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CLASSIFICATION OF CIRCLE ACTIONS ON 4-MANIFOLDS

BY

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ABSTRACT. This article studies locally smooth S^1-actions on closed oriented 4-manifolds in terms of the orbit space, orbit type data, and the characteristic class of the action which lies in $H_1(M^*, S^*)$ where M^* is the orbit space and S^* is the image of a certain collection of singular orbits. It is proved that such actions are determined by their weighted orbit spaces and are in 1-1 correspondence with "legally-weighted" 3-manifolds. The information contained in the weighted orbit space is used to give a presentation of the fundamental group of the 4-manifold, and in certain cases the quadratic form is computed.

The classification of actions of the circle on 3-manifolds due to Seifert, Orlik, and Raymond **[12]**, **[11]**, **[7]** has been known for some years and has been widely applied in the study of other transformation groups on 3-manifolds. More recently, several papers have brought into focus certain aspects of actions of the circle on 4-manifolds. In particular there are the papers of Hsiang **[5]** and Church and Lamotke **[2]** which treat various cases of semifree circle actions on orientable 4-manifolds. Orlik and Wagreich **[9]** have studied the problem from the point of view of algebraic geometry, and Pao **[10]** has proved the existence of nonlinear circle actions on the 4-sphere.

This article is a continuation of **[4]** in which we studied locally smooth actions of the circle group S^1 on simply connected 4-manifolds. In that paper an equivariant classification of these actions was given, and it was shown how to compute the quadratic forms of these 4-manifolds in terms of the classifying data. There resulted the theorem that any simply connected closed 4-manifold which carries a locally smooth S^1-action is homotopy equivalent to a connected sum of the manifolds S^4, CP^2, $-CP^2$, and $S^2 \times S^2$. We improve this in §13.

The present article considers classification of locally smooth effective S^1-actions on closed oriented 4-manifolds in terms of the weighted orbit space, which consists of the orbit space, orbit type data, and the characteristic class of the action, which relates the Euler class of the principal orbit bundle to certain equivariant gluing information. Oriented S^1-manifolds are called

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equivalent if they are orientation-preserving equivariantly homeomorphic. We show that two S^1-actions on closed oriented 4-manifolds are equivalent iff their weighted orbit spaces are isomorphic. Furthermore, any legally weighted 3-manifold is the weighted orbit space of some S^1-action on a closed oriented 4-manifold. This gives a complete account of such actions.

We assume that the reader is familiar with [**4**]. In fact we need to refer to [**4**] so often that we have continued the numbering of [**4**] in this paper. Thus our first section is §9 and references to §§1–8 refer to sections of [**4**]. Rather than review our notation here, we direct the reader to §§2 and 3.

The organization of the paper is as follows. In §9 we show how to extend the classification theorem of [**4**] to the general case. In §§10 and 11 the information contained in the weighted orbit space is used to give a presentation of the fundamental group of the 4-manifold and, in certain cases, a method for computing its quadratic form. In §12 techniques due to Church and Lamotke are employed to show that a condition which is weaker than isomorphism of weighted orbit spaces is often sufficient to imply equivalence.

9. Extension of the classification theorem. Let M be an oriented connected closed 4-manifold carrying a locally smooth S^1-action. The orbit space M^* is then an orientable 3-manifold which can be oriented according to (3.2). The structure of $E^* \cup F^* \subseteq M^*$ is similar to that obtained in the simply connected case (3.1), with one essential difference, namely there may be circles contained in the 1-manifold E^* (and of course M^* need not be simply connected). It is easy to see that the slice representation must be constant along such a circle C^* in E^*; so as in (3.3) we may orient C^* and assign a single pair of Seifert invariants (α, β) to C^*. We call such circles *simply-weighted.* If a weighted circle contains points of F^*, we call it *multiply-weighted.*

Let C^* be a simply weighted circle in M^* and let $Q^* = D^2 \times C^*$ be a regular neighborhood of C^*. Choose $x^* \in C^*$ and let $V^* = D^2 \times x^*$. Consider the action of S^1 on $V \times C^*$ where S^1 acts trivially on C^*. Notice that it also has orbit space Q^*.

PROPOSITION (9.1). *Q is equivalent to $V \times C^*$ by a homeomorphism inducing the identity map on Q^*.*

PROOF. Split Q^* along V^* to obtain $V^* \times I$. If Q' denotes the corresponding splitting of Q, [**1**, II.7.1] provides an equivalence of Q' with $V \times I$ over $V^* \times I$. The proposition now follows since any equivalence $g\colon V \to V$ over V^* (in particular, the homeomorphism repairing the cut) is equivariantly isotopic over V^* to the identity of V. This may be seen as follows. Away from the exceptional orbit set of V the action is free and we have a map Φ:

$V^* - E^* \to S^1$ such that $g(x) = \Phi(p(x)) \cdot x$ (p is the orbit map). Actually Φ extends over all of V^* such that $g(x) = \Phi(p(x)) \cdot x$. This may be seen by an argument which uses the fact that for $x^* \in E^*$ there are only finitely many choices for $\Phi(x^*)$ satisfying the defining formula. Now since $V^* \cong D^2$, Φ is nullhomotopic, and this gives rise to the desired equivariant isotopy. □

(9.2). Let L_i^*, $i = 1, \ldots, n$, be the weighted circles of M^*, and let Q_i^* be a regular neighborhood of L_i^* (see (3.2)) and $Q^* = \bigcup_{i=1}^n Q_i^*$. On each ∂Q_i^* choose a meridional loop m_i^* and a complementary loop l_i^* which together with m_i^* generates $H_1(\partial Q_i^*)$. In ∂Q_i there sits over m_i^* a "canonical" section q_i which is defined in §2 and which, according to (9.1) and the proof of (3.6), defines an element of $H_1(\partial Q_i)$ which is independent of the choice of meridian on ∂Q_i^*. For each i arbitrarily choose a section l_i over l_i^*. Finally, let N^* be a regular neighborhood of $S^* = E^* \cup F^* - \bigcup_{i=1}^n L_i^*$ and let $X^* = \mathrm{Cl}(M^* - (Q^* \cup N^*))$.

The restriction of the orbit map p is a principal S^1-bundle projection $X \to X^*$ which is trivial over ∂Q^* by (3.8) and (9.1). The bundle can be extended by filling in $Q_i^* \times S^1$ over each Q_i^* using a bundle isomorphism $\partial Q_i^* \times S^1 \to \partial Q_i = p^{-1}(\partial Q_i^*)$ for the gluing. Such bundle isomorphisms are classified up to vertical equivariant isotopy by homotopy classes of maps $\partial Q_i^* \to S^1$. That is, an equivariant homeomorphism ϕ_i: $\partial Q_i^* \times S^1 \to \partial Q_i$ inducing $1_{\partial Q^*}$ is classified by the integers (r, s) where in ∂Q_i, $\phi_i(m_i^* \times 0) \sim q_i + rh$ and $\phi_i(l_i^* \times 0) \sim l_i + sh$, h an orbit. However, a bundle automorphism Φ of $Q^* \times S^1$ may be constructed as in (5.4) so that $\Phi(m_i^* \times 0) \sim m_i^* \times 0$ and $\Phi(l_i^* \times 0) \sim (l_i^* \times 0) + sh$ for any $s \in \mathbf{Z}$. Hence the resulting bundle $(Q_i^* \times S^1) \cup_{\phi_i} X^*$ depends only on the integer r. This fact will be used several times. Thus we determine a principal S^1-bundle π: $Z \to X^* \cup Q^* = \mathrm{Cl}(M^* - N^*)$ by specifying that each bundle isomorphism ϕ_i satisfy $\phi_i(m_i^* \times 0) \sim q_i$ in ∂Q_i. (It is easy to see that such bundle isomorphisms actually exist.)

Let $\chi' \in H^2(\mathrm{Cl}(M^* - N^*))$ be the Euler class of the bundle π: $Z \to \mathrm{Cl}(M^* - N^*)$. By Poincaré duality we have

$$H^2(\mathrm{Cl}(M^* - N^*)) \approx H_1(\mathrm{Cl}(M^* - N^*), \partial \mathrm{Cl}(M^* - N^*)) \approx H_1(M^*, S^*).$$

The *characteristic class* of the S^1-action on M, $\chi \in H_1(M^*, S^*)$, is defined to be the Poincaré dual to χ'.

(9.3). The *weighted orbit space* of the S^1-action on M consists of the oriented orbit space M^*, the characteristic class $\chi \in H_1(M^*, S^*)$, the weighted circles and arcs ((3.3)(c), (d)), the simply-weighted circles, and the finite collection of points in $F^* - \partial M^*$. In this formulation it is not necessary to assign Euler numbers to the components of F^* as in (3.3), for this information is obviously carried by χ.

An *isomorphism* f: $M_1^* \to M_2^*$ of weighted orbit spaces is an orientation-preserving homeomorphism which preserves weights and satisfies $f_*(\chi_1) = \chi_2$

where $\chi_i \in H_1(M_i^*, S_i^*)$ is the characteristic class.

PROPOSITION (9.4). *Let h: $M_1 \to M_2$ be an equivalence of closed oriented 4-dimensional S^1-manifolds. Then h^*: $M_1^* \to M_2^*$ is an isomorphism.*

PROOF. By (3.6) and (9.1) h^* preserves weights; so it suffices to show that $(h^*)_*(\chi_1) = \chi_2$. But this follows because $h(X) \to h^*(X^*)$ is bundle equivalent to $X \to X^*$, $h^*(m_i^*)$ are meridians on $h^*(\partial Q_i^*)$, and $h(q_i)$ are canonical sections over $h^*(m_i^*)$. □

(9.5). The components of $Q^* \cup N^*$ correspond in a 1-1 fashion with the components of $E^* \cup F^*$. Suppose the component C^* of $E^* \cup F^*$ has the component K^* of $Q^* \cup N^*$ as regular neighborhood. The Euler number of the restriction of the principal S^1-bundle over ∂K^* (oriented as a boundary component of X^*) will be called the *index* of C^*. For example, a weighted arc $[b'; (\alpha_1, \beta_1), \ldots, (\alpha_k, \beta_k); b'']$ has index $b'' - b' = \pm 1$ or 0 (2.7), an isolated fixed point has index ± 1, and a weighted circle has index 0. Also we have shown in (5.1) that if S^* has m components with indices ε_i, $i = 1, \ldots, m$, then $\partial\chi = (\varepsilon_1, \ldots, \varepsilon_m) \in H_0(S^*) = \bigoplus_m \mathbf{Z}$ and $\sum_1^m \varepsilon_i = 0$.

Define Σ to be the subset of elements $\sigma \in H_1(M^*, S^*)$ such that if $\partial\sigma = (\zeta_1, \ldots, \zeta_m) \in H_0(S^*)$, then for each isolated fixed point C_i^*, $\zeta_i = \pm 1$, and for each weighted arc $[b'; (\alpha_1, \beta_1), \ldots, (\alpha_k, \beta_k); b''] = C_j^*$, $\zeta_j = b'' - b'$. Thus Σ contains the characteristic class χ of the action. It is clear that in general Σ need not be a subgroup.

A *legally weighted 3-manifold* Y^* is an oriented compact connected 3-manifold along with the data:

(i) A finite collection of weighted arcs and circles in int Y^* as in (3.3) and (9.1) satisfying the criteria of (3.3).

(ii) A finite set of distinguished points in int Y^* disjoint from the sets of (i).

(iii) A class $\chi \in \Sigma \subseteq H_1(Y^*, S^*)$ where S^* is the union of ∂Y^*, the points of (ii) and the weighted arcs of (i).

The concept of isomorphism carries over in an obvious way to legally weighted 3-manifolds.

THEOREM (9.6). *To each legally weighted 3-manifold Y^* there corresponds a closed oriented 4-manifold with S^1-action and weighted orbit space isomorphic to Y^*.*

PROOF. Let N^* be a regular neighborhood of S^* and Q^* a regular neighborhood of the weighted circles of Y^*. As in (9.2) there is a $\chi' \in H^2(\mathrm{Cl}(Y^* - N^*))$ which is Poincaré dual to χ. Let π: $Z \to \mathrm{Cl}(Y^* - N^*)$ be the principal S^1-bundle with Euler class χ'. The bundle projection π is trivial over Q^*.

Next use (4.4) and (9.1) to construct a 4-manifold Q with S^1-action and orbit space Q^* with weights induced from Y^*. Remove $\pi^{-1}(\mathrm{int}\, Q^*)$ from Z

and replace it with Q by gluing each component Q_i of Q to $\pi^{-1}(\mathrm{Cl}(Y^* - (N^* \cup Q^*)))$ with an equivariant homeomorphism ϕ_i: $\partial Q_i \to \pi^{-1}(\partial Q_i^*)$ over ∂Q_i^* which takes the loop q_i to a loop $\phi_i(q_i) \sim 0$ in $\pi^{-1}(Q_i^*)$. An argument similar to that used in (9.2) shows that the resulting S^1-manifold W over $\mathrm{Cl}(Y^* - N^*)$ is well defined up to equivalence.

The required 4-dimensional S^1-manifold M is obtained from W in a straightforward fashion. For each component of N^*, N_j^*, which is the regular neighborhood of a weighted arc, use (5.3)(b) to construct the 4-manifold N_j with S^1-action and orbit space N_j^* with weights induced from Y^*. By (2.7) the orbit map $\partial N_j \to \partial N_j^*$ is the S^1-bundle projection $S^3 \to S^2$ or $S^2 \times S^1 \to S^2$, and by definition of Σ this bundle is equivalent to the restriction over ∂N_j^* of p: $X \to X^*$. These bundles admit a unique automorphism up to vertical equivariant isotopy. Thus up to equivalence there is a unique way to equivariantly attach N_j to W over ∂N_j^*.

If the component N_k^* of N^* is a regular neighborhood of a point x^* of (9.5)(ii) the action extends over N_k^* by coning the action on ∂N_k so that the cone point is fixed by the action and is sent by the orbit map to x^*. Finally, spread out the principal orbit bundle over the collar on ∂Y^* and collapse orbits over ∂Y^* to points. □

THEOREM (9.7). *Closed oriented 4-dimensional S^1-manifolds are equivalent if and only if their weighted orbit spaces are isomorphic.*

PROOF. If M_1 and M_2 are equivalent, (9.4) shows that M_1^* and M_2^* are isomorphic. If M_1^* and M_2^* are isomorphic, we may identify $M_1^* = M_2^* = Q^* \cup N^* \cup X^*$. By (3.6) and (9.1) Q and N are well defined up to equivalence by the orbit data, and clearly X is specified by χ.

Now suppose that we choose sections s_1 and s_2 over ∂Q^* and let π_j: $Z_j \to Z^* = Q^* \cup X^*$, $j = 1, 2$, be the extension of the bundle $X \to X^*$ obtained by equivariantly attaching $Q^* \times S^1$ with a bundle isomorphism ϕ_j so that $\partial Q^* \times 0$ is attached to $s_j(\partial Q^*)$. In (9.2) we saw that the resulting bundles depend only on the class $\Sigma_i^k s_j(m_i^*) \in H_1(\partial Q)$. Let η_j be the Euler class of Z_j. We have the following commutative diagram with exact top row and left-hand column.

$$\begin{array}{ccccc}
H^1(Q^*) & \xrightarrow{\delta_1} & H^2(Z^*, Q^*) & \xrightarrow{j_1^*} & H^2(Z^*) \\
{\scriptstyle i^*}\downarrow & & \lambda \downarrow \approx & & \\
H^1(\partial Q^*) & \xrightarrow{\delta_2} & H^2(X^*, \partial Q^*) & & \\
{\scriptstyle \delta_3}\downarrow & \swarrow {\scriptstyle j_2^*} & & & \\
H^2(Q^*, \partial Q^*) & & & &
\end{array}$$

Let $\theta_j \in H^2(X^*, \partial Q^*)$ be the obstruction to extending the section s_j to a global section for the bundle $X \to X^*$. Certainly s_j extends to a section s_j' over Q^* for the bundle π_j: $Z_j \to Z^*$. If θ_j' is the obstruction to extending s_j' to a global section for π_j then $\lambda(\theta_j') = \theta_j$ and $j_1^*(\theta_j') = \eta_j$. So if $d \in H^1(\partial Q^*)$ is the primary difference of s_1 and s_2 then $j_1^*\lambda^{-1}\delta_2(d) = \eta_1 - \eta_2$. Thus if $\eta_1 = \eta_2$ there is a $d' \in H^1(Q^*)$ such that $\delta_1 d' = \lambda^{-1}\delta_2 d$, and $\delta_3(d - i^*d') = 0$. It follows that $d \in i^*(H^1(Q^*))$, so the Kronecker product $\langle d, m^* \rangle = 0$ for each meridian m^* on ∂Q^*. Hence $s_1(m_i^*) \sim s_2(m_i^*)$ in ∂Q_i for each i.

It is easily seen that there is a section s_1 of $X \to X^*$ over ∂Q^* such that $s_1(m_i^*) \sim q_i$ for each i, and so the Euler class χ' of π_1: $Z_1 \to Z^*$ is the Poincaré dual to the characteristic class χ. If s_2 is any other section over ∂Q^* such that the Euler class of π_2: $Z_2 \to Z^*$ is χ', the above paragraph shows that also $s_2(m_i^*) \sim q_i$ for each i. Thus again an argument as in (9.2) shows that as S^1-manifolds $Q \cup_{\phi_1} X$ and $Q \cup_{\phi_2} X$ are equivalent. I.e. there is essentially one way to sew the pieces together to obtain the characteristic class χ. □

(9.8). Suppose $H_1(M^*) = 0$, then ∂: $H_1(M^*, S^*) \to H_0(S^*)$ is injective. Since by (9.5) $\partial\chi = (\varepsilon_1, \ldots, \varepsilon_m)$, χ is determined by the indices, and the present context yields an easier proof of (5.6).

10. Fundamental groups. Let S^1 act on the closed oriented 4-manifold M; fix a regular neighborhood R^* in M^* of $E^* \cup F^*$, and let X^* be the closure of the complement. Triangulate X^* and for each component Q_j^* of R^* which is a regular neighborhood of a weighted circle fix a meridian m_j^* and a complementary curve l_j^* in the 1-skeleton of ∂Q_j^*. Thus m_j^* and l_j^* generate $\pi_1(\partial Q_j^*)$. Let M_-^* denote the closure in M^* of the complement of the components of R^* which are regular neighborhoods of simply-weighted circles.

From a maximal tree in the 1-skeleton of X^* we obtain a presentation of $\pi_1(X^*)$, $(a_1^*, \ldots, a_k^* | r_1^*, \ldots, r_g^*)$, where loops a_i^* arise from edges not lying in the maximal tree, and there is one relation r_j^* corresponding to each 2-simplex τ_j.

THEOREM (10.1). *If $F \neq \varnothing$ then $\pi_1(M)$ is obtained from $\pi_1(M_-^*)$ by adding generators c_n, $n = 1, \ldots, t$, one for each simply-weighted circle of M^*, and relations $c_n^{\alpha_n}$, $[c_n, l_n^*]$, and $m_n^* c_n^{\beta_n}$ where the nth simply-weighted circle has weight (α_n, β_n).*

PROOF. There is a section s to the free action over the 1-skeleton of X^*, and if c is the obstruction cocycle obtained from s then c represents the Euler class of p: $X \to X^*$. Let $b_j = c(\tau_j)$, h an oriented orbit, and $a_i = s(a_i^*)$. It then follows from Van Kampen's Theorem that

$$\pi_1(X) = \left(a_i, h | r_j h^{b_j}, [a_i, h] : i = 1, \ldots, k; j = 1, \ldots, g\right).$$

If there is a component Q_j^* of R^* corresponding to a multiply-weighted circle k_j^* then $\pi_1(\partial Q_j) = (q_j) \times (l_j) \times (h)$ and $\pi_1(Q_j) = (k_j)$, where k_j is a section over k_j^* and $l_j = s(l_j^*)$. In $\pi_1(X \cup Q_j)$ one has the relations $h = 1$, $k_j = l_j$, $q_j = 1$ (see [3, §3]) and $q_j = m_j = s(m_j^*)$. These correspond to the obvious relations $k_j^* = l_j^*$ and $m_j^* = 1$ in $\pi_1(X^* \cup Q_j^*)$. Hence $\pi_1(X \cup Q_j) = \pi_1(X^* \cup Q_j^*)$. Similar arguments show that if $F \neq \varnothing$, $\pi_1(M_-) = \pi_1(M_-^*)$.

It remains to consider a component Q_n^* of R^* corresponding to the simply-weighted circle k_n^*. Let c_n be an exceptional orbit in Q_n. Then with notation as before, $\pi_1(Q_n) = (c_n) \times (k_n)$ and $\pi_1(\partial Q_n) = (q_n) \times (l_n) \times (h)$. In $\pi_1(M_- \cup Q_n)$ there are the relations $h = c_n^{\alpha_n}$, $l_n = k_n$, $c_n^{-\beta_n} = q_n = m_n h^{u_n}$ for some integer u_n, and $[c_n, k_n] = 1$. Thus if $F \neq \varnothing$, $h = 1$, and $\pi_1(M_- \cup Q_n)$ is obtained from $\pi_1(M_-)$ by adding the generator c_n and the relations $c_n^{\alpha_n}$, $m_n c_n^{\beta_n}$, and $[c_n, l_n]$. Since we have the isomorphism $\pi_1(M_-) \approx \pi_1(M_-^*)$ when $F \neq \varnothing$, this proves the theorem. □

COROLLARY (10.2). *If $F \neq \varnothing$ and M^* has no simply-weighted circles then* $\pi_1(M) = \pi_1(M^*)$. □

COROLLARY (10.3). *If $F \neq \varnothing$ then $H^1(M) \approx H^1(M^*)$.*

PROOF. It suffices to show that $H_1(M)/\text{Torsion}$ is isomorphic to $H_1(M^*)/\text{Torsion}$. Let m_n^*, $n = 1, \ldots, t$, be meridians on the boundaries of regular nieghborhoods of the simply-weighted circles. Note that $H_1(M^*)$ is $H_1(M_-^*)$ modulo the subgroup generated by the m_j^*. By (10.1) $H_1(M)$ is $H_1(M_-^*) \oplus (\bigoplus_{n=1}^t \mathbf{Z}_{\alpha_n})$ modulo the subgroup generated by the elements $m_n^* + \beta_n c_n$ where c_n generates $\mathbf{Z}_{\alpha_n}$. In $H_1(M)$, $m_n^* = -\beta_n c_n$ is a torsion element, so $H_1(M)/\text{Torsion}$ is isomorphic to $H_1(M_-^*)$ modulo the subgroup generated by the torsion subgroup of $H_1(M_-^*)$ and the m_j^*. By our previous remarks this is also isomorphic to $H_1(M^*)/\text{Torsion}$. □

In case $F = \varnothing$ our formulation of $\pi_1(M)$ is not as satisfactory since it involves the choice of the section s over the 1-skeleton of X^*. For convenience we choose s so that $m_n = s(m_n^*) = q_n$ for each $n = 1, \ldots, t$.

THEOREM (10.4). *If $F = \varnothing$ then*

$$\pi_1(M) = \big(a_i, c_n, h \,|\, [a_i, h], r_j h^{b_j}, [c_n, l_n], c_n^{\alpha_n} h^{-1}, c_n^{\beta_n} m_n:$$
$$i = 1, \ldots, k; j = 1, \ldots, g; n = 1, \ldots, t\big). \quad □$$

EXAMPLE (10.5). Let M have weighted orbit space $M^* = S^2 \times S^1$ with simply-weighted circle, (pt) $\times S^1$, with weight (α, β). Using notation established in §9 we have $X^* \cong D^2 \times S^1$, $Q^* \cong D^2 \times S^1$, $\partial X^* = \partial Q^*$, and $S^* = \varnothing$; so $\Sigma = H_1(M^*) \approx \mathbf{Z}$. The principal bundle $X \to X^*$ is trivial so we may choose a section s over X^*. The meridian m^* on ∂Q^* bounds a disk D in X^*. If $t_1 = t(m^*)$ then $\chi' = n \in H^2(\text{Cl}(M^* - N^*)) = H^2(M^*) = \mathbf{Z}$ means that in $\pi_1(\partial Q)$ we have $q = t_1 h^n = h^n$ since t_1 bounds the disk $t(D)$. So using

(10.4) we obtain

$$\pi_1(M) = \left(a, c, h \mid [a, c], c^\alpha h^{-1}, c^\beta h^n\right) \quad \text{if } \chi' = n.$$

I.e.

$$\pi_1(M) = \left(a, c \mid [a, c], c^{\alpha n+\beta}\right) \approx \mathbf{Z} \oplus \mathbf{Z}_{\alpha n+\beta}.$$

In fact it is not difficult to see that $M \cong L(\alpha n + \beta, \alpha) \times S^1$. Note that different choices of χ (up to sign) give nonhomeomorphic manifolds M.

11. Quadratic forms. Let M be a closed oriented 4-manifold carrying an S^1-action and suppose:

(i) The fixed point set is nonempty.

(ii) Each simply-weighted circle is $\mathbf{Z}_2$-nullhomologous in M^*.

In this section we shall indicate a procedure for computing a matrix representing the quadratic form determined by the intersection pairing on $H_2(M)/\text{Torsion}$.

(11.1). In §8 we showed that when M is simply connected there are $\chi(F) - 2$ "equivariant" 2-cycles in M whose intersection matrix C represents the quadratic form of M. ($\chi(F)$ denotes the Euler characteristic of F.) Also we showed that C is congruent to a matrix N which is a direct sum of matrices of the form (1), (-1), and $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

The argument of §8 involves a neighborhood of $E \cup F$ which is obtained by plumbing 2-disk bundles over S^2. In the present situation a similar neighborhood of $E \cup F$ may be obtained by plumbing 2-disk bundles over S^2 and over the boundary components of M^* which need not be 2-spheres. Since the arguments of §8 involve only the matrix of the plumbing, the results carry over. If M^* has m boundary components and n points in $F^* - \partial M^*$ this technique yields $2m + n - 2$ 2-cycles in M with homology classes in $H_2(M)/\text{Torsion}$ and intersection matrix C as above.

(11.2). Let β_i and β_i^* denote the Betti numbers of M and M^*. Using Smith Theory, duality, and (10.3) we obtain $\beta_2 = \chi(F) - 2 + 2\beta_1^*$. Let $b = \frac{1}{2}(\beta_2 - (2m + n - 2))$. If $\partial M^* = \varnothing$, i.e., if $m = 0$, then $\chi(F) = n$ and $b = \beta_1^* = \beta_2^*$. By duality we may choose 1-cycles $\zeta_1, \ldots, \zeta_b$ and 2-cycles $\eta_1, \ldots, \eta_b$ of M^* whose homology classes generate $H_1(M^*)/\text{Torsion}$ and $H_2(M^*)/\text{Torsion}$ and whose intersection numbers are $\eta_i \cdot \zeta_j = \delta_{ij}$ for all i, j.

If $\partial M^* \neq \varnothing$ let g_i be the genus of the ith component of ∂M^*, and let $G = \sum_{i=1}^m g_i$. Then $\chi(F) = 2m + n - 2G$ so $b = \beta_1^* - G$. Examination of the exact sequence

$$0 \to H_3(M^*, \partial M^*) \xrightarrow{\partial} H_2(\partial M^*) \xrightarrow{i^*} H_2(M^*) \xrightarrow{j^*} H_2(M^*, \partial M^*)$$

shows that $j^*(H_2(M^*))$ has rank $\beta_2^* - m + 1$. However, $1 - \beta_1^* + \beta_2^* = \chi(M^*) = \frac{1}{2}\chi(\partial M^*) = m - G$; so $\beta_2^* - m + 1 = \beta_1^* - G =$

b. Choose 2-cycles $\eta_1, \ldots, \eta_b$ of M^* whose corresponding homology classes generate $j^*(H_2(M^*))$. As above we can find 1-cycles $\zeta_1, \ldots, \zeta_b$ with homology classes in $H_1(M^*)/\text{Torsion}$ such that $\eta_i \cdot \zeta_j = \delta_{ij}$ for all i, j.

In either case each ζ_j may be represented by a loop Z_j^* and each η_i^* by an embedded closed surface S_i^* in the interior of M^*. After shifting into general position we may assume that the Z_j^* are disjoint from each other and also from $E^* \cup F^*$ and that the surfaces S_i^* meet the Z_j^* and $E^* \cup F^*$ in a finite number of points in int M^*.

We shall now replace the surfaces S_i^* by (perhaps singular) surfaces T_i^* over which there is a section to the orbit map p: $M \to M^*$. First of all, we remove intersections with the simply-weighted circles. Since simply-weighted circles are assumed to be $\mathbf{Z}_2$-nullhomologous in M^*, an S_i^* which intersects such a circle does so at an even number of points. Take a small regular neighborhood R^* of the given weighted circle and remove intersections with S_i^* in pairs, replacing a pair of disks in S_i^* with an annulus on ∂R^*. Since this procedure leaves the homology class of S_i^* unchanged, we may suppose that no S_i^* meets a simply-weighted circle.

If $S_i^* \cap (E^* \cup F^*) = \varnothing$ choose an arc whose interior is disjoint from $(\bigcup Z_j^*) \cup E^* \cup F^*$ and which runs from a point in S_i^* to a point $x_i^* \in F^*$. Thicken the arc to a 3-disk D^* and replace $S_i^* \cap \partial D^*$ by $\mathrm{Cl}(\partial D^* - S_i^*)$ to obtain T_i^*, a surface homologous to S_i^*, such that $T_i^* \cap (E^* \cup F^*) = x_i^*$.

If S_i^* intersects a weighted arc or circle A^* use a similar procedure, i.e. if $S_i^* \cap (A^* - F^*) \neq \varnothing$ choose an x^* in this intersection such that there is an arc from x^* to F^* whose interior lies in $E^* - S_i^*$; thicken the arc to a 3-disk and interchange boundary components as before. If we proceed carefully, dealing with one intersection at a time, we obtain a (perhaps singular) surface T_i^* homologous to S_i^* such that $T_i^* \cap (E^* \cup F^*) \subseteq F^*$, and the singular points of T_i^* lie in F^*. In all other cases let $T_i^* = S_i^*$.

There is a section to the orbit map over each T_i^*; let T_i be the image of such a section. For each j let $Z_j = p^{-1}(Z_j^*)$. The T_i and Z_j, $i, j = 1, \ldots, b$, are 2-cycles in M which represent elements of $H_2(M)/\text{Torsion}$ since the classes of η_i and ζ_j lie in $H_*(M^*)/\text{Torsion}$. There is exactly one point of $T_i \cap Z_j$ over each point of $T_i^* \cap Z_j^*$, and according to our orientation conventions (3.2) the intersection number in M is the same as the intersection number in M^*. Hence $T_i \cdot Z_j = \delta_{ij}$. Furthermore, $Z_j \cap Z_k = \varnothing$ if $j \neq k$, so the intersection matrix of the 2-cycles Z_j and T_i, $i, j = 1, \ldots, b$, is the $2b \times 2b$ matrix $\left(\begin{smallmatrix} X & I \\ I & 0 \end{smallmatrix}\right)$ where I is the $b \times b$ identity matrix and X is the intersection matrix of the T_i.

THEOREM (11.3). *Let S^1 act with nonempty fixed point set on the closed oriented 4-manifold M, and suppose that each simply-weighted circle in M^* is $\mathbf{Z}_2$-nullhomologous. Then the quadratic form of M is represented by the matrix*

$$Q = \begin{pmatrix} N & & 0 \\ & E & I \\ 0 & & \\ & I & 0 \end{pmatrix}$$

where E is the diagonal $b \times b$ *matrix with* $E_{ii} = 0$ *or* 1 *according to the parity of* $T_i \cdot T_i$ (*and N is the matrix described in* (11.1)).

PROOF. The matrix C of (8.5) is the intersection matrix of $\beta_2 - 2b$ 2-cycles whose image in M^* is a union of arcs and boundary components; so we may as well have chosen the Z_j^* to avoid these. The intersection matrix of these $\beta_2 - 2b$ 2-cycles along with the T_i and Z_j thus has the form

$$Q' = \begin{bmatrix} C & Y & 0 \\ Y & X & I \\ 0 & I & 0 \end{bmatrix}.$$

Since N is congruent to C over $\mathbf{Z}$, we may replace C by N in Q' (also changing Y).

The operation of adding an integral constant times row i to row j and then that constant times column i to column j preserves the congruence class over $\mathbf{Z}$ of an integral matrix. Such operations on the first $\beta_2 - b$ rows and columns of Q' by the last b rows and columns yield a congruent matrix Q in the required form. Since the determinant of Q is $-\det N = \pm 1$, $H_2(M)/\text{Torsion}$ is indeed generated by the classes of the above β_2 2-cycles, and Q represents the quadratic form of M. □

COROLLARY (11.4). *Assume the hypothesis of* (11.3). *Then the signature of M is the signature of the matrix N* (*which is easily computed from the weighted orbit space* M^*). □

EXAMPLE (11.5). Let N and S denote the hemispheres of S^2 with poles n and s respectively. Suppose M is a closed oriented 4-manifold carrying an S^1-action with orbit space $M^* = S^2 \times S^1$ with $E^* \cup F^* = n \times S^1 = \{(3, 2), (2, 1)\}$ a weighted circle. If $p\colon M \to M^*$ is the orbit map then $p^{-1}(S \times S^1) \cong (S \times S^1) \times S^1$. Now q is a section over $\partial N \times 1$, and if the characteristic class of the action $\chi = b \in \mathbf{Z} = H_1(M^*)$ ($S^* = \varnothing$), then b is the obstruction to extending q to a section over $S \times 1$. Hence q is homologous to $-m + bh$ on $p^{-1}(\partial N) = p^{-1}(\partial S)$, where $m = \partial S \times 1 \times 1$.

Suppose $n \times 1 \in F^*$ and let $T^* = S^2 \times 1$ and $Z^* = s \times S^1$. Then $\beta_2 = \chi(F) - 2 + 2\beta_1^* = 2$, so as above T and Z generate $H_2(M)/\text{Torsion}$. Let B be a small 4-disk slice about the point $p^{-1}(n \times 1)$. View $\partial B \cong S^3$ as the union of solid tori V' and V'' with S^1-actions with Seifert invariants (2, 1) and (3, 1) and core orbits O' and O''. On $\partial V' = \partial V''$ we have $q' \sim -q'' -$

h'' and $h' \sim h''$. We may assume that $C = T \cap \partial B$ lies on $\partial V'$; then in $\partial V'$ we have $C \sim -q' + bh'$. Let $\hat{V}' \subset V'$ be a smaller invariant solid torus with core orbit O' and let $\hat{C} \sim q' + b\hat{h}'$ on $\partial \hat{V}'$. Then $T \cdot T$ is the linking number of C and $\hat{C}$ in ∂B. In $\hat{V}'$ we have $\hat{C} \sim (2b+1)O'$ and in V'' we have $C \sim (3b+2)O''$ (cf. [6, p. 12]). It follows that $T \cdot T = (2b+1)(3b+2) \equiv b$ (mod 2). The matrix of the quadratic form of M is thus $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ if b is even and

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \approx \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

if b is odd. Hence topologically, at least two distinct manifolds M are involved in this example. Of course by (10.2) each has $\pi_1(M) \approx \mathbf{Z}$.

12. Weak isomorphisms. In this section a technique due to Church and Lamotke [2] is employed to show that weighted orbit spaces are often isomorphic under conditions seemingly less stringent than those of (9.3). We say that legally weighted 3-manifolds M_1^* and M_2^* are *weakly isomorphic* if there is a homeomorphism of M_1^* onto M_2^* preserving orientation, indices (9.5), and the orbit data of (9.5)(i), and (ii). Examples (10.5) and (11.5) demonstrate that weakly isomorphic orbit spaces need not be isomorphic.

THEOREM (12.1). *Let S^1 act on the closed oriented 4-manifolds M_1 and M_2, and suppose that there is a component of $E^* \cup F^*$ in M_1^* with nonzero index. Then M_1^* and M_2^* are weakly isomorphic if and only if they are isomorphic (so M_1 and M_2 are equivalent).*

PROOF. According to (9.5) the sum of the indices in M_1^* is 0 so we can find a collection of disjoint oriented polyhedral embedded arcs $\{l_i\}$ with endpoints in F^* such that a component of $E^* \cup F^*$ with index ε contains ε terminal points if $\varepsilon > 0$, $-\varepsilon$ initial points if $\varepsilon < 0$, and meets $\bigcup l_i$ in no other points. If there is a nonzero index in int M_1^* we may number the l_i so that l_1 has its initial point x_0 or terminal point x_1 lying on a weighted arc or point. Suppose for convenience that it is x_0.

Let $h: M_1^* \to M_2^*$ be a weak isomorphism which we may assume to be PL. The sums Σl_i and $\Sigma h(l_i)$ define relative 1-cycles with homology classes $\zeta \in H_1(M_1^*, S_1^*)$ and $h_*(\zeta) \in H_1(M_2^*, S_2^*)$, and by (9.5) $\partial\zeta = \partial\chi_1$ and $\partial h_*(\zeta) = \partial\chi_2$ where χ_k is the characteristic class of the action on M_k. Thus there are $\psi_k \in H_1(M_k^*)$ such that under $j_{k*}: H_1(M_k^*) \to H_1(M_k^*, S_k^*)$, $j_{1*}(\psi_1) = \chi_1 - \zeta$ and $j_{2*}(\psi_2) = \chi_2 - h_*(\zeta)$. Represent ψ_1 and ψ_2 by loops l_0 and k_0 based at x_0 and $h(x_0)$. The product paths $l_0 \cdot l_1$ and $k_0 \cdot h(l_1)$ are homotopic keeping endpoints fixed to embedded polyhedral arcs l_1' and k_1' which are disjoint from $\bigcup_{i \geqslant 2} l_i$ and $\bigcup_{i \geqslant 2} h(l_i)$. For $i > 1$ let $l_i' = l_i$, and choose disjoint regular neighborhoods B_i of l_i' meeting ∂M_1^* regularly. We may suppose that each B_i meets only the components of $E^* \cup F^*$ met by l_i' and that B_i contains each weighted arc that it meets. Let $D_i = h(B_i)$ for $i \geqslant 2$ and let D_1 be an analogous regular neighborhood of k'_1. Then $\Sigma l'_i$ and $k'_1 + \Sigma_{i \geqslant 2} h(l_i)$ are relative 1-cycles with homology classes $\omega_1 \in H_1(\bigcup B_i, \bigcup B_i \cap S_1^*)$ and $\omega_2 \in H_1(\bigcup D_i, \bigcup D_i \cap S_2^*)$. If i_1: $(\bigcup B_i, \bigcup B_i \cap S_1^*) \to (M_1^*, S_1^*)$ is the inclusion then $i_{1*}(\omega_1) = j_{1*}(\psi_1) + \zeta_1 = \chi_1$, and similarly, $i_{2*}(\omega_2) = \chi_2$.

Taking advantage of the fact that $h(x_0)$ lies on a weighted arc or point, we can find an ambient isotopy of M_2^* which fixes $\bigcup_{i \geqslant 2} D_i$ and whose finishing homeomorphism f preserves orbit data, fixes $h(x_0)$ and $h(x_1)$, and takes the pair $(h(B_1), h(B_1) \cap S_2^*)$ to $(D_1, D_1 \cap S_2^*)$. Now let $g = f \circ h$ and $g' = g| \bigcup B_i$. Since $g'_*(\omega_1)$ is represented by $g'(l'_1) + \Sigma_{i \geqslant 2} h(l_i)$, we have $\partial g'_*(\omega_1) = \partial \omega_2$. But ∂ on $H_1(\bigcup D_i, \bigcup D_i \cap S_1^*)$ is injective so $g'_*(\omega_1) = \omega_2$. Hence

$$g_*(\chi_1) = g_* i_{1*}(\omega_1) = i_{2*} g'_*(\omega_1) = i_{2*}(\omega_2) = \chi_2.$$

Thus g is an isomorphism.

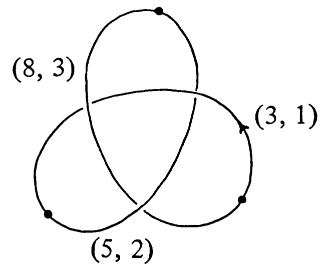
The converse follows from (9.7). □

13. Addendum to §8. Recent work of Peter Sie Pao has pointed the way towards sharpening some of the results of §8. Pao has shown **[10]** that if M^4 is a homotopy 4-sphere with S^1-action and orbit space $M^* \cong S^3$ then $M \cong S^4$. We shall use his ingenious "replacement trick" to prove that modulo the 3-dimensional Poincaré conjecture, a simply connected 4-manifold carrying a locally smooth S^1-action must be a connected sum of copies of S^4, CP^2, $-CP^2$, and $S^2 \times S^2$. This gives rise to many interesting examples of nonstandard S^1-actions. The result of Pao's replacement trick is described in the next proposition.

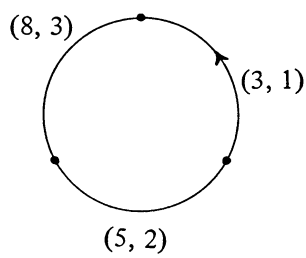
PROPOSITION (13.1) (PAO **[10]**). *Let M be a 4-manifold with an S^1-action whose weighted orbit space contains a weighted circle C^* with exactly two fixed points. Then M admits a different S^1-action whose weighted orbit space is either M^* with C^* replaced by a pair of fixed points or M^*-*int D^3 *with C^* removed.* □

THEOREM (13.2). *Let S^1 act locally smoothly on the simply connected 4-manifold M, and suppose that M^* is not a counterexample to the 3-dimensional Poincaré conjecture. Then M is a connected sum of copies of S^4, CP^2, $-CP^2$, and $S^2 \times S^2$.*

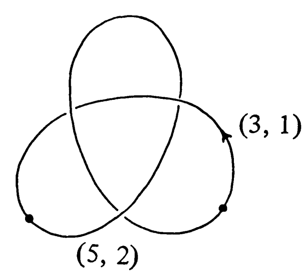
PROOF. It follows from the equivariant classification theorem (6.2) that M is the equivariant connected sum $M_1 \# N_1 \# \cdots \# N_k$ of simply connected 4-manifolds where each $N_i^* \cong S^3$ and has one unknotted multiply-weighted circle as $E^* \cup F^*$, and $M_1^* = M^*$ with the same $E^* \cup F^*$ (as sets) except that each weighted circle in M_1^* contains exactly 2 fixed points. For example, if M^* contains the weighted circle



then M is the equivariant connected sum of N_1 whose weighted orbit space is S^3 with



and M' where $M'^* = M^*$ except that the previous weighted circle has been replaced by



The S^1-actions on the manifolds N_i all extend to actions of T^2 (7.1). Hence by [**8**] these manifolds are connected sums of copies of S^4, CP^2, $-CP^2$, and $S^2 \times S^2$. The proof is completed by showing that M_1 also admits a T^2-action. Since each weighted circle in M_1^* has exactly two fixed points, (13.1) implies that M_1 admits another S^1-action whose weighted orbit space is M^* or M^* with some disks removed and in which $E^* \cup F^*$ contains no weighted circles. So by (7.1) M_1 admits a T^2-action. □

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