

THE ASYMPTOTIC BEHAVIOUR OF CERTAIN INTEGRAL FUNCTIONS

BY

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ABSTRACT. Let $f(z)$ be an integral function satisfying

$$\int^{\infty} \{ \log m(r, f) - \cos \pi \rho \log M(r, f) \}^+ \frac{dr}{r^{\rho+1}} < \infty$$

and

$$0 < \lim_{r \rightarrow \infty} \frac{\log M(r, f)}{r^{\rho}} < \infty$$

for some $\rho: 0 < \rho < 1$. It is shown that such functions have regular asymptotic behaviour outside a set of circles with centres ζ_i and radii t_i for which

$$\sum_{i=1}^{\infty} \frac{t_i}{|\zeta_i|} < \infty.$$

1. Introduction. For an integral function $f(z)$ let

$$M(r, f) = \max_{|z|=r} |f(z)|, \quad m(r, f) = \min_{|z|=r} |f(z)|$$

and let $n(r, f)$ be the number of zeros of f in $|z| \leq r$. The order ρ of f is

$$\rho = \lim_{r \rightarrow \infty} \frac{\log \log M(r, f)}{\log r}.$$

The following result appears in [6].

THEOREM A. *Let ρ be a positive number less than one and let $f(z)$ be an integral function of order ρ satisfying the following conditions:*

(i) *there is a finite constant K such that*

$$\overline{\lim}_{\substack{r_2 > r_1 \\ r_1 \rightarrow \infty}} \int_{r_1}^{r_2} \{ \log m(r, f) - \cos \pi \rho \log M(r, f) \} \frac{dr}{r^{\rho+1}} \leq K;$$

(ii) *there are numbers α and β , with $0 < \alpha < \beta < \infty$, such that, for all large r ,*

$$\alpha r^{\rho} \leq n(r, f) \leq \beta r^{\rho}.$$

Let

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$$(1.1) \quad k = \left\{ \frac{2\beta}{\alpha} \right\}^{1/\rho}.$$

Then there is a curve $C: z = re^{i\phi(r)}$, where $\phi(r)$ is a continuous function satisfying

$$(1.2) \quad |\phi(R_1) - \phi(R_2)| = o \left| \log \frac{R_2}{R_1} \right|^{1/2} \quad \text{as } \min(R_1, R_2) \rightarrow \infty,$$

a function $\epsilon(t)$ satisfying $\pi \geq \epsilon(t) \geq 0$ and $\epsilon(t) \rightarrow 0$ as $t \rightarrow \infty$, and a function $\nu(t)$ satisfying $1 \geq \nu(t) \geq 0$, $\nu(t) \rightarrow 0$ as $t \rightarrow \infty$ and

$$(1.3) \quad \int_0^\infty \frac{\nu(t)}{t} dt < \infty,$$

for which the following is true. If ζ is any point on C , then the set

$$\{z: k^{-1}|\zeta| < |z| \leq k|\zeta| \text{ and } |\arg z\zeta^{-1}| \geq \epsilon(|\zeta|)\}$$

contains at most $\nu(|\zeta|)N(|\zeta|)$ zeros of f , where $N(|\zeta|)$ is the number of zeros of f in

$$\{z: k^{-1}|\zeta| < |z| \leq k|\zeta|\}.$$

The equation (1.2) is a consequence of the following: there is a constant $A = A(k)$ and a function $\Delta(t)$ satisfying $\pi \geq \Delta(t) \geq 0$, $\Delta(t) \rightarrow 0$ and

$$(1.4) \quad \int_0^\infty \frac{\Delta(t)^2}{t} dt < \infty$$

for which

$$(1.5) \quad |\phi'(t)| \leq A \frac{\Delta(t)}{t} \quad \text{for all large } t.$$

The reader is referred to [6] for details.

It will be shown here that this result leads to a precise description (outside a small exceptional set) of the asymptotic behavior of a certain class of integral functions. To be specific, let ρ be a positive number less than one and suppose that f is an integral function satisfying

(i)' with the convention that $a^+ = \max(0, a)$ for any real number a ,

$$\int_0^\infty \{\log m(r, f) - \cos \pi\rho \log M(r, f)\}^+ \frac{dr}{r^{\rho+1}} < \infty;$$

(ii)' there is a finite nonzero constant β such that

$$0 < \beta = \lim_{r \rightarrow \infty} \frac{\log M(r, f)}{r^\rho} < \infty.$$

We shall prove here

THEOREM 1. Let ρ be a positive number less than one and let $f(z)$ be an integral function satisfying conditions (i)' and (ii)' above. Then $f(z)$ satisfies the hypotheses of Theorem A and with $\phi(r)$ as in that theorem we have

$$(1.6) \quad |r^{-\rho} \log |f(re^{i(\phi(r)+\theta-\pi)})| - \beta \cos \rho\theta| = o(1)$$

as r tends to infinity outside a set of discs with centres ζ_i , radii t_i , for which

$$(1.7) \quad \sum_{i=1}^{\infty} \frac{t_i}{|\zeta_i|} < \infty.$$

The exceptional set of Theorem 1 may be described briefly, following Hayman [7], as an E -set. Theorem 1 has much in common with results of Essén [4] and Essén and Lewis [5] on subharmonic functions. In [4] Essén is concerned with functions subharmonic in the plane slit along the negative real axis while [5] generalizes the considerations of [4] to functions subharmonic in d -dimensional cones and also establishes an improved estimate of the exceptional set. When restricted to integral functions the result of [4] combined with the estimate of the exceptional set of [5] may be viewed as a special case of Theorem 1, when $|f(r)| = M(r, f)$, $|f(-r)| = m(r, f)$ and $\log |f(-r)| \leq \cos \pi\rho \log |f(r)|$.

The condition (i)' cannot be replaced with

$$(1.8) \quad \overline{\lim}_{r_1, r_2 \rightarrow \infty} \int_{r_1}^{r_2} \{ \log m(r, f) - \cos \pi\rho \log M(r, f) \} \frac{dr}{r^{\rho+1}} \leq 0,$$

a condition arising in the work of Anderson [1]. For in [1], Anderson shows that

$$\int_0^{\infty} \{ \log |f(-r)| - \cos \pi\rho \log |f(r)| \} \frac{dr}{r^{\rho+1}}$$

exists (so that (1.8) certainly holds) for an integral function $f(z)$ with real negative zeros if

$$(1.9) \quad \frac{\log f(r)}{r^{\rho}} \rightarrow A \quad (0 < A < \infty)$$

for some ρ : $0 < \rho < 1$. It will be shown in §9, however, that there exists an integral function $f(z)$ with real negative zeros satisfying (1.9) and such that, for some $\varepsilon > 0$,

$$\log |f(-r)| < (A \cos \pi\rho - \varepsilon)r^{\rho}$$

for all r in a set of infinite logarithmic measure. Since an E -set intersects every ray through the origin in a set of finite logarithmic measure (1.6) cannot hold outside an E -set.

2. Preliminaries. From (i)' it follows that

$$\overline{\lim}_{r_1, r_2 \rightarrow \infty} \int_{r_1}^{r_2} \{ \log m(r, f) - \cos \pi \rho \log M(r, f) \} \frac{dr}{r^{\rho+1}} \leq 0$$

and from this together with (ii)' and the theorem of Anderson already mentioned [1, p. 154] we deduce that

$$(2.1) \quad \log M(r, f) \sim \beta r^\rho,$$

$$(2.2) \quad \log f_1(r) \sim \beta r^\rho,$$

where

$$(2.3) \quad f_1(z) = \prod_1^\infty \left(1 + \frac{z}{|a_n|} \right),$$

the numbers a_n , $n = 1, 2, 3, \dots$, being the nonzero zeros of f arranged in order of increasing magnitude. A well-known consequence of (2.2) is that

$$(2.4) \quad n(r, f) = n(r, f_1) \sim \pi^{-1} \beta \sin \pi \rho r^\rho,$$

so that functions satisfying (i)' and (ii)' are of order ρ and satisfy (i) and (ii) of Theorem A.

In the course of the proof of Theorem 1 we shall find it convenient to refer to a result due to Kolomiiceva [9]. A complete discussion of Kolomiiceva's theorem would involve us in needless complications but a simple consequence of it is

LEMMA 1. *Let $g(z)$ be an integral function satisfying*

$$\lim_{r \rightarrow \infty} \frac{\log M(r, g)}{r^\rho} = \beta,$$

where $0 < \rho < 1$ and $0 < \beta < \infty$, which is such that, for each $\eta > 0$, the number of zeros of g in

$$\{|z| \leq r\} \cap \{|\arg z| \leq \pi - \eta\}$$

is $o(r^\rho)$ as $r \rightarrow \infty$. Then a necessary and sufficient condition that

$$\log |g(re^{i\theta})| = (\beta \cos \rho\theta + o(1))r^\rho$$

outside a set E is the following: given $\varepsilon > 0$, there exist $\delta = \delta(\varepsilon) > 0$ and $r(\varepsilon)$ such that for all z outside E satisfying $|z| > r(\varepsilon)$,

$$(2.5) \quad \int_0^{\delta r} \frac{n_z(t, g)}{t} dt < \varepsilon r^\rho,$$

where $n_z(t, g)$ is the number of zeros of g contained in the open disc with centre z and radius t .

3. An auxiliary function. We suppose without loss of generality that $f(0) = 1$ so that

$$(3.1) \quad f(z) = \prod_1^{\infty} \left(1 - \frac{z}{a_n}\right).$$

As was mentioned before the results of Theorem A hold for functions satisfying the hypotheses of Theorem 1. Choose

$$(3.2) \quad k = (1/2)^{1/\rho}$$

and let C and $\phi(t)$ be as in Theorem A. We relabel C as C_{π} and for every θ satisfying $-\pi < \theta < \pi$ we define C_{θ} by

$$(3.3) \quad C_{\theta}: z = re^{i(\phi(r) + \theta - \pi)}.$$

Let us rearrange the zeros of f in the following way: if a_n is a zero of f lying on the curve C_{θ} say, we transfer it to the point $a'_n = |a_n|e^{i\theta}$ and define

$$(3.4) \quad F(z) = \prod_1^{\infty} \left(1 - \frac{z}{a'_n}\right).$$

Our first concern is to show that $\log|F(|z|e^{i\theta})|$ and $\log|f(z)|$ do not greatly differ. Later we shall show that $\log|F(z)|$ and $\log|f_1(z)|$ have similar asymptotic behavior and then, after estimating $\log|f_1(z)|$, we shall appeal to the intermediate character of F to estimate $\log|f(z)|$.

4. Comparison of f and F . We shall prove

LEMMA 2. *Given any number $\epsilon > 0$, there exists a number $R(\epsilon)$ such that, if $f(z) \neq 0$,*

$$(4.1) \quad |\log|f(z)| - \log|F(|z|e^{i\theta})| < \epsilon r^{\rho}$$

whenever $|z| > R(\epsilon)$, where θ satisfies $-\pi < \theta \leq \pi$ and is such that z lies on C_{θ} .

Throughout the proof we suppose that $z = re^{i\psi}$ is not a zero of f . We have, from (3.1) and (3.4),

$$(4.2) \quad \log \left| \frac{f(z)}{F(re^{i\theta})} \right| = \sum_1^{\infty} \log \left| \left(1 - \frac{z}{a_n}\right) \left(1 - \frac{re^{i\theta}}{a'_n}\right)^{-1} \right|$$

and we examine the sum of (4.2) in three parts. First, with $a_n = r_n e^{i\phi_n}$ consider, for $p > 1$,

$$\begin{aligned}
 S_1 &= \prod_{r_n > k^{pr}} \log \left| \left(1 - \frac{z}{a_n}\right) \left(1 - \frac{re^{i\theta}}{a'_n}\right)^{-1} \right| \\
 &\leq \sum_{r_n > k^{pr}} \log \left(1 + \frac{r}{r_n}\right) \left(1 - \frac{r}{r_n}\right)^{-1} \\
 &\leq 2r(1 - k^{-1})^{-1} \sum_{r_n > k^{pr}} r_n^{-1} \\
 &= 2r(1 - k^{-1})^{-1} \int_{k^{pr}}^{\infty} \frac{dn(t)}{t}.
 \end{aligned}$$

Integrating by parts we obtain

$$(4.3) \quad S_1 = O(k^{p(\rho-1)}r^\rho).$$

Next consider

$$\begin{aligned}
 S_2 &= \sum_{r_n < k^{-pr}} \log \left| \left(1 - \frac{z}{a_n}\right) \left(1 - \frac{re^{i\theta}}{a'_n}\right)^{-1} \right| \\
 &\leq \sum_{r_n < k^{-pr}} \log \left\{ \left(1 + \frac{r_n}{r}\right) \left(1 - \frac{r_n}{r}\right)^{-1} \right\} \\
 (4.4) \quad &\leq 2r^{-1}(1 - k^{-1})^{-1} \sum_{r_n < k^{-pr}} r_n \\
 &= 2r^{-1}(1 - k^{-1})^{-1} \int_0^{k^{-pr}} t \, dn(t) \\
 &= O(k^{-p(\rho+1)}r^\rho).
 \end{aligned}$$

Finally we consider the remaining part of the sum, that for which $k^{-pr} \leq r_n \leq k^{pr}$. Since $\theta = \pi + \psi - \phi(r)$ and $a'_n = r_n e^{i(\pi + \phi_n - \phi(r_n))}$,

$$\begin{aligned}
 J_n &= \left| \left(1 - \frac{z}{a_n}\right) \left(1 - \frac{re^{i\theta}}{a'_n}\right)^{-1} \right|^2 \\
 &= \frac{\left(1 - \frac{r}{r_n}\right)^2 + \frac{4r}{r_n} \sin^2\left(\frac{\psi - \phi_n}{2}\right)}{\left(1 - \frac{r}{r_n}\right)^2 + \frac{4r}{r_n} \sin^2\left(\frac{\psi - \phi_n - \phi(r) + \phi(r_n)}{2}\right)}
 \end{aligned}$$

Let us write $t_n = r/r_n$, $\psi - \phi_n = \psi_n$, $\phi(r) - \phi(r_n) = \nu_n$. Then

$$\begin{aligned}
 J_n &= \frac{(1 - t_n)^2 + 4t_n \sin^2(\psi_n/2)}{(1 - t_n)^2 + 4t_n \sin^2((\psi_n - \nu_n)/2)} \\
 &= 1 + \frac{4t_n \sin(\psi_n - \frac{1}{2}\nu_n) \sin \frac{1}{2}\nu_n}{(1 - t_n)^2 + 4t_n \sin^2((\psi_n - \nu_n)/2)}.
 \end{aligned}$$

Hence

$$\begin{aligned}
 \log J_n &\leq \frac{4t_n \left| \sin(\psi_n - \frac{1}{2}\nu_n) \sin \frac{1}{2}\nu_n \right|}{(1 - t_n)^2 + 4t_n \sin^2\left(\frac{\psi_n - \nu_n}{2}\right)} \\
 (4.5) \quad &\leq \frac{8t_n \left| \sin\left(\frac{\psi_n - \nu_n}{2}\right) \sin \frac{1}{2}\nu_n \right| + 8t_n \left| \sin \frac{1}{4}\nu_n \sin \frac{1}{2}\nu_n \right|}{(1 - t_n)^2 + 4t_n \sin^2\left(\frac{\psi_n - \nu_n}{2}\right)}
 \end{aligned}$$

since, for any real numbers a and b ,

$$\begin{aligned}
 \left| \sin\left(a - \frac{1}{2}b\right) \right| &\leq 2 \left| \sin\left(\frac{1}{2}(a - b) + \frac{1}{4}b\right) \right| \\
 &\leq 2 \left| \sin \frac{1}{2}(a - b) \right| + 2 \left| \sin \frac{1}{4}b \right|.
 \end{aligned}$$

Further, from (1.5),

$$\begin{aligned}
 |\nu_n| &= |\phi(r) - \phi(r_n)| \leq \left| \int_{r_n}^r |\phi'(t)| dt \right| \\
 (4.6) \quad &\leq \left| \int_{r_n}^r A \frac{\Delta(t)}{t} dt \right| \leq A \left| \log \frac{r}{r_n} \right| \left\{ \sup_{t > k^{-pr}} \Delta(t) \right\} \\
 &\leq Ak^p \left| 1 - \frac{r}{r_n} \right| \left\{ \sup_{t > k^{-pr}} \Delta(t) \right\}.
 \end{aligned}$$

Substituting (4.6) into (4.5) we obtain

$$\begin{aligned}
 \log J_n &\leq \frac{4Ak^p t_n |1 - t_n| \left| \sin\left(\frac{\psi_n - \nu_n}{2}\right) \right|}{(1 - t_n)^2 + 4t_n \sin^2\left(\frac{\psi_n - \nu_n}{2}\right)} \left\{ \sup_{t > k^{-pr}} \Delta(t) \right\} \\
 &\quad + A^2 k^{2p} t_n \left\{ \sup_{t > k^{-pr}} \Delta(t)^2 \right\} \\
 &\leq Ak^p t_n^{1/2} \left\{ \sup_{t > k^{-pr}} \Delta(t) \right\} + A^2 k^{2p} t_n \left\{ \sup_{t > k^{-pr}} \Delta(t)^2 \right\} \\
 &\leq A_1 k^{3p} \left\{ \sup_{t > k^{-pr}} \Delta(t) \right\},
 \end{aligned}$$

where $A_1 = A + \pi A^2$. Hence, since from (2.4) the number of zeros of f in $|z| < k^p r$ is at most $2\alpha k^{p\rho} r^\rho$ for large r , where $\alpha = \beta\pi^{-1}\sin \pi\rho$,

$$(4.7) \quad \begin{aligned} S_3 &= \sum_{k^{-p}r < r_n < k^p r} \log \left| \left(1 - \frac{z}{a_n}\right) \left(1 - \frac{re^{i\theta}}{a'_n}\right)^{-1} \right| \\ &\leq 2\alpha A_1 k^{p(\rho+3)} r^\rho \left\{ \sup_{t > k^{-p}r} \Delta(t) \right\}. \end{aligned}$$

Given $\epsilon > 0$ we may choose p sufficiently large that $S_1 + S_2 < \epsilon r^\rho$ for all large r and with this p we may choose $r_0(\epsilon)$ so that $S_3 < \epsilon r^\rho$ for $r > r_0(\epsilon)$, since $\Delta(t) \rightarrow 0$ as $t \rightarrow \infty$, which proves one half of Lemma 1. The second half, that

$$\log \left| \frac{F(|z|e^{i\theta})}{f(z)} \right| < \epsilon |z|^\rho,$$

is proved similarly.

5. The zeros of $F(z)$. We shall prove

LEMMA 3. *Let δ be a fixed positive number less than π and let $n_z(t, F, \delta)$ be the number of zeros of F contained in*

$$(5.1) \quad \left\{ \zeta: |\arg \zeta| < \pi - \frac{1}{2} \delta \right\} \cap \left\{ \zeta: |\zeta - z| < t \right\}.$$

Then given any positive number $\epsilon < \frac{1}{2}$ there exists a number $R(\epsilon, \delta)$ such that, with $|z| = r$,

$$(5.2) \quad \int_0^{\epsilon r} n_z(t, F, \delta) \frac{dt}{t} < \epsilon r^\rho$$

for all z outside a set H_1 (where H_1 depends only on δ) and such that $|z| > R(\epsilon, \delta)$. Moreover H_1 is covered by a set of discs C_i , centres ζ_i , radii t_i , $i = 1, 2, 3, \dots$, such that $\sum_1^\infty t_i/|\zeta_i| < \infty$.

Throughout the proof of Lemma 3 we write $n_z(t)$, $n_z(t, \delta)$ instead of $n_z(t, F)$, $n_z(t, F, \delta)$.

We shall make use of an argument of Azarin [2] in which the following lemma is used.

LEMMA 4 ([10, Lemma 3.2]). *If a set E in the complex plane is covered by discs of bounded radii such that each point of the set is the centre of a disc, then from this one may select a subsystem of discs which covers the set, each point of the plane being covered no more than ν times by the discs of this subsystem, where ν is an absolute constant.*

Let $R_1 = R_1(\delta)$ be such that, for $r \geq R_1$ we have $\epsilon(r) < \frac{1}{2} \delta$, where $\epsilon(r)$ is

the function occurring in Theorem A. (We may note that, if $z \in S(\delta, R_1)$, where

$$(5.3) \quad S(\delta, R_1) = \{z: |z| \geq R_1 \text{ and } |\arg z| \leq \pi - \delta\},$$

then $n_z(t, \delta) = n_z(t)$ certainly for $0 < t < \frac{1}{4}\delta|z|$.) Let H_1 be the set of points z in $|z| \geq R_1$ at which, for some $t = t(z)$ satisfying $0 < t < \frac{1}{2}|z|$, we have

$$(5.4) \quad n_z(t, \delta) \geq t|z|^{\rho-1}.$$

Let E_n be the subset of H_1 contained in the annulus

$$\{z: 4^{n+1} > |z| \geq 4^n\}, \quad n = 0, 1, 2, \dots$$

We surround each point z of H_1 by a disc of radius $t(z)$ and from the set of such discs surrounding points of E_n we select a subsystem K_n which covers E_n , while covering each point of the plane at most ν times. This can be done, by Lemma 4. We note that the members of K_m do not intersect the members of K_n if $|n - m| \geq 2$, and therefore $K = \bigcup_{n=1}^{\infty} K_n$ is a set of discs the members of which cover each point of the plane at most 2ν times.

Now, K is a countable set the members of which may be ordered: C_i , $i = 1, 2, 3, \dots$, where C_i is a disc with centre ζ_i and radius t_i , where $0 < t_i < \frac{1}{2}|\zeta_i|$, $i = 1, 2, 3, \dots$; moreover, from (5.4) we have

$$n_{\zeta_i}(t_i, \delta) \geq t_i |\zeta_i|^{\rho-1}, \quad i = 1, 2, 3, \dots$$

Hence

$$(5.5) \quad \sum_1^{\infty} \frac{t_i}{|\zeta_i|} \leq \sum_1^{\infty} \frac{n_{\zeta_i}(t_i, \delta)}{|\zeta_i|^{\rho}}.$$

Now, if z_n is one of the zeros of F contained in $S(\frac{1}{2}\delta, R_1)$ and also in one of the discs, say C_i , then $|z_n| - |\zeta_i| \leq |z_n - \zeta_i| < t_i < \frac{1}{2}|\zeta_i|$ so $|z_n| < \frac{3}{2}|\zeta_i|$. Hence, from (5.5) and the fact that K covers any point in the plane at most 2ν times,

$$(5.6) \quad \sum_1^{\infty} \frac{t_i}{|\zeta_i|} \leq 2\nu \left(\frac{3}{2}\right)^{\rho} \sum \frac{1}{|z_n|^{\rho}},$$

where the sum on the right-hand side is taken over those zeros of F which are contained in $S(\frac{1}{2}\delta, R_1)$. We proceed to show that this sum is finite.

Let n be a nonnegative integer, and let b_n be a positive number satisfying $k^n R_1 < b_n < k^{n+1} R_1$ at which

$$(5.7) \quad \nu(b_n) \log k \leq \int_{k^n R_1}^{k^{n+1} R_1} \nu(t) \frac{dt}{t},$$

where $\nu(t)$ is the function occurring in Theorem A and k is given by (3.2). The number of zeros of F in

$$\{z: k^n R_1 \leq |z| < k^{n+1} R_1 \text{ and } |\arg z| \leq \pi - \frac{1}{2}\delta\}$$

is no more than $\nu(b_n)N(b_n)$, where $N(b_n)$ is the number of zeros of F in $\{z: k^{-1}b_n \leq |z| < kb_n\}$. Hence, making use of (5.7) we have, for some constant A ,

$$\begin{aligned} \sum \frac{1}{|z_n|^\rho} &\leq \sum_0^\infty \nu(b_n)N(b_n)(k^n R_1)^{-\rho} \\ &\leq \sum_0^\infty \nu(b_n)A b_n^\rho (k^n R_1)^{-\rho} \leq A k^\rho \sum_0^\infty \nu(b_n) \\ &\leq A k^\rho (\log k)^{-1} \int_{R_1}^\infty \nu(t) \frac{dt}{t} < \infty, \end{aligned}$$

from Theorem A. The sum on the left-hand side of (5.6) is thus finite.

Suppose that z is a point outside H_1 and satisfying $|z| \geq R_1$. Then, given any positive number $\varepsilon < \frac{1}{2}$, $\int_0^{\varepsilon|z|} n_z(t, \delta) dt/t < \varepsilon|z|^\rho$. This proves Lemma 3.

6. The behaviour of $f_1(z)$. Let $f_1(z)$ be the function (2.3). Since

$$\log m(r, f) - \cos \pi \rho \log M(r, f) \geq \log m(r, f_1) - \cos \pi \rho \log M(r, f_1)$$

it follows from (i)' and Kjellberg's Lemma [8, p. 193, formula (21)] that

$$(6.1) \quad \int_0^\infty |\log m(r, f_1) - \cos \pi \rho \log M(r, f_1)| \frac{dr}{r^{\rho+1}} < \infty.$$

Given a positive number $\varepsilon > 0$, it follows from (6.1) and (2.2) that

$$\log m(r, f_1) > \left(\beta \cos \pi \rho - \frac{1}{2} \varepsilon \right) r^\rho$$

for r outside a set $E = E(\varepsilon)$ of finite logarithmic measure. Hence, for $\delta = \varepsilon/2\beta\rho$,

$$(6.2) \quad \log |f_1(re^{i\theta})| > \log m(r, f_1) > (\beta \cos \rho\theta - \varepsilon)r^\rho$$

for $\pi \geq |\theta| \geq \pi - \delta$ and for r outside E .

It is well known (see e.g. [12, p. 272]) that

$$(6.3) \quad |r^{-\rho} \log |f_1(re^{i\theta})| - \beta \cos \rho\theta| \rightarrow 0$$

as $r \rightarrow \infty$, uniformly for $|\theta| \leq \pi - \delta$. In particular

$$|r^{-\rho} \log |f_1(re^{i(\pi-\delta)})| - \beta \cos \rho(\pi - \delta)| \rightarrow 0$$

as $r \rightarrow \infty$. Hence, for $\pi \geq |\theta| \geq \pi - \delta$ and for sufficiently large r

$$\begin{aligned} (6.4) \quad r^{-\rho} \log |f_1(re^{i\theta})| &\leq r^{-\rho} \log |f_1(re^{i(\pi-\delta)})| \\ &\leq \beta \cos \rho(\pi - \delta) + \frac{1}{2} \varepsilon \\ &< \beta \cos \rho\theta + 2\beta \sin \frac{1}{2} \rho(|\theta| - \pi + \delta) + \frac{1}{2} \varepsilon \\ &< \beta \cos \rho\theta + \varepsilon. \end{aligned}$$

Taking (6.2) and (6.4) together, and taking account of (6.3) we obtain

LEMMA 5. Let $f_1(z)$ be the integral function (2.3). Given $\varepsilon > 0$

$$|r^{-\rho} \log |f_1(re^{i\theta})| - \beta \cos \rho\theta| < \varepsilon \quad (-\pi < \theta \leq \pi)$$

for all large r outside E , a set of finite logarithmic measure.

From this together with Lemma 1 we deduce

LEMMA 6. Given any $\varepsilon > 0$ there exist positive numbers $\delta = \delta(\varepsilon)$ and $r(\varepsilon)$ such that

$$(6.5) \quad \int_0^{\delta r} n_{-r}(t, f_1) \frac{dt}{t} < \varepsilon r^\rho$$

for all $r > r(\varepsilon)$ lying outside a set E_0 of finite logarithmic measure, where $n_{-r}(t, f_1)$ is the number of zeros of f_1 in $(-r-t, -r+t)$. Further, E_0 is independent of ε and is a union of disjoint intervals each of which contains more than one point.

We need verify only that E_0 may be taken to be a union of disjoint intervals each containing more than one point; Lemma 6 certainly holds for some set E_1 of finite logarithmic measure and some functions $\delta(\varepsilon)$, $r(\varepsilon)$, by Lemma 1.

To this end, given $\varepsilon > 0$, let $\eta = \delta(\frac{1}{3}\varepsilon)$, where δ is the function known to exist and suppose that r_1 and r_2 are two points outside E_1 with $2r_1 \geq r_2 > r_1 > r(\frac{1}{3}\varepsilon)$ and such that f_1 has no zeros in $[r_1, r_2]$. Then, for $r_1 < r < r_2$,

$$(6.6) \quad \begin{aligned} I &= \int_0^{\eta r} n_{-r}(t, f_1) \frac{dt}{t} \\ &= \sum_1^{n_1} \log \frac{\eta r}{r + x_n} + \sum_1^{n_2} \log \frac{-\eta r}{r + y_n}, \end{aligned}$$

where x_1, \dots, x_{n_1} are the zeros of f_1 in $(-r, -r + \eta r)$ and y_1, \dots, y_{n_2} are the zeros of f_1 in $(-r - \eta r, -r)$. Now

$$\frac{r}{r + x_n} = 1 - \frac{x_n}{r + x_n} < 1 - \frac{x_n}{r_1 + x_n} = \frac{r_1}{r_1 + x_n},$$

so, in view of Lemma 1 and Lemma 5

$$(6.7) \quad \begin{aligned} \sum_1^{n_1} \log \frac{\eta r}{r + x_n} &< \sum_1^{n_1} \log \frac{\eta r_1}{r_1 + x_n} \\ &\leq \int_0^{\eta r_1} n_{-r_1}(t, f_1) \frac{dt}{t} < \frac{1}{3} \varepsilon r_1^\rho. \end{aligned}$$

Similarly, $-r/(r + y_n) < -r_2/(r_2 + y_n)$, so

$$(6.8) \quad \sum_1^{n_2} \log \frac{-\eta r}{r + y_n} \leq \int_0^{\eta r_2} n_{-r_2}(t, f_1) \frac{dt}{t} < \frac{1}{3} \varepsilon r_2^\rho < \frac{2}{3} \varepsilon r_1^\rho.$$

From (6.6), (6.7) and (6.8),

$$(6.9) \quad \int_0^{\eta r} n_{-r}(t, f_1) \frac{dt}{t} < \varepsilon r_1^p < \varepsilon r^p$$

for r in (r_1, r_2) .

Let E_0 be the set obtained by removing from E_1 all points which are limit points both from the left and from the right of the complement of E_1 . Then E_0 is a union of disjoint intervals each containing more than one point and is contained in E_1 . Moreover, if $r > r(\frac{1}{3}\varepsilon)$ and r lies outside E_0 , then (6.9) holds, with $\eta = \eta(\varepsilon) = \delta(\frac{1}{3}\varepsilon)$. This completely proves Lemma 6.

7. Further consideration of the zeros of F . We first observe that the set of intervals the union of which is E_0 is countable since the logarithmic measure of E_0 is finite and the logarithmic measure of each interval is positive. We may therefore regard each interval as closed without affecting the value of the logarithmic measure of E_0 . We write $E_0 = \cup_1^\infty J_i$, where

$$J_i = [c_i, d_i], \quad c_{i+1} > d_i > c_i, \quad i = 1, 2, 3, \dots$$

Let ε be any positive number. With Lemmas 3 and 6 in view, let $\tau = \tau(\varepsilon) = \min(\frac{1}{2}, \varepsilon, \delta(\varepsilon))$ (where $\delta(\varepsilon)$ is the function of Lemma 6) and let $r_0(\varepsilon)$ be a number at least as large as $\max(r(\varepsilon), R(\tau, \frac{1}{8}\tau))$ (where $r(\varepsilon)$, $R(x, y)$ are respectively the functions of Lemmas 6 and 3) such that $r_0(\varepsilon) \notin E_0$ (E_0 is the set of Lemma 6) and for which $d_i < (1 + \frac{1}{4}\tau)c_i$ whenever $c_i > r_0(\varepsilon)$. Since E_0 is of finite logarithmic measure this choice of $r_0(\varepsilon)$ is possible.

Define, for $J_i = [c_i, d_i]$ where $c_i > r_0(\varepsilon)$,

$$B_i = \{z: |z| \in J_i\} \cap \{z: \pi \geq |\arg z| > \pi - \frac{1}{8}\tau\},$$

$$D_i = \{z: |z| \in J_i\} \cap \{z: \pi \geq |\arg z| > \pi - \frac{1}{8}\tau\} \setminus H_1$$

where H_1 is the exceptional set of Lemma 3 corresponding to $\delta = \frac{1}{8}\tau$. $H_1 = H_1(\tau) = H_1(\varepsilon)$.

Let z be any point in D_i . Then, with $r = |z|$,

$$\begin{aligned} I(z) &= \int_0^{\tau r} n_z(t, F) \frac{dt}{t} \\ &= \log \frac{(\tau r)^{m+n+p+q}}{\prod_1^m |z - a_j| \prod_1^n |z - b_j| \prod_1^p |z - u_j| \prod_1^q |z - v_j|}, \end{aligned}$$

where the a_j, b_j, u_j, v_j are zeros of F in $|\xi - z| < \tau r$ which are respectively in B_i , in $\{z: |z| \in J_i\} \setminus B_i$, in $|z| < c_i$ and in $|z| > d_i$. Then we have, recalling Lemma 3 and Lemma 6,

$$\begin{aligned}
I(z) &\leq \log \frac{(\tau r)^m}{\prod_1^m |z - a_j|} + \log \frac{(\tau r)^n}{\prod_{j=1}^n (r - |u_j|)} + \log \frac{(\tau r)^p}{\prod_{j=1}^p (|v_j| - r)} \\
&\quad + \int_0^{\tau r} n_z(t, F, \frac{1}{8}\tau) \frac{dt}{t} \\
&\leq \log \frac{(\tau r)^m}{\prod_1^m |z - a_j|} + \log \frac{(\tau c_i)^n}{\prod_{j=1}^n (c_i - |u_j|)} + \log \frac{(\tau d_i)^p}{\prod_{j=1}^p (|v_j| - d_i)} \\
&\quad + \int_0^{\tau r} n_z(t, F, \frac{1}{8}\tau) \frac{dt}{t} \\
(7.1) \quad &\leq \log \frac{(\tau r)^m}{\prod_1^m |z - a_j|} + \int_0^{\tau c_i} n_{-c_i}(t, f_1) \frac{dt}{t} + \int_0^{\tau d_i} n_{-d_i}(t, f_1) \frac{dt}{t} \\
&\quad + \int_0^{\tau r} n_z(t, F, \frac{1}{8}\tau) \frac{dt}{t} \\
&\leq \log \frac{(\tau r)^m}{\prod_1^m |z - a_j|} + \int_0^{\delta(\epsilon)c_i} n_{-c_i}(t, f_1) \frac{dt}{t} + \int_0^{\delta(\epsilon)d_i} n_{-d_i}(t, f_1) \frac{dt}{t} \\
&\quad + \int_0^{\tau r} n_z(t, F, \frac{1}{8}\tau) \frac{dt}{t} \\
&\leq \log \frac{(\tau r)^m}{\prod_1^m |z - a_j|} + \epsilon [c_i^p + d_i^p] + \tau r^p \\
&\leq \log \frac{(\tau r)^m}{\prod_1^m |z - a_j|} + 4\epsilon r^p.
\end{aligned}$$

Now, B_i is contained in a rectangle the sides of which have length $\frac{1}{4}\tau d_i < \frac{3}{8}\tau c_i$ and $d_i - c_i \cos \frac{1}{8}\tau$. Hence for any point z in D_i the circle $|\zeta - z| < \tau|z|$ contains all of B_i and so all the zeros of F in B_i (i.e. all the a_j) appear in (7.1). We can thus apply Cartan's Lemma [3, p. 75] to estimate (7.1) and obtain

$$\prod_1^m |z - a_j| \geq \left\{ \tau d_i \exp\left(-\frac{\epsilon d_i^p}{m}\right) \right\}^m$$

outside a set of at most m discs C'_j , $j = 1, 2, \dots, m$, the sum of the radii of which is at most $A = 2\epsilon r d_i \exp(-\epsilon d_i^p/m)$. Hence, for all z in D_i outside these discs we have, from (7.1),

$$(7.2) \quad \int_0^{\tau r} n_z(t, F) \frac{dt}{t} < 4\epsilon r^p + \log\left(\frac{r}{d_i}\right)^m + \epsilon d_i^p < 6\epsilon r^p.$$

We must have $A < 2e(d_i - c_i)$. For suppose that $A > 2e(d_i - c_i)$. Then

$$\begin{aligned}
 I(\varepsilon, d_i) &= \int_0^{\delta(\varepsilon)d_i} n_{-d_i}(t, f_1) \frac{dt}{t} \geq \int_0^{\tau d_i} n_{-d_i}(t, f_1) \frac{dt}{t} \\
 &\geq \log \frac{(\tau d_i)^m}{\prod_{j=1}^m (d_i - |a_j|)} \\
 &\geq \log \frac{(\tau d_i)^m}{(d_i - c_i)^m} \\
 &> \log \frac{(2e\tau d_i)^m}{A^m} = \varepsilon d_i^p,
 \end{aligned}$$

a contradiction, since $I(\varepsilon, d_i) \leq \varepsilon d_i^p$, d_i being a boundary point of E_0 . Hence

$$(7.3) \quad A \leq 2e(d_i - c_i).$$

Suppose that C'_j has radius t'_j and centre $\zeta'_j, j = 1, 2, \dots, m$.

$$\sum_{j=1}^m \frac{t'_j}{|\zeta'_j|} \leq \frac{1}{c_i} \sum_{j=1}^m t'_j \leq 2e \left(\frac{d_i - c_i}{c_i} \right).$$

Also, since $d_i < (1 + \frac{1}{4}\tau)c_i < 2c_i$ and since, for $x \geq 1, \log x \geq (x - 1)/x$,

$$\frac{d_i - c_i}{c_i} = \frac{d_i}{c_i} \frac{d_i - c_i}{d_i} < 2 \log \frac{d_i}{c_i},$$

so

$$(7.4) \quad \sum_{j=1}^m \frac{t'_j}{|\zeta'_j|} < 4e \log \frac{d_i}{c_i}.$$

We are thus able to prove

LEMMA 7. Let ε be any positive number, and let $\tau = \min(\frac{1}{2}, \varepsilon, \delta(\varepsilon))$, where $\delta(\varepsilon)$ is the function of Lemma 6. Let $r_0(\varepsilon)$ be a positive number greater than $\max(r(\varepsilon), R(\tau, \frac{1}{8}\tau))$ such that $r_0(\varepsilon) \notin E_0$ and for which $d_i < (1 + \frac{1}{4}\tau)c_i$ whenever $c_i > r_0(\varepsilon)$, where $r(\varepsilon), R(\tau, \frac{1}{8}\tau)$ are respectively the functions of Lemmas 6, 3, and E_0 is the set of Lemma 6. Then for all z in

$$T\left(\frac{1}{8}\tau, r_0(\varepsilon)\right) = \left\{ z: |z| \geq r_0(\varepsilon) \text{ and } \pi \geq |\arg z| \geq \pi - \frac{1}{8}\tau \right\}$$

we have, with $|z| = r$,

$$(7.5) \quad \int_0^{\tau r} n_z(t, F) \frac{dt}{t} < 6\varepsilon r^p$$

except when z belongs to an E -set, $H_2 = H_2(\varepsilon)$.

Suppose first that z , in $T(\frac{1}{8}\tau, r_0(\varepsilon))$, lies in $\cup \{z: |z| \in J_i\}$, where the union is over those $J_i = [c_i, d_i]$ for which $c_i > r_0(\varepsilon)$. Then for all z outside H_1 , the E -set of Lemma 3, and outside a set of discs centres ζ , radii t for which

$$\sum \frac{t}{|z|} < 4e \sum \log \frac{d_i}{c_i} < 4e \log \text{meas } E_0 < \infty,$$

(7.5) holds. This follows from (7.2) and (7.4).

Suppose next that z , in $T(\frac{1}{8}\tau, r_0(\epsilon))$, lies outside $\cup \{z: |z| \in J_i\}$. Then, with $|z| = r$, we have from Lemma 6

$$\int_0^{\tau r} n_z(t, F) \frac{dt}{t} \leq \int_0^{\delta(\epsilon)r} \eta_{-r}(t, f_1) \frac{dt}{t} < \epsilon r^p.$$

(7.5) thus holds for z in $T(\frac{1}{8}\tau, r_0(\epsilon))$ outside an E -set, and Lemma 7 is proved.

We prove

LEMMA 8. Let ϵ be any positive number and let $\sigma = \sigma(\epsilon) = \frac{1}{32}\tau(\epsilon)$, where $\tau(\epsilon)$ is the function of Lemma 7. There exists a number $r_1(\epsilon)$ and an E -set, $H_3 = H_3(\epsilon)$, such that

$$(7.6) \quad \int_0^{\sigma r} n_z(t, F) \frac{dt}{t} < 6\epsilon r^p,$$

whenever $|z| = r > r_1(\epsilon)$ and z lies outside H_3 .

For z in $T(\frac{1}{8}\tau, r_0(\epsilon))$ and outside $H_2(\epsilon)$, (7.6) certainly holds, by Lemma 7.

Consider z outside $T(\frac{1}{8}\tau, r_1(\epsilon))$, where $r_1(\epsilon) = \max(r_0(\epsilon), R(\frac{1}{32}\tau, \frac{1}{8}\tau))$. Let $H_4(\epsilon)$ be the E -set $H_1(\frac{1}{8}\tau)$ of Lemma 3. Then, with $\sigma = \frac{1}{32}\tau$ and $r = |z|$, and z outside $H_4(\epsilon)$, we have from Lemma 3

$$\int_0^{\sigma r} n_z(t, F, \frac{1}{8}\tau) \frac{dt}{t} < \sigma r^p < \epsilon r^p.$$

But for $0 < t < \frac{1}{32}\tau r$ and $r \geq R(\frac{1}{32}\tau, \frac{1}{8}\tau)$, $n_z(t, F) = n_z(t, F, \frac{1}{8}\tau)$ for z outside $T(\frac{1}{8}\tau, r_1(\epsilon))$. Hence $\int_0^{\sigma r} n_z(t, F) \frac{dt}{t} < \epsilon r^p$ for z outside $T(\frac{1}{8}\tau, r_1(\epsilon))$ and outside $H_4(\epsilon)$, with $|z| = r > R(\frac{1}{32}\tau, \frac{1}{8}\tau)$.

Lemma 8 then follows with $H_3(\epsilon) = H_2(\epsilon) \cup H_4(\epsilon)$.

The following is an immediate consequence of Lemma 8.

LEMMA 9. Let ϵ be any positive number. There exist positive numbers $\alpha(\epsilon)$, $r_2(\epsilon)$ and an E -set H_5 , independent of ϵ , such that

$$\int_0^{\alpha(\epsilon)r} n_z(t, F) \frac{dt}{t} < \epsilon r^p$$

when $r = |z| > r_2(\epsilon)$ and z lies outside H_5 .

For z such that $r = |z| > r_1(\frac{1}{6})$ and z lies outside $H_3(\frac{1}{6})$, where r_1 and H_3 are as in Lemma 8, we have, with $\sigma = \sigma(\frac{1}{6})$, $\int_0^{\sigma r} n_z(t, F) \frac{dt}{t} < r^p$. Given any integer $n \geq 1$, suppose that $H_3(1/6n)$ is covered by the discs $C_i(n)$, radii $t_i(n)$ and centres $\zeta_i(n)$, $i = 1, 2, 3, \dots$. Let $i_0 = i_0(n)$ be the smallest integer such that

$$(7.7) \quad \sum_{i=i_0}^{\infty} \frac{t_i(n)}{|\xi_i(n)|} \leq 2^{-n} \sum_{i=1}^{\infty} \frac{t_i(1)}{|\xi_i(1)|}, \quad n = 2, 3, \dots$$

Let $r_2(1) = r_1(\frac{1}{6})$ and, supposing $r_2(m)$ defined, $m \geq 1$, let $r_2(m + 1)$ be the smallest number which is no less than $\max\{r_2(1/m) + 1, r_1(1/6(m + 1))\}$ and such that

$$C_i(m + 1) \subset \{z: |z| \leq r_2(1/(m + 1))\}, \quad i = 1, 2, \dots, i_0(m + 1) - 1.$$

Let H_4 be given by

$$H_4 = \left\{ \bigcup_1^{\infty} C_i(1) \right\} \cup \left\{ \bigcup_{n=2}^{\infty} \bigcup_{i=i_0(n)}^{\infty} C_i(n) \right\}.$$

From (7.7), H_4 is an E -set.

Given any number ϵ , $0 < \epsilon < 1$, let m be the integer such that

$$(7.8) \quad \frac{1}{m + 1} < \epsilon \leq \frac{1}{m};$$

define $r_2(\epsilon) = r_2(1/(m + 1))$, $\alpha(\epsilon) = \sigma(1/6(m + 1))$. Let ϵ be any positive number, $0 < \epsilon < 1$, and let z be outside H_4 and such that $r = |z| > r_2(\epsilon)$. Then, if m is the integer satisfying (7.8), z lies outside $H_3(1/6(m + 1))$ and $r = |z| > r_1(1/6(m + 1))$ so by Lemma 8,

$$\begin{aligned} \int_0^{\alpha(\epsilon)r} n_z(t, F) \frac{dt}{t} &= \int_0^{\sigma\left(\frac{1}{6(m+1)}\right)'} n_z(t, F) \frac{dt}{t} \\ &< \frac{1}{m + 1} r^\rho < \epsilon r^\rho. \end{aligned}$$

Lemma 9 is thus proved.

8. Completion of the proof of Theorem 1. By Lemma 1 and Lemma 9,

$$(8.1) \quad |r^{-\rho} \log|F(re^{i\theta})| - \beta \cos \rho\theta| \rightarrow 0$$

as $z = re^{i\theta}$ tends to infinity outside H_4 . From (8.1) and Lemma 2, Theorem 1 follows.

9. A counterexample. Let $f(z)$ be an integral function with real negative zeros. In [11] Titchmarsh proves that if

$$(9.1) \quad \overline{\lim}_{r \rightarrow \infty} \frac{\log|f(r)|}{r^\rho} = A \quad (0 < A < \infty)$$

for some ρ such that $0 < \rho < 1$ then

- (i) $\overline{\lim}_{r \rightarrow \infty} \log|f(-r)|/r^\rho = A \cos \pi\rho$; and
- (ii) given $\epsilon > 0$,

$$(9.2) \quad \log|f(-r)| > (A \cos \pi\rho - \epsilon)r^\rho$$

for all r outside a set of linear density zero.

We shall show that the exceptional set of (ii) cannot be replaced with a set of finite logarithmic measure by constructing an integral function satisfying (9.1) for which (9.2) fails for some $\epsilon > 0$ on a set of infinite logarithmic measure. The construction depends on Lemma 1.

Let A be any fixed positive number. Let (R_m) be an increasing sequence of positive numbers, let $\eta_m = (\log m)^{-1}$ and let $\delta_m = m^{-1/2}$, $m = 2, 3, \dots$. Let $f(z)$ be an integral function with real negative zeros for which the counting function $n(r, f)$ satisfies

$$n(r, f) \sim Ar^\rho.$$

We introduce an integral function $g(z)$ obtained from $f(z)$ by placing $1 + [\eta_m R_m^\rho]$ additional zeros at $-R_m$. It is clear that the sequence (R_m) may be chosen sparsely enough that $n(r, g)$, the counting function of the zeros of $g(z)$, satisfies $n(r, g) \sim Ar^\rho$.

With Lemma 1 in view let us consider, for $R_m < r < (1 - 1/m)^{-1}R_m$,

$$(9.3) \quad \int_0^{\delta_m r} \frac{n_{-r}(t, g)}{t} dt.$$

Since $r - R_m < r - r(1 - m^{-1}) < \delta_m r$, each zero at $-R_m$ contributes $\log\{\delta_m r / r - R_m\}$ to the integral (9.3). Hence we have, for $R_m < r < (1 - 1/m)^{-1}R_m$,

$$(9.4) \quad \begin{aligned} \int_0^{\delta_m r} \frac{n_{-r}(t, g)}{t} dt &\geq \eta_m R_m^\rho \log\left\{ \frac{\delta_m r}{r - R_m} \right\} \\ &\geq \eta_m R_m^\rho \log(m\delta_m) = \frac{1}{2} R_m^\rho \\ &> \frac{1}{4} r^\rho \end{aligned}$$

for all large m , taking account of the definitions of η_m and δ_m . Further

$$E = \bigcup_{m=3}^{\infty} \left\{ r: R_m < r < \left(1 - \frac{1}{m}\right)^{-1} R_m \right\}$$

is a set of infinite logarithmic measure.

Now we appeal to Lemma 1 to conclude that there must be a number $\epsilon > 0$ such that

$$(9.5) \quad \log|f(-r)| < (A \cos \pi\rho - \epsilon)r^\rho$$

for all large r in E . For suppose that there were a sequence (r_n) tending to infinity through E such that

$$\log|f(-r_n)| \geq (A \cos \pi\rho - o(1))r_n^\rho.$$

From (i) of Titchmarsh's result, then,

$$(9.6) \quad \log|f(-r_n)| = (A \cos \pi\rho + o(1))r_n^\rho$$

and so, from Lemma 1, there must exist $\delta > 0$ such that

$$\int_0^{\delta r_n} \frac{n - r_n(t, g)}{t} dt < \frac{1}{4} r_n^p$$

for all large n , which contradicts (9.4). (9.5) thus holds for all large r in E .

Theorem 1 is an improved version of a result which forms part of a thesis submitted for the degree of Ph.D at the University of London. It is a pleasure to express my gratitude to Professor W. K. Hayman of Imperial College, London, for his generous advice and encouragement.

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