

PROJECTIVE MODULES FOR FINITE CHEVALLEY GROUPS

BY

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ABSTRACT. The purpose of this paper is to obtain character formulas for certain indecomposable projective modules for a finite Chevalley group. It is shown that these modules are also modules for the corresponding semi-simple algebraic group.

Introduction. Let $G(q)$ be a finite universal Chevalley group constructed over a field with $q = p^n$ elements. Srinivasan [13, Remark 3] has conjectured that certain generalized characters of $G(q)$ are principal indecomposable characters (for p) of $G(q)$. The objective of this paper is to provide, under certain conditions, an affirmative answer to this conjecture.

The contents of the paper are as follows. In §2 we obtain a Z -basis for the ring spanned by the Brauer characters of the principal indecomposable modules (PIM's). This basis consists of generalized characters which are products of functions considered by Wong [18] and the Steinberg character. The PIM's of $G(q)$ are indexed by a certain set of weights and in §3 we consider Harish-Chandra's definition for the length of a weight. This is applied in §4 to show that if the length is small, with respect to q , then the corresponding element of the above Z -basis is the Brauer character of a $G(q)$ -module. This is accomplished by applying Weyl's character formula in characteristic 0 and then transferring to the field of characteristic p .

§5 is devoted to obtaining a lower bound for the dimension of each PIM in terms of the weight which characterizes the module. In §6 we show that each $G(q)$ -PIM is a summand of an indecomposable G -module, where G is the corresponding simply connected semisimple algebraic group. By applying a result of Humphreys [7], we then find an upper bound for the dimension of each PIM.

While the dimensional bounds of §§5 and 6 are not always precise, conditions are found in §7 which insure equality. In Theorem 7.4 we apply these results to obtain character formulas for those PIM's which correspond to weights whose length is small with respect to p . We also show that they are modules for the algebraic group G . With certain restrictions, the Brauer

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characters of these PIM's are the generalized characters of Srinivasan's conjecture.

1. Preliminaries. In this section we introduce the notation to be used throughout the paper.

1.1. *Lie algebras.* Our basic reference for results concerning Lie algebras will be [11].

Let \mathfrak{g} be a finite dimensional simple Lie algebra over the field of complex numbers, \mathfrak{h} a Cartan subalgebra, and Δ the set of roots determined by \mathfrak{h} . Let W be the Weyl group and let $\Pi = \{\alpha_1, \dots, \alpha_n\}$ be a set of simple roots for Δ . If \mathfrak{h}_0^* is the rational vector space spanned by Δ , then Π is a basis for \mathfrak{h}_0^* . We order \mathfrak{h}_0^* by the lexicographical ordering determined by Π . That is, $\sum m_i \alpha_i$ is positive if the first nonzero coefficient is positive.

The fundamental weights $\lambda_1, \dots, \lambda_n$ are the elements of \mathfrak{h}_0^* determined by $2(\lambda_i, \alpha_j)/(\alpha_j, \alpha_j) = \delta_{ij}$, where the inner product is the one given by the Killing form of \mathfrak{g} . Let X be the set of all integer linear combinations of the λ_i . Since X is contained in \mathfrak{h}_0^* it inherits the lexicographical ordering. The notation $>$ is used to denote the resulting total ordering on X .

Denote by X^+ the set of all nonnegative integer linear combinations of the λ_i . If $\lambda \in X^+$, then λ is positive in the ordering of X and there are but a finite number of elements of X^+ which are less than λ . So we may carry out induction on the elements of X^+ .

If V is a finite dimensional \mathfrak{g} -module, $V = \sum V_\lambda$ denotes the weight space decomposition of V with respect to \mathfrak{h} . For $\mu \in X^+$ the irreducible module with μ as highest weight will be designated by $V(\mu)$ and $\Delta(\mu)$ will denote the set of weights for this module.

1.2. *Chevalley groups.* We shall use [15] as our reference for the construction and properties of Chevalley groups.

Let p be a prime, $GF(p)$ the finite field with p -elements, and K the algebraic closure of $GF(p)$. Let G be the universal Chevalley group constructed from \mathfrak{g} over K . As an algebraic group, G is a simply connected and semisimple group which is defined over $GF(p)$. Let B denote the standard Borel subgroup and U the unipotent subgroup of B . If T is the standard maximal torus of G , then each element of T is of the form $h(\chi)$, where χ is a homomorphism of X into the multiplicative group of K .

Any semisimple element of G is conjugate to an element of T and any element of T is of finite order prime to p . So the two terms, semisimple and p -regular, are the same for elements of G . In particular, if Y is a finite dimensional module for G over the field K , then we may define the Brauer character of Y in the usual fashion.

Let $\{X_\alpha, \alpha \in \Delta; H_i, 1 \leq i \leq n\}$ be a fixed Chevalley basis for \mathfrak{g} and U_Z the corresponding Z -form for the universal enveloping algebra of \mathfrak{g} [15, §2]. If

V is a \mathfrak{g} -module and M an admissible lattice (with respect to U_Z), then $\bar{V} = M \otimes_Z K$ is a rational G -module.

Moreover, if $V = \sum V_\lambda$ is the weight space decomposition of V , then there is a corresponding direct sum decomposition $\bar{V} = \sum \bar{V}_\lambda$ with

$$h(\chi)\bar{v} = \chi(\lambda)\bar{v}$$

for $h(\chi) \in T$ and $\bar{v} \in \bar{V}_\lambda$. We call \bar{V} the derived G -module.

1.3. *Finite groups.* For notation and results concerning the modular representation theory of finite groups we refer to [5].

Let q be a power of the prime p and let $G(q)$ be the finite universal Chevalley group constructed from \mathfrak{g} over $GF(q)$. We view $G(q)$ as a subgroup of G and let $T(q)$, $U(q)$ and $B(q)$ denote the corresponding subgroups of $G(q)$.

Let $K[G(q)]$ be the group algebra of $G(q)$ over K . As a left $G(q)$ -module, $K[G(q)]$ may be written as the direct sum of indecomposable modules. These are the principal indecomposable modules (PIM's) of the group $G(q)$, for the prime p . The objective of this paper is to study the Brauer characters of these modules.

In keeping with the notation of algebraic groups, a p -regular element of $G(q)$ will be referred to as semisimple. All other notation is standard.

The notation of this section will remain fixed. All modules for G and $G(q)$ are assumed to be finite dimensional and over the field K . Modules for \mathfrak{g} are also assumed to be finite dimensional and over the field of complex numbers C .

2. Brauer characters of PIM's. The objective of this section is to show that certain naturally occurring generalized characters form a Z -basis for the ring spanned by the Brauer characters of the principal indecomposable modules of $G(q)$, for the prime p . These generalized characters are products of functions considered by Wong [18] and the Steinberg character.

2.1. Recall that K is the algebraic closure of $GF(p)$. Choose a fixed embedding of the multiplicative group of K into the multiplicative group of C . For ε a nonzero element of K , let ε' denote the corresponding complex p' -root of 1. We assume that the Brauer characters of K -modules are defined with respect to this embedding.

For each $\tau \in X^+$ we define a complex valued class function s_τ on the semisimple classes of $G(q)$ by

$$s_\tau(g) = \sum_\nu \chi(\nu)'$$

where $g \in G(q)$ is G -conjugate to $h(\chi) \in T$ and the sum is over all ν in the W -orbit of τ . We adopt the convention that $s_\tau(g) = 0$ if g is not semisimple.

Let X_q be the set of all elements $\mu = \sum m_i \lambda_i$ of X^+ such that $0 < m_i < q -$

1, for $i = 1, \dots, n$. Wong [18, Corollary 3B] has shown that the q^n functions $s_\mu, \mu \in X_q$, form a basis for the vector space of complex valued functions on the semisimple classes of $G(q)$. In fact, if τ is an arbitrary element of X^+ , then

$$s_\tau = \sum_{\mu} a_\mu s_\mu$$

where the a_μ are integers and the sum is over $\mu \in X_q$ with $\mu < \tau$ (see [18]).

2.2. Let $\mu \in X^+$ and let $V(\mu)$ be the irreducible \mathfrak{g} -module with highest weight μ . An admissible lattice for $V(\mu)$ may be obtained by setting $M = U_{\mathbb{Z}}x$, where x is a weight vector of weight μ in $V(\mu)$. If $\bar{V}(\mu)$ is the derived module, with respect to this lattice, then $\bar{V}(\mu)$ has a unique maximal submodule N and so $F(\mu) = \bar{V}(\mu)/N$ is an irreducible G -module. The various $F(\mu)$, for $\mu \in X^+$, form a complete set of rational irreducible modules for G . The module $F(\mu)$ is uniquely determined as the rational irreducible G -module which has a nonzero vector v such that v is fixed by U and $h(\chi)v = \chi(\mu)v$ for all $h(\chi) \in T$ (see [15, Theorem 39, p. 209]).

The $F(\mu)$, for $\mu \in X_q$, remain irreducible when viewed as $G(q)$ -modules and in fact form a complete set of absolutely irreducible $G(q)$ -modules in characteristic p [15, Theorem 43, p. 230]. For $\mu \in X_q$, let ϕ_μ be the Brauer character of $F(\mu)$ as a $G(q)$ -module. Then by results of Wong [18, Theorem 3E], [19, Theorem 2D],

$$\phi_\mu = s_\mu + \sum_{\nu} b_\nu s_\nu \tag{2.2.1}$$

where the b_ν are integers (possibly negative) and the sum is over $\nu \in X_q$ with $\nu < \mu$.

Let $U(\mu)$ denote the principal indecomposable $G(q)$ -module whose unique top and bottom composition factor is $F(\mu)$ ($\mu \in X_q$). Let η_μ be the character of this module. Our first objective is to obtain an expression for η_μ which mirrors the above expression for ϕ_μ . To accomplish this we first recall certain facts about the Steinberg module for $G(q)$.

2.3. Let $\sigma \in X^+$ be given by

$$\sigma = (q - 1) \sum_{i=1}^n \lambda_i.$$

Then σ is the highest element of X_q and $\bar{V}(\sigma)$, the Steinberg module, is irreducible both as G and as $G(q)$ -module. That is, in the previous notation, $\bar{V}(\sigma) = F(\sigma)$. The dimension of this module is q^N , where N is the number of positive roots, and all other absolutely irreducible modules for $G(q)$ in characteristic p have lower dimension (see [18, Lemma 3D]).

Since $U(q)$ is a Sylow p -subgroup of $G(q)$ and $|U(q)| = q^N$, a standard result on modular representations shows that ϕ_σ is the unique character in its

p -block and that $\phi_\sigma = \eta_\sigma$. In particular, this character vanishes on elements of $G(q)$ which are not semisimple and is an ordinary irreducible character of $G(q)$.

To simplify the notation we let $\phi_\sigma = \phi$. Note that by [16, Theorem 15.5], $\phi(s)$ is (up to a sign) the order of a Sylow p -subgroup of the centralizer of s , for s a semisimple element of $G(q)$.

2.4. If f and h are complex valued functions on a finite group, let (f, h) denote the usual inner product of f and h .

LEMMA 2.1. *If $\mu \in X_q$, then (i) $(\phi, s_\mu) = 0$ for $\mu < \sigma$, and (ii) $(\phi, s_\mu) = 1$ for $\mu = \sigma$.*

PROOF. First note that if $\mu = 0$ then $s_\mu = \phi_0$, the principal Brauer character. Since ϕ and ϕ_0 lie in different p -blocks, $(\phi, \phi_0) = 0$.

Assume that $(\phi, s_\nu) = 0$ for all $\nu \in X_q$ with $\nu < \mu$. If $\mu \neq \sigma$, then $\phi_\mu \neq \phi$ and again since ϕ_μ, ϕ lie in different p -blocks, we have $(\phi, \phi_\mu) = 0$. Now by (2.2.1), $\phi_\mu = s_\mu + \sum b_\nu s_\nu$ with the sum over $\nu \in X_q$ and $\nu < \mu$. The induction hypothesis then shows that $(\phi, s_\mu) = 0$.

Now if $\mu = \sigma$, then $(\phi, \phi) = (\phi, s_\sigma)$ and since $(\phi, \phi) = 1$ the lemma is proved.

Let r denote the opposition involution of W . Since $r\Pi = -\Pi$, for each τ in X_q , $-r\tau$ and $\sigma + r\tau$ are also elements of X_q . Note that $r\sigma = -\sigma$.

LEMMA 2.2. *Let $\mu, \tau \in X_q$. Then (i) $(s_\mu\phi, s_\tau) = 0$ for $\tau < \sigma + r\mu$, and (ii) $(s_\mu\phi, s_\tau) = 1$ for $\tau = \sigma + r\mu$.*

PROOF. First note that $s_\mu(g^{-1}) = s_{-r\mu}(g)$, for g a semisimple element of $G(q)$. Using this we see that

$$(s_\mu\phi, s_\tau) = (\phi, s_{-r\mu}s_\tau).$$

Now if γ and ρ are arbitrary elements of X_q , then by [18, formula 9]

$$s_\gamma s_\rho = s_{\gamma+\rho} + \sum_\nu b_\nu s_\nu$$

where the b_ν are integers and the sum is over $\nu \in X_q$ with $\nu < \gamma + \rho$. So

$$s_{-r\mu}s_\tau = s_{-r\mu+\tau} + \sum_\nu b_\nu s_\nu$$

with the sum over $\nu \in X_q$ and $\nu < -r\mu + \tau$.

If $\tau < \sigma + r\mu$, then $-r\mu + \tau < \sigma$ and so the ν appearing in the above sum are all less than σ . Lemma 2.1(i) then shows that $(s_\mu\phi, s_\tau) = 0$.

If $\tau = \sigma + r\mu$, then $-r\mu + \tau = \sigma$ and so by Lemma 2.1(ii), $(s_\mu\phi, s_\tau) = 1$, which completes the proof of the lemma.

The shift from μ to $\sigma + r\mu$ will occur throughout the paper, so we define a function $f: X_q \rightarrow X_q$ by

$$f(\mu) = \sigma + r\mu.$$

The remarks of 2.3 show that $\bar{V}(\sigma)$ is a projective $G(q)$ -module. So for M any $G(q)$ -module, $M \otimes_K \bar{V}(\sigma)$ is also a projective module (see [5, p. 426, ex. 2]), hence is the direct sum of certain uniquely determined principal indecomposable modules.

LEMMA 2.3. *If $\mu \in X_q$, then*

$$s_\mu \phi = \eta_{f(\mu)} + \sum_{\tau} a_\tau \eta_\tau$$

where the a_τ are integers and the sum is over $\tau \in X_q$ for which $\tau > f(\mu)$.

PROOF. The preceding remarks show that $F(\tau) \otimes \bar{V}(\sigma)$ may be decomposed as the direct sum of PIM's. Since $\phi_\tau \phi$ is the Brauer character of this module it may be written as a positive integer linear combination of the η_ν , for $\nu \in X_q$. (2.2.1) shows that s_μ may be expressed as an integer linear combination of the various ϕ_τ and so

$$s_\mu \phi = \sum_{\tau} a_\tau \eta_\tau$$

where the sum is over $\tau \in X_q$ and the a_τ are (possibly negative) integers.

The orthogonality relations for modular characters show that $a_\tau = (s_\mu \phi, \phi_\tau)$. From (2.2.1) and Lemma 2.2

$$\begin{aligned} (s_\mu \phi, \phi_\tau) &= 0 && \text{for } \tau < f(\mu), \\ &= 1 && \text{for } \tau = f(\mu), \end{aligned}$$

which completes the proof of the lemma.

The next lemma yields stronger conditions on the τ which appear in the above expression for $s_\mu \phi$.

LEMMA 2.4. *If $\mu \in X_q$, then*

$$s_\mu \phi = \eta_{f(\mu)} + \sum_{\nu} a_{f(\nu)} \eta_{f(\nu)}$$

where the $a_{f(\nu)}$ are integers and the sum is over $\nu \in X_q$ with $\nu < \mu$ and $f(\nu) > f(\mu)$.

PROOF. For each $\tau \in X_q$ we may choose $\nu \in X_q$ with $f(\nu) = \tau$ and so by Lemma 2.3

$$s_\mu \phi = \eta_{f(\mu)} + \sum_{\nu} a_{f(\nu)} \eta_{f(\nu)} \tag{2.4.1}$$

where the sum is over $\nu \in X_q$ with $f(\nu) > f(\mu)$.

Note that $f(\nu) > f(\mu)$ implies $r\nu > r\mu$, so if $r = -1$ then $\nu < \mu$. However, if $r \neq -1$ then for arbitrary elements of X_q , $r\nu > r\mu$ does not imply $\nu < \mu$. So

a further argument is needed to show that the ν which appear in (2.4.1) with $a_{f(\nu)} \neq 0$ are such that $\nu < \mu$.

Since ϕ is an integer valued character and $s_\mu(g^{-1}) = s_{-r\mu}(g)$ we have

$$(s_\mu\phi)(g^{-1}) = (s_{-r\mu}\phi)(g) \quad \text{for all } g \in G(q). \quad (2.4.2)$$

If Y is a $G(q)$ -module, let Y^* denote the contragredient module. Then for $\lambda \in X_q$, $U(\lambda)^*$ is a PIM which has $F(\lambda)^*$ as its top and bottom composition factors. By [19, Lemma 2B], $F(\lambda)^* = F(-r\lambda)$ and so $U(\lambda)^* = U(-r\lambda)$. In particular, $\eta_\lambda(g^{-1}) = \eta_{-r\lambda}(g)$ for all $g \in G(q)$. If we note that $-rf(\lambda) = f(-r\lambda)$, then the formulas (2.4.1) and (2.4.2) show

$$s_{-r\mu}\phi = \eta_{f(-r\mu)} + \sum_{\nu} a_{f(\nu)}\eta_{f(-r\nu)}.$$

The η_λ are linearly independent, so this expression is unique. Applying Lemma 2.3, with μ replaced by $-r\mu$, shows that if $a_{f(\nu)} \neq 0$ then $f(-r\nu) > f(-r\mu)$ and hence $\nu < \mu$.

In the next result we apply this lemma to obtain an expression for $\eta_{f(\mu)}$ which mirrors that of (2.2.1).

THEOREM 2.5. *If $\mu \in X_q$, then*

$$\eta_{f(\mu)} = s_\mu\phi + \sum_{\nu} c_\nu s_\nu\phi$$

where the c_ν are (possibly negative) integers and the sum is over $\nu \in X_q$ with $\nu < \mu$ and $f(\nu) > f(\mu)$.

PROOF. The proof is by induction on μ . If $\mu = 0$, then $f(\mu) = \sigma$ and $\eta_{f(\mu)} = \phi = s_0\phi$.

Assume that the result is true for all $\rho \in X_q$ with $\rho < \mu$. From Lemma 2.4

$$s_\mu\phi = \eta_{f(\mu)} + \sum_{\rho} a_{f(\rho)}\eta_{f(\rho)}$$

where the $a_{f(\rho)}$ are integers and the sum is over $\rho \in X_q$ with $f(\rho) > f(\mu)$ and $\rho < \mu$. Hence

$$\eta_{f(\mu)} = s_\mu\phi - \sum_{\rho} a_{f(\rho)}\eta_{f(\rho)}.$$

Now by the induction hypothesis, the $s_\nu\phi$ terms which occur in the expression for $\eta_{f(\rho)}$ are such that $\nu < \rho$ and $f(\nu) > f(\rho)$. So $\nu < \mu$ and $f(\nu) > f(\mu)$, which proves the theorem.

Before applying this theorem we first prove a reformulation of a result of Wong [18].

LEMMA 2.6. *Let $\mu \in X_q$ and let θ_μ be the Brauer character of the $G(q)$ -module $\bar{V}(\mu)$. Then*

$$\theta_\mu = s_\mu + \sum_\nu b_\nu s_\nu$$

where the b_ν are integers and the sum is over $\nu \in X_q$ with $\nu < \mu$ and $f(\nu) > f(\mu)$.

PROOF. By [18, formula 10]

$$\theta_\mu = s_\mu + \sum_\nu b_\nu s_\nu \tag{2.4.3}$$

where the b_ν are integers and the sum is over $\nu \in X_q$ with $\nu < \mu$. So it suffices to show that if $b_\nu \neq 0$, then $f(\nu) > f(\mu)$.

If $\theta_{-r\mu}$ is the character of $\bar{V}(-r\mu)$, then again by [18]

$$\theta_{-r\mu} = s_{-r\mu} + \sum_\tau a_\tau s_\tau \tag{2.4.4}$$

where the sum is now over $\tau < -r\mu$. The \mathfrak{g} -module which is contragredient to $V(\mu)$ is $\bar{V}(-r\mu)$, so by [19, Lemma 1A] there exists a lattice for $V(-r\mu)$ such that $\bar{V}(-r\mu)$ is isomorphic to $\bar{V}(\mu)^*$. Hence $\theta_{-r\mu}(g) = \theta_\mu(g^{-1})$. (2.4.3) then shows

$$\theta_{-r\mu} = s_{-r\mu} + \sum_\nu b_\nu s_{-r\nu}$$

Comparing this with (2.4.4) we have $b_\nu = a_{-r\nu}$ for all $\nu \in X_q$. In particular, $b_\nu \neq 0$ implies $-r\nu < -r\mu$ and so $f(\nu) > f(\mu)$.

COROLLARY 2.7. *Let $\mu \in X_q$. If $U(f(\tau))$, for $\tau \in X_q$, occurs as a summand of either $\bar{V}(\mu) \otimes \bar{V}(\sigma)$ or of $F(\mu) \otimes \bar{V}(\sigma)$, then $\tau < \mu$ and $f(\tau) > f(\mu)$.*

The principal indecomposable module $U(f(\mu))$ occurs exactly once as a summand of $\bar{V}(\mu) \otimes \bar{V}(\sigma)$ and as a summand of $F(\mu) \otimes \bar{V}(\sigma)$.

PROOF. By Lemma 2.6 the character of $\bar{V}(\mu) \otimes \bar{V}(\sigma)$ is

$$\theta_\mu \phi = s_\mu \phi + \sum_\nu b_\nu s_\nu \phi$$

where $\nu < \mu$ and $f(\nu) > f(\mu)$. Now by Lemma 2.4, $s_\nu \phi$ may be expressed as a sum of certain $\eta_{f(\tau)}$ with $\tau < \nu$ and $f(\tau) > f(\nu)$. Since $\nu < \mu$ and $f(\nu) > f(\mu)$ it follows that

$$\theta_\mu \phi = \eta_{f(\mu)} + \sum_\tau c_\tau \eta_{f(\tau)}$$

where the c_τ are integers and the sum is over $\tau \in X_q$ with $\tau < \mu$ and $f(\tau) > f(\mu)$. Since this is the character of a projective module, the c_τ are positive integers.

The principal indecomposable modules which occur as summands are determined by the character and so the result for $\bar{V}(\mu) \otimes \bar{V}(\sigma)$ follows.

Using (2.2.1) and applying a similar argument yields the result for $F(\mu) \otimes \bar{V}(\sigma)$.

REMARK. We shall obtain a precise decomposition of $\bar{V}(\mu) \otimes \bar{V}(\sigma)$ in §7, under the hypothesis that μ is 'small' with respect to the prime p .

Let $X(G(q))$, the ring of generalized characters, be the set of all integer linear combinations of the ordinary irreducible characters of $G(q)$. Denote by $Y(G(q))$ the ring generated by the characters of the principal indecomposable modules of $G(q)$.

Theorem 2.5 shows that the $s_\mu\phi$, for $\mu \in X_q$, form a \mathbb{Z} -basis for $Y(G(q))$. It also shows that every element of $Y(G(q))$ is divisible by ϕ . This may be stated more precisely as follows.

COROLLARY 2.8. *If $F: X(G(q)) \rightarrow Y(G(q))$ is defined by $F(\chi) = \chi\phi$, then F is an epimorphism of \mathbb{Z} -modules.*

PROOF. If $\chi \in X(G(q))$, then $\chi\phi$ is a generalized character that vanishes on elements of $G(q)$ which are not semisimple. By [3, Theorem 17], $\chi\phi$ is an element of $Y(G(q))$. So F is a \mathbb{Z} -linear map of $X(G(q))$ into $Y(G(q))$.

From [3, Theorem 15], we see that for each ϕ_r there exists $\chi \in X(G(q))$ such that $\phi_r(g) = \chi(g)$ for all semisimple elements g . Using (2.2.1) we may express s_μ in terms of the ϕ_r and so there exists $\chi \in X(G(q))$ such that $s_\mu(g) = \chi(g)$ for all semisimple elements g . Then $F(\chi) = s_\mu\phi$ and since the $s_\mu\phi$ form a \mathbb{Z} -basis for $Y(G(q))$, F is an epimorphism.

REMARKS. (1) This result has been obtained independently, and in greater generality, by G. Lusztig [12].

(2) The corollary shows that $Y(G(q))$ is a principal ideal of the ring $X(G(q))$. It would be interesting to know what conditions a finite group H must satisfy to insure that $Y(H)$ is a principal ideal of $X(H)$, for some prime p .

The results of [1] show that, with certain restrictions, each $s_\mu\phi$ may be constructed as the alternating sum of characters induced from the Cartan subgroups of $G(q)$. These are the generalized characters of Srinivasan's conjecture. So an affirmative solution to this conjecture would follow by obtaining conditions which insure that $s_\mu\phi$ is a principal indecomposable character. The remainder of this paper is concerned with finding such conditions. Note that the concluding remarks of [1] yield examples of weights μ for which $s_\mu\phi$ is not a principal indecomposable character.

3. A length function. In [6, p. 306], Harish-Chandra has given a definition for the length of a Lie algebra module. The objective of this section is to obtain a slight reformulation of this definition and to record several elementary results for later use.

3.1. Recall that \mathfrak{g} is a simple Lie algebra, Δ is a set of roots and Π is a set of simple roots. For $\mu \in X$ and $\alpha \in \Delta$, let $\langle \mu, \alpha \rangle = 2(\mu, \alpha)/(\alpha, \alpha)$ and note that this is an integer. If $\mu \in X^+$, let $\Delta(\mu)$ be the set of weights of $V(\mu)$.

DEFINITION [6, p. 306]. For $\mu \in X^+$, define the length $\ln(\mu)$ of μ by

$$\ln(\mu) = \max_{\alpha, \nu} |\langle \nu, \alpha \rangle|$$

where the maximum is over all $\alpha \in \Delta$ and all $\nu \in \Delta(\mu)$.

Our first objective is to show that $\ln(\mu) = \langle \mu, \beta_0 \rangle$, where β_0 is the highest short root of Δ .

LEMMA 3.1. *If $\mu \in X^+$, then $\ln(\mu) = \max_{\alpha} \langle \mu, \alpha \rangle$ where the maximum is over $\alpha \in \Delta$.*

PROOF. Choose $\nu \in \Delta(\mu)$ and $\alpha \in \Delta$ such that $\ln(\mu) = \langle \nu, \alpha \rangle$. We first show that there exists a frontier weight $\nu' \in \Delta(\mu)$ with $\langle \nu', \alpha \rangle = \langle \nu, \alpha \rangle$. Recall that ν' is a frontier weight if there is no root γ such that $\nu' + \gamma$ and $\nu' - \gamma$ are both elements of $\Delta(\mu)$.

Let S be the set of all $\beta \in \Delta^+$ for which $(\beta, \alpha) = 0$. Since $\Delta(\mu)$ is a finite set, there are but finitely many elements of $\Delta(\mu)$ which can be expressed in the form $\nu + \sum m_{\beta} \beta$ where the sum is over $\beta \in S$ and the m_{β} are positive integers. Choose one with $\sum m_{\beta}$ maximal and let $\nu' = \nu + \sum m_{\beta} \beta$. Then $\langle \nu, \alpha \rangle = \langle \nu', \alpha \rangle$.

Let $\gamma \in \Delta$. If $\langle \gamma, \alpha \rangle = 0$, then either γ or $-\gamma$ is an element of S . So the maximality of $\sum m_{\beta}$ shows that not both of $\nu' + \gamma, \nu' - \gamma$ lie in $\Delta(\mu)$. Now suppose that $\langle \gamma, \alpha \rangle \neq 0$. Then one of $\langle \nu' + \gamma, \alpha \rangle, \langle \nu' - \gamma, \alpha \rangle$ is greater than $\langle \nu, \alpha \rangle$ and so not both $\nu' + \gamma$ and $\nu' - \gamma$ are elements of $\Delta(\mu)$.

So ν' is a frontier weight and hence there exists $w \in W$ such that $\nu' = w\mu$ (see [11, p. 263, ex. 3]). Since $\langle w\mu, \alpha \rangle = \langle \mu, w^{-1}\alpha \rangle$ the result follows.

Define a partial order on X by $\mu \gg \tau$ if and only if $\mu - \tau$ is the sum of positive roots. The notation \gg is not standard. It is used here simply because $>$ has been used to indicate the total ordering of X given by the lexicographical ordering.

For $\alpha \in \Delta$, let $\alpha^v = 2\alpha/(\alpha, \alpha)$. Then by [2, Chapter VI], $\Delta^v = \{\alpha^v | \alpha \in \Delta\}$ is also an irreducible root system. So Δ^v has a unique element β_0^v , which is maximal with respect to the partial order \gg . Since the highest root of any root system is always long and v interchanges long and short roots, it follows that $\beta_0 \in \Delta$ is short.

The root β_0 is referred to as the highest short root of Δ . Note that if all roots of Δ have the same length, then β_0 is simply the highest root of Δ .

Using Lemma 3.1 we now have the following reformulation for the definition of $\ln(\mu)$.

PROPOSITION 3.2. *If $\mu \in X^+$, then $\ln(\mu) = \langle \mu, \beta_0 \rangle$.*

PROOF. If $\gamma \in \Delta$ then $\gamma^v \ll \beta_0^v$ and so

$$\beta_0^v - \gamma^v = \sum_{i=1}^n c_i \alpha_i^v$$

where the c_i are nonnegative integers. Then

$$\langle \mu, \beta_0 \rangle - \langle \mu, \gamma \rangle = (\mu, \beta_0^v) - (\mu, \gamma^v) = \sum c_i (\mu, \alpha_i^v) \geq 0.$$

So $\langle \mu, \gamma \rangle \leq \langle \mu, \beta_0 \rangle$ for all $\gamma \in \Delta$. Lemma 3.1 then shows that $\ln(\mu) = \langle \mu, \beta_0 \rangle$.

If τ is an arbitrary element of X , then there exists a unique $\mu \in X^+$ such that $w\tau = \mu$, for some $w \in W$. We extend the definition of \ln to all of X by setting $\ln(\tau) = \ln(\mu)$. Note that if $\mu \in X^+$ and $\nu \in \Delta(\mu)$, then by definition $\ln(\nu) < \ln(\mu)$.

3.2. We conclude this section by recording two elementary results for later use.

LEMMA 3.3. *Let m be a positive integer and let $\mu \in X^+$ with $\ln(\mu) < m$. Suppose that $\tau \in X$ and $\ln(\tau) < m$. If $\mu \equiv \tau \pmod{mX}$, then either $\mu = \tau$ or $\mu - \tau = m\lambda_j$ for some fundamental weight λ_j , α_j is a short root, and $\langle \lambda_j, \beta_0 \rangle = 1$.*

PROOF. From the hypothesis we have $\langle \mu - \tau, \alpha_i \rangle = a_i m$, where a_i is an integer. Since $\langle \mu, \alpha_i \rangle < m$ and $|\langle \tau, \alpha_i \rangle| < m$, it follows that a_i is either 0 or 1. So $\mu - \tau = m\lambda$, where $\lambda \in X^+$ and $\langle \lambda, \alpha_i \rangle \in \{0, 1\}$ for each $\alpha_i \in \Pi$. Now $\langle \mu - \tau, \beta_0 \rangle < 2m$ and so $\langle \lambda, \beta_0 \rangle < 2$. If $\lambda = \sum n_i \lambda_i$ and $\beta_0 = \sum b_i \alpha_i$, then

$$\langle \lambda, \beta_0 \rangle = \sum_i n_i b_i (\alpha_i, \alpha_i) / (\beta_0, \beta_0).$$

Since β_0 is a short root, the ratio $(\alpha_i, \alpha_i) / (\beta_0, \beta_0)$ is a positive integer. Moreover, each of the b_i are nonzero positive integers. So $\langle \lambda, \beta_0 \rangle = 0$ implies that $\mu = \tau$. If $\langle \lambda, \beta_0 \rangle = 1$, then $\lambda = \lambda_j$, α_j is a short root, and $\langle \lambda_j, \beta_0 \rangle = b_j = 1$. This completes the proof of the lemma.

Let X_r denote the \mathbb{Z} -lattice spanned by the elements of Π . Then X_r is a subgroup of X and X/X_r is a finite group. The order of this group is referred to as the index of connection for Δ . The indices for the various root systems are given in [2].

Using [2, Planche I-IX] to find the expressions for the fundamental weights in terms of the simple roots and referring to the table of [8, p. 66] for β_0 , we are led to the following observation.

OBSERVATION. Let $\beta_0 = \sum b_i \alpha_i$. If λ_j is a fundamental weight, α_j is a short root, and $\lambda_j \in X_r$, then $b_j > 1$.

This empirical observation is precisely what is needed for the proof of the following lemma.

LEMMA 3.4. *Assume that p does not divide the index of connection for Δ and let $\mu \in X^+$ with $\ln(\mu) < p$. If $\tau \in \Delta(\mu)$ is such that $\mu \equiv \tau \pmod{pX}$, then $\mu = \tau$.*

PROOF. First note that since $\tau \in \Delta(\mu)$ the definition of \ln shows $\ln(\tau) < \ln(\mu)$. So if $\mu \neq \tau$, then by Lemma 3.3, $\mu - \tau = p\lambda_j$ where α_j is short and $b_j = 1$. But $\tau \in \Delta(\mu)$ implies that $\tau \ll \mu$ and so $p\lambda_j \in X_r$. Since p is prime to $|X/X_r|$, this implies $\lambda_j \in X_r$ and thus contradicts the preceding observation. So $\mu = \tau$.

4. An application. In this section our goal is to show that if μ is an element of X^+ with $\ln(\mu) < q - 1$, then $s_\mu\phi$ is the Brauer character of a certain $G(q)$ -module.

4.1. Let $\delta \in X^+$ be given by

$$\delta = \sum_{i=1}^n \lambda_i.$$

Let m be a positive integer and let $\mu \in X^+$ with $\ln(\mu) < m$. We first obtain a decomposition of $V(\mu) \otimes V(m\delta)$ into the direct sum of irreducible \mathfrak{g} -modules. This will be accomplished by means of Weyl's character formula. By transferring to the field K and letting $m = q - 1$, we then obtain a $G(q)$ -module whose Brauer character is $s_\mu\phi$.

We first recall the statement of Weyl's character formula.

If $\lambda \in X$, then λ is a complex valued function on \mathfrak{h} and so we may define a function $e(\lambda)$ by

$$e(\lambda)(H) = \exp(\lambda(H))$$

for $H \in \mathfrak{h}$. If V is a finite dimensional \mathfrak{g} -module with $V = \sum V_\lambda$ the weight space decomposition, then the character Φ of V is the function on \mathfrak{h} given by

$$\Phi = \sum_{\lambda} m_{\lambda} e(\lambda)$$

where m_{λ} is the dimension of V_{λ} . The module V is uniquely determined (up to isomorphism) by the character Φ .

For $\lambda \in X$, let A_{λ} denote the function on \mathfrak{h} given by

$$A_{\lambda} = \sum_w \varepsilon(w)e(w(\lambda + \delta))$$

where the sum is over all $w \in W$ and ε is the alternating character of W . Then for $\mu \in X^+$ and Φ_{μ} the character of $V(\mu)$, Weyl's character formula [11, p. 225] states that

$$\Phi_{\mu} = A_{\mu}/A_0. \tag{4.1.1}$$

Where the denominator vanishes we agree to read this formula as $\Phi_{\mu}A_0 = A_{\mu}$. Since A_0 is nonzero on a dense set, (4.1.1) uniquely determines Φ_{μ} .

For $\tau \in X^+$, define a function S_{τ} by

$$S_{\tau} = \sum_{\nu} e(\nu)$$

where the sum is over all $\nu \in W\tau$. Now for each $\lambda \in X$, the W -orbit of λ contains exactly one element of X^+ . So for Φ the character of V , we may use $m_\lambda = m_{w\lambda}$ to combine terms in the previous expression for Φ to obtain

$$\Phi = \sum_{\tau} m_{\tau} S_{\tau}$$

where the sum is now over all weights τ of V which lie in X^+ .

LEMMA 4.1. *Let $\tau, \rho \in X^+$. Then*

$$S_{\tau} A_{\rho} = \sum_{\nu} A_{\rho+\nu}$$

where the sum is over $\nu \in W\tau$.

PROOF.

$$S_{\tau} A_{\rho} = \left(\sum_{\nu} e(\nu) \right) \left(\sum_w \varepsilon(w) e(w(\rho + \delta)) \right) = \sum_w \varepsilon(w) \left(\sum_{\nu} e(\nu + w\rho + w\delta) \right).$$

For a fixed w , $w\nu$ varies over $W\tau$ with ν , so we may replace ν by $w\nu$ and interchange the order of summation to yield

$$S_{\tau} A_{\rho} = \sum_{\nu} \sum_w \varepsilon(w) e(w(\nu + \rho + \delta)) = \sum_{\nu} A_{\rho+\nu}$$

which is the desired result.

LEMMA 4.2. *Let m be a positive integer and let $\tau \in X^+$ with $\ln(\tau) < m$. Then*

$$S_{\tau} \Phi_{m\delta} = \sum_{\nu} \Phi_{m\delta+\nu}$$

where the sum is over all $\nu \in W\tau$.

PROOF. Note that since $\ln(\tau) < m$ we have

$$\langle m\delta, \alpha_i \rangle + \langle \nu, \alpha_i \rangle \geq 0$$

for all $\alpha_i \in \Pi$ and $\nu \in W\tau$, so $m\delta + \nu \in X^+$.

By (4.1.1)

$$\Phi_{m\delta+\nu} = A_{m\delta+\nu}/A_0$$

and by Lemma 4.1

$$\sum_{\nu} A_{m\delta+\nu} = S_{\tau} A_{m\delta}.$$

Dividing by A_0 then yields the result.

Lemma 4.2 shows that $S_{\tau} \Phi_{m\delta}$ is the character of the \mathfrak{g} -module $\sum_{\nu} \nu(m\delta + \nu)$, where the sum is a direct sum over the W -orbit of τ .

For $\mu \in X^+$ let $\Delta^+(\mu)$ denote the set of all weights in $\Delta(\mu)$ which are also elements of X^+ and let $m(\tau, \mu)$ be the multiplicity of τ as a weight of $V(\mu)$.

PROPOSITION 4.3. *Let m be a positive integer and let $\mu \in X^+$ with $\ln(\mu) <$*

m. Then

$$\Phi_\mu \Phi_{m\delta} = \sum_\tau m(\tau, \mu) \sum_\nu \Phi_{m\delta + \nu}$$

where the sum is over $\tau \in \Delta^+(\mu)$ and $\nu \in W\tau$.

PROOF. Since

$$\Phi_\mu = \sum_\tau m(\tau, \mu) S_\tau$$

with the sum over $\tau \in \Delta^+(\mu)$, we have

$$\Phi_\mu \Phi_{m\delta} = \sum_\tau m(\tau, \mu) S_\tau \Phi_{m\delta}.$$

Lemma 4.2 shows that this is equal to

$$\sum_\tau m(\tau, \mu) \sum_\nu \Phi_{m\delta + \nu}$$

where the sum is over $\tau \in \Delta^+(\mu)$ and $\nu \in W\tau$, which establishes the result.

A restatement of Proposition 4.3 in terms of \mathfrak{g} -modules shows that

$$V(\mu) \otimes V(m\delta) \cong \sum_\tau m(\tau, \mu) \sum_\nu V(m\delta + \nu)$$

where the sum is a direct sum of \mathfrak{g} -modules.

4.2. Let V be a \mathfrak{g} -module and let $V = \sum V_\lambda$ be the weight space decomposition. If \bar{V} is the derived G -module, then $\bar{V} = \sum \bar{V}_\lambda$ (see 1.2) so for θ the Brauer character of \bar{V} ,

$$\theta(h(\chi)) = \sum m_\lambda \chi(\lambda)'$$

where $h(\chi) \in T$ and m_λ is the dimension of V_λ .

Let K^\times be the multiplicative group of K and let $\chi \in \text{Hom}(X, K^\times)$. Then $\chi(\lambda_i)'$ is a root of unity. Write $\chi(\lambda_i)' = \exp(r_i)$, for some complex number r_i . Define $H_\chi \in \mathfrak{h}$ by

$$H_\chi = \sum_{i=1}^n r_i H_i$$

where the H_i are elements of the Chevalley basis (see 1.2). Now $\lambda_i(H_j) = \delta_{ij}$, so

$$e(\lambda_i)(H_\chi) = \exp(\lambda_i(H_\chi)) = \exp(r_i) = \chi(\lambda_i)'$$

and it follows that

$$e(\lambda)(H_\chi) = \chi(\lambda)'$$

for all $\lambda \in X$. Hence for each $\chi \in \text{Hom}(X, K^\times)$ there exists $H_\chi \in \mathfrak{h}$ such that

$$\Phi(H_\chi) = \theta(h(\chi))$$

where Φ is the character of V as g -module. Note that H_χ is not unique.

In 2.1 the function s_μ was defined on the semisimple classes of $G(q)$. The same definition yields a class function on the semisimple classes of G , which we also denote by s_μ . We then have

$$S_\mu(H_\chi) = s_\mu(h(\chi)) \quad \text{for } \mu \in X^+.$$

These remarks allow us to transfer the previous results into statements concerning the characters of certain G -modules.

For $\mu \in X^+$, let θ_μ be the Brauer character of $\bar{V}(\mu)$ as a G -module.

PROPOSITION 4.4. *Let m be a positive integer and let $\mu \in X^+$ with $\ln(\mu) < m$. Then*

$$s_\mu \theta_{m\delta} = \sum_\nu \theta_{m\delta+\nu}$$

where the sum is over all ν in the W -orbit of μ .

PROOF. If $g \in G$ is semisimple, then g is conjugate to some $h(\chi) \in T$ and so it suffices to show that

$$s_\mu \theta_{m\delta}(h(\chi)) = \sum_\nu \theta_{m\delta+\nu}(h(\chi))$$

for all $h(\chi) \in T$.

By Lemma 4.2 and the preceding remarks,

$$s_\mu \theta_{m\delta}(h(\chi)) = S_\mu \Phi_{m\delta}(H_\chi) = \sum_\nu \Phi_{m\delta+\nu}(H_\chi) = \sum_\nu \theta_{m\delta+\nu}(h(\chi))$$

and the result follows.

A statement of this proposition with $m = q - 1$ shows the following.

COROLLARY 4.5. *Let $\mu \in X^+$ with $\ln(\mu) < q - 1$. Then $s_\mu \phi$ is the Brauer character of the G -module (and hence of the $G(q)$ -module) $\sum_\nu \bar{V}(\sigma + \nu)$ where the sum is a direct sum over $\nu \in W\mu$.*

5. A dimensional lower bound. The purpose of this section is to obtain a lower bound for the dimension of a principal indecomposable module of $G(q)$. This will be accomplished by showing that certain $B(q)$ -PIM's occur in the decomposition of $U(\mu)|B(q)$.

5.1. We first give a brief description of the principal indecomposable modules of $B(q)$.

Note that $U(q)$ is a normal Sylow p -subgroup of $B(q)$, $T(q)$ is an abelian subgroup of order $(q - 1)^n$ and $B(q) = T(q)U(q)$.

Let $\text{Irr}(T(q))$ denote the set of ordinary irreducible characters of $T(q)$. Since $(|T(q)|, p) = 1$, $\text{Irr}(T(q))$ may be viewed as the set of Brauer characters of the irreducible $K[T(q)]$ modules. For $\zeta \in \text{Irr}(T(q))$, let ζ be the character of the one-dimensional $K[T(q)]$ module whose Brauer character is ζ and let

$\epsilon(\zeta)$ be given by

$$\epsilon(\zeta) = |T(q)|^{-1} \sum_t \bar{\zeta}(t^{-1})t$$

where the sum is over $t \in T(q)$. Then $\epsilon(\zeta)$ is a primitive idempotent of $K[T(q)]$ such that $K[T(q)]\epsilon(\zeta)$ has ζ as Brauer character. So

$$K[T(q)] = \sum_{\zeta} K[T(q)]\epsilon(\zeta)$$

a direct sum over $\zeta \in \text{Irr}(T(q))$. Let ζ^* denote the character of $B(q)$ induced from ζ . Then ζ^* is the Brauer character of the $K[B(q)]$ module $K[B(q)]\epsilon(\zeta)$ and

$$K[B(q)] = \sum_{\zeta} K[B(q)]\epsilon(\zeta). \tag{5.1.1}$$

This shows that $K[B(q)]\epsilon(\zeta)$ is a direct sum of certain principal indecomposable $B(q)$ -modules. Now the dimension of any $B(q)$ -PIM is divisible by $|U(q)| = q^N$, and since the dimension of $K[B(q)]\epsilon(\zeta)$ is q^N , it follows that (5.1.1) expresses $K[B(q)]$ as the direct sum of PIM's.

For each $\zeta \in \text{Irr}(T(q))$, define $\hat{\zeta}$ by

$$\hat{\zeta}(tu) = \zeta(t)$$

for $t \in T(q)$ and $u \in U(q)$. Then $\hat{\zeta}$ is the Brauer character of an irreducible $B(q)$ -module and the corresponding principal indecomposable module of $B(q)$ has $\hat{\zeta}^*$ as Brauer character.

If we define ζ_{μ} , for $\mu \in X$, by

$$\zeta_{\mu}(h(\chi)) = \chi(\mu)'$$

then $\zeta_{\mu} \in \text{Irr}(T(q))$. Let $Z(\mu)$ be the $B(q)$ -PIM with Brauer character ζ_{μ}^* . So

$$Z(\mu) = K[B(q)]\epsilon(\zeta_{\mu}).$$

The unique top and bottom composition factors of $Z(\mu)$ are one-dimensional and have $\hat{\zeta}_{\mu}$ as Brauer character.

Now $h(\chi) \in T(q)$ if and only if χ is a homomorphism of X into $GF(q)^{\times}$ (see 1.3) and so not all of the ζ_{μ} are distinct. It follows that $\zeta_{\mu} = \zeta_{\nu}$ if and only if $\mu \equiv \nu \pmod{(q-1)X}$. So if X_{q-1} denotes the set of $\nu \in X^+$ with $\langle \nu, \alpha_i \rangle \leq q-2$ for all $\alpha_i \in \Pi$, then

$$\text{Irr}(T(q)) = \{ \zeta_{\nu} | \nu \in X_{q-1} \}$$

and

$$K[B(q)] = \sum_{\nu} Z(\nu)$$

a direct sum over $\nu \in X_{q-1}$.

In [16, p. 102], Steinberg has given a basis for the $G(q)$ -module $\bar{V}(\sigma)$.

Using this basis it follows that for $t \in T(q)$, $\phi(t)$ is equal to the order of the $U(q)$ -centralizer of t . Hence

$$\phi|B(q) = \zeta_0^* \quad \text{and} \quad \hat{\zeta}_\mu \zeta_0^* = (\zeta_\mu \zeta_0)^* = \zeta_\mu^*.$$

So a module which has ζ_μ^* as Brauer character may be obtained by tensoring the one-dimensional $K[B(q)]$ module affording $\hat{\zeta}_\mu$ with $\bar{V}(\sigma)|B(q)$.

5.2. Since $U(\mu)$ is a PIM for $G(q)$, $U(\mu)|B(q)$ is the direct sum of principal indecomposable $B(q)$ -modules. We shall show that certain $B(q)$ -PIM's occur as summands of $U(\mu)|B(q)$ and hence obtain a lower bound for the dimension of $U(\mu)$.

LEMMA 5.1. *If $\mu \in X_q$, then $Z(\mu)$ is a direct summand of $U(\mu)|B(q)$.*

PROOF. The characterization of $F(\mu)$ given in 2.2 shows that $F(\mu)$ contains a line which, as $B(q)$ -module, has $\hat{\zeta}_\mu$ as Brauer character. Since $F(\mu)$ is a submodule of $U(\mu)$, it follows that $U(\mu)|B(q)$ contains a line affording $\hat{\zeta}_\mu$. Write

$$U(\mu)|B(q) = \sum a_\nu Z(\nu)$$

a direct sum of PIM's of $B(q)$. Then there exists ν such that $a_\nu \neq 0$ and $Z(\nu)$ contains a line whose Brauer character is $\hat{\zeta}_\mu$. Hence $\hat{\zeta}_\nu = \hat{\zeta}_\mu$ and so $Z(\nu) = Z(\mu)$, which proves the lemma.

Now W acts on the finite abelian group $T(q)$ and hence permutes the characters ζ_τ by $w\zeta_\tau(t) = \zeta_\tau(t^w)$. In particular, $w\zeta_\tau = \zeta_{w\tau}$.

LEMMA 5.2. *If $\tau \in X$, $\mu \in X_q$ and $w \in W$, then*

$$(\eta_\mu|B(q), \hat{\zeta}_\tau) = (\eta_\mu|B(q), \hat{\zeta}_{w\tau}).$$

PROOF. Since $\phi(t) = \phi(t^w)$ is the order of the $U(q)$ -centralizer of $t \in T(q)$, we have $|\text{cl}(t)| = |\text{cl}(t^w)|$ where $\text{cl}(t)$, $\text{cl}(t^w)$ are the $B(q)$ -conjugacy classes of t and t^w .

Now any semisimple element of $B(q)$ is conjugate (in $B(q)$) to a unique element of $T(q)$, so

$$\begin{aligned} (\eta_\mu|B(q), \hat{\zeta}_\tau) &= |B(q)|^{-1} \sum_t |\text{cl}(t)| \eta_\mu(t^{-1}) \zeta_\tau(t) \\ &= |B(q)|^{-1} \sum_t |\text{cl}(t^w)| \eta_\mu(t^{-w}) \zeta_\tau(t^w) \\ &= |B(q)|^{-1} \sum_t |\text{cl}(t)| \eta_\mu(t^{-1}) \zeta_{w\tau}(t) \\ &= (\eta_\mu|B(q), \hat{\zeta}_{w\tau}) \end{aligned}$$

which proves the lemma.

The next proposition is an immediate consequence of the two previous lemmas.

PROPOSITION 5.3. *If $\mu \in X_q$, then $Z(\nu)$ is a direct summand of $U(\mu)|B(q)$ for each $\nu \in W\mu$.*

PROOF. Lemma 5.1 shows that $Z(\mu)$ occurs as a summand of $U(\mu)|B(q)$ with positive multiplicity, say a . The orthogonality relations show that

$$a = (\eta_\mu|B(q), \hat{\zeta}_\mu).$$

So if $\nu = w\mu$, for some $w \in W$, then by Lemma 5.2

$$a = (\eta_\mu|B(q), \hat{\zeta}_\nu).$$

Hence the multiplicity of $Z(\nu)$ as a $B(q)$ -summand of $U(\mu)$ is also a .

Let $W\zeta_\mu$ be the W -orbit of ζ_μ as a character of $T(q)$ and note that if $\zeta_{w\mu} \neq \zeta_\mu$, then $Z(w\mu) \neq Z(\mu)$. Since the dimension of each $B(q)$ -PIM is q^N , we have the following lower bound for the dimension of $U(\mu)$.

COROLLARY 5.4. *If $\mu \in X_q$, then $\eta_\mu(1) \geq |W\zeta_\mu|q^N$.*

If $w\mu = \mu$ then $w\zeta_\mu = \zeta_\mu$, but not conversely. So $|W\zeta_\mu| < |W\mu|$ and equality need not hold. For example, $\zeta_\sigma = 1$ and so $|W\zeta_\sigma| = 1$, while $|W\sigma| = |W|$. A more characteristic example may be obtained as follows. Let \mathfrak{g} be a simple Lie algebra of type A_n with n odd. Let $\mu = (q - 1/2)\lambda_i$ where $i = (n + 1)/2$. Then $r\mu = -\mu$ and since $\mu \equiv -\mu \pmod{(q - 1)X}$, we have $\zeta_\mu = \zeta_{-\mu}$ and hence $|W\zeta_\mu| < |W\mu|$.

Our objective is to find conditions on μ which will insure that $s_\mu\phi$ is the character of a $G(q)$ -PIM, namely $U(f(\mu))$. Since the degree of $s_\mu\phi$ is $|W\mu|q^N$, we seek a condition which will guarantee that the dimension of a PIM is at least $|W\mu|q^N$. The next lemma provides such a condition.

LEMMA 5.5. *Let $\mu \in X_q$ with $\ln(\mu) < (q - 1)/2$. If $\tau \in \Delta(\mu)$ is such that $\zeta_\tau = \zeta_\mu$, then $\tau = \mu$.*

PROOF. Since $\zeta_\tau = \zeta_\mu$, we have $\langle \mu - \tau, \alpha_i \rangle \equiv 0 \pmod{(q - 1)}$, for all $\alpha_i \in \Pi$. But $\ln(\tau) < \ln(\mu)$ since $\tau \in \Delta(\mu)$ and so

$$|\langle \mu - \tau, \alpha_i \rangle| \leq |\langle \mu, \alpha_i \rangle| + |\langle \tau, \alpha_i \rangle| < q - 1,$$

hence $\langle \mu - \tau, \alpha_i \rangle = 0$ for all $\alpha_i \in \Pi$. Thus, $\mu = \tau$.

This lemma yields the following corollary to Proposition 5.3.

COROLLARY 5.6. *Let $\mu \in X_q$ with $\ln(\mu) < (q - 1)/2$. Then (i) $\eta_\mu(1) \geq |W\mu|q^N$, and (ii) $\eta_{f(\mu)}(1) \geq |W\mu|q^N$.*

PROOF. By Corollary 5.4, $\eta_\mu(1) \geq |W\zeta_\mu|q^N$. So to prove the first inequality it suffices to show that $|W\zeta_\mu| = |W\mu|$. Now $w\mu \in \Delta(\mu)$ and so by Lemma 5.5, $\zeta_{w\mu} = \zeta_\mu$ if and only if $w\mu = \mu$. Since $w\zeta_\mu = \zeta_{w\mu}$, we have $|W\zeta_\mu| = |W\mu|$. The result for $\eta_{f(\mu)}$ follows from the fact that $\zeta_{f(\mu)} = \zeta_{r\mu}$.

REMARK. Examples indicate the conclusion of Corollary 5.6 may be

obtained under less restrictive conditions on μ . For example, if \mathfrak{g} is of type E_7 , E_8 , F_4 or G_2 , then it suffices to assume that $\ln(\mu) < q - 1$.

6. A dimensional upper bound. In this section we obtain an upper bound for the dimension of a principal indecomposable module of $G(q)$. This is done by showing that each $U(\mu)$ occurs as a $G(q)$ -summand of a certain G -module and obtaining an upper bound for the dimension of this module.

6.1. We first recall Steinberg's tensor product theorem [14], which expresses each of the irreducible G -modules $F(\lambda)$ in terms of the $F(\mu)$ for $\mu \in X_p$.

For i a nonnegative integer, let γ be the automorphism of K given by $t^\gamma = t^{p^i}$, $t \in K$. Then γ induces a surjective endomorphism of G which we also denote by γ . If Y is a rational G -module, then we may define a new action of G on Y by

$$g \cdot y = g^\gamma y \quad (y \in Y, g \in G).$$

Let $Y^{(p^i)}$ denote the rational G -module obtained from Y in this fashion. For $y \in Y$, we let $y^{(p^i)}$ be the corresponding element of $Y^{(p^i)}$.

THEOREM (STEINBERG). *Let $\mu \in X^+$ and write*

$$\mu = \sum_{i=0}^r p^i \mu_i$$

where $\mu_i \in X_p$. Then $F(\mu)$ is isomorphic, as G -module, to

$$F(\mu_0) \otimes F(\mu_1)^{(p)} \otimes \cdots \otimes F(\mu_r)^{(p^r)}.$$

This tensor product will be abbreviated as $\otimes_{i=0}^r F(\mu_i)^{(p^i)}$. The next lemma is simply a restatement of the tensor product theorem.

LEMMA 6.1. *Suppose that $q = p^k$. Let $\mu \in X^+$ and write*

$$\mu = \sum_{i=0}^r q^i \mu_i$$

where $\mu_i \in X_q$. Then $F(\mu)$ is isomorphic, as a G -module, to $\otimes_{i=0}^r F(\mu_i)^{(q^i)}$.

6.2. Let M be a rational G -module and let $\tau \in X$. A nonzero vector $m \in M$ such that $h(\chi)m = \chi(\tau)m$ for all $h(\chi) \in T$, is said to be a T -weight vector of weight τ .

Let $\mu, \lambda \in X^+$. If $\bar{V}(\mu) = \sum \bar{V}_\nu$ and $\bar{V}(\lambda) = \sum \bar{V}_\tau$ (see 1.2), then

$$\bar{V}(\mu) \otimes \bar{V}(\lambda) = \sum \bar{V}_\nu \otimes \bar{V}_\tau$$

expresses $\bar{V}(\mu) \otimes \bar{V}(\lambda)$ as the direct sum of T -weight spaces with weights $\nu + \tau$, for $\nu \in \Delta(\mu)$ and $\tau \in \Delta(\lambda)$.

Now $m(\mu, \mu) = 1$, $m(\lambda, \lambda) = 1$, and $\nu + \tau \ll \mu + \lambda$ in the above sum. So the T -weight $\mu + \lambda$ occurs with multiplicity one in this tensor product. That is, there is a unique line in $\bar{V}(\mu) \otimes \bar{V}(\lambda)$ which affords the T -weight $\mu + \lambda$.

Let $\mu \in X_q$ and write

$$\bar{V}(\mu) \otimes \bar{V}(\sigma) = Q_1 \oplus \cdots \oplus Q_m \tag{6.2.1}$$

a decomposition as the direct sum of indecomposable G -modules. Let x (y) be an element of $\bar{V}(\mu)$ ($\bar{V}(\sigma)$) of weight μ (σ). Then $x \otimes y$ is a T -weight vector of weight $\mu + \sigma$ and hence so is its projection on each Q_i . Since this weight is of multiplicity one, it determines a unique summand, say Q_1 , with $x \otimes y \in Q_1$. Let $\underline{Q}(\mu) = Q_1$.

Now $\bar{V}(\mu) \otimes \bar{V}(\sigma)$ is a projective $G(q)$ -module, so each Q_i is the direct sum of PIM's. Our objective is to first show that $\underline{Q}(\mu)|G(q)$ has $U(f(\mu))$ as a direct summand and then to obtain an upper bound for the dimension of $\underline{Q}(\mu)$.

6.3. In the next two lemmas let $M = \bar{V}(\mu) \otimes \bar{V}(\sigma)$.

LEMMA 6.2. *Let $\mu \in X_q$ with $\ln(\mu) < (q - 1)/2$. Then the $B(q)$ -PIM $Z(\mu)$ occurs exactly once as a direct summand of $M|B(q)$.*

PROOF. The character of M as a $G(q)$ -module is $\sum_{\tau} m(\tau, \mu) s_{\tau} \phi$ where the sum is over $\tau \in \Delta^+(\mu)$. So for a the multiplicity of $Z(\mu)$ as a summand of $M|B(q)$,

$$a = \sum_{\tau} m(\tau, \mu) (s_{\tau} \phi|B(q), \hat{\xi}_{\mu}).$$

Now $s_{\tau}|T(q) = \sum_{\nu} \zeta_{\nu}$, with the sum over $\nu \in W\tau$ and $\phi|B(q) = \zeta_{\delta}^*$, so (see 5.1)

$$s_{\tau} \phi|B(q) = \sum \zeta_{\nu}^*.$$

Hence

$$a = \sum_{\tau} m(\tau, \mu) \sum_{\nu} (\zeta_{\nu}^*, \hat{\xi}_{\mu}).$$

By the orthogonality relations for Brauer characters, $(\zeta_{\nu}^*, \hat{\xi}_{\mu}) = 1$ if $\zeta_{\nu} = \zeta_{\mu}$ and 0 otherwise. Since $\nu \in \Delta(\mu)$, Lemma 5.5 shows that $\zeta_{\nu} = \zeta_{\mu}$ if and only if $\nu = \mu$. So $a = m(\mu, \mu) = 1$.

REMARK. If we do not assume that $\ln(\mu) < (q - 1)/2$, then $Z(\mu)$ is a summand of $M|B(q)$, but the multiplicity may be greater than 1.

LEMMA 6.3. *Let $\mu \in X_q$ with $\ln(\mu) < (q - 1)/2$. Then $U(f(\mu))$ is a summand of $\underline{Q}(\mu)|G(q)$.*

PROOF. By Corollary 2.7, $U(f(\mu))$ is a $G(q)$ -summand of M . The decomposition of (6.2.1) shows that it occurs as a summand of Q_i , for some i . So by Lemma 5.1, $Z(f(\mu))$ is a summand of $Q_i|B(q)$.

Now $f(\mu) = (q - 1)\delta + r\mu$, so $Z(f(\mu)) = Z(r\mu)$ and hence by Proposition 5.3, $Z(\mu)$ is a summand of $Q_i|B(q)$.

Note that the line $K(x \otimes y)$ has $\hat{\xi}_\mu \hat{\xi}_\sigma = \hat{\xi}_\mu$ as Brauer character. So $Q(\mu)|B(q)$ has a composition series whose bottom factor has $\hat{\xi}_\mu$ as Brauer character. It follows, as in the proof of Lemma 5.1, that $Z(\mu)$ is a direct summand of $Q(\mu)|B(q)$.

So $Z(\mu)$ is a $B(q)$ -summand of both Q_i and $Q(\mu)$. Lemma 6.2 then shows that $Q_i = Q(\mu)$, which completes the proof.

The general case will follow from Lemma 6.3 applied to the group $G(q^a)$, for a an appropriately large integer. Toward this end we introduce some notation for this group.

The group $G(q^a)$ is the universal Chevalley group constructed over the field $GF(q^a)$, so its PIM's are indexed by the elements of X_{q^a} . For $\mu \in X_{q^a}$, let $\hat{U}(\mu)$ denote the corresponding PIM for $G(q^a)$. Then

$$\bar{V}((q^a - 1)\delta) = \hat{U}((q^a - 1)\delta)$$

is the Steinberg module for $G(q^a)$ and we denote its character by ϕ_a .

If Y is a finite dimensional G -module, let $\text{br}(Y)$ be the Brauer character of Y .

PROPOSITION 6.4. *If $\mu \in X_q$, then $U(f(\mu))$ is a direct summand of $Q(\mu)|G(q)$.*

PROOF. By Corollary 2.7, $\bar{V}(\mu) \otimes \bar{V}(\sigma)$, and hence $Q(\mu)|G(q)$, is a direct sum of certain $U(f(\tau))$ for $\tau \in X_q$ with $\tau < \mu$. So

$$\text{br}(Q(\mu)|G(q)) = \sum_{\tau} a_{f(\tau)} \eta_{f(\tau)}$$

where the $a_{f(\tau)}$ are positive integers and the sum is over $\tau \in X_q$ with $\tau < \mu$.

By Theorem 2.5, $\eta_{f(\tau)}$ may be expressed in terms of the $s_\nu \phi$ with $\nu < \tau$. So to show $a_{f(\mu)} \neq 0$ in the above sum, it suffices to show

$$\text{br}(Q(\mu)|G(q)) = s_\mu \phi + \sum_{\nu} b_\nu s_\nu \phi$$

where the b_ν are integers and the sum is over $\nu \in X_q$ with $\nu < \mu$.

Step 1. Choose an integer a such that $\ln(\mu) < (q^a - 1)/2$. If $a = 1$, then the result follows from Lemma 6.3 and so we may assume $a > 2$. Now

$$(q^a - 1)\delta = \sum_{i=0}^{a-1} q^i \sigma$$

and so by Lemma 6.1

$$\bar{V}((q^a - 1)\delta) \cong \bigotimes_{i=0}^{a-1} \bar{V}(\sigma)^{(q^i)} \tag{6.3.1}$$

an isomorphism of G -modules.

If we let

$$A = \bigotimes_{i=1}^{a-1} \bar{V}(\sigma)^{(q^i)}$$

then

$$\bar{V}(\mu) \otimes \bar{V}((q^a - 1)\delta) \cong \bar{V}(\mu) \otimes \bar{V}(\sigma) \otimes A$$

and by definition of the Q_i ,

$$\bar{V}(\mu) \otimes \bar{V}(\sigma) \otimes A \cong \sum_{i=1}^m Q_i \otimes A$$

where the isomorphisms are of G -modules.

Now write

$$Q_1 \otimes A = N_1 \oplus \dots \oplus N_n$$

a direct sum of indecomposable G -modules. Note that

$$z = x \otimes y \otimes y^{(q)} \otimes \dots \otimes y^{(q^{a-1})}$$

is an element of $Q_1 \otimes A$ and is a T -weight vector of weight $\mu + (q^a - 1)\delta$. This weight occurs with multiplicity one in $\bar{V}(\mu) \otimes \bar{V}((q^a - 1)\delta)$, so we may choose the notation so that $z \in N_1$.

Since $\ln(\mu) < (q^a - 1)/2$, we may apply Lemma 6.3 to the group $G(q^a)$ to see that $\hat{U}((q^a - 1)\delta + r\mu)$ is a $G(q^a)$ summand of N_1 and hence is also a summand of $Q_1 \otimes A$.

Step 2. Applying Corollary 2.7 to $\bar{V}(\mu) \otimes \bar{V}((q^a - 1)\delta)$ and the group $G(q^a)$, shows that it is the direct sum of various $\hat{U}((q^a - 1)\delta + r\lambda)$ for $\lambda \in X_{q^a}$ with $\lambda < \mu$. Hence we have a similar statement for $Q_1 \otimes A$.

By Theorem 2.5 and Step 1 above, we see that

$$\text{br}(Q_1)\text{br}(A)|G(q^a) = s_\mu\phi_a + \sum_\lambda c_\lambda s_\lambda\phi_a$$

where the c_λ are integers and the sum is over $\lambda \in X_{q^a}$ with $\lambda < \mu$. Now

$$\bar{V}(\sigma)^{(q^i)}|G(q) \cong \bar{V}(\sigma)$$

and so $\text{br}(A)|G(q) = \phi^{a-1}$. (6.3.1) shows that $\phi_a|G(q) = \phi^a$. Hence

$$\text{br}(Q_1)\phi^{a-1} = s_\mu\phi^a + \sum_\lambda c_\lambda s_\lambda\phi^a$$

and since ϕ does not vanish on semisimple elements of $G(q)$

$$\text{br}(Q_1)|G(q) = s_\mu\phi + \sum_\lambda c_\lambda s_\lambda\phi$$

where the sum is over $\lambda \in X_{q^a}$ with $\lambda < \mu$.

Step 3. As a function on $G(q)$, $s_\lambda = \sum_\nu d_\nu s_\nu$, where the d_ν are integers and the sum is over $\nu \in X_q$ with $\nu < \lambda$ (see 2.1). This yields

$$\text{br}(Q_1)|G(q) = s_\mu\phi + \sum_\nu b_\nu s_\nu\phi$$

where the b_ν are integers and the sum is over $\nu \in X_q$ with $\nu < \mu$.

So $a_{f(\mu)} \neq 0$. Since the PIM's which occur as summands of $Q_1 = Q(\mu)$ are uniquely determined by the Brauer character, it follows that $U(f(\mu))$ is a summand of $Q(\mu)|G(q)$.

A similar result has also been obtained by J. E. Humphreys and D. N. Verma [10].

6.4. We now turn to the problem of finding an upper bound for the dimension of $Q(\mu)$. This will be accomplished by applying a result of Humphreys [7, Theorem 5.1], which we now state.

DEFINITION (See [7]). Let μ and τ be two elements of X^+ . Say that μ and τ are linked, and write $\mu \sim \tau$, if there exists $w \in W$ such that $\mu + \delta \equiv w(\tau + \delta) \pmod{pX}$.

(6.4.1). THEOREM (HUMPHREYS). *Assume that p is greater than the Coxeter number of W . If M is a rational indecomposable G -module, then the highest weights of any two composition factors of M are linked.*

If w is the product of the reflections corresponding to the elements of Π (the simple reflections of W), then the Coxeter number of W is the order of w . The Coxeter numbers for the various root systems may be found in [2].

Let \mathfrak{g}_Z be the Z -span of the Chevalley basis for \mathfrak{g} and let $\mathfrak{g}_K = \mathfrak{g}_Z \otimes_Z K$. Then the Lie algebra of the algebraic group G is precisely \mathfrak{g}_K [15, p. 64]. So, by taking differentials, any rational G -module becomes a module for the Lie algebra \mathfrak{g}_K . We record several elementary remarks which make use of this passage from G to \mathfrak{g}_K -modules.

(1) If B is a rational G -module, then $B^{(p^i)}$ (for $i > 1$) is a trivial \mathfrak{g}_K -module. This is immediate since the differential of a p th power is zero.

(2) Let A, B be rational G -modules and let $B' = B^{(p^i)}$ ($i > 1$). Then, as a \mathfrak{g}_K -module, $A \otimes B'$ is isomorphic to the direct sum of $\dim B$ copies of A .

If ρ_1 (ρ_2) is the representation of G acting on A (B') and $d\rho_i$ is the differential of ρ_i , then

$$d(\rho_1 \otimes \rho_2) = d\rho_1 \otimes 1_{B'} + 1_A \otimes d\rho_2.$$

By (1), $d\rho_2 = 0$ so $d(\rho_1 \otimes \rho_2) = d\rho_1 \otimes 1_{B'}$ and the statement follows.

(3) If $\mu \in X^+$, then $F(\mu)$ is completely reducible as a \mathfrak{g}_K -module.

In fact, if $\mu = \sum_{i=0}^n p^i \mu_i$ with $\mu_i \in X_p$, then

$$F(\mu) \cong \bigotimes_{i=0}^n F(\mu_i)^{(p^i)}$$

and so by (2), $F(\mu)$ is \mathfrak{g}_K -isomorphic to the direct sum of $m = \prod_1^n \dim F(\mu_i)$ copies of $F(\mu_0)$.

(4) Let $\mu, \tau \in X^+$ and let $a = p^i$ with $i > 1$. Then any G -composition factor of $F(\mu) \otimes F(\tau)^{(a)}$ has highest weight linked to μ . Let

$$F(\mu) \otimes F(\tau)^{(a)} = M_1 \supset M_2 \supset \dots \supset M_{n+1} = 0$$

be a G -composition series with $M_i/M_{i+1} \cong F(\nu_i)$. Now each M_i is a \mathfrak{g}_K -module, so as a \mathfrak{g}_K -module $F(\mu) \otimes F(\tau)^{(a)}$ has the same composition factors as $\sum_{i=1}^n F(\nu_i)$ where the sum is a direct sum. By (3), $F(\nu_i)$ is \mathfrak{g}_K -isomorphic to the direct sum of a certain number of copies of $F(\nu_{i,0})$, where $\nu_i = \sum_j p^j \nu_{ij}$ with $\nu_{ij} \in X_p$.

On the other hand, (2) and (3) show that $F(\mu) \otimes F(\tau)^{(a)}$ is \mathfrak{g}_K -isomorphic to the direct sum of copies of $F(\mu_0)$, where $\mu = \sum_j p^j \mu_j$ with $\mu_j \in X_p$.

Now the $F(\lambda)$, for $\lambda \in X_p$, are irreducible and nonisomorphic as \mathfrak{g}_K -modules [4], so it follows that $\mu_0 = \nu_{i,0}$ ($i = 1, \dots, n$) and hence $\mu \sim \nu_i$ for all i .

(5) Assume that p is greater than the Coxeter number of W . Let Q be a rational indecomposable G -module and let $\tau \in X^+$. If $a = p^i$ ($i > 1$), then any two G -composition factors of $Q \otimes F(\tau)^{(a)}$ have highest weights linked.

Let $F(\nu_1), \dots, F(\nu_n)$ be the G -composition factors of Q . Then by (6.4.1), the ν_i are all linked. Now $Q \otimes F(\tau)^{(a)}$ has the same G -factors as $\sum_{i=1}^n F(\nu_i) \otimes F(\tau)^{(a)}$ a direct sum of G -modules. By (4), the G -factors of $F(\nu_i) \otimes F(\tau)^{(a)}$ have highest weights linked to ν_i . Since linkage is a transitive relation, the statement follows.

Let A and B be modules for a group H . Write $A \ll_H B$ if the multiplicity of any irreducible module C as an H -composition factor of A is less than or equal to the multiplicity of C as an H -composition factor of B . Write $A \leftrightarrow_H B$ if A and B have exactly the same H -composition factors, counting multiplicities.

For $\mu \in X^+$, let $L(\mu)$ denote the set of all $\tau \in \Delta^+(\mu)$ such that $\mu \equiv w\tau \pmod{pX}$ for some $w \in W$.

PROPOSITION 6.5. *Assume that p is greater than the Coxeter number of W . If $\mu \in X_q$, then*

$$\dim Q(\mu) < \left(\sum_{\tau} m(\tau, \mu) |W\tau| \right) q^N$$

where the sum is over $\tau \in L(\mu)$.

PROOF. We present the proof of this proposition in a series of steps.

Write

$$\bar{V}(\mu) \otimes \bar{V}(\sigma) = Q_1 \oplus \dots \oplus Q_m$$

a direct sum of indecomposable G -modules, with the numbering chosen so that $Q_1 = Q(\mu)$.

Step 1. Choose a positive integer a such that $\ln(\mu) < q^a - 1$. Then

$$\bar{V}(\mu) \otimes \bar{V}((q^a - 1)\delta) \leftrightarrow \sum_G \sum_{\tau} m(\tau, \mu) \sum_{\nu} \bar{V}((q^a - 1)\delta + \nu)$$

where the sum is a direct sum of G -modules and is over $\tau \in \Delta^+(\mu)$, $\nu \in W\tau$.

PROOF. Since the composition factors of any finite dimensional rational G -module are uniquely determined by the Brauer character, it suffices to show that the two modules have the same Brauer character. The Brauer character of $\bar{V}(\mu)$ is given by $\sum_{\tau} m(\tau, \mu) s_{\tau}$, where the sum is over $\tau \in \Delta^+(\mu)$. So $\bar{V}(\mu) \otimes \bar{V}((q^a - 1)\delta)$ has

$$\sum_{\tau} m(\tau, \mu) s_{\tau} \theta_a \quad (6.4.2)$$

as Brauer character, where θ_a is the character of $\bar{V}((q^a - 1)\delta)$ as G -module.

If $\tau \in \Delta^+(\mu)$, then $\ln(\tau) < \ln(\mu)$. So $\ln(\tau) < q^a - 1$ and Proposition 4.4 shows that $s_{\tau} \theta_a$ is the Brauer character of $\sum_{\nu} \bar{V}((q^a - 1)\delta + \nu)$ where the sum is over $\nu \in W\tau$. So (6.4.2) is also the Brauer character of

$$\sum_{\tau} m(\tau, \mu) \sum_{\nu} \bar{V}((q^a - 1)\delta + \nu)$$

and the statement is proved.

Step 2. Let

$$A = \bigotimes_{i=0}^{a-2} \bar{V}(\sigma)^{(q^i)}.$$

(In case $a = 1$, we agree that $A = K$ is the trivial G -module.) Then

$$\bar{V}(\mu) \otimes \bar{V}((q^a - 1)\delta) \cong \sum_{i=1}^m Q_i \otimes A^{(q)}.$$

PROOF. By (6.3.1)

$$\bar{V}((q^a - 1)\delta) \cong \bigotimes_{i=0}^{a-1} \bar{V}(\sigma)^{(q^i)}$$

and so the definition of A shows that

$$\bar{V}((q^a - 1)\delta) \cong \bar{V}(\sigma) \otimes A^{(q)}.$$

The result then follows from the definition of the Q_i .

Step 3. Any G -composition factor of $Q_1 \otimes A^{(q)}$ has highest weight linked to $(q^a - 1)\delta + \mu$.

PROOF. We first show that $F((q^a - 1)\delta + \mu)$ occurs as a G -factor of $Q_1 \otimes A^{(q)}$.

Since $F((q^a - 1)\delta + \mu)$ is a G -composition factor of $\bar{V}((q^a - 1)\delta + \mu)$ (see 2.2), Steps 1 and 2 show that it occurs as a factor of some $Q_i \otimes A^{(q)}$. In particular, since any rational T -module is the direct sum of T -weight spaces, $Q_i \otimes A^{(q)}$ must contain a vector of weight $(q^a - 1)\delta + \mu$.

The remarks of 6.2, with $\lambda = (q^a - 1)\delta$, show that the T -weight $(q^a - 1)\delta + \mu$ occurs with multiplicity one in $\bar{V}(\mu) \otimes \bar{V}((q^a - 1)\delta)$. Since $x \otimes y \in Q_1$, $x \otimes y \otimes y^{(q)} \otimes \cdots \otimes y^{(q^{a-1})}$ is an element of $Q_1 \otimes A^{(q)}$. This is a

T -weight vector of weight $(q^a - 1)\delta + \mu$ and so it follows that $F((q^a - 1)\delta + \mu)$ occurs as a G -factor of $Q_1 \otimes A^{(q)}$.

If $\tau = \sum_{i=0}^{a-2} q^i (q - 1)\delta$ then Lemma 6.1 shows that $A \cong F(\tau)$. Now Q_1 is a rational indecomposable G -module, so by (5) any two factors of $Q_1 \otimes A^{(q)}$ have highest weights linked. Since $F((q^a - 1)\delta + \mu)$ is a factor, the result follows.

Step 4.

$$Q_1 \otimes A^{(q)} \ll_G \sum_{\tau} m(\tau, \mu) \sum_{\nu} \bar{V}((q^a - 1)\delta + \nu)$$

where the sum is a direct sum of G -modules over $\tau \in L(\mu), \nu \in W\tau$.

PROOF. First let λ be an arbitrary element of X^+ . Choose a weight vector v of weight λ in $V(\lambda)$. Then $U_Z v$ is an admissible lattice and the derived module $\bar{V}(\lambda)$ (with respect to this lattice) is a cyclic G -module generated by \bar{v} , the image of v in $\bar{V}(\lambda)$ [15, p. 212, ex. (c)]. Since the T -weight space for \bar{v} is one-dimensional, $\bar{V}(\lambda)$ is an indecomposable G -module. So (6.4.1) shows that all composition factors of $\bar{V}(\lambda)$ have highest weights linked. But $F(\lambda)$ is a composition factor (see 2.2), so the highest weights must be linked to λ .

Now let $F(\gamma)$ be a G -factor of $Q_1 \otimes A^{(q)}$. By Step 3, $\gamma \sim (q^a - 1)\delta + \mu$. Steps 1 and 2 show that $F(\gamma)$ is a factor of some $\bar{V}((q^a - 1)\delta + \nu)$. So the preceding paragraph shows that $\gamma \sim (q^a - 1)\delta + \nu$. Write $\nu = w\tau$, for some $\tau \in \Delta^+(\mu)$ and $w \in W$. Since linkage is a transitive relation, it follows that $(q^a - 1)\delta + \mu \sim (q^a - 1)\delta + \tau$. Hence there exists $w \in W$ with

$$(q^a - 1)\delta + \mu + \delta \equiv w((q^a - 1)\delta + \tau + \delta) \pmod{pX}.$$

So $\mu \equiv w\tau \pmod{pX}$ and hence $\tau \in L(\mu)$.

This shows that $F(\gamma)$ is a composition factor of

$$\sum_{\tau} m(\tau, \mu) \sum_{\nu} \bar{V}((q^a - 1)\delta + \nu)$$

where the sum is over $\tau \in L(\mu)$ and $\nu \in W\tau$. The result then follows from Step 1.

Step 5. We now complete the proof of the proposition.

Let $\tau \in \Delta^+(\mu)$. Lemma 4.2 (with $m = q^a - 1$) shows that the dimension of $\sum_{\nu} V((q^a - 1)\delta + \nu)$, where the sum is over $\nu \in W\tau$, is equal to $|W\tau|q^{aN}$. So by Step 4,

$$\dim(Q_1 \otimes A^{(q)}) < \left(\sum_{\tau} m(\tau, \mu) |W\tau| \right) q^{aN}$$

where the sum is over $\tau \in L(\mu)$. Noting that $\dim A^{(q)} = q^{(a-1)N}$ then completes the proof.

Note that if $\ln(\mu) < q - 1$, then we may choose $a = 1$ and so A is the trivial G -module. Hence we may restate Step 4 of the above proof in the following form.

COROLLARY 6.6. *Assume that p is greater than the Coxeter number of W . If $\mu \in X_q$ with $\ln(\mu) \leq q - 1$, then*

$$Q(\mu) \leq \sum_G \sum_{\tau} m(\tau, \mu) \sum_{\nu} \bar{V}(\sigma + \nu)$$

where the sum is a direct sum over $\tau \in L(\mu)$ and $\nu \in W\tau$.

REMARK. Suppose that \mathfrak{g} is of type A_1 and p is an odd prime. Humphreys [9] has shown that the $Q(\mu)$, for $\mu \in X_p$, are the principal indecomposable modules for the restricted universal enveloping algebra of \mathfrak{g}_K .

7. Conclusion. The purpose of this section is to obtain character formulas for the PIM's of $G(q)$ which correspond to weights whose length is small with respect to p . Throughout this section we assume that p is greater than the Coxeter number of W .

LEMMA 7.1. *Let μ be an element of X_q for which*

(1) $\dim U(f(\mu)) \geq |W\mu|q^N$, and

(2) $L(\mu) = \{\mu\}$.

Then $U(f(\mu)) = Q(\mu)|G(q)$ and $\dim U(f(\mu)) = |W\mu|q^N$.

PROOF. Since $L(\mu) = \{\mu\}$, Proposition 6.5 shows that $\dim Q(\mu) < |W\mu|q^N$. By Proposition 6.4, $U(f(\mu))$ is a summand of $Q(\mu)|G(q)$ and the result follows from hypothesis (1).

PROPOSITION 7.2. *Let $\mu \in X_q$ with $\ln(\mu) < (q - 1)/2$ and assume that $L(\mu) = \{\mu\}$. Then $U(f(\mu)) = Q(\mu)|G(q)$ and has $s_\mu\phi$ as its Brauer character. Moreover,*

$$U(f(\mu)) \leftrightarrow \sum_{G(q)} \sum_{\nu} \bar{V}(\sigma + \nu)$$

where the sum is a direct sum of $G(q)$ -modules and is over $\nu \in W\mu$.

PROOF. Since $\ln(\mu) < (q - 1)/2$, Corollary 5.6 shows that $\dim U(f(\mu)) \geq |W\mu|q^N$ and so Lemma 7.1 yields $U(f(\mu)) = Q(\mu)|G(q)$. Now $L(\mu) = \{\mu\}$ by hypothesis, so Corollary 6.6 shows that

$$Q(\mu) \leq \sum_G \sum_{\nu} \bar{V}(\sigma + \nu)$$

where the sum is a direct sum over $\nu \in W\mu$. Since both modules have dimension $|W\mu|q^N$, we see that

$$Q(\mu) \leftrightarrow \sum_G \sum_{\nu} \bar{V}(\sigma + \nu).$$

Corollary 4.5 shows that the above sum has $s_\mu\phi$ as Brauer character and so the result follows by restricting to $G(q)$.

The next lemma yields conditions which will insure that $L(\mu) = \{\mu\}$.

LEMMA 7.3. *If $\mu \in X^+$ is such that $\ln(\mu) < p - 1$, then $L(\mu) = \{\mu\}$.*

PROOF. Since p is assumed to be greater than the Coxeter number of W , using the tables of [2] we see that p is also greater than the index of connection of Δ . So Lemma 3.4 shows that $L(\mu) = \{\mu\}$.

The next result follows immediately from Proposition 7.2 and Lemma 7.3. This result provides an affirmative solution to Srinivasan's conjecture, under the hypothesis of $\ln(\mu) < (p - 1)/2$. It also shows that the $G(q)$ -PIM corresponding to $f(\mu)$ is a G -module and hence yields a partial verification to a conjecture of Humphreys and Verma [10].

THEOREM 7.4. *Let $\mu \in X_q$ with $\ln(\mu) < (p - 1)/2$. Then $U(f(\mu)) = Q(\mu)|G(q)$ and has $s_\mu\phi$ as its Brauer character. Moreover,*

$$U(f(\mu)) \underset{G(q)}{\leftrightarrow} \sum_{\nu} \bar{V}(\sigma + \nu)$$

where the sum is over all ν in the W -orbit of μ .

If \mathfrak{g} is of type E_7, E_8, F_4 or G_2 , then the conclusion of Theorem 7.4 holds with the hypothesis $\ln(\mu) < p - 1$.

We now show that a stronger version of Corollary 2.7 may be obtained when μ is small with respect to p . This result may be viewed as a partial Clebsch-Gordon formula for principal indecomposable modules.

COROLLARY 7.5. *Let $\mu \in X_q$ with $\ln(\mu) < (p - 1)/2$. Then as $G(q)$ -modules*

$$\bar{V}(\mu) \otimes \bar{V}(\sigma) \cong \sum_{\tau} m(\tau, \mu)U(f(\tau))$$

where the sum is a direct sum over $\tau \in \Delta^+(\mu)$.

PROOF. If $\tau \in \Delta^+(\mu)$, then $\ln(\tau) < \ln(\mu)$ and so Theorem 7.4 shows that the Brauer character of $U(f(\tau))$ is $s_\tau\phi$. Hence the Brauer character of the right side is $\sum_{\tau} m(\tau, \mu)s_\tau\phi$. Since the left side is the direct sum of $G(q)$ -PIM's and also has the above expression as Brauer character, the result follows.

REMARK. Note that if $(p - 1)/2 < p - \ln(\delta)$, then the preceding corollary also yields a decomposition of $F(\mu) \otimes \bar{V}(\sigma)$. This follows by noting that if $\ln(\mu) < p - \ln(\delta)$, then $F(\mu) = \bar{V}(\mu)$.

To see this it suffices (by 6.4.1) to show that if $\tau \in \Delta(\mu)$ is linked to μ , then $\tau = \mu$. Now for any $w \in W$, $\delta - w\delta$ is the sum of positive roots, and hence $\mu + \delta - w(\tau + \delta)$ is also the sum of positive roots. So $\tau + \delta \in \Delta(\mu + \delta)$ [11, p. 263, ex. 13]. Since $\ln(\mu + \delta) < p$, Lemma 3.4 shows that if $\mu \sim \tau$, then $\mu + \delta = w(\tau + \delta)$ for some $w \in W$. Hence $\mu = \tau$ [11, p. 248] and the remark follows. For a similar remark, concerning the irreducibility of $\bar{V}(\mu)$, see [17].

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