

HEISENBERG MANIFOLDS AND THETA FUNCTIONS

BY

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ABSTRACT. The algebraic structure of the $(2n + 1)$ -dimensional Heisenberg group naturally induces a special class of differential operators whose solutions ($Df = 0$) are related to classical theta function theory.

1. Algebra. This section is purely algebraic. Its principal goal is to understand some of the algebraic concepts which lie at the heart of classical theta function theory. Firstly, we will recall the definition of the Heisenberg group and describe the structure of its automorphism group.

For an integer $n > 0$, let J_{2n} be the $2n \times 2n$ -matrix

$$(1) \quad J_{2n} = \begin{pmatrix} 0 & E_n \\ -E_n & 0 \end{pmatrix}$$

where E_n is the $n \times n$ -identity matrix. Associated to J_{2n} is the nondegenerate, alternating, R -bilinear form on R^{2n} defined by

$$(2) \quad J_{2n}(\xi, \eta) = \eta' \cdot J_{2n} \cdot \xi, \quad \eta, \xi \in R^{2n}.$$

The *Heisenberg group* N_n is defined as the $(2n + 1)$ -dimensional Lie group whose underlying manifold is $R^{2n} \times R$ and whose group multiplication is defined by the rule

$$(3) \quad (\xi, z)(\eta, w) = \left(\xi + \eta, z + w + \frac{1}{2}J_{2n}(\xi, \eta) \right)$$

where $\xi, \eta \in R^{2n}$, $z, w \in R$. In general, we denote a typical element $v \in N_n$ by $v = (\xi, z)$ with $\xi \in R^{2n}$, $z \in R$. It is straightforward to verify that N_n is a 2-step nilpotent, simply connected, analytic group having 1-dimensional center $Z = \{(0, z): z \in R\}$. The converse statement that every $(2n + 1)$ -dimensional, 2-step nilpotent, simply connected, analytic group having 1-dimensional center is isomorphic to N_n is deeper but also standard. Essentially, it is equivalent to the linear algebra result that for any real, alternating, nonsingular $2n \times 2n$ -matrix A we can write

$$(4) \quad A = G'J_{2n}G, \quad G \in \text{GL}(2n, R).$$

For a proof, see Lang [4, p. 371].

Let \mathcal{Q}_n denote the *automorphism group* of N_n . The 1-parameter subgroups of

Received by the editors May 5, 1976.

AMS (MOS) subject classifications (1970). Primary 22E25, 33A75; Secondary 58G99.

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N_n are all of the form $t \mapsto (t\xi, tz): R \rightarrow N_n$ for fixed $(\xi, z) \in N_n$ and hence are straight lines in the underlying manifold. It follows that every $\alpha \in \mathcal{Q}_n$ defines a linear transformation on this underlying manifold and hence can be described by a real $(2n + 1) \times (2n + 1)$ -matrix. Moreover, since Z is fixed by α we can write

$$(5a) \quad \alpha = \begin{pmatrix} S & 0 \\ \delta' & d \end{pmatrix}$$

where S is a real $2n \times 2n$ -matrix, $\delta' \in R^{2n}$ and $d \neq 0$ in R . Hence, $\alpha(\xi, z) = (S\xi, dz + \delta' \cdot \xi)$. The condition for $\alpha \in \mathcal{Q}_n$ is equivalent to the requirement

$$(5b) \quad J_{2n}(S\xi, S\eta) = S'J_{2n}S(\xi, \eta) = d \cdot J_{2n}(\xi, \eta),$$

for all $\xi, \eta \in R^{2n}$. In matrix language this becomes

$$(5c) \quad S'J_{2n}S = dJ_{2n}.$$

Hence, \mathcal{Q}_n consists of all matrices of the form given in (5a), satisfying (5c). Also, the identity component \mathcal{Q}_n^0 of \mathcal{Q}_n consists of all $\alpha \in \mathcal{Q}_n$ with $d > 0$.

We will now describe \mathcal{Q}_n^0 in greater detail. The symplectic group $\text{Sp}(2n, R)$ is defined by

$$(6) \quad \text{Sp}(2n, R) = \{ S \in \text{GL}(2n, R) : S'J_{2n}S = J_{2n} \}.$$

The map $S \mapsto \alpha = \begin{pmatrix} S & 0 \\ 0 & 1 \end{pmatrix}$ identifies $\text{Sp}(2n, R)$ with a subgroup of \mathcal{Q}_n^0 . We will assume this identification without further comment. Set

$$(7) \quad \begin{aligned} \mathfrak{D}_{2n} &= \left\{ \begin{pmatrix} dE_{2n} & 0 \\ 0 & d^2 \end{pmatrix} : d > 0 \right\}, \\ \mathfrak{R}_{2n} &= \left\{ \begin{pmatrix} dE_{2n} & 0 \\ \delta' & d^2 \end{pmatrix} : d > 0, \delta \in R^{2n} \right\}, \\ \text{inn}(N_n) &= \left\{ \begin{pmatrix} E_{2n} & 0 \\ \delta' & 1 \end{pmatrix} : \delta \in R^{2n} \right\}. \end{aligned}$$

Clearly, the sets defined in (7) are all subgroups of \mathcal{Q}_n^0 . \mathfrak{R}_{2n} can be characterized as the maximal, normal, solvable, analytic subgroup of \mathcal{Q}_n^0 . Also, since $\mathfrak{R}_{2n} \cap \text{Sp}(2n, R) = E_{2n}$, we can form the semidirect product $\text{Sp}(2n, R) \ltimes \mathfrak{R}_{2n}$. It is almost immediate that

$$(8) \quad \mathcal{Q}_n^0 = \text{Sp}(2n, R) \ltimes \mathfrak{R}_{2n}.$$

Denote by $\text{inn}(\xi, z)$ the inner automorphism of N_n given by

$$\text{inn}(\xi, z)((\eta, w)) = (\xi, z)^{-1}(\eta, w)(\xi, z).$$

In matrix form we have

$$(9) \quad \text{inn}(\xi, z) = \begin{pmatrix} E_{2n} & 0 \\ J_{2n}(\xi)' & 1 \end{pmatrix}.$$

Hence, $\text{inn}(N_n)$ denotes the group of all inner automorphisms of N_n . The following results are obvious.

$$(10) \quad \mathfrak{R}_{2n} = \mathfrak{D}_{2n} \times \text{inn}(N_n), \quad \mathfrak{Q}_n^0 = \text{Sp}(2n, R) \times \mathfrak{D}_{2n} \times \text{inn}(N_n).$$

Observe that the elements of $\text{Sp}(2n, R)$ and \mathfrak{D}_{2n} commute with each other. Also set

$$(11) \quad \text{SL}(N_n) = \text{Sp}(2n, R) \times \text{inn}(N_n).$$

This is the subgroup of all $\alpha \in \mathfrak{Q}_n^0$ whose restriction to Z is the identity mapping.

We will now begin to investigate the first of two, essentially equivalent, algebraic concepts. In terms of each of these concepts, we can develop the ‘analytic’ results of the next section.

An element $\alpha \in \mathfrak{Q}_n^0$ is called a *CR-structure* if $\alpha^2 \equiv -E_{2n} \pmod{Z}$. The set of all CR-structures is denoted by $\text{CR}(N_n)$. \mathfrak{Q}_n^0 acts on $\text{CR}(N)$ by conjugation. For $\alpha \in \text{CR}(N_n)$ and $\beta \in \mathfrak{Q}_n^0$ we set $\beta[\alpha] = \beta^{-1}\alpha\beta$. Two CR-structures α_1 and α_2 are called *conjugate* whenever $\beta[\alpha_1] = \alpha_2$ for some $\beta \in \mathfrak{Q}_n^0$. We denote the space of \mathfrak{Q}_n^0 -orbits of $\text{CR}(N_n)$, or, equivalently, the space of conjugacy classes of CR-structures by $\text{CR}(N_n)/\mathfrak{Q}_n^0$, while we denote the conjugacy class containing an $\alpha \in \text{CR}(N_n)$ by $\mathfrak{Q}_n^0[\alpha]$. Observe that $\text{CR}(N_n) \subset \text{SL}(N_n)$ and $\text{SL}(N_n)[\alpha] = \mathfrak{Q}_n^0[\alpha]$.

Eventually, we will describe $\text{CR}(N_n)$ and $\text{CR}(N_n)/\mathfrak{Q}_n^0$ explicitly.

We begin with the following definition. Set

$$(12) \quad \text{CR}^*(N_n) = \text{CR}(N_n) \cap \text{Sp}(2n, R).$$

We claim

$$(13) \quad \text{CR}^*(N_n) = \{S \in \text{Sp}(2n, R) : J_{2n}S \text{ is symmetric}\}.$$

$$(14) \quad \text{CR}(N_n) = \text{inn}(N_n)[\text{CR}^*(N_n)].$$

To show (13) observe, $S'J_{2n}S = J_{2n}$ and $S^2 = -E_{2n}$ imply $-S'J_{2n} = S'J_{2n}' = J_{2n}S$. The converse follows by reversing these steps. To show (14) take $\alpha \in \text{CR}(N_n)$ and write

$$\alpha = \begin{pmatrix} S & 0 \\ \delta' & 1 \end{pmatrix}.$$

Let

$$\beta = \begin{pmatrix} E_{2n} & 0 \\ \gamma' & 1 \end{pmatrix} \in \text{inn}(N_n).$$

To solve $\beta[S] = \alpha$ coincides with solving $\delta = (E_{2n} - S)\gamma$. Since $S^2 =$

– E_{2n} , $E_{2n} - S$ is nonsingular and $\gamma = (E_{2n} - S)^{-1}\delta$. Hence, there is for each $\alpha \in \text{CR}(N_n)$, a unique $S \in \text{CR}^*(N_n)$ and a unique $\beta \in \text{inn}(N_n)$, such that $\alpha = \beta[S]$ and (14) is verified.

The most important CR-structure for this paper is J_{2n} . The following result describes the conjugacy class containing J_{2n} .

LEMMA 1.1. For $\alpha \in \text{CR}^*(N_n)$ to be conjugate to J_{2n} it is necessary and sufficient that $\alpha = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ where B is symmetric and positive definite.

PROOF. Firstly, if α is conjugate to J_{2n} then for some

$$\beta = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \in \text{Sp}(2n, R),$$

$\beta[J_{2n}] = \alpha$. It follows that $B = B_1'B_1 + D_1'D_1$. Since $A_1'D_1 - C_1'B_1 = E_n$, there is no nonzero $\xi \in R^n$ satisfying $B_1\xi = D_1\xi = 0$. Hence, B is positive definite. Clearly, B is symmetric as well and the first half of the lemma is verified. Conversely, if $\alpha = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is in $\text{CR}^*(N_n)$ with B symmetric and positive definite, then $VBV' = E_n$ for some $V \in \text{GL}(n, R)$. The matrix

$$S = \begin{pmatrix} VA & VB \\ -(VB)^* & 0 \end{pmatrix} \in \text{Sp}(2n, R) \quad \text{and} \quad S[\alpha] = J_{2n}.$$

Define

$$(15) \quad \mathcal{P}_{2n} = \mathcal{O}_n^0[J_{2n}].$$

The elements $\alpha \in \mathcal{P}_{2n}$ will be called *positive definite* CR-structures.

We will now describe the space of conjugacy classes $\text{CR}(N_n)/\mathcal{O}_n^0$. Essentially, our main result refines the classical result that every R -quadratic form can be diagonalized. Let

(16) \mathcal{S}_n^+ = the space of all nonsingular, symmetric, real, $n \times n$ -matrices.

$\text{GL}(n, R)$ acts on \mathcal{S}_n^+ by the rule

$$X[F] = X'FX, \quad F \in \mathcal{S}_n^+, \quad X \in \text{GL}(n, R).$$

Two elements F and G in \mathcal{S}_n^+ are called *similar*, and we write $F \sim G$, whenever $X[F] = G$ for some $X \in \text{GL}(n, R)$. Define

$$(17) \quad E_{p,q} = \begin{pmatrix} E_p & 0 \\ 0 & E_q \end{pmatrix}, \quad p, q \geq 0 \text{ in } \mathbb{Z} \text{ and } p + q = n,$$

$$J_{2n}(p, q) = \begin{pmatrix} 0 & E_{p,q} \\ -E_{p,q} & 0 \end{pmatrix}, \quad \mathcal{P}_{2n}(p, q) = (\mathcal{O}_n^0)[J_{2n}(p, q)].$$

It is a well-known result of linear algebra (see Siegel [8, p. 85]) that every $F \in \mathcal{S}_n^+$ determines a unique pair of nonnegative integers p, q satisfying $p + q = n$ and $F \sim E_{p,q}$. The pair (p, q) is called the *signature* of F . We want

to translate this result to CR-structures. Define

$$(18) \quad \lambda G = \begin{pmatrix} 0 & G \\ -G^{-1} & 0 \end{pmatrix}, \quad G \in \mathfrak{S}_n^+.$$

Clearly, $\lambda G \in \text{CR}^*(N_n)$ and $J_{2n}(p, q) = \lambda E_{p,q}$. Moreover, we have the following compatibility between the similarity relation on \mathfrak{S}_n^+ and the conjugacy relation on $\text{CR}(N_n)$.

LEMMA 1.2. *For F_1 and F_2 in \mathfrak{S}_n^+ , F_1 and F_2 are similar if and only if λF_1 and λF_2 are conjugate as CR-structures.*

PROOF. Recall $X^* = (x')^{-1}$, $X \in \text{GL}(n, R)$. Set

$$X' = \begin{pmatrix} X & 0 \\ 0 & X^* \end{pmatrix}, \quad X \in \text{GL}(n, R).$$

Clearly, $X' \in \text{Sp}(2n, R)$. Suppose $F_1 \sim F_2$. Then $F_1 = XF_2X'$ for some $X \in \text{GL}(n, R)$. But this implies $X'[\lambda F_1] = \lambda F_2$. Conversely, if $\lambda F_1 \sim \lambda F_2$ as CR-structures then $\beta[\lambda F_2] = \lambda F_1$ for some $\beta = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}(2n, R)$. Hence,

$$F_2 = BF_1^{-1}B' + AF_1A' = (A + iBF^{-1})F(A' - iF^{-1}B').$$

It follows that F_1 and F_2 are Hermitian similar, and since they are real matrices they are similar.

(For the definition of Hermitian similar and the result stated in the proof, see Lang [4, p. 342].)

From the lemma we have that the orbits $\mathfrak{P}_{2n}(p, q)$ are disjoint and each λF , $F \in \mathfrak{S}_n^+$, is contained in the union of these orbits. Our next theorem tells us that the union of these orbits is all of $\text{CR}(N_n)$.

THEOREM 1.3. $\text{CR}(N_n)/\mathcal{O}_n^0 = \{\mathfrak{P}_{2n}(p, q): p, q \geq 0 \text{ in } \mathbb{Z} \text{ and } p + q = n\}$.

PROOF. It suffices to show that each

$$S = \begin{pmatrix} A & B \\ C & -A' \end{pmatrix} \in \text{CR}^*(N_n)$$

(as in (13)) is conjugate to some λF , $F \in \mathfrak{S}_n^+$. For B nonsingular we have

$$\beta[S] = \lambda B \quad \text{where } \beta = \begin{pmatrix} -A & -B \\ B^{-1} & 0 \end{pmatrix} \in \text{Sp}(2n, R).$$

The general case will be reduced to the case above. Since S generates a compact subgroup of $\text{Sp}(2n, R)$, by a classical result S is conjugate in $\text{Sp}(2n, R)$ to an element in $\text{Sp}(2n, R) \cap \mathfrak{O}(2n, R)$. For an explicit proof see Igusa [2, p. 23]. Hence, we can assume S is of the form $S = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}$ where B is symmetric, A is alternating, $A^2 - B^2 = -E_n$ and $AB + BA = 0$. We want to find a

$$T = \begin{pmatrix} E_n & G \\ 0 & E_n \end{pmatrix}$$

with G symmetric satisfying

$$T[S] = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}$$

with B_1 nonsingular. This is equivalent to solving

$$(a) \quad B_1 = -AG + GBG + B + GA$$

with G symmetric and B_1 nonsingular. Since B is symmetric it can be diagonalized by some orthogonal matrix O , i.e., $O'BO$ is diagonal. Condition (a) is independent of such a change, hence we can assume B is diagonal to begin with. Write

$$B = \left(\begin{array}{c|c} \delta(b) & 0 \\ \hline 0 & 0 \end{array} \right)$$

where

$$\delta(b) = \begin{bmatrix} b_1 & & 0 \\ & \cdot & \\ 0 & & b_l \end{bmatrix}$$

is a diagonal $l \times l$ matrix and $b_j \neq 0$, $1 \leq j \leq l$. From $A^2 - B^2 = -E_n$ it follows that

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_4 \end{pmatrix}$$

where A_1 is an $l \times l$ -matrix, $A_4' = -A_4$ and $A_4^2 = -E_{n-l}$. It follows that A_4 is orthogonal and is conjugate by an orthogonal matrix to a matrix consisting of the 2×2 blocks J_2 down the diagonal. A symmetric matrix $S_4 \in \mathfrak{S}_{n-l}$ can then be found satisfying $-A_4 S_4 + S_4 A_4$ is nonsingular. Taking

$$G = \begin{pmatrix} 0 & 0 \\ 0 & S_4 \end{pmatrix}$$

satisfies condition (a).

We will now consider the automorphism group of a CR-structure α . For $\alpha \in \text{CR}(N_n)$, put

$$\text{aut}(\alpha) = \{ \beta \in \mathcal{O}_n^0 : \beta^{-1} \alpha \beta = \alpha \}.$$

The elements of $\text{aut}(\alpha)$ will be called α -automorphisms. If $\beta[\alpha_1] = \alpha_2$ for $\alpha_j \in \text{CR}[N_n]$ and $\beta \in \mathcal{O}_n^0$, then clearly $\beta^{-1} \text{aut}(\alpha_1) \beta = \text{aut}(\alpha_2)$. Hence, to compute the automorphism groups of CR-structures it suffices to compute the automorphism groups of the CR-structures $J_{2n}(p, q)$.

We begin by considering $\text{aut}(J_{2n})$. Firstly, set

$$(20) \quad K = \text{Sp}(2n, R) \cap \mathcal{O}(2n, R).$$

Obviously, for $S \in \text{Sp}(2n, R)$, we have $S \in K$ if and only if $S'J_{2n} = J_{2n}S'$, which implies

$$(21) \quad K = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in \text{Sp}(2n, R) \right\}.$$

As we have mentioned in the proof of Theorem 1.3, K has been previously studied. It is not hard to show that K is a maximal compact subgroup of $\text{Sp}(2n, R)$ and any maximal compact subgroup of $\text{Sp}(2n, R)$ is conjugate in $\text{Sp}(2n, R)$ to K . The next lemma is almost obvious.

LEMMA 1.4. $\text{aut}(J_{2n}) = K \not\propto \mathfrak{D}_{2n}$.

PROOF. For

$$\alpha = \begin{pmatrix} S & 0 \\ \delta' & d \end{pmatrix} \in \text{aut}(J_{2n})$$

we have

$$S^{-1}J_{2n}S = S'J_{2n}S = J_{2n} \quad \text{and} \quad \delta' = \delta'S^{-1}J_{2n}S.$$

Hence, $\delta = 0$ and

$$\alpha = \begin{pmatrix} 1/\sqrt{d} & S & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{d} E_{2n} & 0 \\ 0 & d \end{pmatrix}.$$

This proves the lemma.

In the same way we can verify the next lemma.

LEMMA 1.5. $\text{aut}(J_{2n}(p, q)) = K_{p,q} \not\propto \mathfrak{D}_{2n}$ where

$$K_{p,q} = \left\{ \begin{pmatrix} A & B \\ -E_{p,q}BE_{p,q} & E_{p,q}AE_{p,q} \end{pmatrix} \in \text{Sp}(2n, R) \right\}.$$

Define

$$(22) \quad \text{aut}(p, q) = \text{aut}(J_{2n}(p, q)), \quad G(p, q) = \text{aut}(p, q) \not\propto N_n,$$

and, in general, if $\alpha \in \text{CR}(N_n)$,

$$G(\alpha) = \text{aut}(\alpha) \not\propto N_n.$$

For congruent CR-structures α_1 and $\alpha_2 = \beta[\alpha_1]$ the mapping

$$(\gamma, v) \mapsto (\beta^{-1}\gamma\beta, \beta^{-1}(v))$$

defines an isomorphism from $G(\alpha_1)$ onto $G(\alpha_2)$. We will now consider the converse. Our main tools come from Jacobson [3], primarily Theorem 10, p. 81 and the theorem of Malcev and Harish-Chandra, p. 92. Corollary 2 on p. 72 is used as well. These results will be assumed without further comment.

The case $n = 2$ is trivial, hence we assume until the end of the next theorem that $n \geq 3$. In this case both the orthogonal group $\mathfrak{O}(p, q)$ of $E_{p,q}$ and its

commutator subgroup $\Omega(p, q)$ are irreducible (see O'Meara [7, p. 111]). The map

$$A \mapsto A' = \begin{pmatrix} A & 0 \\ 0 & E_{p,q} A E_{p,q} \end{pmatrix}$$

defines an isomorphism of $\Theta(p, q)$ into $\text{aut}(p, q)$. Since $J_{2n} \in \text{aut}(p, q)$ it follows that $\text{aut}(p, q)$ is irreducible. Thus N_n is the nil-radical of $G(p, q)$. Also, $\text{aut}(p, q) = [\text{aut}(p, q), \text{aut}(p, q)] \not\propto z_{p,q}$, where $z_{p,q}$ is the center of $\text{aut}(p, q)$.

THEOREM 1.6. *Two CR-structures α_1 and α_2 are conjugate, up to multiplication by $-E_{2n}$, if and only if the corresponding groups $G(\alpha_1)$ and $G(\alpha_2)$ are Lie group isomorphic.*

PROOF. Without loss of generality we can assume $\alpha_1 = J_{2n}(p, q)$ and $\alpha_2 = J_{2n}(r, s)$. Assume $\rho: G(p, q) \rightarrow G(r, s)$ is a Lie group isomorphism. Firstly, $\rho(N_n) = N_n$ and, by our discussion above, $\rho([\text{aut}(p, q), \text{aut}(p, q)])$ is a Levi-factor of $G(r, s)$. The conjugacy theory of Levi-factors implies we can assume without loss of generality that

$$\rho([\text{aut}(p, q), \text{aut}(p, q)]) = [\text{aut}(r, s), \text{aut}(r, s)].$$

Thus, for some $S \in \text{Sp}(2n, R)$, $S^{-1} \text{aut}(p, q) S \supset [\text{aut}(r, s), \text{aut}(r, s)]$ and, hence, $\Omega(r, s)$ centralizes $S^{-1} J_{2n}(p, q) S$. As a result, $\Omega(r, s) \subset \Theta(F)$, where $F = X E_{p,q} \bar{X}'$, where $X = A' - i C' E_{p,q}$. Since both F and $E_{p,q}$ are real matrices it follows that F and $E_{p,q}$ are similar. Hence, $\Omega(r, s)$ is contained in a conjugate of $\Theta(p, q)$ in $\text{GL}(n, R)$. Thus, $\Omega(G) \subset \Theta(p, q)$ where G is similar to $E_{r,s}$. By Schur's lemma, since $\Omega(G)$ is irreducible and commutes with $E_{p,q} G$, we have $G = \varepsilon \cdot E_{p,q}$, $\varepsilon \neq 0$ in R , and the theorem follows.

Let \mathbb{C}^n denote the n -fold cartesian product of the complex numbers \mathbb{C} . In all that follows we shall fix the identifications

$$R^{2n} \simeq R^n \oplus R^n \simeq \mathbb{C}^n,$$

defined by setting $(\xi_1, \xi_2) = \xi_1 + i\xi_2$, $\xi_1, \xi_2 \in R^n$. Thus $J_{2n}(\xi, \eta) = \text{im } \xi \bar{\eta}$, $\xi, \eta \in R^n$. It is obvious, but important, to note that the symplectic group $\text{Sp}(2n, R)$ does not act as a group of analytic automorphisms of \mathbb{C}^n (assuming the above identification). A simple computation shows that $S \in \text{Sp}(2n, R)$ is an analytic automorphism of \mathbb{C}^n if and only if it satisfies the property $S^+ J_{2n} = J_{2n} S^+$. Hence the group K defined above [20] describes all the elements of $\text{Sp}(2n, R)$ which are analytic automorphisms.

We will now describe the lattices of N_n . A lattice of N_n will be defined as a discrete subgroup Γ of N_n such that the space of right cosets $\Gamma \backslash N_n = \{\Gamma_v : v \in N_n\}$ is compact and $\underline{[\Gamma, \Gamma]} = \Gamma \cap Z$. This last condition is not usual but

permits a simpler exposition. Furthermore, the lattices of N_n can be characterized as the collection of all nonabelian subgroups of N_n on $2n$ -generators. The main tool in structuring the lattices of N_n is \mathcal{O}_n^0 . Set

$$(23) \quad \mathcal{L} = \text{collection of all lattices of } N_n$$

and define the action of \mathcal{O}_n^0 on \mathcal{L} by

$$(24) \quad \Gamma^\alpha = \alpha(\Gamma), \quad \alpha \in \mathcal{O}_n^0 \text{ and } \Gamma \in \mathcal{L}.$$

Since $\Gamma \cap Z$ is a nontrivial discrete subgroup of Z whenever $\Gamma \in \mathcal{L}$, we can find a uniquely determined positive real number $\beta(\Gamma)$ defined by the condition

$$(25) \quad \Gamma \cap Z = \{(0, \beta(\Gamma) \cdot m) : m \in \mathbb{Z}\}.$$

The description of \mathcal{L} requires, as we shall see below, familiarity with those lattices D of \mathbb{C}^n which admit Riemann forms. We will say more about this in the next section. For now we shall consider just what we need for describing \mathcal{L} .

A lattice of \mathbb{C}^n is a discrete subgroup D of \mathbb{C}^n where $D \setminus \mathbb{C}^n$ is compact. The lattices in \mathcal{L} are a source of lattices in \mathbb{C}^n of a very special kind. Consider the short exact sequence

$$(26) \quad 1 \rightarrow Z \rightarrow N_n \xrightarrow{\pi} \mathbb{C}^n \rightarrow 1$$

where $\pi(\xi, z) = \xi$. Recall we are identifying R^{2n} with \mathbb{C}^n . By Malcev [6] whenever $\Gamma \in \mathcal{L}$ we have that $\pi(\Gamma)$ is a lattice of \mathbb{C}^n . Let $S \in \text{Sp}(2n, R)$ and $\Gamma \in \mathcal{L}$. Then $S(\Gamma) \in \mathcal{L}$ and S induces a diffeomorphism between $\Gamma \setminus N_n$ and $S(\Gamma) \setminus N_n$. However, in general, as noted above, S is not an analytic isomorphism of \mathbb{C}^n and, hence, as complex manifolds $\pi(\Gamma) \setminus \mathbb{C}^n$ and $S(\pi(\Gamma)) \setminus \mathbb{C}^n$ are not necessarily analytically isomorphic.

The condition $[\Gamma, \Gamma] = \Gamma \cap Z$ (in fact, simply the fact that $[\Gamma, \Gamma]$ is a nontrivial discrete subgroup of Z) places further conditions on $\pi(\Gamma)$. For $v = (\xi, z), w = (\eta, w) \in N_n, [v, w] = (0, J_{2n}(\xi, \eta))$. Hence, for any $\xi, \eta \in \pi(\Gamma)$ we have

$$(27) \quad J_{2n}(\xi, \eta) \in \beta(\Gamma) \cdot Z.$$

In fact we will show in Lemma 1.7 below that

$$(28) \quad J_{2n}(\pi(\Gamma), \pi(\Gamma)) = \beta(\Gamma) \cdot Z.$$

With this in mind, we call a lattice D of \mathbb{C}^n a *Heisenberg lattice* whenever $J_{2n}(D, D) = l \cdot Z$ for some $l > 0$ in R . Denote the collection of all Heisenberg lattices of \mathbb{C}^n by $H\mathcal{L}(\mathbb{C}^n)$.

For elements $\xi_1, \xi_m \in \mathbb{C}^n$ denote the Z -module of \mathbb{C}^n spanned by these elements by $[\xi_1, \dots, \xi_m]$.

LEMMA 1.7. For a lattice D of C^n satisfying $J_{2n}(D, D) \subset l \cdot Z$, $l > 0$, we have $D \in H\mathcal{L}(C^n)$.

PROOF. Dividing the elements of D by \sqrt{l} we see that there is no loss of generality in assuming $l = 1$. Let l_1 be the smallest positive integer in $J_{2n}(D, D)$ and $l_1 = J_{2n}(\xi_1, \eta_1)$ for $\xi_1, \eta_1 \in D$. Set

$$[\xi_1, \eta_1]^\perp = \{\xi \in D: J_{2n}(\xi, \xi_1) = J_{2n}(\eta_1, \xi) = 0\}.$$

We claim $D = [\xi_1, \eta_1] \oplus [\xi_1, \eta_1]^\perp$. For $\xi \in D$, the Euclidean algorithm implies

$$0 \leq ml_1 + J_{2n}(\xi_1, \xi) = r < l_1, \quad m, r \in Z.$$

Hence, $0 \leq J_{2n}(\xi_1, \xi + m\eta_1) = r < l_1$, which contradicts the definition of l_1 unless $r = 0$. Hence, $\xi + m\eta_1 \in [\xi_1]^\perp$. In the same way, $\xi + m'\xi_1 \in [\eta_1]^\perp$ for some $m' \in Z$. Thus $\xi + m'\xi_1 + m\eta_1 \in [\xi_1, \eta_1]^\perp$ and our claim is verified. We are done once we show that for $\xi, \eta \in [\xi_1, \eta_1]^\perp$, $l_1/J_{2n}(\xi, \eta)$. Again by the Euclidean algorithm

$$0 \leq ml_1 + J_{2n}(\xi, \eta) = r < l_1, \quad m, r \in Z,$$

and, as before, $r = 0$ and $l_1/J_{2n}(\xi, \eta)$.

The proof of the previous lemma is the main step in an induction proof (see Lang [5, p. 64]) of the next result.

LEMMA 1.8. For $D \in H\mathcal{L}(C^n)$ we can write D as the J_{2n} -orthogonal direct sum $D = [\xi_1, \eta_1] \oplus \dots \oplus [\xi_n, \eta_n]$ of 2-dimensional Z -modules $[\xi_j, \eta_j]$. Let $l_j = J_{2n}(\xi_j, \eta_j)$; then the decomposition can be taken so that $l_j \in R_+^x$ and l_j/l_{j+1} , i.e., $l_{j+1}/l_j \in Z$.

Moreover, these numbers l_1, \dots, l_n are uniquely determined by D .

COROLLARY. $\pi\mathcal{L} = H\mathcal{L}(C^n)$.

PROOF. By Lemma 1.7 and the discussion preceding it, whenever $\Gamma \in \mathcal{L}$, $\pi\Gamma \in H\mathcal{L}(C^n)$. Conversely, let $D \in H\mathcal{L}(C^n)$ and $D = [\xi_1, \eta_1] \oplus \dots \oplus [\xi_n, \eta_n]$ as in Lemma 1.8. In N_n choose preimages of ξ_j, η_j and set $\Gamma =$ the group generated by these preimages. Then $\Gamma \in \mathcal{L}$ and $\pi\Gamma = D$.

COROLLARY. For $\Gamma \in \mathcal{L}$, $\beta(\Gamma)Z = J_{2n}(\pi\Gamma, \pi\Gamma)$.

Set

$$(29) \quad Z_n^\# = \{l = (l_1, l_2, \dots, l_n) \in Z_+^x: l_j/l_{j+1}\}.$$

Let e_j , $1 \leq j \leq 2n$, denote the $2n$ -tuple consisting of all zeros except for a one in the j th place. For $l \in Z_n^\#$ define

$$(30) \quad D(l) = [e_1, \dots, e_n, e_{n+1}, l_2e_{n+2}, \dots, l_n e_{2n}].$$

Clearly, $D(l) \in H\mathcal{L}(C^n)$ and $J_{2n}(D(l), D(l)) = Z$.

THEOREM 1.9. *For each $D \in H\mathcal{L}(\mathbb{C}^n)$ there exist a unique $l \in \mathbb{Z}_n^\#$, a unique $d \neq 0$ in \mathbb{R}_+ and an $\alpha \in \text{Sp}(2n, \mathbb{R})$ such that $D = \alpha(d \cdot D(l))$.*

PROOF. We can assume $J_{2n}(D, D) = \mathbb{Z}$. Then $D = [\xi_1, \eta_1] \oplus \dots \oplus [\xi_n, \eta_n]$ as in Lemma 1.8, and if $l_j = [\xi_j, \eta_j]$ we have $l \in \mathbb{Z}_n^\#$. Set $\xi = (\xi_1, \dots, \xi_n)$ and $\eta' = (\eta_1, \eta_2/l_2, \dots, \eta_n/l_n)$. Regarding $\xi, \eta' \in \mathbb{R}^{2n}$, it is easy to see that the matrix $\alpha = (\xi \eta') \in \text{Sp}(2n, \mathbb{R})$ and $D = \alpha(D(l))$. The uniqueness is also contained in Lemma 1.8.

Set $e'_j = (e_j, 0) \in N_n$. For $l \in \mathbb{Z}_n^\#$ define

$$(31) \quad \Gamma(l) = \text{gp}\{e'_1, \dots, e'_n, e'_{n+1}, l_2 e'_{n+2}, \dots, l_n e'_{2n}\}.$$

In particular, set $\Gamma_n = \Gamma(1, 1, \dots, 1)$. For $d \in \mathbb{R}_+^x$ we put

$$dv = \begin{pmatrix} dE_{2n} & 0 \\ 0 & d^2 \end{pmatrix} v, \quad v \in N_n.$$

THEOREM 1.10. *For $\Gamma \in \mathcal{L}$, there exist a unique $l \in \mathbb{Z}_n^\#$, a unique $d \in \mathbb{R}_+^x$ and an $\alpha \in \text{SL}(N_n)$ such that $\Gamma = \alpha(d \cdot \Gamma(l))$.*

PROOF. Without loss of generality we can assume $\beta(\Gamma) = 1$. Then $D = \pi(\Gamma) \in H\mathcal{L}(\mathbb{C}^n)$ and $J_{2n}(D, D) = \mathbb{Z}$. Hence, $D = \alpha(D(l))$, where $\alpha \in \text{Sp}(2n, \mathbb{R})$ and $l \in \mathbb{Z}_n^\#$. Since $\alpha \in \text{aut}(N_n)$ and commutes with π , we can just as well assume $D = D(l)$. There exists $\delta = (\delta_1, \dots, \delta_{2n}) \in \mathbb{R}^{2n}$ so that

$$\Gamma = \text{gp}\{(e_j, \delta_j), (l_j e_{n+j}, l_j \delta_{n+j})\}_{j=1, \dots, n}.$$

Hence, $\Gamma = \text{inn}(J_{2n}(\delta), 0) \times (\Gamma(l))$, which proves the theorem.

Since

$$\begin{pmatrix} dE_{2n} & 0 \\ 0 & d^2 \end{pmatrix} \in \mathcal{O}_n^0$$

it follows that the groups $\Gamma(l), l \in \mathbb{Z}_n^\#$, determine \mathcal{L} up to \mathcal{O}_n^0 .

Let $\Gamma \in \mathcal{L}$. In the next section we shall consider various function spaces on N_n periodic with respect to left multiplication by elements from Γ . The remainder of this chapter is spent developing some of the algebraic machinery which will be applied to analyzing the structure of these function spaces. For $D \in H(\mathbb{C}^n)$ write $D = \alpha(d \cdot D(l))$ as in Theorem 1.9. The $l \in \mathbb{Z}_n^\#$ and $d > 0$ are uniquely determined by D . Define

$$(32) \quad \mathfrak{P}(D) = (1 \cdot l_2 \cdot \dots \cdot l_n)^2.$$

We call $\mathfrak{P}(D)$ the *Pfaffian* of D . Also, we put

$$(33) \quad \beta(D) = d^2.$$

$\beta(D)$ is the unique positive number satisfying

$$(34) \quad J_{2n}\beta(D, D) = \beta(D) \cdot \mathbb{Z}.$$

Analogously, for $\Gamma \in \mathcal{L}$ we can write $\Gamma = \alpha(d \cdot \Gamma(I))$ as in Theorem 1.14. The Pfaffian $\mathcal{P}(\Gamma)$ of Γ is defined by setting

$$(35) \quad \mathcal{P}(\Gamma) = (1 \cdot l_2 \cdots l_n)^2.$$

Clearly, for $\Gamma \in \mathcal{L}$, $\mathcal{P}(\Gamma) = \mathcal{P}(\pi\Gamma)$. Moreover, we have previously defined $\beta(\Gamma) > 0$ by the condition

$$(36) \quad \Gamma \cap Z = \beta(\Gamma) \cdot ZZ$$

where $ZZ = \{(0, m) : m \in Z\}$. Clearly,

$$(37) \quad \beta(\Gamma) = d^2 = \beta(\pi(\Gamma)).$$

A duality theory for $H\mathcal{L}(\mathbb{C}^n)$, and correspondingly for \mathcal{L} , will now be developed. For $D \in H\mathcal{L}(\mathbb{C}^n)$ put

$$(38) \quad \hat{D} = \{\xi \in \mathbb{C}^n : J_{2n}(\xi, \eta) \in \beta(D) \cdot Z \text{ for all } \eta \in D\}.$$

We call \hat{D} the dual of D .

LEMMA 1.11. For $D \in H\mathcal{L}(\mathbb{C}^n)$, $\hat{D} \in H\mathcal{L}(\mathbb{C}^n)$, $D \subset \hat{D}$ and $O(\hat{D}/D) = \mathcal{P}(D)$.

PROOF. Let $D = \alpha(d \cdot D(I))$ as above. Then $\hat{D} = \alpha(d \cdot \hat{D}(I))$. Hence the lemma is proved once we show its validity for $D(I)$. In this case,

$$\hat{D} = \text{gp}\{e_1 e_2 / l_2, \dots, e_n / l_n, e_{n+1}, \dots, e_{2n}\}.$$

Then $J_{2n}(\hat{D}, \hat{D}) = 1/l_n \cdot Z$ and, hence, $\hat{D} \in H\mathcal{L}(\mathbb{C}^n)$. Clearly, $D \subset \hat{D}$ and $O(\hat{D}/D) = (1 \cdot l_2 \cdots l_n)^2 = \mathcal{P}(D)^2$.

A relative duality theory will also be required. For $D \subset D'$, both lattices in $H\mathcal{L}(\mathbb{C}^n)$, put

$$(39) \quad \widehat{D'|D} = \{\xi \in \mathbb{C}^n : J_{2n}(\xi, \eta) \in \beta(D')Z \text{ for all } \eta \in D\}.$$

We say that $\widehat{D'|D}$ is the dual of D with respect to D' . The following result is obvious.

$$\text{LEMMA 1.12. } \widehat{D'|D} = (\beta(D')/\beta(D))\hat{D}.$$

We want to investigate the relationship between D , D' and $\widehat{D'|D}$. A useful fact is the next lemma.

LEMMA 1.13. For $D \in H\mathcal{L}(\mathbb{C}^n)$ with $D \subset D_n$,

$$O(D \setminus D_n) = \beta(D)^n \cdot \mathcal{P}(D)^{1/2}.$$

PROOF. Write $D = d \cdot \alpha(D(I))$ as above. Then $D = T(D_n)$ where

$$T = d \cdot \alpha \cdot \begin{pmatrix} E_n & & & 0 \\ & 1 & & \\ & & l_2 & \\ & & & \ddots \\ 0 & & & & l_n \end{pmatrix}.$$

However, $\det T = d^{2n} \mathfrak{P}(D)^{1/2}$. Since $\beta(D) = d^2$, we are done.

LEMMA 1.14. *Let $D \subset D'$ be two lattices in $H\mathcal{L}(\mathbb{C}^n)$. Then*

- (a) $O(D \setminus D') = (\beta(D)/\beta(D'))^n (\mathfrak{P}(D)/\mathfrak{P}(D'))^{1/2}$;
- (b) $O(\widehat{D'}|D/D') = O(D \setminus D') \cdot \mathfrak{P}(D')$.

PROOF. Write $D' = d' \cdot \alpha(D(I'))$ as above. Then $D'' = (1/d')\alpha^{-1}(D) \subset D(I') \subset D_n$. Since

$$O(D_n/D(I')) = \mathfrak{P}(D')^{1/2}$$

and

$$O(D_n/D'') = \beta(D')^n \mathfrak{P}(D'')^{1/2} = (\beta(D)/\beta(D'))^n \mathfrak{P}(D)^{1/2},$$

we have the first result.

The second result immediately follows from Lemmas 1.11 and 1.12 in view of (a).

The above results translate immediately into a duality theory for \mathcal{L} . Since each $D \in H\mathcal{L}(\mathbb{C}^n)$ determines a $\Gamma \in \mathcal{L}$ only up to conjugation in N_n , what is really considered is a duality theory of conjugacy classes of \mathcal{L} .

Let $\Gamma \in \mathcal{L}$ and set $D = \pi\Gamma$. Choose a lattice Γ' in N_n such that $\pi\Gamma' = \hat{D}$. As we have shown, Γ' is determined up to inner automorphism in N_n . We denote the conjugacy class containing Γ' as an element by $\hat{\Gamma}$. Also, we can also choose a $\Gamma'' \in \hat{\Gamma}$ so that $\Gamma \subset \Gamma''$. By abuse of language we shall also consider $\hat{\Gamma}$ as a lattice itself satisfying $\Gamma \subset \hat{\Gamma}$.

LEMMA 1.15. *For $\Gamma \in \mathcal{L}$, $O(\Gamma[\hat{\Gamma}, \hat{\Gamma}] \setminus \hat{\Gamma}) = \mathfrak{P}(\Gamma)$.*

Suppose $\Gamma, \Gamma' \in \mathcal{L}$ and $\Gamma \subset \Gamma'$. Set $D = \pi(\Gamma)$ and $D' = \pi(\Gamma')$. Choose a lattice Γ'' satisfying $\pi\Gamma'' = \widehat{D'}|D$. The conjugacy class containing Γ'' is denoted by $\widehat{\Gamma'}|\hat{\Gamma}$, and, as above, $\widehat{\Gamma'}|\hat{\Gamma}$ will also denote a lattice contained in $\widehat{\Gamma'}|\hat{\Gamma}$ as a conjugacy class satisfying $\Gamma \subset \Gamma' \subset \widehat{\Gamma'}|\hat{\Gamma}$.

LEMMA 1.16. *Let $\Gamma \subset \Gamma'$ be lattices in N_n . Then*

$$\widehat{\Gamma'}|\hat{\Gamma} = (\beta(\Gamma')/\beta(\Gamma))\hat{\Gamma}$$

where

$$d \cdot \gamma = \begin{pmatrix} dE_{2n} & 0 \\ 0 & d^2 \end{pmatrix} \cdot \gamma$$

when $d \in R$ and $\gamma \in N_n$.

PROOF. Directly from definition.

LEMMA 1.17. For $\Gamma \subset \Gamma'$ lattices in N_n the following results hold.

(a) If $\Gamma \subset \Gamma_n$, then

$$O(\Gamma \setminus \Gamma_n) = \beta(\Gamma)^n \mathcal{P}(\Gamma)^{1/2}.$$

(b) $O(\Gamma[\Gamma', \Gamma'] \setminus \Gamma') = (\beta(\Gamma)/\beta(\Gamma'))^n (\mathcal{P}(\Gamma)/\mathcal{P}(\Gamma'))^{1/2}.$

(c) $O(\widehat{\Gamma|\Gamma}/\widehat{\Gamma'|\Gamma'}[\widehat{\Gamma|\Gamma}, \widehat{\Gamma'|\Gamma'}]) = O(\Gamma'/\Gamma[\Gamma', \Gamma']) \cdot \mathcal{P}(\Gamma').$

PROOF. Follows from the corresponding results for lattices in C^n .

2. Analysis. Let Γ be a lattice in N_n . The space of right cosets $\Gamma \setminus N_n = \{\Gamma v: v \in N_n\}$ admits a unique probability measure invariant under translations by N_n . This measure will be assumed in all that follows. Hence, we get a unitary representation R of N_n of $\mathcal{L}^2(\Gamma \setminus N_n)$ by setting

$$(1) \quad R_v(F(\Gamma w)) = F(\Gamma wv), \quad F \in \mathcal{L}^2(\Gamma \setminus N_n), \quad v, w \in N_n.$$

We will write functions F on $\Gamma \setminus N_n$ as functions on N_n which are fixed under left translations by Γ . In general, if F is a function on N_n and $v \in N_n$ we set

$$(2) \quad L_v F(w) = F(v^{-1}w), \quad w \in N_n.$$

Thus, the condition on a function F to be a function on $\Gamma \setminus N_n$ is given by $L_\gamma F = F$ for all $\gamma \in \Gamma$. We will refer to such functions as Γ -periodic functions.

The beginning of this section recalls some of the basic results concerning the representation R . In particular, we reconsider the distinguished subspace theory of L. Auslander and J. Brezin. Analogous to §1, the central theme is how the automorphism group and the duality theory controls the structure of the distinguished subspaces. Essentially, once we know the multiplicity space $H_1(\Gamma)$, defined below, the theory of distinguished subspaces can be derived algebraically. The next topic is the construction of a special function theory for $\Gamma \setminus N_n$. The guiding force in this theory is the concept of CR-structures. H. Rossi, in 1970, was apparently the first to see how CR-structures played an important role in developing a special function theory. Since both the distinguished subspace theory and the special function theory are built out of the automorphism group, it comes as no surprise that they are intimately related. This relationship is exploited to give an analytic characterization of the distinguished subspaces (see [1] where this result first appears). The final topic is the application of our theory to classical theta function theory.

For each $m \in Z$ put

$$(3) \quad H_m(\Gamma) = \{ F \in \mathcal{L}^2(\Gamma \setminus N_n) : F(\xi, z) = e^{2\pi imz/\beta(\Gamma)} F(\xi, 0) \}.$$

Each $H_m(\Gamma)$ is easily seen to be R -invariant and we have the \mathcal{L}^2 -decomposition

$$(4) \quad \mathcal{L}^2(\Gamma \setminus N_n) = \sum_{m \in \mathbb{Z}} \oplus H_m(\Gamma).$$

Moreover, for each $F \in \mathcal{L}^2(\Gamma \setminus N_n)$ the projection $p_m(F)$ of F into $H_m(\Gamma)$ is given by the formula

$$(5) \quad p_m F(\xi, z) = \int_0^{\beta(\Gamma)} e^{-2\pi i(m \cdot s)/\beta(\Gamma)} F(\xi, z + s) ds.$$

The *Stone-von Neumann theorem* implies $H_1(\Gamma)$ is R -invariant and irreducible, and any two R -invariant and irreducible subspaces \mathcal{H}_1 and \mathcal{H}_2 of $\mathcal{L}^2(\Gamma \setminus N_n)$ are unitarily equivalent if and only if $\mathcal{H}_1, \mathcal{H}_2 \subset H_m(\Gamma)$ for some $m \in \mathbb{Z}$. We assume this result in what follows. Clearly, when $m = 0$, $H_0(\Gamma \setminus N_n) = \mathcal{L}^2(\pi\Gamma \setminus R^{2n})$ and $H_0(\Gamma \setminus N_n)$ decomposes into the infinite \mathcal{L}^2 -direct sum of 1-dimensional R -invariant subspaces. Below we will show that $R|H_m(\Gamma)$ is the product of $|m|^n \mathcal{G}(\Gamma)$ copies of a single irreducible representation of N_n when $m \neq 0$.

Let Γ be a lattice in N_n . For any automorphism $\alpha \in \mathcal{A}_n^0$ it is directly verified that the following results hold.

$$(6)(a) \quad \mathcal{L}^2(\Gamma \setminus N_n) \cdot \alpha^{-1} = \mathcal{L}^2(\Gamma^\alpha \setminus N_n).$$

$$(b) \quad H_m(\Gamma) \cdot \alpha^{-1} = H_m(\Gamma^\alpha).$$

(c) If \mathcal{H} is an R -invariant and irreducible subspace of $\mathcal{L}^2(\Gamma \setminus N_n)$ then $\mathcal{H} \cdot \alpha^{-1}$ is an R -invariant and irreducible subspace of $\mathcal{L}^2(\Gamma^\alpha \setminus N_n)$.

Thus, knowing R on $\mathcal{L}^2(\Gamma \setminus N_n)$ is equivalent to knowing R on $\mathcal{L}^2(\Gamma^\alpha \setminus N_n)$. For this reason it is sufficient as far as representation theory is concerned to restrict our attention to those lattices $\Gamma = \Gamma(I)$, $I \in \mathbb{Z}_n^\#$.

Let $\Gamma = \Gamma(I)$. Then $\Gamma \subset \Gamma_n$ and choose $\Gamma' = \widehat{\Gamma_n | \Gamma}$ to contain Γ_n . As we have said, Γ' is a maximal element from the class of lattices in N_n having the property that $[\Gamma', \Gamma'] \subset \Gamma_n$. It follows by elementary arguments that each $\gamma' \in \Gamma'$ determines a character $\chi(\gamma')$ on Γ_n by the rule

$$(7) \quad \chi(\gamma')(\gamma) = e^{2\pi i[\gamma, \gamma']}, \quad \gamma \in \Gamma_n.$$

Moreover, the mapping $\gamma' \mapsto \chi(\gamma')$ induces an isomorphism

$$(8) \quad \Gamma_n[\Gamma', \Gamma'] \setminus \Gamma' \cong \text{ch}(\Gamma \setminus \Gamma_n).$$

We denote the abstract group in (8) by $\Delta = \Delta(\Gamma)$, where in context it will be clear which of the two groups is meant.

LEMMA 2.1. *For each $\gamma' \in \Gamma$, $L_{\gamma'} H_1(\Gamma_n)$ is an R -invariant and irreducible subspace of $H_1(\Gamma)$ and can be given by*

$$(9) \quad L_{\gamma'} H_1(\Gamma_n) = \{ F \in H_1(\Gamma) : L_{\gamma'} F = \chi(\gamma')(\gamma) F \text{ for all } \gamma \in \Gamma_n \}.$$

PROOF. Since $L_\gamma R_\nu = R_\nu L_\gamma$ and $H_1(\Gamma_n)$ is R -invariant and irreducible, the first part of the lemma is proved once we know $L_\gamma H_1(\Gamma_n) \subset H_1(\Gamma)$. This follows from $[\Gamma', \Gamma] \subset \Gamma_n$. To prove the second part of the lemma, first observe that $H_1(\Gamma_n)$ is the eigenvalue one space in $H_1(\Gamma)$ with respect to the action of Γ_n on $H_1(\Gamma)$ by left-translation. Take $F \in H_1(\Gamma_n)$ and $\gamma' \in \Gamma'$. Then

$$L_\gamma(L_{\gamma'}F)(v) = F(\gamma'^{-1}\gamma^{-1}v) = F([\gamma', \gamma]\gamma^{-1}\gamma'^{-1}v) = e^{2\pi i[\gamma', \gamma]}L_\gamma F(v)$$

for all $\gamma \in \Gamma_n$. Conversely, if $G \in H_1(\Gamma)$ satisfies $L_\gamma G = \chi(\gamma)(\gamma)G$ for all $\gamma \in \Gamma_n$ then $L_{\gamma^{-1}}G \in H_1(\Gamma_n)$. Hence, we have proved the lemma.

COROLLARY. For each $\gamma' \in \Gamma'$, $H_1(\Gamma_n)$ and $L_{\gamma'}H_1(\Gamma_n)$ are unitarily equivalent.

PROOF. Follows from the first sentence of the proof of the lemma.

THEOREM 2.2. Let $\Gamma = \Gamma(I)$. Then

$$(10) \quad H_1(\Gamma) = \sum_{\gamma' \in \Delta(\Gamma)} \oplus L_{\gamma'}H_1(\Gamma_n)$$

is a direct sum decomposition of $H_1(\Gamma)$ into R -invariant, irreducible and unitarily equivalent subspaces.

PROOF. The group $\Gamma \setminus \Gamma_n$ acts on $H_1(\Gamma)$ by left-translations and hence decomposes $H_1(\Gamma)$ into subspaces determined by $\text{ch}(\Gamma \setminus \Gamma_n)$. By (8) and Lemma 2.1 all the characters appear and we are done.

COROLLARY. Let Γ be an arbitrary lattice in N_n . Then $R|H_1(\Gamma) =$ the restriction of R to $H_1(\Gamma)$ is $\mathfrak{P}(\Gamma)^{1/2}$ times a single irreducible representation of N_n .

PROOF. Simply translate by (6) Theorem 2.2.

We will now develop distinguished subspace theory from the point of view of the automorphism group. Define for a lattice $\Gamma = \Gamma(I)$ and an integer $m > 0$,

$$B_m(\Gamma) = \{ \alpha \in \mathfrak{O}_n^0 : \alpha(\Gamma) \subset \Gamma \text{ and } \alpha|Z = m \cdot E_1 \},$$

where E_1 denotes the identity mapping of Z .

We will associate to each $\alpha \in B_m(\Gamma)$ an \mathfrak{L}^2 -decomposition of $H_m(\Gamma)$ into R -invariant and irreducible subspaces.

Set $\Gamma = \Gamma(I)$, $I \in \mathbb{Z}_n^*$, and define

$$(11) \quad \Gamma(m) = \text{gp}\{\Gamma, m^{-1}[\Gamma, \Gamma]\}, \quad m \in \mathbb{Z}_+^*.$$

To each $\alpha \in B_m(\Gamma)$, $\Gamma^\alpha = \alpha\Gamma \subset \Gamma \subset \Gamma_n$. Choose $\Gamma^* = \widehat{\Gamma_n}[\Gamma^\alpha]$ containing Γ_n . To each $\gamma^* \in \Gamma^*$ we associate the character $\chi(\gamma^*)$ of $\Gamma_n^{\alpha^{-1}}$ defined by the rule

$$(12) \quad \chi(\gamma^*)(\alpha^{-1}(\gamma)) = e^{2\pi i m[\gamma^*, \alpha^{-1}(\gamma)]}, \quad \gamma \in \Gamma_n.$$

An elementary exercise shows that the mapping $\gamma^* \mapsto \chi(\gamma^*)$ induces the following isomorphism.

$$(13) \quad \Gamma_n[\Gamma^*, \Gamma^*] \setminus \Gamma^* \cong \text{ch}(\Gamma(m) \setminus \Gamma_n^{\alpha^{-1}}).$$

LEMMA 2.3. *To each $\gamma^* \in \Gamma^*$, $L_{\gamma^*}(H_1(\Gamma_n) \cdot \alpha)$ is an R -invariant and irreducible subspace of $H_m(\Gamma)$ and is given by*

$$(14) \quad L_{\gamma^*}(H_1(\Gamma_n) \cdot \alpha) = \{ F \in H_m(\Gamma) : L_{\alpha^{-1}(\gamma)} F = \chi(\gamma^*)(\alpha^{-1}(\gamma)) F \text{ for all } \gamma \in \Gamma_n \}.$$

PROOF. The first statement follows as Lemma 2.1 using $[\Gamma^*, \Gamma] \subset \Gamma_n(m)$. The second statement follows as Lemma 2.1.

COROLLARY. *For each $\gamma^* \in \Gamma^*$, $L_{\gamma^*}(H_1(\Gamma_n) \cdot \alpha)$ is unitarily equivalent to $H_1(\Gamma_n) \cdot \alpha$.*

THEOREM 2.4. *For each lattice $\Gamma = \Gamma(I)$, $\alpha \in B_m(\Gamma)$,*

$$H_m(\Gamma) = \sum_{\gamma^*} \oplus L_{\gamma^*}(H_1(\Gamma_n) \cdot \alpha)$$

where γ^* runs over $\Gamma_n[\Gamma^*, \Gamma^*] \setminus \Gamma^*$, $\Gamma^* = \widehat{\Gamma_n} | \Gamma_n^\alpha$.

PROOF. $\Gamma(m) \setminus \Gamma_n^{\alpha^{-1}}$ is a finite group and acts on $H_m(\Gamma)$ by left-translation having $H_1(\Gamma_n) \cdot \alpha$ as its eigenvalue one space. This action decomposes $H_m(\Gamma)$ into an ℓ^2 -direct sum (see [1, p. 8]) corresponding to characters on $\Gamma(m) \setminus \Gamma_n^{\alpha^{-1}}$, all of which occur nontrivially by (13) and Lemma 2.3.

COROLLARY. *$R|H_m(\Gamma)$ is $|m|^n \mathcal{P}(\Gamma)^{1/2}$ times a single irreducible representation of N_n .*

PROOF. Combine Lemma 2.3, its corollary and Theorem 2.4 along with $O(\Gamma_n[\Gamma^*, \Gamma^*] \setminus \Gamma^*) = |m|^n \mathcal{P}(\Gamma)^{1/2}$.

We will now use the theory of CR-structures discussed in §1 to pick out a special subspace of functions in N_n . Essentially, we mimic the way holomorphic functions on \mathbb{C}^n are determined by the Cauchy-Riemann equations. To do so we must first describe the Lie theoretic language which allows this translation. Recall, $L_n =$ Lie algebra of N_n . Set $L_n^{\mathbb{C}} =$ the complexification of L_n , which we shall regard as a left-invariant differential operator on $\mathbb{C}^\infty(N_n)$. The vectors

$$(15) \quad \begin{aligned} X_j &= \partial / \partial x_j + (y_j/2) \partial / \partial z, & 1 \leq j \leq n, \\ Y_j &= \partial / \partial y_j - (x_j/2) \partial / \partial z, & 1 \leq j \leq n, \\ Z &= \partial / \partial z \end{aligned}$$

determine a \mathbb{C} -basis of $L_n^{\mathbb{C}}$. The action of \mathcal{Q}_n^0 on $L_n^{\mathbb{C}}$ is given as follows. For $\xi = (x, y) \in \mathbb{C}^{2n}$ and $z \in \mathbb{C}$, put

$$(\xi, z) = \mathbf{x}' \cdot \mathbf{X} + \mathbf{y}'\mathbf{Y} + z\mathbf{Z} = \sum x_j X_j + \sum y_j Y_j + z\mathbf{Z}$$

where

$$\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{C}^n, \quad \mathbf{X} = \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix}$$

and corresponding notation for y 's. Let

$$\alpha = \begin{pmatrix} S & 0 \\ \delta' & d \end{pmatrix},$$

$d > 0$, be in \mathcal{O}_n^0 . Then

$$(16) \quad \alpha(\xi, z) = (S\xi, dz + \delta' \cdot \xi).$$

In particular, we will apply (16) to positive definite CR-structures. Let β be a positive definite CR-structure. Then β determines a direct sum decomposition of $L_n^{\mathbb{C}}$,

$$(17) \quad L_n^{\mathbb{C}} = V_i(\beta) \oplus V_{-i}(\beta) \oplus Z_{\mathbb{C}},$$

where $V_{\pm i}(\beta)$ is the eigenvalue $\pm i$ subspace of $L_n^{\mathbb{C}}$ with respect to β and $Z_{\mathbb{C}}$ = center of $L_n^{\mathbb{C}}$. Clearly, $\overline{V_i(\beta)} = V_{-i}(\beta)$. For reasons which will be made apparent below we will be solely concerned with the space $V_i(\beta)$.

Define

$$(18) \quad \theta(\beta) = \{G \in \mathbb{C}^\infty(N_n) : V_i(\beta)G = 0\}.$$

We call a function G a β -theta function whenever $G \in \theta(\beta)$. The automorphism group \mathcal{O}_n^0 relates the various subspaces of β -theta functions. Firstly, we can write $\beta = \alpha^{-1}J_{2n}\alpha$ where $\alpha \in \mathcal{O}_n^0$. The choice of α is, of course, not unique but is determined uniquely up to left multiplication by an element from $\mathcal{O}(2n, R)$ and multiplication by elements from \mathcal{D}_{2n} . Obviously,

$$(19) \quad V_i(\beta) = \alpha^{-1}(V_i(J_{2n})), \quad \theta(\beta) = \theta(J_{2n}) \cdot \alpha.$$

The first equality is purely formal while the second follows by the chain rule.

We want an explicit basis for $V_i(\beta)$. We begin with $V_i(J_{2n})$. Let $\mathbf{v} = (\xi, z) \in V_i(J_{2n})$, $\xi = (\mathbf{x}, \mathbf{y})$. It follows that $J_{2n}(\mathbf{v}) = i\mathbf{v}$ which implies $\mathbf{y} = i\mathbf{x}$ and $z = 0$. Thus, the n vectors

$$(20) \quad \mathbf{X} + i\mathbf{Y}$$

define a \mathbb{C} -basis of $V_i(J_{2n})$. Let

$$\beta = \begin{pmatrix} S & 0 \\ \gamma' & 1 \end{pmatrix}$$

be an arbitrary positive definite CR-structure where $S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ and $\gamma = (\gamma',$

$\gamma'' \in R^{2n}$. We want to describe a basis of $V_i(\beta)$ in terms of S and γ . For $v = (\xi, z) \in V_i(\beta)$, $\beta(v) = iv$ implies

$$z = \left(\frac{i+1}{2}\right)\gamma' \cdot \xi \quad \text{and} \quad y = B^{-1}(iE_n - A)x = (iE_n - D)^{-1}Cx.$$

Our previous characterization of positive definite CR-structures allows the taking of inverses. Hence, the n vectors

$$(21) \quad X + iG(\beta)Y + ((i+1)/\sqrt{2})(\gamma' + i\gamma'')Z,$$

where $G(\beta) = B^{-1}(iA + E_n)$, define a \mathbb{C} -basis of $V_i(\beta)$. Observe that B is symmetric and positive definite and AB is symmetric, implying that $G(\beta)$ is symmetric and has a positive definite real part.

For a positive definite CR-structure β we can write $\beta = \alpha^{-1}J_{2n}\alpha$, $\alpha \in \mathcal{Q}_n^0$. We want to write (21) in terms of α . Firstly, we will digress to give an interpretation of $\mathcal{Q}_n^0[J_{2n}] = \{\alpha^{-1}J_{2n}\alpha : \alpha \in \mathcal{Q}_n^0\}$ which put our eventual formulas into their proper context. Let \mathfrak{Z}_n be the space of $n \times n$ complex symmetric matrices $T = X + iY$ where T is positive definite. The space \mathfrak{Z}_n is called the n -dimensional Siegel upper half plane. $\text{Sp}(2n, R)$ acts on \mathfrak{Z}_n under fractional linear transformations defined by the rule

$$(22) \quad (T)M = (A + TC)^{-1}(B + TD),$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}(2n, R)$, $T \in \mathfrak{Z}_n$. A standard fact is $(T)M \in \mathfrak{Z}_n$ whenever $T \in \mathfrak{Z}_n$. Also, $\text{Sp}(2n, R)$ acts transitively on \mathfrak{Z}_n and, hence, the mapping $M \mapsto (iE_n)M$ induces a bijection between $K \setminus \text{Sp}(2n, R)$ and \mathfrak{Z}_n , where $K = \text{Sp}(2n, R) \cap \mathfrak{o}(2n, R)$. Recall, we have set $\mathcal{P}_{2n}^* = \text{Sp}(2n, R)[J_{2n}]$, and it follows immediately that the mapping $M \mapsto M^{-1}J_{2n}M$ induces a bijection between $K \setminus \text{Sp}(2n, R)$ and \mathcal{P}_{2n}^* .

LEMMA 2.5. Let $\beta = M^{-1}J_{2n}M$, $M \in \text{Sp}(2n, R)$. Then $iG(\beta)^{-1} = (iE_n)M$.

PROOF. Setting

$$M = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

then

$$C = -(A_1' - iC_1')(A_1 + iC_1), \quad B = (D_1' + iB_1')(D_1 - iB_1)$$

and

$$iA + E_n = (D_1' + iB_1')(A_1 + iC_1).$$

The lemma follows from (21) and (22).

An important property about $\theta(\beta)$ which follows directly by the chain rule is asserted in the next lemma.

LEMMA 2.6. $L_v\theta(\beta) = \theta(\beta)$ for all $\beta \in \mathcal{P}_{2n}$ and $v \in N_n$.

We are mainly interested in those functions $F \in \theta(\beta)$ which are Γ -periodic where Γ is a lattice. For $\beta \in \mathcal{P}_{2n}$ and $\Gamma \in \mathcal{L}$ define

$$(23) \quad \theta(\beta, \Gamma) = \theta(\beta) \cap \mathcal{L}^2(\Gamma \setminus N_n), \quad \theta_m(\beta, \Gamma) = \theta(\beta) \cap H_m(\Gamma).$$

We will eventually describe these spaces in terms of β and Γ .

LEMMA 2.7. For $F \in \theta(\beta, \Gamma)$, the projection $p_m(F)$ into $H_m(\Gamma)$ is in $\theta_m(\beta, \Gamma)$.

PROOF. It is straightforward to verify that

$$V_i(\beta)p_m(F)(\xi, z) = \int_0^{\beta(\Gamma)} e^{-2\pi i(m \cdot s)/\beta(\Gamma)} V_i(\beta)F(\xi, z + s) ds,$$

which immediately implies the lemma.

COROLLARY. $\theta(\beta, \Gamma) = \sum_{m \in \mathbb{Z}} \oplus \theta_m(\beta, \Gamma)$ in the \mathcal{L}^2 -norm.

Again, the automorphism group will be utilized to relate the various spaces. Suppose $\beta \in \mathcal{P}_{2n}$ and $\Gamma \in \mathcal{L}$. Then $\beta = \alpha^{-1}J_{2n}\alpha$, $\alpha \in \mathcal{O}_n^0$, and $\Gamma = \gamma(\Gamma(I))$, where $\gamma \in \mathcal{O}_n^0$ and $I \in \mathbb{Z}_n^\#$. Then

$$(24) \quad \theta(\beta, \Gamma) = \theta(\gamma\alpha^{-1}J_{2n}\alpha\gamma^{-1}, \Gamma(I)) \cdot \gamma^{-1}.$$

The first use we make of (24) is to reduce the problem of computing the subspaces $\theta(\beta, \Gamma)$ to the special case where $\Gamma = \Gamma(I)$ and β is arbitrary or the special case where Γ is arbitrary and $\beta = J_{2n}$.

LEMMA 2.8. $\theta_0(\beta, \Gamma) = \mathbb{C}$.

PROOF. Assume $\beta = J_{2n}$. Then if $F \in \theta_0(\beta, \Gamma)$ it is easy to see that $F(\xi)$, $\xi = x + iy \in \mathbb{C}^n$, is entire. Since it is also periodic, it must be a constant.

Before we consider $\theta_m(\beta, \Gamma)$ for $m > 0$ we remark on the following description of $H_1(\Gamma_n)$. Each $F \in H_1(\Gamma_n)$ can be written

$$(25) \quad F(\xi, z) = e^{2\pi iz} e^{+\pi ix \cdot y} F_1(\xi), \quad \xi = (x, y),$$

where

$$F_1(\xi) = \sum_{l \in \mathbb{Z}^n} f(y + l) e^{2\pi i l \cdot x} \quad \text{and} \quad f(y) = \int_0^{E_n} F_1(\xi) dx.$$

The function $F_1(\xi)$ will be denoted by $P(f)(\xi)$ to show its dependence on f . Clearly, $f \in \mathcal{L}^2(\mathbb{R}^n)$ and is not uniquely determined by the condition $F = e^{2\pi iz} e^{-\pi ix \cdot y} P(f)$.

For

$$\beta = \begin{pmatrix} S & 0 \\ \gamma^t & 1 \end{pmatrix} \in \mathcal{P}_{2n}$$

define the function on N_n by the rule

$$(26) \quad \psi(\beta, \Gamma_n)(\xi, z) = e^{2\pi iz} e^{-\pi i x y} P(f)(\xi),$$

where

$$f(y) = \exp\left\{\pi i\left(iG(\beta)^{-1}[y] + (i - 1)G(\beta)^{-1}(\gamma' + i\gamma'') \cdot y\right)\right\}.$$

THEOREM 2.9. $\theta_1(\beta, \Gamma_n)$ is a 1-dimensional \mathbb{C} -vector space and the function $\psi(\beta, \Gamma_n)$ exists and is in $\theta_1(\beta, \Gamma_n)$.

PROOF. The real part of $G(\beta)$ being symmetric and positive definite immediately implies the function in (26) exists and hence is in $H_1(\Gamma_n)$. We shall now show that it is, in fact, up to constant multiple the unique function in $\theta_1(\beta, \Gamma_n)$. Assume $F \in \theta_1(\beta, \Gamma_n)$. Since $F \in C^\infty(N_n)$ it follows $P(f)$ is also infinitely differentiable. Hence, $f(y) = \int_0^E P(f) dx$ is infinitely differentiable and its derivatives may be computed by differentiating inside the integral. Using $Y(e^{2\pi iz} e^{-\pi i x y}) = 0$, condition (21) becomes

$$iG(\beta)YP(f) = -2\pi iy P(f) - XP(f) + (i + 1)\pi i(\gamma' + i\gamma'')P(f).$$

However, $\int_0^E XP(f) dx = 0$ by periodicity in x . Hence,

$$iG(\beta)\partial f(y)/\partial y = (-2\pi iy + (i + 1)\pi i(\gamma' + i\gamma''))f(y),$$

and, as is standard knowledge, this equation has a unique solution, up to constant multiple, given by the formula in (26).

COROLLARY. Let $\Gamma = \Gamma(I)$ and $\alpha \in B_m(\Gamma)$, $m > 0$. Then

$$\theta_m(\beta, \Gamma) = \sum_{\gamma^*} L_{\gamma^*}(\theta(\alpha\beta\alpha^{-1}, \Gamma_n) \cdot \alpha)$$

where γ^* is as in Theorem 2.4. In particular,

$$\dim_{\mathbb{C}} \theta_m(\beta, \Gamma) = m^n \mathfrak{P}(\Gamma).$$

PROOF. Simply apply Lemma 2.7, Theorem 2.4 and (19) to the previous theorem.

COROLLARY. To each distinguished subspace \mathcal{G} of $H_m(\Gamma_n)$,

$$\dim_{\mathbb{C}}(\mathcal{G} \cap \theta_m(\beta, \Gamma_n)) = \mathfrak{P}(\Gamma_n) = 1.$$

Before continuing we will make several observations about these results. Firstly, although we gave an explicit function $\psi_1(\beta, \Gamma_n)$ to describe $\theta_1(\beta, \Gamma_n)$, this function is, in fact, explicitly associated to the pair (β, Γ_n) . Also, the question of convergence, i.e. uniform convergence on compact subsets of the series in (26), immediately indicates $\theta_m(\beta, \Gamma_n) = 0$ whenever $m < 0$.

Thus far we have proceeded by describing the function theory related to a fixed choice of a lattice in N_n . The final topic of this section involves the following general question. What information does the function theory give us about the lattices in N_n ?

LEMMA 2.10. *Let Γ be a lattice in N_n and choose $\beta \in \mathcal{P}_{2n}$. Then*

$$\Gamma = \{v \in N_n: L_v F = F \text{ whenever } F \in \theta_1(\beta, \Gamma)\}.$$

PROOF. Suppose to begin with $\Gamma = \Gamma_n$. Then it is easy to see by direct computation that $\psi_1(\beta, \Gamma_n)$ can be fixed by $L_v, v \in N_n$, if and only if $v \in \Gamma_n$. For $\Gamma = \Gamma(I)$ we consider the functions $L_{\gamma^*}(\psi_1(\beta, \Gamma_n)), \gamma^*$ chosen as in Theorem 2.4 with $\alpha \equiv \text{identity}$. The lemma follows by definition in light of our remark above concerning Γ_n .

COROLLARY. *Let Γ be a lattice in N_n . Then*

$$\Gamma(m) = \{v \in N_n: L_v|H_m(\Gamma) = \text{identity}\}.$$

Conversely, $H_m(\Gamma) = \{F \in \mathcal{L}^2(\Gamma \setminus N_n): L_\gamma F = F \text{ whenever } \gamma \in \Gamma(m)\}$.

PROOF. It is not difficult to see that the smallest \mathcal{R} -invariant subspace containing $\theta_1(\beta, \Gamma)$ is $H_1(\Gamma)$. Hence, since \mathcal{L} and \mathcal{R} commute, the previous lemma implies $\Gamma = \{v \in N_n: L_v F = F \text{ whenever } F \in \theta_1(\beta, \Gamma)\}$. The corollary follows.

COROLLARY. *For two lattices Γ_1, Γ_2 in N_n , we have $\Gamma_1(m) = \Gamma_2(m)$ if and only if $H_m(\Gamma_1) = H_m(\Gamma_2)$.*

3. Classical theta function theory. Let D be a lattice in \mathbb{C}^n . The central problem is the question of the existence of nontrivial meromorphic functions on the complex torus $D \setminus \mathbb{C}^n$. As is well known, this problem is equivalent to the question of the existence of a nontrivial theta function (defined below) on $D \setminus \mathbb{C}^n$. In this section we will indicate how the classical theory can be developed in the framework of the Heisenberg group. We begin by recalling some of the essential features of the classical theory. For a complete treatment see Lang [4].

For a lattice $D \subset \mathbb{C}^n$, by a *Riemann form* on \mathbb{C}^n with respect to D we mean an R -bilinear form $\tilde{A}: \mathbb{C}^n \times \mathbb{C}^n \rightarrow R$ which satisfies the following properties.

- (1)(a) The form \tilde{A} is alternating.
- (b) It takes integral values on $D \times D$.
- (c) The form $(x, y) \rightarrow \tilde{A}(ix, y)$ is symmetric and positive semidefinite.

We shall work solely with *nondegenerate* Riemann forms which have the added condition that the form in (c) is both symmetric and positive definite. Not every lattice $D \subset \mathbb{C}^n, n > 1$, admits a nondegenerate Riemann form. For those lattices D which do, we call the corresponding complex torus $D \setminus \mathbb{C}^n$ an *abelian manifold*. For the purpose of this work we shall assume that if $D \setminus \mathbb{C}^n$ has n algebraically independent meromorphic functions defined on it, then D admits a nondegenerate Riemann form. The question we pose is the converse: Show that if D admits a nondegenerate Riemann form, then $D \setminus \mathbb{C}^n$ has defined on it n algebraically independent meromorphic functions. The proof

we give is essentially that of C. L. Siegel except in a new framework. We will also investigate the structure of these function spaces.

Recalling the identification $\mathbb{C}^n \cong \mathbb{R}^{2n}$ above we can associate to the form \tilde{A} a $2n \times 2n$ \mathbb{R} -matrix, also denoted by A , uniquely determined by the condition $\tilde{A}(\xi, \eta) = (\eta_1, \eta_2)'A(\xi_1, \xi_2)$, where $\xi = \xi_1 + i\xi_2$, $\eta = \eta_1 + i\eta_2$. We will now describe those lattices D for which $D \setminus \mathbb{C}^n$ is an abelian manifold.

The condition for A to be a nondegenerate Riemann form with respect to a lattice D is given in matrix language by

- (2)(a) $A' = -A$.
- (b) $\eta' A \xi \in \mathbb{Z}$ whenever $\xi, \eta \in D$.
- (c) AJ'_{2n} is symmetric and positive definite.

LEMMA 3.1. *Let A be a $2n \times 2n$ \mathbb{R} -matrix satisfying (a) and (c) of (2). Then there exists an \mathbb{R} -matrix C satisfying $A = C'J_{2n}C = C'CJ_{2n}$.*

PROOF. Since $A' = -A$ and AJ'_{2n} is symmetric, A has the property that $AJ_{2n} = J_{2n}A$. Since, moreover, $J'_{2n}A = AJ'_{2n}$ is positive definite, it follows that $Y + iX = \bar{C}'_1C_1$, where $A = \begin{pmatrix} X & Y \\ -Y & X \end{pmatrix}$ and $C_1 = C'_1 + iC''_1$ (see Lang [4, p. 374]). Put

$$C = \begin{pmatrix} C'_1 & -C''_1 \\ C''_1 & C'_1 \end{pmatrix}$$

and the lemma follows.

The condition on the $2n \times 2n$ real matrix C , $J_{2n}C = CJ_{2n}$, implies that C induces a \mathbb{C} -linear mapping on \mathbb{C}^n . It follows that if we are given a lattice D in \mathbb{C}^n admitting a Riemann form A , then a \mathbb{C} -linear mapping C of \mathbb{C}^n exists such that $C(D)$ is a lattice in \mathbb{C}^n and $J_{2n} = (C^{-1})'AC^{-1}$ is the corresponding Riemann form. In particular, we can restrict our attention to the Heisenberg lattices D of \mathbb{C}^n considered in §1. We can summarize this discussion as follows.

LEMMA 3.2. *Let D be a lattice admitting a Riemann form A . Then there is a \mathbb{C} -linear mapping C of \mathbb{C}^n having determinant 1 such that $C(D) \in hL(\mathbb{C}^n)$ and $J_{2n} = (C^{-1})'AC^{-1}$.*

Eventually, we will relate function theory of $\Gamma \setminus N_n$ to that of $D \setminus \mathbb{C}^n$ where $\pi\Gamma = D$. Essentially, Γ -periodic functions on the 'nonabelian' object $\Gamma \setminus N_n$ are transformed into functions on \mathbb{C}^n which, although not D -periodic, satisfy a functional equation with respect to the action of D .

Recall the short exact sequence $1 \rightarrow Z \rightarrow N_n \xrightarrow{\pi} \mathbb{C}^n \rightarrow 1$ where $\pi(\xi, z) = \xi$. For purposes of the classical theory, it is most natural to consider the spaces

$$(3) \quad \theta_m(\Gamma) = \theta_m(J_{2n}, \Gamma), \quad m \in \mathbb{Z}^*_+.$$

We begin by considering results which do not depend on a lattice. Put

$$(4) \quad \theta_d = \theta(J_{2n}) \cap C_d^\infty(N_n)$$

where $C_d^\infty(N_n)$ consists of all $F \in C^\infty(N_n)$ of the form $F(\xi, z) = e^{2midz}F_1(\xi)$, $d > 0$ and real. Set

$$(5) \quad M_d = M_d(\xi, z) = e^{-2midz}e^{-midxy}e^{\pi dy \cdot y}, \quad \xi = (x, y) \in R^{2n}.$$

LEMMA 3.3. *The mapping $F(\xi, z) \rightarrow M_d(\xi, z)F(\xi, z)$ defines an isomorphism from θ_d onto $E(C^n)$, the entire function on C^n .*

PROOF. Let $F(\xi, z) = e^{2nidz}F_1(\xi) \in \theta_d$. Since $(X + iY)F = 0$, we conclude F_1 satisfies $\pi d(x + iy)F_1 + (\partial/\partial x + i\partial/\partial y)F_1 = 0$. This latter result implies that

$$M_d(\xi, z)F(\xi, z) = e^{\pi dy \cdot y}e^{-midxy}F_1(\xi)$$

is a solution of the Cauchy-Riemann equations. The converse follows by reversing the steps.

COROLLARY. $M_m(\xi, z)\theta_m(\Gamma) \subset E(C^n)$ for all lattices Γ in N_n .

For $F \in \theta_m(\Gamma)$, $M_m(\xi, z)F(\xi, z)$ is entire. We will now translate the Γ -periodicity of F to the entire function $M_m(\xi, z)F(\xi, z)$. To each lattice Γ of N_n , $\pi(\Gamma)$ is a Heisenberg lattice in C^n . Set $D = \pi(\Gamma)$. There are many lattices in N_n which determine D in this way. These problems have already been accounted for in the classical theory. We will see how they translate into our language.

Let D be a Heisenberg lattice and set $d = \beta(D)$. Recall $J_{2n}(D \times D) = d \cdot Z$, $d > 0$. Let $K: C^n \rightarrow R$ be an R -linear mapping and $B: C^n \times C^n \rightarrow R$ an R -bilinear form satisfying

- (6)(a) $B - B' = J_{2n}$.
- (b) $B(D \times D) \subset dZ$.

Corresponding to the pair K and B we shall describe a lattice Γ in N_n satisfying $\pi\Gamma = D$. Firstly, condition (a) implies $B = G + \frac{1}{2}J_{2n}$ where G is a symmetric R -bilinear form. Define the section $\rho: C^n \rightarrow N_n$ to π by the formula

$$(7) \quad \rho(\xi) = (\xi, K(\xi) + \frac{1}{2}B(\xi, \xi)), \quad \xi \in C^n.$$

LEMMA 3.4. *Let $K: C^n \rightarrow R$ be an R -linear map and $B: C^n \times C^n \rightarrow R$ an R -bilinear form satisfying (6). Let ρ be the associated section. Then $\Gamma = \text{gp}(\rho(D))$ is a lattice in N_n and $\pi\Gamma = D$.*

PROOF. Without loss of generality, we can assume $D = D(I)$, $I = (I_1, I_2, \dots, I_n) \in Z_n^\#$. Put

$$\Gamma' = \text{gp}(\rho(e_j), \rho(I_j e_{j+n}))_{1 \leq j \leq n}.$$

We must show $\rho(D) \subset \Gamma'$. Take $\gamma \in \rho(D)$ and write

$$\gamma = (\xi, K(\xi) + \frac{1}{2}G(\xi, \xi)), \quad \xi \in \mathbb{C}^n.$$

Write $\xi = (x, y)$ and put

$$\gamma' = (\xi, K(\xi) + \frac{1}{2}(G(x, x) + G(y, y) + J_{2n}(x, y)))$$

which clearly lies in Γ' . Since $\gamma^{-1}\gamma' = (0, G(x, y) - \frac{1}{2}J_{2n}(x, y))$ by (b), $\gamma^{-1}\gamma' \in [\Gamma', \Gamma'] \subset \Gamma'$ and we are done.

Define for a lattice Γ in N_n ,

$$(8) \quad C_m^{(r)}(\Gamma) = C^{(r)}(N_n) \cap H_m(\Gamma).$$

LEMMA 3.5. *Let $D \in HL(\mathbb{C}^n)$ and choose K and B and Γ as in the previous lemma. Then $M_m(\xi, z)C_m^{(r)}(\Gamma)$ consists of the space of all $C^{(r)}$ -functions θ on \mathbb{C}^n which satisfy the following functional equation:*

$$(9) \quad \theta(\xi + \eta) = \theta(\xi)e^{-2\pi im(L(\xi, \eta) + K(\eta) + B(\eta, \eta)/2)}e^{-\pi imL(\eta, \eta)},$$

where $\xi = x + iy \in \mathbb{C}^n$, $\eta = r + is \in D$, $L(\xi, \eta) = s \cdot \xi$.

PROOF. Let $F \in C_m^{(r)}(\Gamma)$ and write

$$\theta(\xi) = e^{-\pi imx \cdot y}e^{\pi my \cdot y}F(\xi, 0).$$

Then

$$\theta(\xi + \eta) = e^{-\pi im(x+r)(y+s)}e^{\pi m(y+s)(y+s)}F(\xi + \eta, 0).$$

However, $(\eta, K(\eta)) + \frac{1}{2}(B(\eta, \eta)) \in \Gamma$ and, using the Γ -periodicity of F , we get

$$F(\eta + \xi, K(\eta) + \frac{1}{2}B(\eta, \eta) + \frac{1}{2}J(\eta, \xi)) = F(\xi, 0).$$

Hence,

$$\theta(\xi + \eta) = \theta(\xi) \cdot e^{-\pi im(r\gamma + sx + rs)}e^{\pi m(2sy + s)}e^{-2\pi im(K(\eta) + B(\eta, \eta)/2 + J(\eta, \xi)/2)},$$

which can be simplified to give the lemma.

Define for K, B satisfying (6) with respect to the lattice D ,

$$(10) \quad \text{Th}^{(m)}(K, B; D) = \text{the space of all entire functions } \theta \text{ on } \mathbb{C}^n \text{ satisfying (9).}$$

Lemmas 3.3 and 3.5 immediately imply the following.

THEOREM 3.6. *Let $D \in H\mathcal{L}(\mathbb{C}^n)$ and choose K, B to satisfy (6) with respect to D . Let Γ be the corresponding lattice with respect to K, B and D . Then*

$$(11) \quad M_m(\xi, z)\theta_m(\Gamma) = \text{Th}^{(m)}(K, B; D).$$

COROLLARY. $\dim_{\mathbb{C}}\text{Th}^{(m)}(K, B; D) = |m|^n \cdot \mathcal{P}(D)^{1/2}$.

We close this section with a discussion of how our methods have determined all classes of theta functions usually considered, as, for example,

in Lang [5]. There the following definition is made.

Let D_1 be a lattice admitting a nondegenerate Riemann form A . Choose:

(12)(a) An R -bilinear form $B_1: \mathbb{C}^n \times \mathbb{C}^n \rightarrow R$ satisfying $B_1(D_1 \times D_1) \subset \mathbb{Z}$ and $A = B_1 - B_1'$.

(b) An R -bilinear form $L_1: \mathbb{C}^n \times \mathbb{C}^n \rightarrow \mathbb{C}$ which is \mathbb{C} -linear in its first variable and $A = L_1 - L_1'$.

(c) An R -linear map $K_1: \mathbb{C}^n \rightarrow R$.

We denote by $\text{Th}(L_1, K_1, B_1; D_1)$ the space of all entire functions $\theta(\xi)$ on \mathbb{C}^n satisfying the following functional equation.

$$(13) \quad \theta(\xi + \eta) = \theta(\xi) \exp\left\{-2\pi i\left(L_1(\xi, \eta) + \frac{1}{2}L_1(\eta, \eta) + K_1(\eta) + \frac{1}{2}B(\eta, \eta)\right)\right\},$$

$$\xi \in \mathbb{C}^n, \quad \eta \in D.$$

By Lemma 3.2 there exists a complex linear isomorphism C of \mathbb{C}^n such that the mapping $\theta \mapsto \theta \cdot C$ defines an isomorphism from $\text{Th}(L', K, B; D)$ onto $\text{Th}(L_1, K_1, B_1; D_1)$, where $L' = C'L_1C$, $K = C'K_1C$, $B = C'B_1C$ and $D = C^{-1}(D_1)$ is a lattice in \mathbb{C}^n having J_{2n} as its corresponding Riemann form. Let $L(\xi, \eta) = s \cdot (x + iy)$, $\xi = x + iy$, $\eta = r + is$. Then since $L' - L'' = L'' - L'$ we have $L = L'' + L'$, where $L'': \mathbb{C}^n \times \mathbb{C}^n \rightarrow \mathbb{C}$ is a symmetric \mathbb{C} -bilinear form. Hence $e^{\pi i L''(\xi, \xi)}$ is entire. Also

$$e^{\pi i L''(\xi, \xi)} \text{Th}(L', K, B; D) = \text{Th}(L, K, B; D).$$

The next theorem summarizes this discussion.

THEOREM 3.7. *Let D_1 be a lattice admitting a nondegenerate Riemann form A . Choose B_1, L_1, K_1 as in (12). Then there exists a \mathbb{C} -linear map C of \mathbb{C}^n and a symmetric \mathbb{C} -bilinear form L'' such that the mapping $\theta \rightarrow e^{\pi i L''(\xi, \xi)}\theta(C\xi)$ defines a \mathbb{C} -isomorphism between $\text{Th}(L_1, K_1, B_1; D_1)$ and $\text{Th}^{(1)}(K, B; D)$, where $D = C^{-1}(D_1)$ is a Heisenberg lattice in \mathbb{C}^n and K, B satisfy (6) with respect to D .*

Suppose now that D is a fixed lattice admitting a Riemann form. To each Riemann form A_1 on D and choice of L_1, K_1, B_1 satisfying condition (12) with respect to D and A_1 we can form the space $Q(\text{Th}(L_1, K_1, B_1; D))$ of quotients of elements from $\text{Th}(L_1, K_1, B_1; D)$. Define

$$A(D \setminus \mathbb{C}^n) = \sum Q(\text{Th}(L_1, K_1, B_1; D))$$

where we sum over all A_1, L_1, K_1, B_1 as above. We call $A(D \setminus \mathbb{C}^n)$ the space of abelian functions on $D \setminus \mathbb{C}^n$.

THEOREM 3.8. *Let D be a lattice admitting a nondegenerate Riemann form and Γ a lattice satisfying $\pi\Gamma = D$. Then*

$$A(D \setminus \mathbb{C}^n) = \sum_{j>0} Q(\theta_j(\Gamma)).$$

PROOF. Fix K, B satisfying (6) with respect to D . For any nondegenerate Riemann form A_1 on D and L_1, K_1, B_1 satisfying conditions (12) relative to D, A_1 , there is an integer $m > 0$ such that $L_2 = mL - L_1$ is an R -bilinear form, C -linear in its first variable and $A_2 = L_2 = L_2^+ = mJ_{2n} - A_1$ is a Riemann form relative to D . Clearly we can form $\text{Th}(mL - L_1, mK - K_1, mB - B_1; D)$ and choose a nonzero α in this space. For any $f, g \in \text{Th}(L_1, K_1, B_1; D)$ both $\alpha f, \alpha g \in \text{Th}(mL, mK, mB; D)$. Let Γ be the lattice determined by K, B as in Theorem 3.6. Then $\text{Th}(mL, mK, mB; D) = \text{Th}^{(m)}(K, B; D)$ which equals $Mm(\xi, z)\theta_m(\Gamma)$.

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