TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 241, July 1978

ON THE GROWTH OF THE INTEGRAL MEANS OF SUBHARMONIC FUNCTIONS OF ORDER LESS THAN ONE[1]

BY

FARUK F. ABI-KHUZAM

ABSTRACT. Let u be a subharmonic function of order λ $(0 < \lambda < 1)$, and let $m_s(r, u) = \{(1/2\pi)\int_{-\pi}^{\pi} |u(re^{i\theta})|^s \, d\theta\}^{1/s}$. We compare the growth of $m_s(r, u)$ with that of the Riesz mass of u as measured by $N(r, u) = (1/2\pi)\int_{-\pi}^{\pi} u(re^{i\theta}) \, d\theta$. A typical result of this paper states that the following inequality is sharp:

$$\liminf_{r\to\infty} \frac{m_s(r, u)}{N(r, u)} \leqslant m_s(\psi_\lambda) \tag{$*$}$$

where $\psi_\lambda(\theta) = (\pi\lambda/\sin \pi\lambda)\cos \lambda\theta$.

The case $s = 1$ is due to Edrei and Fuchs, the case $s = 2$ is due to Miles and Shea and the case $s = \infty$ is due to Valiron.

Introduction. Let f be a meromorphic function of finite order λ and let $\log M(r, f)$, $T(r, f)$, $N(r, 0)$, $N(r, \infty)$ be the basic functionals in Nevanlinna theory associated with f. The problems of finding sharp asymptotic inequalities for ratios of these functionals originated and were investigated by Valiron [**14**], Polya, Nevanlinna and others.

Recently, Miles and Shea [**10**, p. 377] used Fourier series techniques to obtain sharp bounds for an L_2 version of these problems. They used their result to get the best bounds yet in the L_1 case–very close to the conjectured sharp bound for this still open problem due to Nevanlinna [**11**, p. 54]. When the order of f is less than one the Nevanlinna problem was completely solved by Edrei and Fuchs [**5**] and for entire functions of any finite order with zeroes on a ray by Hellerstein and Williamson [**8**].

In this paper we consider an L_s $(1 \leqslant s < \infty)$ version of these problems:

Let u be a subharmonic function in the plane. Put

$$m_s(r, u) = \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} |u(re^{i\theta})|^s \, d\theta \right\}^{1/s}, \tag{1}$$

$$N(r, u) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(re^{i\theta}) \, d\theta \tag{2}$$

Received by the editors July 31, 1975 and, in revised form, February 4, 1977.

AMS (MOS) subject classifications (1970). Primary 30A64, 31A10; Secondary 30A68.

Key words and phrases. Riesz mass, order of a subharmonic function, proximate order.

[1] This research was supported in part by NSF Grant A78-8319.

and consider the following

Problem 1. *Determine sharp upper bounds for*

$$\liminf_{r\to\infty} \frac{m_s(r, u)}{N(r, u)} \qquad (1 \leqslant s < \infty) \tag{3}$$

where u is a subharmonic function of finite nonintegral order λ.

Problem 2. *Determine sharp lower bounds for*

$$\limsup_{r\to\infty} \frac{m_s(r, u)}{N(r, u)} \qquad (1 \leqslant s < \infty) \tag{4}$$

where u is a subharmonic function whose Riesz mass is distributed along a ray and whose order is finite.

By combining the methods of Edrei and Fuchs and Miles and Shea, I have obtained a complete solution of Problem 1 for subharmonic functions of order less than one, and for a class of Δ-subharmonic functions of order less than one. I have also obtained a complete solution of Problem 2. For functions of order greater than one, Problem 1 remains unsolved.

In concluding this introduction, I wish to express my sincere gratitude to Professor Albert Edrei; most of the ideas in this paper were developed while I was a student under his guidance. I am also grateful to the referee for various suggestions and comments to improve this paper.

1. Summary and notation. Consider a function $w = u - v$, where u and v are subharmonic in the plane and harmonic in a neighbourhood of the origin. Let μ and ν be the Riesz masses of u and v respectively, and let

$$n(t, u) = \int_{|a|<t} d\mu(a), \qquad n(t, v) = \int_{|a|<t} d\nu(a). \tag{1.1}$$

Define $N(r, \cdot\,)$ by

$$N(r, \cdot\,) = \int_0^r n(t, \cdot\,)t^{-1}\, dt \tag{1.2}$$

and put

$$T(r) = T(r, w) = \frac{1}{2\pi} \int_{-\pi}^{\pi} w^+(re^{i\theta})\, d\theta + N(r, v). \tag{1.3}$$

$T(r)$ is called the characteristic of w; the order λ and the lower order μ of w are defined in terms of $T(r)$ by:

$$\lambda = \limsup_{r\to\infty} \frac{\log T(r)}{\log r}, \qquad \mu = \liminf_{r\to\infty} \frac{\log T(r)}{\log r}. \tag{1.4}$$

(This double use of the letter μ should not give rise to any confusion.)

Although $T(r)$ as defined by (1.3) is not unique, it is an easy matter to obtain a unique characteristic for the function w. Indeed, all that is needed is to be able to construct subharmonic functions u and v such that their Riesz masses are respectively equal to the positive and negative parts of the Riesz mass of w and then define the characteristic from this special representation $w = u - v$.

From now on we shall assume that this has been done. Thus under consideration is a (δ-subharmonic) function $w = u - v$ where,

(i) u and v are subharmonic in the plane and harmonic in a neighbourhood of the origin with $w(0) = 0$;

(ii) the Riesz mass μ of u equals the positive part of the Riesz mass of w, and the Riesz mass ν of v equals the negative part of the Riesz mass of w;

(iii) the order λ of w is finite and nonintegral.

We denote by $\mathfrak{D}$ the class of all functions w satisfying (i), (ii) and (iii).

For $w \in \mathfrak{D}$, define α_m by

$$w(z) = \mathrm{Re}\left\{ \sum_{m=1}^{\infty} \alpha_m z^m \right\} \tag{1.5}$$

for z near 0.

Since the order λ of w is finite, we may then write

$$\begin{aligned} w(z) = \mathrm{Re}(p(z)) &+ \int_{|a|<\infty} \log|E(z/a, q)|\, d\mu(a) \\ &- \int_{|a|<\infty} \log|E(z/a, q)|\, d\nu(a); \end{aligned} \tag{1.6}$$

where $q = [\lambda]$, $p(z) = \alpha_q z^q + \cdots + \alpha_1 z$, and $E(x, q)$ is the Weierstrass primary factor of genus q.

The characteristic of w as defined in (1.3) was introduced by Privaloff **[12]** who also established the following:

(a) $T(r, w)$ is a nondecreasing function of r;

(b) $T(r, w) = o(\log r)$ implies that w is a constant;

(c) $T(r, w) = O(\log r)$ is a necessary and sufficient condition for w to have the form

$$w(z) = \int \log|z - a|\, d\mu(a) - \int \log|z - a|\, d\nu(a) + \text{constant}$$

where the mass distributions μ and ν are bounded.

The properties (a), (b) and (c) show that $T(r)$ gives a great deal of information about the function w and lead naturally to the consideration of functions w of finite order defined by (1.4).

In order to state our results we use the notation $\psi_\lambda(\theta) = \pi\lambda \csc \pi\lambda \cos(\lambda\theta)$; then we have

THEOREM 1. *Let* $w \in \mathfrak{D}$ *be subharmonic of order* λ $(0 < \lambda < 1)$; *then*

$$\liminf_{r\to\infty} \frac{m_s(r, u)}{N(r, u)} \leqslant m_s(\psi_\lambda) \qquad (1 \leqslant s < \infty). \tag{1.7}$$

This inequality is sharp.

THEOREM 2. *Let $w \in \mathfrak{D}$ be subharmonic. If the Riesz mass of w is distributed along a ray and if its order λ is nonintegral, then*

$$\limsup_{r\to\infty} \frac{m_s(r, u)}{N(r, u)} \geqslant m_s(\psi_\lambda) \qquad (1 \leqslant s < \infty). \tag{1.8}$$

This inequality is sharp.

Theorems 1 and 2 both hold true when w and ψ_λ are replaced by w^+ and ψ_λ^+ respectively. Also, both theorems hold true for $w \in \mathfrak{D}$, $(0 < \lambda < 1)$ satisfying the condition $N(r, u) = N(r, v)$; the result corresponding to Theorem 2 requiring the additional assumption that the masses μ and ν be distributed along the negative and positive x-axes respectively. Since the condition $N(r, u) = N(r, v)$ is somewhat artificial we omit the proofs.

For a general Δ-subharmonic function the following result will follow easily from theorems of Hardy and Littlewood [7]:

THEOREM 3. *Let $w \in \mathfrak{D}$ be of nonintegral order λ. Denote by $\{c_m(r)\}$ the Fourier coefficients of w and put $\tilde{w}(z) = w(z) - \sum_{|m| \leqslant q} c_m(r) e^{im\theta}$. Then*

$$\liminf_{r\to\infty} \frac{m_s(r, \tilde{w})}{N(r)} \leqslant m_s(\tilde{\psi}_\lambda) \tag{1.9}$$

where $N(r) = N(r, u) + N(r, v)$, $q = [\lambda]$, $s = 2k$, $k = 1, 2, 3, \ldots$ and $\tilde{\psi}_\lambda$ is defined analogously to $\tilde{w}$. Furthermore (1.9) *is sharp.*

We point out that the proof of Theorem 2 carries over to higher dimensions provided that appropriate restrictions are put on the index s. For example, if u is subharmonic in R^m where $m = 3$ or 4, then an analogue of Theorem 2 may be obtained for the range $1 \leqslant s \leqslant 2$.

2. **Preliminary lemmas.** Let $w \in \mathfrak{D}$ and denote by $c_m(r) = c_m(r, w)$ the Fourier coefficients of w, i.e.,

$$c_m(r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} w(re^{i\theta}) e^{-im\theta}\, d\theta. \tag{2.1}$$

Using (1.6) it is possible to compute these coefficients **[10]**, and one finds that

$$c_m(r) = \frac{1}{2}\alpha_m r^m + \frac{1}{2m} \int_{|a|<r} \left\{ \left(\frac{r}{a}\right)^m - \left(\frac{\bar{a}}{r}\right)^m \right\} d\mu(a)$$
$$- \frac{1}{2m} \int_{|a|<r} \left\{ \left(\frac{r}{a}\right)^m - \left(\frac{\bar{a}}{r}\right)^m \right\} d\nu(a) \tag{2.2}$$

for $m \geqslant 1$ and, for $m \geqslant q + 1$, also

$$c_m(r) = \frac{1}{2m}\left\{\int_{|a|>r}\left(\frac{r}{a}\right)^m d\mu(a) - \int_{|a|>r}\left(\frac{r}{a}\right)^m d\nu(a) + \int_{|a|<r}\left(\frac{\bar{a}}{r}\right)^m d\mu(a) - \int_{|a|<r}\left(\frac{\bar{a}}{r}\right)^m d\nu(a)\right\}, \quad (2.3)$$

where $q = [\lambda]$ and λ is the order of w.

For $m < 0$, $c_m(r) = \overline{c_{-m}(r)}$, and $c_0(r) = N(r, u) - N(r, v)$.

From (2.2) and (2.3) one easily obtains **[10]**

$$|c_0(r)| \leqslant N(r),$$

$$|c_m(r)| \leqslant \tfrac{1}{2}|\alpha_m|r^m + \tfrac{1}{2}m\int_0^r \{(r/t)^m - (t/r)^m\}N(t)\,dt/t + N(r) \quad (2.4)$$

$1 \leqslant m \leqslant q$, $q \neq 0$, and when $m \geqslant q + 1$

$$|c_m(r)| \leqslant \tfrac{1}{2}m\left\{\int_0^r (t/r)^m N(t)\,dt/t + \int_r^\infty (r/t)^m N(t)\,dt/t\right\} - N(r),$$

where $N(r) = N(r, u) + N(r, v)$.

LEMMA 2.1. *Let $c_m(r)$ be the Fourier coefficients of a function $w \in \mathfrak{D}$ of nonintegral order λ. Then there exists a slowly varying function L, a sequence r_n increasing to infinity, and two absolute constants M and r_0, such that*

(a) $$N(t) \leqslant t^\lambda L(t) = A(t) \quad (0 < t < \infty);$$

(b) $$N(r_n) = A(r_n);$$

(c) $$\frac{|c_m(r)|}{A(r)} \leqslant \frac{M}{|m| + 1} \qquad (r \geqslant r_0).$$

Here slowly varying means that L is positive and satisfies

$$\lim_{r\to\infty} L(\sigma r)/L(r) = 1$$

for every $\sigma > 0$.

LEMMA 2.2 **[6]**. *Let there be given two functions $\varphi_1(x)$ and $\varphi_2(x)$ defined on the interval $0 \leqslant x < \infty$, with $\varphi_2(x) \geqslant 0$. Let there also be given two numbers $\lambda \geqslant 0$ and $\varepsilon > 0$ such that*

(a) *both of the integrals*

$$\int_0^\infty \frac{|\varphi_1(x)|}{x^{\sigma+1}}\,dx \quad \textit{and} \quad \int_0^\infty \frac{\varphi_2(x)}{x^{\sigma+1}}\,dx$$

converge for $\lambda < \sigma < \lambda + \varepsilon$, and the second integral diverges for $\sigma < \lambda$;

(b) *there exists a function $\Psi(z)$, holomorphic in $|z - \lambda| < \varepsilon$ and real for real z, such that for $\lambda < \sigma < \lambda + \varepsilon$*

$$\int_0^\infty \frac{\varphi_1(r)}{r^{\sigma+1}}\, dr = \Psi(\sigma)\int_0^\infty \frac{\varphi_2(r)}{r^{\sigma+1}}\, dr.$$

Then

$$\limsup_{r\to\infty} \frac{\varphi_1(r)}{\varphi_2(r)} \geqslant \Psi(\lambda) \geqslant \liminf_{r\to\infty} \frac{\varphi_1(r)}{\varphi_2(r)}.$$

DEFINITION [2, p. 149]. Let g be a real valued integrable function on $[-\pi, \pi]$. The "star function" of g is defined by

$$g^*(\theta) = \sup_{|E|=2\theta} \int_E g \qquad (0 \leqslant \theta \leqslant \pi, |E| = \text{Lebesgue measure of } E).$$

LEMMA 2.3 [2, p. 150]. *For $g, h \in L_1[-\pi, \pi]$ the following statements are equivalent.*

(a) *For every convex nondecreasing function Φ on $(-\infty, \infty)$*

$$\int_{-\pi}^{\pi} \Phi(g(x))\, dx \leqslant \int_{-\pi}^{\pi} \Phi(h(x))\, dx,$$

(b) $$g^*(\theta) \leqslant h^*(\theta) \qquad (0 \leqslant \theta \leqslant \pi).$$

PROOF OF LEMMA 2.1. The existence of the function $A(t)$ and the sequence r_n is Theorem 16 of [9, p. 35]. The proof of (c) is given in [1]. We add here that, since w is harmonic in a neighborhood of the origin (property (i) of the class $\mathfrak{D}$), $N(t) = N(t, u) + N(t, v)$ vanishes in a neighborhood of the origin, and so, we may and do take $A(t)$ to be vanishing in the same neighborhood of the origin.

3. **Proof of Theorem 1.** Let $w \in \mathfrak{D}$ be subharmonic of order λ $(0 < \lambda < 1)$, and let $N(r) = N(r, w)$. Then the order of $N(r)$ is also equal to λ. Let $\{r_n\}$ be the sequence increasing to infinity and satisfying parts (b) and (c) of Lemma 2.1. By part (c), there is a subsequence (which we also denote by $\{r_n\}$) and numbers ξ_m such that

$$\frac{c_m(r_n)}{A(r_n)} \to \xi_m \qquad (r_n \to \infty, \text{all } m). \tag{3.1}$$

Clearly then $\xi_m = O(m^{-1})$, and so, by the Riesz-Fisher theorem, there is a function $\varphi \in L_2(-\pi, \pi)$ such that

$$\varphi(\theta) \sim \sum_{m=-\infty}^{\infty} \xi_m e^{im\theta}. \tag{3.2}$$

If $1 \leqslant s \leqslant 2$, then using the fact that the $L_s(-\pi, \pi)$ norm is a nondecreasing function of s, together with Parseval's identity, (3.1) and part (c) of Lemma 2.1, we have

$$\limsup_{r_n\to\infty}\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\frac{w(r_ne^{i\theta})}{A(r_n)}-\varphi(\theta)\right|^s d\theta\right\}^{1/s}$$

$$\leqslant \limsup_{r_n\to\infty}\left\{\sum_{m=-\infty}^{\infty}\left|\frac{c_m(r_n)}{A(r_n)}-\xi_m\right|^2\right\}^{1/2}=0. \tag{3.3}$$

If $2 \leqslant s < \infty$, then applying the Hausdorff-Young theorem, (3.1), and taking into account part (c) of Lemma 2.1 we obtain:

$$\limsup_{r_n\to\infty}\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\frac{w(r_ne^{i\theta})}{A(r_n)}-\varphi(\theta)\right|^s d\theta\right\}^{1/s}$$

$$\leqslant \limsup_{r_n\to\infty}\left\{\sum_{m=-\infty}^{\infty}\left|\frac{c_m(r_n)}{A(r_n)}-\xi_m\right|^{s'}\right\}^{1/s'}=0; \tag{3.4}$$

where $1/s + 1/s' = 1$.

From (3.3), (3.4), Minkowski's inequality and part (b) of Lemma 2.1, we conclude:

$$\lim_{r_n\to\infty}\frac{m_s(r_n, w)}{N(r_n)}=m_s(|\varphi|)\quad (1\leqslant s<\infty); \tag{3.5}$$

$$\lim_{r_n\to\infty}\frac{m_s(r_n, w^+)}{N(r_n)}=m_s(\varphi^+)\quad (1\leqslant s<\infty); \tag{3.6}$$

and

$$\lim_{r_n\to\infty}\left\{\frac{1}{2\pi}\int_E\left|\frac{w(r_ne^{i\theta})}{A(r_n)}-\varphi(\theta)\right|^s d\theta\right\}^{1/s}=0 \tag{3.7}$$

for any measurable set $E\ (\subset [-\pi, \pi])$ and any $s\ (1 \leqslant s < \infty)$.

It follows from (3.5) and (3.6) that Theorem 1 will be established if we show that

$$m_s(\varphi^+) \leqslant m_s(\psi_\lambda^+) \tag{3.8}$$

and

$$m_s(|\varphi|) \leqslant m_s(|\psi_\lambda|). \tag{3.9}$$

Proof of (3.8). Let $E \subset [-\pi, \pi]$ be a Lebesgue measurable set of measure 2β $(0 < \beta < \pi)$. From (3.7) we have

$$\frac{1}{2\pi}\int_E \varphi(\theta)\,d\theta = \lim_{r_n\to\infty}\frac{1}{A(r_n)}\cdot\frac{1}{2\pi}\int_E w(r_n e^{i\theta})\,d\theta. \tag{3.10}$$

To estimate the right-hand side of (3.10) we follow the well-established methods of Edrei and Fuchs [5]:

Since the function $\log|1 + re^{i\theta}/|a||$ is an even function of θ which decreases steadily as θ varies from 0 to π, we have [3, p. 15],

$$\frac{1}{2\pi}\int_E \log\left|1 - \frac{re^{i\theta}}{a}\right| d\theta \leqslant \frac{1}{\pi}\int_0^{\beta} \log\left|1 + \frac{re^{i\theta}}{|a|}\right| d\theta.$$

Using this in (1.6) with $q = 0$ and $d\nu(a) \equiv 0$, we deduce

$$\begin{aligned}\frac{1}{2\pi}\int_E w(re^{i\theta})\,d\theta &\leqslant \int_{|a|<\infty} d\mu(a)\,\frac{1}{\pi}\int_0^{\beta}\log\left|1+\frac{re^{i\theta}}{|a|}\right| d\theta \\ &= \int_0^{\infty} N(t)P(t, r, \beta)\,dt \end{aligned}\tag{3.11}$$

where $P(t, r, \beta) = r\sin\beta/(t^2 + 2tr\cos\beta + r^2)$, $0 < \beta < \pi$.

Using parts (a) and (b) of Lemma 2.1 and properties of proximate orders we deduce from (3.10) and (3.11), that

$$\begin{aligned}\frac{1}{2\pi}\int_E \varphi(\theta)\,d\theta &\leqslant \lim_{r_n\to\infty}\frac{1}{A(r_n)}\int_0^{\infty} A(t)P(t, r_n, \beta)\,dt \\ &= \lim_{r\to\infty}\frac{1}{A(r)}\int_0^{\infty} A(t)P(t, r, \beta)\,dt \\ &= \frac{\sin\lambda\beta}{\sin\pi\lambda} = \frac{1}{2\pi}\int_{-\beta}^{\beta}\psi_\lambda(\theta)\,d\theta. \end{aligned}\tag{3.12}$$

In view of the definition of the star function and the fact that ψ_λ decreases steadily from 0 to π, (3.12) implies

$$\varphi^*(\beta) \leqslant \psi_\lambda^*(\beta), \qquad 0 < \beta < \pi. \tag{3.13}$$

It is easily seen that (3.13) remains true for $\beta = 0$ and $\beta = \pi$; thus, applying Lemma 2.2 with $g = \varphi$, $h = \psi_\lambda$ and $\Phi(x) = (\max(x, 0))^s$ we see that (3.8) is an immediate consequence of (3.13).

PROOF OF (3.9). The proof of (3.9) will be along the same lines as the proof of (3.8) but a little extra care is needed.

Let $E \subset [-\pi, \pi]$ be a Lebesgue measurable set of measure 2β ($0 < \beta < \pi$). Put $E_1 = E_1(r) = \{\theta \in E\colon w(re^{i\theta}) \geqslant 0\}$, $E_2 = E_2(r) = \{\theta \in E\colon w(re^{i\theta}) < 0\}$, and $2\beta_1(r) = |E_1|$, $2\beta_2(r) = |E_2|$. Thus $\beta_1(r) + \beta_2(r) = \beta$ for all r.

Again from (3.7) we have

$$\frac{1}{2\pi}\int_E |\varphi(\theta)|\,d\theta = \lim_{r_n\to\infty}\frac{1}{A(r_n)}\cdot\frac{1}{2\pi}\int_E |w(r_n e^{i\theta})|\,d\theta. \tag{3.14}$$

Now we write $\int_E |w(re^{i\theta})|\, d\theta = \int_{E_1} w(re^{i\theta})\, d\theta - \int_{E_2} w(re^{i\theta})\, d\theta$ and follow the same steps that led to (3.11). At one point we need to use the fact that

$$\frac{1}{2\pi}\int_{E_2} -\log\left|1-\frac{re^{i\theta}}{a}\right| d\theta \leqslant \frac{1}{\pi}\int_0^{\beta_2} -\log\left|1-\frac{re^{i\theta}}{|a|}\right| d\theta$$

$$= -\log^+(r/|a|) + \frac{1}{\pi}\int_0^{\pi-\beta_2} \log\left|1+\frac{re^{i\theta}}{|a|}\right| d\theta,$$

and we are led to

$$\frac{1}{2\pi}\int_E |w(re^{i\theta})|\, d\theta \leqslant \int_0^\infty N(t)P(t, r, \beta_1(r))\, dt - N(r)$$
$$+ \int_0^\infty N(t)P(t, r, \pi - \beta_2(r))\, dt. \tag{3.15}$$

It is our intention to set $r = r_n$, but before doing so we select (if necessary) subsequences and assume that $\beta_1(r_n) \to \beta_1$, $\beta_2(r_n) \to \beta_2$. We shall also assume that $\beta_1 > 0$, $\beta_2 > 0$, so that for large n, $0 < \beta_1(r_n) < \pi$, $0 < \beta_2(r_n) < \pi$ and of course, $\beta_1(r_n) + \beta_2(r_n) = \beta$. Now in (3.15) putting $r = r_n$ and taking account of (3.14) and properties of proximate orders we are led to the following inequality:

$$\frac{1}{2\pi}\int_E |\varphi(\theta)|\, d\theta$$
$$\leqslant \lim_{r_n\to\infty} \frac{1}{A(r_n)}\left\{\int_0^\infty A(t)\big(P(t, r_n, \beta_1(r_n))\right.$$
$$\left. + P(t, r_n, \pi - \beta_2(r_n))\big)\, dt - A(r_n)\right\}$$
$$= \frac{\sin\lambda\beta_1}{\sin\pi\lambda} + \frac{\sin\lambda(\pi-\beta_2)}{\sin\pi\lambda} - 1$$
$$= \frac{1}{\pi}\int_0^{\beta_1} \psi_\lambda(\theta)\, d\theta + \frac{1}{\pi}\int_0^{\pi-\beta_2} \psi_\lambda(\theta)\, d\theta - 1$$
$$= \frac{1}{\pi}\int_0^{\beta_1} \psi_\lambda(\theta)\, d\theta - \frac{1}{\pi}\int_{\pi-\beta_2}^{\pi} \psi_\lambda(\theta)\, d\theta$$
$$\leqslant \frac{1}{\pi}\int_0^{\beta_1} |\psi_\lambda(\theta)|\, d\theta + \frac{1}{\pi}\int_{\pi-\beta_2}^{\pi} |\psi_\lambda(\theta)|\, d\theta$$
$$\leqslant \frac{1}{2\pi}|\psi_\lambda|^*(\beta_1+\beta_2) = \frac{1}{2\pi}|\psi_\lambda|^*(\beta). \tag{3.16}$$

From (3.16) follows that

$$|\varphi|^*(\beta) \leqslant |\psi_\lambda|^*(\beta), \qquad 0 < \beta < \pi. \tag{3.17}$$

The inequality (3.17) was established under the assumption $\beta_1\beta_2 \neq 0$. If $\beta_2 = 0$, then it is possible to have $\beta_2(r_n) = 0$ for infinitely many values of n. In this case $\beta_1(r_n) = \beta$, $(1/2\pi)\int_E |w(r_n e^{i\theta})|\,d\theta = (1/2\pi)\int_{E_1} w(r_n e^{i\theta})\,d\theta$ and then (3.17) follows from (3.13).

If $\beta_1 = 0$ and $\beta_1(r_n) = 0$ for infinitely many values of n, then

$$\begin{aligned}\frac{1}{2\pi}\int_E |w(r_n e^{i\theta})|\,d\theta &= \frac{1}{2\pi}\int_{E_2} - w(r_n e^{i\theta})\,d\theta \\ &= -A(r_n) + \frac{1}{2\pi}\int_{E_2^c} w(r_n e^{i\theta})\,d\theta \\ &\leqslant -A(r_n) + \int_0^\infty N(t)P(t, r_n, \pi - \beta)\,dt.\end{aligned}$$

This gives

$$\frac{1}{2\pi}\int_E |\varphi(\theta)|\,d\theta \leqslant -1 + \frac{1}{2\pi}\int_0^{\pi-\beta} \psi_\lambda(\theta)\,d\theta = -\frac{1}{\pi}\int_{\pi-\beta}^{\pi} \psi_\lambda(\theta)\,d\theta$$

which is only possible if $\frac{1}{2} < \lambda < 1$ in which case it again leads to $|\varphi|^*(\beta) \leqslant |\psi_\lambda|^*(\beta)$.

The validity of (3.17) for $\beta = 0$ is trivial; also, since for all $g \geqslant 0$, g is increasing and convex, it follows from inequality (3.17) that $|\varphi|^*(\pi) \leqslant |\psi_\lambda|^*(\pi)$.

Applying Lemma 2.3 with $g = |\varphi|$, $h = |\psi_\lambda|$ and $\Phi(x) = (\max(x, 0))^s$, (3.9) follows immediately from (3.17).

4. **Proof of Theorem 2.** Let $w \in \mathfrak{D}$ be subharmonic of nonintegral order λ and assume that the Riesz mass of w is distributed along the negative real axis. In this case the Fourier coefficients $\gamma_m(r)$ of w are given by:

$$\begin{aligned}(-1)^m\gamma_m(r) &= \frac{(-1)^m}{2}\alpha_m r^m \\ &\quad + \tfrac{1}{2}m\int_0^r \left\{\left(\frac{r}{t}\right)^m - \left(\frac{t}{r}\right)^m\right\} N(t)\,\frac{dt}{t} + N(r) \qquad (4.1)\end{aligned}$$

where $1 \leqslant m \leqslant q$, $q \neq 0$ and $q = [\lambda]$; and

$$(-1)^m\gamma_m(r) = N(r) - \frac{m}{2}\left\{\int_0^r \left(\frac{t}{r}\right)^m N(t)\,\frac{dt}{t} + \int_r^\infty \left(\frac{r}{t}\right)^m N(t)\,\frac{dt}{t}\right\}$$

$$(m \geqslant q + 1). \qquad (4.2)$$

A result of Edrei and Fuchs [4, p. 308] implies that functions w satisfying the hypothesis of Theorem 2, also satisfy

$$r^q = o(T(r, w)) \qquad (r \to \infty). \qquad (4.3)$$

From (4.3) follows easily that

$$r^q = o(m_s(r, w)), \qquad r^q = o(m_s(r, w^+)) \qquad (1 \leqslant s < \infty, r \to \infty) \quad (4.4)$$

and (4.4) together with some standard computations implies that we may assume, in the proof of Theorem 2, that $\alpha_m = 0$ for $1 \leqslant m \leqslant q$. From now on we make this assumption.

Let σ be any real number satisfying $\lambda < \sigma < q + 1$. Then a simple integration by parts of (4.1) and (4.2) gives:

$$\int_0^\infty \frac{\gamma_m(r)}{r^{\sigma+1}}\, dr = \frac{(-1)^m \sigma^2}{\sigma^2 - m^2} \int_0^\infty \frac{N(r)}{r^{\sigma+1}}\, dr \tag{4.5}$$

from which follows

$$\int_0^\infty \frac{w(re^{i\theta})}{r^{\sigma+1}}\, dr = \psi_\sigma(\theta) \int_0^\infty \frac{N(r)}{r^{\sigma+1}}\, dr \tag{4.6}$$

where $N(r) = N(r, w)$ and $-\pi \leqslant \theta \leqslant \pi$.

Let s be a real number satisfying $1 < s < \infty$ (the case $s = 1$ is well known) and let s' be the index conjugate to s, i.e., $1/s' + 1/s = 1$. Let $g(\theta)$ be a real-valued function, continuous on $[-\pi, \pi]$ and such that

$$m_{s'}(g) = \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} |g(\theta)|^{s'}\, d\theta \right\}^{1/s'} = 1. \tag{4.7}$$

From (4.6) we calculate

$$\begin{aligned} &\int_0^\infty \frac{(1/2\pi)\int_{-\pi}^{\pi} w(re^{i\theta})\, g(\theta)\, d\theta}{r^{\sigma+1}}\, dr \\ &\qquad = \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \psi_\sigma(\theta)\, g(\theta)\, d\theta \right) \int_0^\infty \frac{N(r)}{r^{\sigma+1}}\, dr \\ &\qquad = \Psi(\sigma) \int_0^\infty \frac{N(r)}{r^{\sigma+1}}\, dr. \end{aligned} \tag{4.8}$$

The continuity of $g(\theta)$ ensures that $\Psi(\sigma) = \Psi(\sigma; g)$ is holomorphic in a sufficiently small neighbourhood of λ. All the other conditions of Lemma 2.2 are satisfied and so

$$\limsup_{r\to\infty} \frac{1}{N(r)} \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} w(re^{i\theta})\, g(\theta)\, d\theta \geqslant \Psi(\lambda). \tag{4.9}$$

Minkowski's inequality, (4.7) and (4.9) give

$$\limsup_{r\to\infty} \frac{m_s(r, w)}{N(r)} \geqslant \Psi(\lambda) = \Psi(\lambda; g) \tag{4.10}$$

valid for any continuous function $g(\theta)$ that satisfies (4.7). Since the continuous functions are dense in $L_{s'}[-\pi, \pi]$ we conclude from (4.10) that

$$\limsup_{r\to\infty} \frac{m_s(r, w)}{N(r)} \geq \sup_{m_s'(g)=1} |\Psi(\lambda; g)| = m_s(\psi_\lambda) \tag{4.11}$$

and the proof of (1.8) is complete.

Now let $g(\theta)$ be a real-valued function, continuous and nonnegative on $[-\pi, \pi]$ and satisfying (4.7). Let $E_\lambda = \{\theta: \psi_\lambda(\theta) \geq 0\}$. From (4.6) we calculate

$$\begin{aligned}\int_0^\infty & \frac{(1/2\pi)\int_{E_\lambda} w(re^{i\theta}) g(\theta)\, d\theta}{r^{\sigma+1}}\, dr \\ &= \left(\frac{1}{2\pi}\int_{E_\lambda} \psi_\lambda(\theta) g(\theta)\, d\theta\right)\int_0^\infty \frac{N(r)}{r^{\sigma+1}}\, dr \\ &= \Psi(\sigma)\int_0^\infty \frac{N(r)}{r^{\sigma+1}}\, dr. \end{aligned}\tag{4.12}$$

From (4.12) and Lemma 2.2 we obtain

$$\limsup_{r\to\infty} \frac{1}{N(r)} \frac{1}{2\pi}\int_{E_\lambda} w(re^{i\theta}) g(\theta)\, d\theta \geq \Psi(\lambda). \tag{4.13}$$

Since

$$\begin{aligned}\frac{1}{2\pi}\int_{E_\lambda} w(re^{i\theta}) g(\theta)\, d\theta &\leq \frac{1}{2\pi}\int_{E_\lambda} w^+(re^{i\theta}) g(\theta)\, d\theta \\ &\leq \left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \left(w^+(re^{i\theta})\right)^s d\theta\right\}^{1/s}\end{aligned}$$

we conclude from (4.13) that

$$\limsup_{r\to\infty} \frac{m_s(r, w^+)}{N(r)} \geq \Psi(\lambda) = \Psi(\lambda; g) \tag{4.14}$$

valid for any continuous nonnegative function $g(\theta)$ that satisfies (4.7). Recalling the definition of E_λ we deduce

$$\limsup_{r\to\infty} \frac{m_s(r, w^+)}{N(r)} \geq m_s(\psi_\lambda^+)$$

and the proof of Theorem 2 is completed.

5. **Proof of Theorem 3.** Let $w \in \mathfrak{D}$ be of nonintegral order λ and let $N(r) = N(r, u) + N(r, v)$. Let r_n be the sequence of Lemma 2.1 relative to $N(r)$; then there exist numbers ξ_m such that for a subsequence of r_n (also denoted by r_n)

$$\frac{c_m(r_n)}{N(r_n)} \to \xi_m \qquad (\text{all } m, r_n \to \infty) \tag{5.1}$$

and $\xi_m = O(m^{-1})$ and $|\xi_m| \leqslant \lambda^2/|\lambda^2 - m^2|$.

Now if we let $\varphi(\theta) \sim \sum_{m=-\infty}^{\infty} \xi_m e^{im\theta}$ and $\tilde{\varphi}(\theta) = \varphi(\theta) - \sum_{|m| \leqslant q} \xi_m e^{im\theta}$ then it follows that, as in the previous theorem,

$$\lim_{r_n \to \infty} \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \frac{\tilde{w}(r_n e^{i\theta})}{N(r_n)} - \tilde{\varphi}(\theta) \right|^s d\theta \right\}^{1/s} = 0 \qquad (1 \leqslant s < \infty), \tag{5.2}$$

which in turn implies that

$$\lim_{r_n \to \infty} \frac{1}{N(r_n)} m_s(r_n, \tilde{w}) = m_s(\tilde{\varphi}). \tag{5.3}$$

If s is a positive even integer, then a result of Hardy and Littlewood [7] together with $|\xi_m| \leqslant \lambda^2/(m^2 - \lambda^2)$ $(m \geqslant q + 1)$ gives

$$m_s(\tilde{\varphi}) \leqslant m_s\left(\sum_{|m| \geqslant q+1} \frac{\lambda^2}{m^2 - \lambda^2} e^{im\theta} \right) = m_s(\tilde{\psi}_\lambda).$$

This completes the proof of (1.9).

REMARKS. (1) Let u be subharmonic of nonintegral order λ with its Riesz mass distributed regularly along the negative x-axis, i.e., $N(r) \sim r^\lambda L(r)$ for some slowly varying function L. It can be shown [1] then, that the Fourier coefficients $\{c_m(r)\}$ of u satisfy

$$c_m(r) \sim \frac{(-1)^m \lambda^2}{\lambda^2 - m^2} r^\lambda L(r)$$

which implies that

$$\lim_{r \to \infty} \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \frac{u(re^{i\theta})}{N(r)} - \psi_\lambda(\theta) \right|^s d\theta \right\}^{1/s} = 0 \qquad (1 \leqslant s < \infty)$$

and also

$$\lim_{r \to \infty} \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \frac{\tilde{u}(re^{i\theta})}{N(r)} - \tilde{\psi}_\lambda(\theta) \right|^s d\theta \right\}^{1/s} = 0 \qquad (1 \leqslant s < \infty).$$

This establishes the sharpness of Theorems 1, 2 and 3.

(2) We wish to note the work of Hayman [15] for the case $s = 1$ and dimensions $m \geqslant 2$, as well as Gariepy and Lewis [16] for space analogues of some theorems on meromorphic functions.

REFERENCES

1. Faruk F. Abi-Khuzam, *An abelian theorem for a class of subharmonic functions*, Proc. Amer. Math. Soc. **27** (1977), 253–259.

2. Albert Baernstein, *Integral means, univalent functions and circular symmetrization*, Acta Math. **133** (1974), 139–169.

3. Albert Edrei, *The deficiencies of meromorphic functions of finite lower order*, Duke Math. J. **31** (1964), 1–22.

4. Albert Edrei and W. H. J. Fuchs, *On the growth of meromorphic functions with several deficient values*, Trans. Amer. Math. Soc. **93** (1959), 292–328.

5. ______, *The deficiencies of meromorphic functions of order less than one*, Duke Math. J. **27** (1960), 233–250.

6. A. M. Gorky, *Concerning certain asymptotic properties of entire functions with real negative zeros*, Kharkov Math. Soc. (4) **28** (1961). (Russian)

7. G. H. Hardy and J. E. Littlewood, *Notes on the theory of series*. (XIX): *A problem concerning majorants of Fourier series*, Quart. J. Math. Oxford **6** (1935), 304–315.

8. Simon Hellerstein and Jack Williamson, *Entire functions with negative zeros and a problem of R. Nevanlinna*, J. Analyse Math. **22** (1969), 233–267.

9. B. Ja. Levin, *Distribution of zeros of entire functions*, Transl. Math. Monographs, Vol. 5, Amer. Math. Soc., Providence, R. I., 1964.

10. Joseph Miles and Daniel F. Shea, *An extremal problem in value-distribution theory*, Quart. J. Math. Oxford (2) **24** (1973), 377–383.

11. R. Nevanlinna, *Le théorème de Picard-Borel et la théorie des fonctions meromorphes*, Gauthier-Villars, Paris, 1930.

12. I. I. Privaloff, *Sur la généralisation d'une formule de Jensen*, Bull. Acad. Sciences URSS (1935), 837–847. (Russian)

13. I. P. Proskurnja, *On the growth of meromorphic functions of infinite lower order*, Dokl. Akad. Nauk. SSSR **209** (1973).

14. G. Valiron, *Sur un théorème de M. Wiman*, Opuscula math. A. Wiman dedicata (1930), 1–12.

15. W. Hayman, *Subharmonic functions in R^m*, Actes Congrès Internat. Math. 1970, vol. 2, Gauthier-Villars, Paris, 1971, pp. 601–605.

16. D. Gariepy and R. Lewis, *Space analogues of some theorems for subharmonic and meromorphic functions*, Ark. Mat. **13** (1975), 91–105.

DEPARTMENT OF MATHEMATICS, CORNELL UNIVERSITY, ITHACA, NEW YORK 14853