$f_n = 0 \quad \text{on } K_n \text{ and}$$

(iii) $$\|f_n\|_{\Omega,p} = 2^{q(s+1)n}\phi(2^{-n})^{-q}\Gamma_q(A_n \setminus X)^{1/p}.$$

Let $g_j(z) = \phi(|z|)z^{s+1}\sum_{n=M_j}^{N_j} f_n(z)$. We will show that $\|g_j\|_{X,p} \leqslant C$ for all j. In the following discussion C will denote any constant that is independent of n and j. Lemma (II.1.2) implies that for $z \in A_k$, $k < n-1$,

$$|f_n(z)| \leqslant C2^{q(s+1)n+k}\phi(2^{-n})^{-q}\Gamma_q(A_n \setminus X),$$

and for $z \in A_k$, $k > n+1$,

$$|f_n(z)| \leqslant C2^{q(s+1)n+n}\phi(2^{-n})^{-q}\Gamma_q(A_n \setminus X).$$

We may assume that $X \subset \{|z| \leqslant 1\}$. Then for $z \in A_k \cap X$, $k < n-1$,

$$\phi(|z|)|z|^{s+1}|f_n(z)| \leqslant C2^{q(s+1)n}\phi(2^{-n})^{-q}\Gamma_q(A_n \setminus X).$$

For $z \in A_k$, $k > n+1$,

$$\begin{aligned}\phi(|z|)|z|^{s+1}|f_n(z)| &\leqslant C2^{q(s+1)n+n-(s+1)n}\phi(2^{-n})^{1-q}\Gamma_q(A_n \setminus X)\\ &\leqslant C2^{q(s+1)n}\phi(2^{-n})\Gamma_q(A_n \setminus X).\end{aligned}$$

Now

$$\begin{aligned}\int_X |g_j(z)|^p\, dm &= \sum_{k=0}^{\infty}\int_{A_k\cap X}\left|\sum_{n=M_j}^{N_j}\phi(|z|)z^{s+1}f_n(z)\right|^p dm\\ &\leqslant C\sum_{k=0}^{\infty}\int_{A_k\cap X}\left\{\left[\sum_{n=M_j;n\neq k-1,k,k+1}^{N_j}\phi(|z|)|z|^{s+1}|f_n(z)|\right]^p\right.\\ &\qquad\left.+\sum_{n=k-1}^{k+1}\left(\phi(|z|)|z|^{s+1}|f_n(z)|\right)^p\right\} dm.\end{aligned}$$

By the above estimates and the choice of M_j, N_j, we have for $z \in A_k$

$$\sum_{n=\max(k+2,M_j)}^{N_j}\phi(|z|)|z|^{s+1}|f_n(z)| \leqslant C\sum_{n=M_j}^{N_j}2^{q(s+1)n}\phi(2^{-n})^{-q}\Gamma_q(A_n \setminus X) \leqslant C.$$

Similarly,

$$\sum_{n=M_j}^{\min(k-2,N_j)}\phi(|z|)|z|^{s+1}|f_n(z)| \leqslant C\sum_{n=M_j}^{N_j}2^{q(s+1)n}\phi(2^{-n})^{-q}\Gamma_q(A_n \setminus X) \leqslant C.$$

Thus

$$\sum_{k=0}^{\infty}\int_{A_k\cap X}\left[\sum_{n=M_j;n\neq k-1,k,k+1}^{N_j}\phi(|z|)|z|^{s+1}|f_n(z)|\right]^p dm \leqslant C.$$

Next, we estimate

$$\sum_{k=0}^{\infty} \int_{A_k \cap X} \sum_{n=k-1}^{k+1} \left(\phi(|z|)|z|^{s+1}|f_n(z)|\right)^p \, dm.$$

For each k,

$$\begin{aligned} \int_{A_k \cap X} &\left(\phi(|z|)|z|^{s+1}|f_{k-1}(z)|\right)^p \, dm \\ &\leqslant C\left(\phi(2^{-k+1})^p 2^{-p(k-1)} \|f_{k-1}\|_{X,p}^p\right) \\ &\leqslant C\phi(2^{-k+1})^{p-pq} 2^{(k-1)[-p+pq(s+1)]} \Gamma_q(A_{k-1} \setminus X) \\ &\leqslant C 2^{q(s+1)(k-1)} \phi(2^{-k+1})^{-q} \Gamma_q(A_{k-1} \setminus X) \end{aligned}$$

and similarly for f_k and f_{k+1}. Thus

$$\begin{aligned} \sum_{k=0}^{\infty} \int_{A_k \cap X} \sum_{n=k-1}^{k+1} &\left(\phi(|z|)|z|^{s+1}|f_n(z)|\right)^p \, dm \\ &\leqslant C \sum_{k=M_j}^{N_j} 2^{q(s+1)k} \phi(2^{-k})^{-q} \Gamma_q(A_k \setminus X) \\ &\leqslant C \quad \text{by choice of } M_j \text{ and } N_j. \end{aligned}$$

Combining the above estimates, we obtain

$$\int_X |g_j|^p \, dm \leqslant C \quad \text{for all } j.$$

Next we pass to a subsequence of the $\{g_j\}$ that converges weakly to $g \in L^p(X)$. Denote the subsequence also by $\{g_j\}$. We form $h_j(z) = z\phi(|z|)^{-1}g_j(z)$ and $F_j(z) = z^{-s-1}h_j(z)$, which are analytic in $\mathbf{C} \setminus \Delta(0, 2^{-M_j})$. By the above estimates the functions h_j and F_j are uniformly bounded on compact subsets of $\mathbf{C} \setminus \{0\}$. Hence, there are subsequences that converge uniformly on compact subsets of $\mathbf{C} \setminus \{0\}$ to $h(z) = z\phi(|z|)^{-1}g(z)$ and $F(z) = z^{-s-1}h(z)$ respectively.

We claim that h is a polynomial of degree $s + 1$ with $h(0) = 0$. The above estimates show that there is a number $M > 0$ that bounds the h_j in the following sense: to any $z \in \Delta(0, 1) \setminus \{0\}$ there corresponds an integer J such that for $j > J$ and $|\zeta| \geqslant |z|$, $|h_j(\zeta)| < M$. This implies that h is bounded near 0, so h is entire and $\lim_{z \to 0} h(z) = 0$. To show that h is a polynomial we consider

$$\lim_{z \to \infty} z^{-s-1}h(z) = F(\infty) = \lim_{j \to \infty} F_j(\infty).$$

For all j, $F_j(\infty) = \sum_{n=M_j}^{N_j} f_n(\infty)$ lies in $[C_1/2, 3C_2]$ where C_1 and C_2 are the constants of Lemma (1.1). Therefore, we have that $\lim_{j \to \infty} F_j(\infty) = \beta \in [C_1, 2C_2]$, and

$$h(z) = \beta z^{s+1} + \sum_1^s \beta_i z^i \quad \text{where } \beta_i \text{ is a constant for each } i.$$

Thus

$$g_j = \phi(|z|)z^{-1}h_j \to \phi(|z|)z^{-1}h = \beta\phi(|z|)z^s + \sum_1^s \beta_i\phi(|z|)z^{i-1}$$

weakly and pointwise on each bounded subset of $\mathbf{C} \setminus \{0\}$.

This means that if $u \in L^q(X)$, then

$$\int g_j u\, dm \to \int \beta\phi(|z|)z^s u\, dm + \sum_1^s \beta_i \int \phi(|z|)z^{i-1}u\, dm.$$

Wilkin's lemma (Lemma (I.4.1)) and the original hypothesis imply that there is a function $v_s \in L^q(X)$ which is a linear combination of the functions $z^{-j}v$, $0 \leqslant j \leqslant s$, such that

$$\int f v_s\, dm = \frac{f^{(s)}(0)}{s!}$$

for all $f \in R_0(X)$. Taking $u = \phi(|z|)^{-1}v_s$, we get a contradiction.

The next theorem may be proved in a similar way, and we omit many of the details.

THEOREM (1.2). *Let ϕ be an admissible function and s a nonnegative integer. Suppose that there is a function $v \in L^q(X)$ representing* 0 *for $R^p(X)$ such that $|z|^{-s}\phi(|z|)^{-1}v \in L^q(X)$. Then*

$$\lim_{r\to 0} r^{-qs-q}\phi(r)^{-q}\Gamma_q(\Delta(0,r) \setminus X) = 0.$$

PROOF. Suppose that there is a sequence $r_n \to 0$ and a $b > 0$ such that

$$r_n^{-qs-q}\phi(r_n)^{-q}\Gamma_q(\Delta(0,r_n) \setminus X) > b \quad \text{for all } r_n.$$

We may assume as before that

$$2^{q(s+1)n}\phi(2^{-n})^{-q}\Gamma_q(A_n \setminus X) \leqslant 1 \quad \text{for all } n.$$

Note that if $2^{-k} > r_n$, and $|2^{-k} - r_n| < 2^{-k-1}$,

$$2^{q(s+1)} \sum_{n=k}^{\infty} 2^{q(s+1)n}\phi(2^{-n})^{-q}\Gamma_q(A_n \setminus X) > b.$$

Thus there is a sequence of integers $M_1 \leqslant N_1 < M_2 \leqslant N_2 < \cdots$ such that

$$2 > \sum_{n=M_j}^{N_j} 2^{q(s+1)n}\phi(2^{-n})^{-q}\Gamma_q(A_n \setminus X) > 2^{-q(s+1)}b$$

for all j. The proof then proceeds as before.

2. **Density at bounded point evaluations.** We will get an estimate for Γ_q capacity in terms of the measure m. The following lemma is in [4].

LEMMA (2.1). *Let μ be a measure of total mass* 1 (*i.e.* $\int d\mu = 1$). *If* $1 < q < 2$ *and* $p = q/(q-1)$, *then*

$$\int_{\mathbf{C}}\left\{\int|\zeta - z|^{-1}\,d\mu(\zeta)\right\}^{p} dm \leqslant C\left\{\sup_{z\in\mathbf{C}}\int|\zeta - z|^{q-2}\,d\mu(\zeta)\right\}^{p-1}$$

where C is some constant depending only on p.

LEMMA (2.2). *For each q,* $1 < q < 2$, *there is a positive constant C such that*

$$\Gamma_q(X) \geqslant Cm(X)^{(2-q)/2}$$

for all compact sets $X \subset \mathbf{C}$.

PROOF. Define $f = m(X)^{-1}\int_X(z-\zeta)^{-1}\,dm(\zeta)$. Then f is analytic in $\mathbf{C}\setminus X$ and $f'(\infty) = 1$. To estimate $\|f\|_{p,\mathbf{C}\setminus X}$ we apply Lemma (II.2.1) with $\mu = m(X)^{-1}\chi_X$ where χ_X is the characteristic function of X. We get

$$\|f\|_{\mathbf{C}\setminus X,p} \leqslant C\left\{\sup_{z\in\mathbf{C}} m(X)^{-1}\int_X|z-\zeta|^{q-2}\,dm(\zeta)\right\}^{1/q}$$

We will use C to denote any constant depending only on p. Choose $R > 0$ so that $R^2 = m(X)$, and let $D = \Delta(\zeta, R)$. Then since r^{q-2} is a decreasing function of r,

$$\begin{aligned} m(X)^{-1}\int_X|z-\zeta|^{q-2}\,dm(\zeta) &\leqslant \pi^{-1}R^{-2}\int_D|z-\zeta|^{q-2}\,dm(\zeta) \\ &= \pi^{-1}R^{-2}\int_0^{2\pi}\int_0^R r^{q-2}r\,dr\,d\theta \\ &= 2R^{-2}\int_0^R r^{q-1}\,dr \\ &= 2(q-1)^{-1}R^{-2}R^q = 2(q-1)^{-1}R^{q-2}. \end{aligned}$$

Applying the above inequality for $\|f\|_{\mathbf{C}\setminus X,p}$, we have

$$\|f\|_{\mathbf{C}\setminus X,p} \leqslant CR^{(q-2)/q}.$$

Define $g = f/\|f\|_{\mathbf{C}\setminus X,p}$. Then g is analytic in $\mathbf{C}\setminus X$ and $\|g\|_{\mathbf{C}\setminus X,p} = 1$. Moreover,

$$g'(\infty) = f'(\infty)/\|f\|_{\mathbf{C}\setminus X,p} \geqslant CR^{(2-q)/q} \geqslant Cm(X)^{(2-q)/2q}.$$

By Lemma (II.1.1) we conclude that

$$\Gamma_q(X) \geqslant Cm(X)^{(2-q)/2},$$

and the proof is complete.

COROLLARY (2.1). *Let ϕ be an admissible function and s a nonnegative integer. Suppose that there is a function* $v \in L^q(X)$ *representing* 0 *for* $R^p(X)$,

$p > 2$, *such that* $|z|^{-s}\phi(|z|)^{-1}v \in L^q(X)$. *Then*

$$m\big(\Delta(0, n^{-1}) \setminus X\big) = o\Big(\phi(n^{-1})^{2t}(n^{-1})^{2t(s+1)}\Big), \quad \textit{where } t = q/(2-q).$$

PROOF. This follows from Theorem (II.1.2) and Lemma (II.2.2).

3. **An example.** In this section we use Hedberg's capacity theorems to construct a Swiss cheese Y such that $\bigcap_{p>2}^{\infty} S^p(Y) = \{0\}$. Let X_0 be the closure of a set having positive measure whose boundary consists of finitely many analytic curves. The first step is to show that for a given $\varepsilon > 0$ and $p > 2$ one can construct a Swiss cheese $X = X_0 \setminus \bigcup_{i=1}^{\infty} D_i$ such that:

(1) $\sum_{i=1}^{\infty} r_i^{2-q} < \varepsilon$ where r_i is the radius of D_i; and

(2) for some p', $p > p' > 2$, $S^{p'}(X) = \varnothing$. For $n = 1, 2, \ldots$ we define X_n inductively by letting $X_n = X_{n-1} \setminus G_n$ where $G_n = \bigcup\{\Delta(t2^{-n}, (\varepsilon 2^{-n})^{3/(2-q)})$, where the summation is taken over all Gaussian integers t such that $|t2^{-n}| \leqslant 1\}$. Then set $X = \bigcap_{n=0}^{\infty} X_n$. Since each G_n consists of $\leqslant 2^{2n}$ disks

$$\sum_{i=1}^{\infty} r_i^{2-q} < \sum_{i=1}^{\infty} 2^{2i}\Big[(\varepsilon 2^{-i})^{3/(2-q)}\Big]^{2-q} = \varepsilon.$$

Now choose q', $q < q' < 2$, so that $3(2-q')/(2-q) \leqslant q'$. Let $x \in X$. We claim that $x \notin S^{p'}(X)$ where $1/p' + 1/q' = 1$. Within any disk centered at x and having radius 2^{-n}, there is a disk in $\mathbf{C} \setminus X$ having radius at least $4^{-1}(\varepsilon 2^{-n})^{3/(2-q)}$. Hence

$$\lim_{n\to\infty} 2^{nq'}\Gamma_{q'}(\Delta(x, 2^{-n}) \setminus X) \geqslant 4^{q'-2} \cdot \lim_{n\to\infty} 2^{nq'}(\varepsilon 2^{-n})^{3(2-q')/(2-q)} > 0.$$

Thus by Theorem (II.1.2), $x \notin S^{q'}(X)$, and X is the desired set.

Given $\varepsilon_j \downarrow 0$ and $p_j \downarrow 2$, it is possible by the above construction to remove open disks D_{jk} of radius r_{jk} from $A_j(0)$ to obtain a Swiss cheese Y_j such that $\sum_{k=1}^{\infty} r_{jk}^{2-q_j} < \varepsilon_j$ $(1/p_j + 1/q_j = 1)$, and $S^{p_j'}(Y_j) = \varnothing$ for some p_j', $p_j > p_j' > 2$. Choose the ε_j so that $\sum_{j=1}^{\infty} 2^{2j}\varepsilon_j < \infty$, and define $Y = \bigcup_{j=0}^{\infty} Y_j \cup \{0\}$.

We will use Hedberg's theorem [9] to prove that for any $p > 2$, $0 \in S^p(Y)$. Let $p > 2$. There is an integer J such that $p > p_j > 2$ for $j \geqslant J$. Hence,

$$\sum_{j=J}^{\infty} 2^{jq}\Gamma_q(A_j(0) \setminus X) < C\sum_{j=J}^{\infty} 2^{jq}\sum_{k=1}^{\infty} r_{jk}^{2-q} < C\sum_{j=J}^{\infty} 2^{jq}\varepsilon_j < \infty.$$

By Hedberg's theorem $0 \in S^p(Y)$, and since $p > 2$ was arbitrary, $0 \in \bigcap_{p>2} S^p(Y)$. That 0 is the only point in $\bigcap_{p>2} S^p(Y)$ follows from the construction of Y and the fact that $x \in S^p(Y)$ if and only if $x \in S^p(Y \cap \overline{\Delta(x, r)})$ for any $r > 0$.

Given any compact set X it would be interesting to find necessary and sufficient conditions for $\bigcap_{p>2} S^p(X)$ to have positive measure. Lemma (I.2.3) implies that a sufficient condition is that there exist a single g which represents 0 for $R^p(X)$ for all $p > 2$.

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