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EXAMPLES OF NONCATENARY RINGS

BY

RAYMOND C. HEITMANN[1]

ABSTRACT. A technique is developed for constructing a new family of noetherian integral domains. To each domain, there naturally corresponds its poset (partially ordered set) of prime ideals. The resulting family of posets has the following property: every finite poset is isomorphic to a saturated subset of some poset in the family. In the process, it is determined when certain power series may be adjoined to noetherian rings without destroying the noetherian property.

Throughout, all rings are assumed to be commutative with identity. The prime ideals of a noetherian ring with the inclusion relation comprise a partially ordered set (poset). To date, the only properties discovered to hold for posets arising in this fashion are those which guarantee the existence or nonexistence of an infinite set of elements obeying some restriction(s). So it is natural to wonder if there are any restrictions at all on finite subsets of these posets. In particular, we are concerned with saturated subsets, i.e., subsets with the property that whenever P and Q are adjacent points in the subset ($P < Q$ and there are no points between them), then they are also adjacent in the entire poset. It is fairly obvious and not particularly interesting that "not necessarily saturated" subsets can be totally arbitrary. The main theorem of this paper, Theorem 2.1, asserts that every finite poset occurs as a saturated subset of the poset of some noetherian ring.

A ring is called catenary if, for every pair of primes $P < Q$, the length of a saturated chain of primes from P to Q is independent of the choice of chain. Nagata has shown that a noetherian ring need not be catenary [N, p. 203]. Theorem 2.1 may be regarded as a generalization of his result; loosely speaking, there is no finite bound on the "noncatenariness" of noetherian rings in general. The theorem is proved by exhibiting a general procedure for constructing examples. The construction is, of necessity, more complex than that of Nagata but is really no more than a generalization of his basic example.

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One intermediate stage is worthy of note. The construction requires us to adjoin to a noetherian ring a power series in one of its elements. This maneuver requires an infinite ring extension which preserves the noetherian property. Such extensions are comparatively rare and so the technique exhibited here may be of some interest in itself.

In §1, we will define the procedure for adjoining a power series element to a ring. Theorem 1.4 gives conditions (not necessarily easy to check) which tell when an extension of this type preserves the noetherian property. In §2, we describe the constructions which establish the main theorem.

The rings we shall use in the constructions will be commutative algebras over a field K.

1. Suppose we have a K-algebra R and a nonzero divisor $x \in R$. How does one go about adjoining a transcendental (over R) power series in x to the ring R? For simplicity, we restrict our attention to power series with coefficients in K. To adjoin the formal power series $Z = \sum_{i=1}^{\infty} a_i x^i$ to R, we must adjoin not only an indeterminate z but also enough additional elements to make z agree with Z in the completion with respect to the (x)-adic topology. To achieve this, let $z_n = (z - \sum_{i=1}^{n} a_i x^i)/x^n$ and take the extension $T = R[z, z_1, \ldots, z_n, \ldots]$. The element z will coincide with Z in the (xT)-adic completion of T; further, no smaller extension has this property. It should be noted that this type of extension occurs in [N, p. 203]. Since we will use it frequently, the following definition is convenient.

Definition 1.1. With the notation as above, $T = R[z, \ldots, z_n, \ldots]$ is called a *simple PS-extension* of R for x.

Note 1.2. It should quickly be pointed out that $R[z, z_1, \ldots, z_n] = R[z_n]$ since $z_{n-1} = (z_n + a_n)x$ and so T is in fact a direct union of simple transcendental extensions, i.e., $T = \lim_{\rightarrow} R[z_n]$. From this note, it is also clear that each $z_n \in xT$.

A simple PS-extension of R is an infinite extension and in general will not preserve the noetherian property. However, when R is noetherian, it is possible to develop a procedure for determining whether a particular simple PS-extension is noetherian. The procedure seems impractical to apply but can often be used to prove existence theorems for noetherian simple PS-extensions. The test will be to check a family of maps to see that all are monic–a standard idea. So we shall now proceed to define the maps.

Assume that R is a noetherian K-algebra and $x \in R$ is not a zero divisor. Fix a prime P of R which satisfies $x \notin P$ and $(P, x) \neq R$. Then $\bar{x}$ is a nonzero nonunit of R/P and so the map $K[x] \to K[\bar{x}] \subset R/P$ is an injection. Next, let $(R/P)\hat{}$ denote the completion of R/P with respect to the $(\bar{x})$-adic topology. The injection $K[x] \rightarrowtail R/P$ canonically extends to an injection $K[[x]] \rightarrowtail (R/P)\hat{}$ and so we have the following commuting diagram of canonical maps:

$$\begin{array}{ccccc} K[x] & \rightarrowtail & R & \rightarrowtail & (R/P) \\ \downarrow & & & & \downarrow \\ K[[x]] & & \rightarrowtail & & (R/P)\hat{} \end{array}$$

The indeterminate z was chosen to correspond to a formal power series Z and clearly each z_n has a corresponding power series Z_n. Clearly the diagram

$$\begin{array}{ccc} K[x] & \rightarrow & R \\ \downarrow & & \downarrow \\ \lim_{\rightarrow} K[x][Z_n] & \rightarrow & T = \lim_{\rightarrow} R[z_n] \end{array}$$

is a pushout and so we obtain a unique map θ_p: $T \rightarrow (R/P)\hat{}$. As $P \subset \ker(\theta_p)$, θ_p induces a map $\bar{\theta}_p$: $T/pT \rightarrow (R/P)\hat{}$. Finally, note that $(R/P)\hat{}$ is noetherian [**B**, p. 204, Proposition 8] and so has only finitely many associated primes of (0), say $Q_1, \ldots, Q_m$. For each Q_i, we define a map $\psi_{P,i}$: $T/PT \rightarrow (R/P)\hat{}/Q_i$ to be the canonical projection of $\bar{\theta}_p$.

DEFINITION 1.3. Hereafter, $\{\psi_{P,i}\}$ will refer to the collection of maps defined in the procedure above. The subscripts range over all primes P such that $x \notin P$ and $(P, x) \neq R$ and over a finite set of integers for each P which corresponds to the set of associated primes of (0) in $(R/P)\hat{}$.

THEOREM 1.4. *If R is noetherian and T is a simple PS-extension of R for some $x \in R$, then T is noetherian if and only if $\{\psi_{P,i}\}$ is a set of monomorphisms.*

PROOF. To prove the reverse direction, ($\Leftarrow$), a special case (Lemma 1.5) will first be shown. Then it will be demonstrated that if the theorem ever fails, it fails in the special case.

LEMMA 1.5. *Let R be a noetherian domain and let T be a simple PS-extension of R for some $x \in R$. If $\{\psi_{(0),i}\}$ is a set of monomorphisms and every infinitely generated prime of T contracts to* (0) *in R, then T is noetherian.*

PROOF. We assume T is not noetherian and shall derive a contradiction. By Cohen's Theorem [**N**, p. 8], T must have an infinitely generated prime ideal P. By the hypothesis, $P \cap R = (0)$.

Recall that $T = \lim_{\rightarrow} R[z_n]$ is a direct union of simple polynomial rings over the domain R. Clearly T is a domain. Noting that $x^n z_n \in R[z]$ for each n, we may conclude that for each $g \in T$, there exists an integer $b(g)$ such that $x^{b(g)}g \in R[z]$. Further, as $x^{b(g)} \notin P$, $g \in P \Leftrightarrow x^{b(g)}g \in P \cap R[z]$. Hence $P = \{g \in T | x^{b(g)}g \in P \cap R[z]\}$. Passing to $R_{(0)}$, the quotient field of R, we note that $R_{(0)}[z]$ is a principal ideal domain and therefore $(P \cap$

$R[z])R_{(0)}[z] = fR_{(0)}[z]$ for some fixed polynomial $f \in P \cap R[z]$. Therefore, $P \cap R[z] = \{g \in R[z] | ug \in fR[z]$ for some $u \in R\}$. $R[z]$ is noetherian and so $P \cap R[z]$ is finitely generated. This enables us to select a fixed $u \in R$ so that $P \cap R[z] = \{g \in R[z] | ug \in fR[z]\}$ and thus $P = \{g \in T | ux^{b(g)}g \in fR[z]\} = \{g \in T | ux^{b(g)}g \in fT\}$.

Next we employ the fact that $\{\psi_{(0),i}\}$ is a set of monomorphisms. This says that for each i, $\psi_{(0),i}$: $T/(0) \to \hat{R}/Q_i$ is an injection, i.e., $\theta_{(0)}$: $T \rightarrowtail \hat{R}$ is not only an injection but in fact $\theta_{(0)}(g)$ is never a zero divisor unless $g = 0$ (for $\theta_{(0)}(T)$ intersects each associated prime of (0) trivially). Hence, without loss of generality, we may regard T as a subring of $\hat{R}$ which contains no nonzero divisors of zero. Now, $\hat{R}$ is noetherian and so obeys the Artin-Rees Lemma. We employ this on the ideals $f\hat{R}$ and $x\hat{R}$ to obtain a fixed integer N so that whenever $m \geqslant N$, $f\hat{R} \cap (x\hat{R})^m = (x\hat{R})^{m-N}[f\hat{R} \cap (x\hat{R})^N]$. There is no generality lost by assuming $b(g) \geqslant N$ for every $g \in T$. Also noting $(x\hat{R})^m = x^m\hat{R}$, we obtain $f\hat{R} \cap x^{b(g)}\hat{R} = x^{b(g)-N}[f\hat{R} \cap x^N\hat{R}]$.

Suppose $g \in P$; then $ux^{b(g)}g \in fT$ and so $ux^{b(g)}g \in f\hat{R} \cap x^{b(g)}\hat{R}$. Therefore, $ux^{b(g)}g = x^{b(g)-N}(ux^Ng)$ where $ux^Ng \in x^N\hat{R} \cap f\hat{R}$ (because x is not a zero divisor in $\hat{R}$). Thus we write $ux^Ng = ft$ for some $t \in \hat{R}$. Since $ux^{b(g)}g \in fT$, $x^{b(g)-N}ft \in fT$ and so $x^{b(g)-N}t \in T$ (as f is not a zero divisor in $\hat{R}$). Now $x^{b(g)-N}t = s \in R[z_n]$ for all sufficiently large n. Choose one such $n = j > b(g) - N$. We may write $s = r + z_jh$ for some $r \in R$, $h \in R[z_j]$. Examining this equation, we note: (1) z_j is divisible by $x^{b(g)-N}$ in the ring T and so also in $\hat{R}$; (2) s is divisible by $x^{b(g)-N}$ in $\hat{R}$; (3) thus, r is also divisible by $x^{b(g)-N}$ in $\hat{R}$; (4) this guarantees r is in fact divisible by $x^{b(g)-N}$ in R because $\hat{R}$ is an inverse limit and so $R/x^mR \cong \hat{R}/x^m\hat{R}$ for any m; and (5) finally s is divisible by $x^{b(g)-N}$ in T because both r and z_j are. This proves $t \in T$. Hence, for every $g \in P$, $ux^Ng \in fT$. So, $P = \{g \in T | ux^Ng \in fT\} = (f/ux^N)(ux^NT\colon fT)$. As T is a domain, P is module-isomorphic to $(ux^NT\colon fT)$. But $(ux^NT\colon fT)$ is an ideal of T which does not contract to (0) for it contains ux^N. So it, and consequently P as well, is finitely generated–the desired contradiction yielding (1.5).

Proof of Theorem 1.4. ($\Leftarrow$) Now assume that we have a pair of rings $R \subset T$ which contradict the theorem. T contains infinitely generated ideals and so R has ideals which are contractions of infinitely generated ideals. Choose an ideal I of R which is maximal with respect to the property of being the contraction of an infinitely generated ideal. Clearly T/IT is a non-noetherian extension of the noetherian ring R/I which has the property that every infinitely generated ideal contracts to (0) in R/I. Again using Cohen's Theorem, T must have an infinitely generated prime which contracts to I. Thus, I is prime and R/I is a domain. Further, as each $z_n \in xT$, $T/xT \cong R/xR$ and it may be concluded that x is not contained in any infinitely generated ideals of T, i.e., $x \notin I$. Next, $T/IT = \lim_{\rightarrow}(R/I)[\bar{z}_n]$ and so T/IT will be a simple PS-extension of R/I provided $\bar{z}$ is transcendental over (R/I). However, by inverting $\bar{x}$, we obtain $(T/IT)_{\bar{x}} = (R/I)_{\bar{x}}[\bar{z}]$ and so $\bar{z}$ is transcendental. Also, noting $(R/I)_{\bar{x}}[\bar{z}]$ is noetherian, we may conclude $(1/\bar{x}) \notin T/IT$ and so $(I, x) \neq R$. Therefore, by the hypothesis, $\{\psi_{I,i}\}$ is a set of monomorphisms. Recalling $\psi_{I,i}$: $T/IT \rightarrow (R/I)\hat{}/Q_i$, we note that $\{\psi_{I,i}\}$ coincides with the set $\{\psi_{I/I,i}\}$ obtained when considering the extension $(R/I) \subset (T/IT)$. So this new extension satisfies the entire hypothesis of Lemma 1.5. Therefore T/IT must be noetherian–a contradiction.

($\Rightarrow$) Conversely, we assume T is noetherian but some $\ker(\psi_{P,i}) \neq (0)$. As $x \notin P$, T/PT is a simple PS-extension of R/P (as argued above). Kernel$(\psi_{P/P,i}) = \ker(\psi_{P,i}) \neq (0)$ and T/PT is clearly noetherian. Therefore, without loss of generality, we may assume that R is a domain and $\ker(\psi_{(0),i}) \neq (0)$ for some $\psi_{(0),i}$.

Let $f \in \mathrm{Ker}(\psi_{(0),i})$. Since T is noetherian, we may invoke Artin-Rees to the ideals xT and fT to obtain the existence of an integer N such that whenever $m > N$, $fT \cap x^mT = x^{m-N}(fT \cap x^NT)$. Now, as $f \in \mathrm{Ker}(\psi_{(0),i})$, $\bar{f} = \theta_{(0)}(f)$ is a zero divisor in $\hat{R}$. (It may actually be zero.) As x is not a zero divisor [**B**, p. 204, Corollary 2], we can find $h \in \hat{R} - x\hat{R}$ such that $\bar{f}h = 0$. Next choose an integer $m > N$. There exists $r \in R$, $v \in \hat{R}$ such that $h = r + x^mv$. Of course, x cannot divide r. Thus $0 = \bar{f}h = \bar{f}r + \bar{f}x^mv$ and so x^m divides $\bar{f}r$ in $\hat{R}$. This forces fr to be divisible by x^m in T. But now, since $m > N$, our formulation of the Artin-Rees Lemma says x divides r–a contradiction. This concludes the proof of Theorem 1.4.

In practice, direct application of Theorem 1.4 may not be practical. However, we may deduce the following corollary which shall be an important ingredient in the constructions of §2.

COROLLARY 1.6. *If R is a countable noetherian ring and $x \in R$ is not a zero divisor, then there exists a noetherian simple PS-extension of R for x. Specifically, there are uncountably many simple PS-extensions and all but countably many are noetherian.*

PROOF. It suffices to prove the second statement. It is obvious that there are uncountably many extensions because $K[[x]]$ is uncountable. A given extension is noetherian unless some $\psi_{P,i}$ fails to be monic. A countable noetherian ring has only countably many primes so it suffices to show that each $\psi_{P,i}$ can only fail to be monic countably often.

Suppose $f \in \mathrm{Ker}(\psi_{P,i})$. As $f \in \lim_{\rightarrow} R[z_n]$, $f \in R[z_n]$ for some n. Hence there exists an integer m such that $u = x^mf \in \ker(\psi_{P,i}) \cap R[z]$. As $\psi_{P,i}$ maps a polynomial in z to 0, we may conclude that $\psi_{P,i}(z)$ is algebraic over

$\psi_{P,i}(R/P)$ in $(R/P)\hat{}/Q_i$. Because (R/P) is countable, it must have countable algebraic closure in $(R/P)\hat{}/Q_i$ and so $\psi_{P,i}(z)$ must belong to a countable set. Further, $\psi_{P,i}(z)$ was defined in accordance with the injection $K[[x]] \to (R/P)\hat{} \to (R/P)\hat{}/Q_i$ and so the chosen formal power series Z must belong to a countable set. Therefore, all but countably many choices of Z will yield a monic $\psi_{P,i}$, which completes the proof.

2. In this section, we take a finite partially ordered set $\mathfrak{A}$ and construct a noetherian ring A such that $\mathfrak{A}$ is isomorphic to a saturated subset of Spec A. (In a mild abuse of notation, the prime spectrum of a ring will be regarded as a poset rather than a topological space.) Before going ahead with the construction, we would like to introduce some terminology.

Consider a finite poset $\mathfrak{A}$. The elements of $\mathfrak{A}$ will be called *points* and the partial order will be denoted by $<$. A totally ordered subset will be called a *chain*; sometimes, the phrase "chains in $\mathfrak{A}$" will be used to stress that the points in the chain belong to $\mathfrak{A}$. The length of chains is defined in the usual manner and saturated chains of length 1 will be called *links*. The height function, which is usually defined on the prime spectrum of a ring, can easily be extended to this setting. If $q \in \mathfrak{A}$, define height $q = \sup\{m | \exists$ a chain of length m whose maximal element is $q\}$.

We now state and prove the main theorem.

THEOREM 2.1. *Given any finite poset $\mathfrak{A}$, there exists a noetherian ring A such that $\mathfrak{A}$ may be embedded in* Spec A *via an embedding that preserves saturated chains.*

PROOF. Without loss of generality, we may assume $\mathfrak{A}$ has unique maximal and minimal elements for each $\mathfrak{A}$ may be embedded (as a saturated subset) in a poset of this type. Utilizing the height function, it is possible to index the points in $\mathfrak{A}-\{q_0, \ldots, q_m\}$-in such a way that $i < j$ implies height $q_i \leqslant$ height q_j. With the points so labelled, it is clear that $q_i < q_j$ implies $i < j$. The notation indicates $\mathfrak{A}$ has $m + 1$ elements. Throughout the proof, m will remain fixed; for notational ease, we assume $m > 0$.

The construction will be somewhat lengthy and accordingly will be divided into three steps. In the first, we construct a domain R_0 with a finite poset $B_0 \subset \operatorname{Spec} R_0$ together with a poset map φ_0: $B_0 \to \mathfrak{A}$ such that for each maximal chain in $\mathfrak{A}$, i.e., a saturated chain from q_0 to q_m, there is a unique chain in B_0 which maps onto it. If B_0 were a saturated subset of Spec R_0 and if φ_0 were 1-1, this step would prove the theorem. Hence, in Step 2, R_0 is extended (inside its quotient field) to a domain R in such a way that primes in the poset B_0 extend to primes of R (all distinct) and the resulting poset B is a saturated subset of Spec R. Then, in Step 3, the second problem is remedied by taking a subring A of R. R will be a finite integral extension of A. Step 3 is merely an extension of a key idea in Nagata's construction [N, p. 203].

Step 1. Let K be a countable field of characteristic zero and let $y_0, \ldots, y_{m-1}$ be indeterminates. Set $R_0 = K[y_0, \ldots, y_{m-1}]$, a countable m-dimensional noetherian domain. Suppose C is a maximal chain in $\mathfrak{A}$. Whenever $q_i \in C$, there is a unique $q_j \in C$ such that $q_i < q_j$ is a link. Therefore, we may define a function from the nonnegative integers less than m to K by

$$\gamma_C(i) = \begin{cases} 0 & \text{if } q_i \notin C, \\ j & \text{if } q_i, q_j \in C \text{ and } q_i < q_j \text{ is a link.} \end{cases}$$

We obtain such a function for every maximal chain. Then, for each maximal chain C and each $q_t \in C$, define an ideal $I(C, t) = \{y_i - \gamma_C(i) | q_i \not> q_t\}$. It is obvious that each $I(C, t)$ is prime and so $B_0 = \{I(C, t)\} \subset \operatorname{Spec} R_0$. Noting that $I(C, t)$ depends only on that portion of the chain C from q_0 to q_t, it is easy to see that $I(C_1, t_1) = I(C_2, t_2) \Leftrightarrow t_1 = t_2$ and C_1, C_2 coincide from q_0 to q_{t_1}. Further, $I(C_1, t_1) \subseteq I(C_2, t_2) \Leftrightarrow I(C_1, t_1) = I(C_2, t_1)$ and $t_1 \leqslant t_2$. Therefore, there is a well-defined poset map from B_0 to $\mathfrak{A}$ given by $\varphi_0(I(C, t)) = q_t$.

Step 2. To convert B_0 into a saturated subset of the spectrum, a technique is needed which makes primes disappear. This can be achieved by making certain elements of the domain power series in other elements. Precisely, we say an element d in a K-algebra D is a power series in x provided there is a sequence of elements $\{a_i\}$ in K such that $\{(d - \sum_{i=1}^{n} a_i x^i)/x^n\} \subset D$. With this convention, d will be a multiple of x. It is possible that d be a power series in x_1 and also in x_2; in fact, every element is a power series in every unit.

Our exact goal is the following: for every maximal chain C and every point $q_u \notin C$, we want y_u to be a power series in $y_v - \gamma_C(v)$, where v is the index of q_v, the greatest point in C which is less than q_u. (Such a q_v exists since $q_0 < q_u$ and C is totally ordered.) The number of these objectives is finite in number, say α. We put the objectives in some order. Then we proceed to construct $R = R_\alpha$ by the following inductive procedure.

We want a sequence of domains $R_0 \subset R_1 \subset \cdots \subset R_\alpha$ such that each R_k satisfies:

(1) R_k is a noetherian extension of R_0 inside the quotient field of R_0;

(2) R_k satisfies the first k (power series) objectives;

(3) for each $I(C, t)$, $I(C, t)R_k$ is prime and $R_k/I(C, t)R_k$ is canonically isomorphic to $S(t, k)$, a subring of R_k independent of C; and

(4) $(R_k)_{I(C, t)R_k}$ is a regular local ring.

Further, $\{y_i - \gamma_C(i) | q_i \not> q_t\}$ will be a regular system of parameters for $(R_0)_{I(C,t)}$ and for $k > 0$, a regular system of parameters for $(R_k)_{I(C, t)R_k}$ may be obtained from a regular system of parameters for $(R'_{k-1})_{I(C,t)R_{k-1}}$ by deleting at most one element. Precisely, if the kth objective is "y_u should be a power series in $y_v - y_{C^*}(v)$" and if $(y_v - \gamma_{C^*}(v)) \in I(C, t)$, we delete y_u; otherwise we leave the system of parameters unchanged. Note that R_0 satisfies these four conditions with $S(t, 0) = K[\{y_i | q_i \geqslant q_t\}]$.

Now assume that R_k has been constructed to satisfy the above conditions. Any extension of R_k will satisfy the first k (power series) objectives. So we need only be concerned with the $(k + 1)$st objective. For some fixed choice of u, v, C^*, that objective is "y_u should be a power series in $y_v - \gamma_{C^*}(v)$". For notational ease, we set $x = y_v - \gamma_{C^*}(v)$. Throughout the construction of R_{k+1}, u, v, C^*, x will remain fixed.

By Corollary 1.6, there is a noetherian domain T_k which is a simple PS-extension of R_k for x. Adopting the notation of §1, write $z = z_0 = a_1x + a_2x^2 + \dots$ and $T_k = \lim_{\rightarrow} R_k[z_n]$. Define a mapping σ from T_k into the quotient field of R_k (which is also the quotient field of R_0) which is the identity on R_k and satisfies $\sigma(z) = y_u$ and $\sigma(z_n) = (y_u - \sum_{i=1}^{n} a_i x^i)/x^n$. Since T_k is a direct limit of polynomial rings and $\sigma(z_{n-1}) = x(\sigma(z_n) + a_n)$, σ is a well-defined homomorphism. Then set $R_{k+1} = \sigma(T_k)$. R_{k+1} is noetherian because T_k is and so trivially R_{k+1} satisfies induction hypotheses (1) and (2).

To verify hypothesis (3), we must define a family of subrings $\{S(t, k + 1)\}$ of R_{k+1}.

$$S(t, k+1) = \begin{cases} S(t, k)\left[\{\sigma(z_n)\}\right] & \text{if } q_v \geqslant q_t, \\ S(t, k) & \text{if } q_v \ngeqslant q_t. \end{cases}$$

First we shall consider those ideals $I(C, t)R_{k+1}$ such that $x \notin I(C, t)$. If π: $R_k \to S(t, k)$ denotes the canonical homomorphism with kernel $I(C, t)R_k$, then $\pi(x) \neq 0$. This enables us to extend π to a mapping π^* on $(R_k)_x$ by $\pi^*(r/x^n) = \pi(r)/\pi(x^n)$. The image of π^* will be contained in the quotient field of $S(t, k)$. As $R_{k+1} \subset (R_k)_x$, we may restrict π^* to R_{k+1}. Next we claim $\pi^*(R_{k+1}) = S(t, k + 1)$. If $q_v \geqslant q_t$, then $q_u > q_v \Rightarrow q_u > q_t$ also and consequently $y_u, y_v \in S(t, 0) \subset S(t, k)$. So $x = y_v - \gamma_{C^*}(v) \in S(t, k)$ and this means $\sigma(z_n) = (y_u - \sum_{i=1}^{n} a_i x^i)/x^n$ is actually in the quotient field of $S(t, k)$. Hence, $\pi^*(\sigma(z_n)) = \sigma(z_n)$ and clearly $\pi^*(R_{k+1}) = S(t, k + 1)$. On the other hand, if $q_v \ngeqslant q_t$, then $y_v - \gamma_C(v) \in I(C, t)$. Here, $\pi(x) = \pi(y_v - \gamma_{C^*}(v)) = \gamma_C(v) - \gamma_{C^*}(v)$ is a unit of K and is therefore invertible in $S(t, k)$. Thus $\pi^*(R_{k+1}) = S(t, k) = S(t, k + 1)$ as desired. This completes the verification of the claim. Next, we must show kernel $\pi^* = I(C, t)R_{k+1}$. Again, we have two cases. If x is invertible modulo $I(C, t)R_{k+1}$, the second case above, kernel $\pi^* = \{r/x^n \in R_{k+1} | r \in I(C, t)R_k\} = I(C, t)R_{k+1}$. In the first case above, we note that $I(C, t)R_{k+1}$ will be the entire kernel provided $R_{k+1} \subset I(C, t)R_{k+1} + S(t, k + 1)$. Since $R_k = I(C, t)R_k + S(t, k)$ and $\sigma(z_n) \in S(t, k + 1)$ for each n, this is true. This completes the verification of hypothesis (3) for those $I(C, t)$ which do not contain x.

Now presume $x \in I(C, t)$. $x = y_v - \gamma_{C^*}(v)$ and $\gamma_{C^*}(v) \neq 0$ for in none of our objectives is the constant term ever zero. As $x \in I(C, t)$, $\gamma_C(v) = \gamma_{C^*}(v)$ and therefore $q_v \in C$. As $q_v \not> q_t$, $q_v < q_t$. $q_{\gamma_C(v)} \in C$ also; thus $q_{\gamma_{C^*}(v)} = q_{\gamma_C(v)} \leqslant q_t$. Since $q_u \not> q_{\gamma_{C^*}(v)}$, $q_u \not> q_t$. Further, as $q_u > q_v$, $q_u \notin C$ and hence $y_u \in I(C, t)$. Finally, as C is totally ordered, q_v is the unique element in C maximal with respect to being less than q_u. Therefore, "q_u should be a power series in $q_v - \gamma_C(v)$" is the unique objective on our list which suggests that q_u should be a power series in an element of $I(C, t)$. Looking at induction hypothesis (4), we see that y_u was a member of a system of parameters for $(R_0)_{I(C,t)}$ and could not have been deleted in the first k steps by this uniqueness observation. The element x clearly also belongs to the same system of parameters for $(R_k)_{I(C, t)R_k}$.

Consider the ring T_k. Since each $z_n \in xT_k$, $I(C, t)T_k$ is the unique prime of T_k lying over $I(C, t)R_k$. Noting $I(C, t)R_{k+1} = \sigma(I(C, t)T_k)$, $I(C, t)R_{k+1}$ will be prime provided $I(C, t)T_k$ contains the kernel of σ.

The kernel of σ is a prime of T_k which contracts to (0) in R_k. Because T_k is a direct union of simple polynomial rings, kernel σ is a direct union of height one primes, i.e., kernel $\sigma = \lim_{\rightarrow}(\text{kernel } \sigma) \cap R_k[z_n]$. Clearly, $\sigma(z - y_u) = 0$. In $R_k[z_n]$, $z = a_1x + \cdots + a_nx^n + z_nx^n$. Thus, $(\text{kernel } \sigma) \cap R_k[z_n]$ contains the linear polynomial $x^nz_n + (a_1x + \cdots + a_nx^n - y_u)$. Let β denote the constant term and K^* denote the quotient field of R_k. Recalling that height one primes of a polynomial ring which contract to (0) correspond to the primes of $K^*[z_n]$, we see that $(\text{kernel } \sigma) \cap R_k[z_n] = (x^nz_n + \beta)K[z_n] \cap R_k[z_n]$. Above, we observed that $(R_k)_{I(C, t)R_k}$ had a system of parameters containing x and y_u. Since $y_u \equiv \beta$ modulo xR_k, there is also a system of parameters containing x and β. (Simply delete y_u and insert β.) Hence, letting $\mathfrak{R}$ denote the regular local ring $(R_k)_{I(C, t)R_k}$, we observe that $(x^n, \beta)\mathfrak{R}$ is a height two ideal of $\mathfrak{R}$. It is an easy exercise to see that this implies $(x^nz_n + \beta)\mathfrak{R}[z_n]$ is prime in $\mathfrak{R}[z_n]$. Therefore, as $(x^nz_n + \beta)\mathfrak{R}[z_n] \subseteq I(C, t)\mathfrak{R}[z_n]$ and $(\text{kernel } \sigma) \cap R_k[z_n] = (x^nz_n + \beta)\mathfrak{R}[z_n] \cap R_k[z_n]$, we have $(\text{kernel } \sigma) \cap R_k[z_n] \subseteq I(C, t)R_k[z_n]$. This proves $(\text{kernel } \sigma) \subseteq I(C, t)T_k$. Therefore, $I(C, t)R_{k+1}$ is prime. Further, $I(C, t)R_{k+1} \cap R_k = I(C, t)R_k$ and so the sum $I(C, t)R_{k+1} + S(t, k)$ is direct. Since each $\sigma(z_n) \in I(C, t)R_{k+1}$, this sum is all of R_{k+1}. As $q_v \not> q_t$, $S(t, k + 1) = S(t, k)$ and so we have $R_{k+1} = I(C, t)R_{k+1} \oplus S(t, k + 1)$, which yields hypothesis (3).

Finally, we must check hypothesis (4). Again, there are two cases. If $x \notin I(C, t)$, $(R_{k+1})_{I(C,t)R_{k+1}} = (R_k)_{I(C, t)R_k}$ is a regular local ring. The same system of parameters again works, exactly as desired. On the other hand, suppose $x \in I(C, t)$. Again letting $\mathfrak{R} = (R_k)_{I(C,t)R_k}$, we may regard $(R_{k+1})_{I(C,t)R_{k+1}}$ as a localization of $\lim_{\to} \mathfrak{R}[\sigma(z_n)] = \sigma(\lim_{\to} \mathfrak{R}[z_n])$. Suppose height $I(C, t)\mathfrak{R} = w$. Then

$$\operatorname{height}\bigl(I(C, t)\lim_{\to} \mathfrak{R}[z_n]\bigr) = \operatorname{height}\bigl(I(C, t)\mathfrak{R}[z_n]\bigr) = w.$$

Now, since kernel σ is a principal prime of height one (in $\lim_{\to} \mathfrak{R}[z_n]$, as noted above), the Krull altitude theorem allows us to conclude that height $I(C, t)R_{k+1} = w - 1$. We have a system of parameters for $(R_k)_{I(C,t)R_k}$ consisting of w elements, among them x and y_u. As $y_u \in xR_{k+1}$, it is superfluous and so we have $(w - 1)$ elements which generate $I(C, t)R_{k+1}$. This proves (4).

Set $R = R_\alpha$. Considering hypothesis (4), we observe that height $I(C, m)R$ = length C. Therefore, letting C be the chain $q_0 < q_{t_1} < \cdots < q_{t_m}$, $(0) = I(C, 0)R \subset I(C, t_1)R \subset \cdots \subset I(C, m)R$ has maximal length and is therefore saturated. This completes Step 2.

Step 3. In the previous step, we obtained a family of subrings $\{S(t, \alpha)\}$. The second subscript is now superfluous and so we will simply write S_t. Note that the third conclusion of the induction in Step 2 yields $R = S_t + I(C, t)R$ for every t, $0 \leqslant t \leqslant m$, and every chain C containing q_t.

We now define a new sequence of rings. Let $A_0 = R$; let $A_k = S_k + (\bigcap_{q_k \in C} I(C, k)R) \cap A_{k-1}$ for each k, $0 < k \leqslant m$. Next we will prove by induction on k that for each $t > k$, $A_k = S_t + (I(C, t)R) \cap A_k$. The decomposition of R just noted shows this is true for $k = 0$. For the induction step, we assume $A_{k-1} = S_t + (I(C, t)R) \cap A_{k-1}$ for each $t \geqslant k$. In particular, $A_{k-1} = S_k + (I(C, k)R) \cap A_{k-1}$. For each maximal chain C containing q_k, this induces a projection π_C: $A_{k-1} \to S_k$. Now note that $A_k = S_k + \bigcap_{q_k \in C}(I(C, k)R) \cap A_{k-1} = \{r \in A_{k-1} | \pi_C(r) \text{ is independent of } C\}$. Next consider any $t > k$. If $y_i \in S_t$, then clearly $i \geqslant t$. Therefore $i > k$ and $q_i \nless q_k$. If $q_i \geqslant q_k$, $\pi_C(y_i) = y_i$; if $q_i \ngeqslant q_k$, $\pi_C(y_i) = 0$. In either case, $\pi_C(y_i)$ is independent of the choice of C. Moreover, the restriction of π_C to S_t is completely determined by its action on the y_i's and so therefore $S_t \subset A_k$ as desired.

As an immediate consequence of this result, we find that $A_0 \supset A_1 \supset \cdots \supset A_m$. Next we claim that A_0 is a finite integral extension of A_m. This will be true provided A_{k-1} is a finite integral extension of A_k. To prove this, let $C_1, C_2, \ldots, C_s$ denote the chains containing q_k. Then we may define a map π from A_{k-1} to $\bigoplus_{j=1}^{s}(S_k)$ where the jth coordinate map is π_{C_j}. Clearly, kernel $\pi = (\bigcap_{q_k \in C} I(C, k)R) \cap A_{k-1} \subset A_k$. Also $\pi(A_{k-1})$ is a submodule of $\bigoplus_{j=1}^{s}(S_k)$. Since S_k is noetherian, $\pi(A_{k-1})$ is a noetherian (and therefore finite) A_k-module. Therefore, A_{k-1}/A_k is a finite A_k-module and so A_{k-1} is also. Now, since the noetherian ring A_0 is a finite integral extension of A_m, A_m is also noetherian [**E**, p. 281].

The domain $A = A_m$ is the domain required by the theorem. Because $I(C, t)R$ is prime, $I(C, t)R \cap A$ is prime. We denote this prime by P_t. Since $A \subset A_t$, P_t is independent of the particular chain C. The set $\{P_t | 0 \leqslant t \leqslant m\} \subset \operatorname{Spec} A$ is the desired subset. If $q_t \leqslant q_w$, there is a maximal chain C containing q_t and q_w. Hence, $I(C, t) \subset I(C, w)$ and so $P_t \subseteq P_w$. For the reverse direction, it is easy to see that $r_w = \prod_{j=0}^{m}(y_w - j) \in A$. Further $r_w \in P_t \Leftrightarrow q_w \not\geqslant q_t$. Therefore $q_t \nleqslant q_w$ implies $r_w \in P_t - P_w$, i.e., $P_t \nsubseteq P_w$. Therefore, $\{P_t | 0 \leqslant t \leqslant m\}$ is order-isomorphic to $\mathfrak{A}$.

Lastly, we must show that this is a saturated subset of $\operatorname{Spec} A$. To do this, we first observe that $A = \{r \in R | \pi_{C,t}(r) = \pi_{C',t}(r) \ \forall C, C', t\}$, where $\pi_{C,t}$: $R \to S_t$ is the projection with kernel $I(C, t)R$. Now suppose $P_u \subset P_v$ are adjacent primes in the subset. As before, we define $r_v = \prod_{j=0}^{m}(y_v - j)$ if $v \neq m$. If $v = m$, we set $r_v = 1$. $r_v \in A$ and unless $q_t \leqslant q_v$, $\pi_{C,t}(r_v) = 0$. We next define a set of integers $J = \{e | q_e < q_v$ is a link$\}$; note $u \in J$. Define $s = \prod_{e \in J}(y_e - v)$. If $q_t < q_v$, then $q_t \not> q_e$ for any $e \in J$. In this case, $\pi_{C,t}(y_e) = \pi_{C',t}(y_e)$ and so $\pi_{C,t}(r_v s) = \pi_{C',t}(r_v s)$. If $q_t = q_v$, $\pi_{C,t}(y_e - v) = 0$ for some e and so $\pi_{C,t}(r_v s) = 0$. Finally, if $q_t \nleqslant q_v$, $\pi_{C,t}(r_v s) = 0$. Thus, $r_v s \in A$. Now, since $r_v s \in (y_u - v)R$, $r_v s \in P_v$. Therefore, any prime of R lying over P_v contains $r_v s$. Since $r_v s$ is a product of monomials, it must contain one of them. Also, $r_v \notin P_v$ and so it must be one of the $(y_e - v)$'s. Let C^* be a chain in $\mathfrak{A}$ containing q_u and q_v. $I(C^*, u)R$ is a prime of R lying over P_u. Since q_u, q_e are incomparable for any $u \neq e \in J$, $y_e \in I(C^*, u)$. This means that a prime of R lying over P_v and containing $I(C^*, u)R$ must contain $(y_u - v)$. Now, $I(C^*, u)R + (y_u - v)R = I(C^*, v)R$; thus the only prime lying over P_v which contains $I(C^*, u)R$ is $I(C^*, v)R$. Finally, since R is an integral extension of A, we may apply the Going-Up Theorem [N, p. 30] and prove $P_u \subset P_v$ is a link in $\operatorname{Spec} A$. For, if $P_u \subset P \subset P_v$, we could complete the diagram:

$$\begin{array}{ccccc} P_u & \subset & P & \subset & P_v \\ \cap & & \cap & & \cap \\ I(C^*, u)R & \subset & — & \subset & — \end{array}$$

However, we have just seen the second blank must be filled by $I(C^*, v)R$ and no intervening prime fits in the bottom line. Thus we have a saturated subset and the theorem has been proved.

NOTE 2.2. It is possible to perform this construction in somewhat more generality, I believe. More than one maximal can be allowed if we require that only the minimal point can be contained in more than one maximal. In this case, several indeterminates must do the work of y_0. In this framework, it becomes possible to take direct unions of such rings. However, as the resulting examples do not seem to exhibit any previously unknown behaviour, the more elementary construction is employed here.

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ADDED IN PROOF. Recently, closely related results have been obtained independently by Ada Maria de Souza Doering using a somewhat different construction.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TEXAS, AUSTIN, TEXAS 78712