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SOME INFINITE FREE BOUNDARY PROBLEMS

BY

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ABSTRACT. Let Γ be the boundary of an unbounded simply connected region $\mathfrak{D}$, and let $\mathcal{C}(\Gamma)$ denote the family of all simply connected regions $\Delta \subset \mathfrak{D}$ such that $\partial\Delta = \Gamma \cup \gamma$ where $\gamma \cap \Gamma$ contains only the infinite point. For $\Delta \in \mathcal{C}(\Gamma)$ we call γ the free boundary of Δ. Given a positive constant λ, we seek to find a region $\Delta_\lambda \in \mathcal{C}(\Gamma)$ with free boundary γ_λ such that there is a bounded harmonic function V in Δ_λ with the properties that (i) $V = 0$ on Γ, (ii) $V = 1$ on γ, (iii) $|\text{grad } V(z)| = \lambda$ for $z \in \gamma_\lambda$. We give sufficient conditions for existence and uniqueness of Δ_λ. We also give quantitative properties of γ_λ.

1. Introduction. Suppose a connected set Γ containing more than one point is the boundary of a region $\mathfrak{D}$ which is simply connected on the Riemann sphere and is unbounded in the complex plane. We define $\mathcal{C}(\Gamma)$ to be the class of regions ω with the following properties.

(a) $\partial\omega = \Gamma \cup \alpha$ where α is a nonempty connected subset of $\mathfrak{D}$ which does not meet Γ in the finite plane.

(b) If Γ is compact, then α separates Γ from the point at infinity.

(c) If Γ is not compact, then $\alpha \cup \Gamma$ is connected on the Riemann sphere.

Given Q, a continuous positive function in $\mathfrak{D}$, we seek a region $\omega \in \mathcal{C}(\Gamma)$ with the property that there is a bounded harmonic function V_ω in ω satisfying the following three conditions.

(1) $V_\omega = 0$ on Γ,

(2) $V_\omega = 1$ on α,

(3) $|\text{grad } V_\omega| = Q$ on α.

For $\omega \in \mathcal{C}(\Gamma)$ the bounded harmonic function denoted V_ω satisfying conditions (1) and (2) will be called the stream function of ω and the set α will be called the free boundary of ω. For the case where Γ is compact, sufficient conditions for existence and uniqueness of a free boundary such that V_ω satisfies condition (3) are given in [1], [4], [5] and [6]. Also, qualitative properties of the free boundary α are given in these papers. It should be pointed out that many of the ideas are outgrowths of Beurling's existence theorem in [3].

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We say Γ is starlike if for $w \in \Gamma$, $\rho w \notin \mathfrak{D}$ for $0 < \rho < 1$. We will assume that Γ is either starlike or in some sense is the limit of starlike curves. Although we will be mainly interested in the case where Q is a positive constant, we will have to apply conformal mapping to reformulate the problem in order to obtain some of our results; hence the problem is stated in the more general form. We remark that many of our results remain true if we consider the case where Γ is starlike and $\rho Q(\rho z)$ is nondecreasing for $0 < \rho < \infty$; see **[1]**.

Since results from **[4]**, **[5]**, and **[6]** will be frequently used, we now summarize some of the results which are either included in or are easy consequences of the results in those papers. Let Γ be compact and Q a positive constant which we denote by λ, then:

(i) There exists $\omega \in \mathcal{C}(\Gamma)$ such that V_ω satisfies (3) on α_λ the free boundary of ω. Furthermore, α_λ is an analytic Jordan curve.

We say Γ is convex if $\mathbf{C} - \mathfrak{D}$ is a convex set.

(ii) If Γ is convex, then α_λ is unique and convex, and for $z \in \alpha_\lambda$ the distance to Γ denoted $d(z, \Gamma)$ is less than $1/\lambda$.

(iii) If Γ is starlike, then α_λ is unique and starlike.

(iv) If Γ_1 and Γ_2 are each starlike and are each the boundary of respective unbounded regions $\mathfrak{D}_1$ and $\mathfrak{D}_2$, with $\mathfrak{D}_1 \subset \mathfrak{D}_2$ and if α_1 and α_2 are the respective free boundaries in (iii) corresponding to Γ_1 and Γ_2, then α_2 lies inside α_1.

In the cases where Γ is convex or starlike then we call α_λ the solution free boundary for Γ. We will need the following result for the general problem. The proof is given in **[4]**.

(v) Suppose Q is continuous in $\mathfrak{D}$ and as $z \in \mathfrak{D}$ tends to Γ, let Q have the property that:

$$Q(z) = o\left(\frac{|\text{grad } u|}{u}\right)$$

where u is any positive harmonic function defined in some $\Omega \in \mathcal{C}(\Gamma)$ and where $u = 0$ on Γ. Then, if there is an $\omega' \in \mathcal{C}(\Gamma)$ with the property that for all $z \in '$, for which

$$\limsup_{\substack{\zeta \to z \\ \zeta \in \omega'}} \frac{|\text{grad } V_{\omega'}(\zeta)|}{Q(\zeta)} < 1, \tag{4}$$

then there exists $\omega \in \mathcal{C}(\Gamma)$ such that $\omega \subset \omega'$ and V_ω satisfies (1), (2) and (3).

2. Convexity. If Γ is convex, then we use facts (i)–(v) to construct a solution which is unique in a certain sense. We begin with the following theorem.

THEOREM 1. *If Γ is convex and unbounded and λ is a positive constant, then there exists $\omega \in \mathcal{C}(\Gamma)$ with the following properties*:

(a) V_ω *satisfies* (3) *for* $Q \equiv \lambda$.

(b) *If* α_λ *is the free boundary of* ω, *then* α_λ *is of distance less than* $1/\lambda$ *from* Γ.

PROOF. Without loss of generality we suppose that Γ is contained in the lower half plane and the origin lies on Γ. For $t > 0$ let H_t denote the half plane $\operatorname{Im} z \leqslant -t$ and $\mathfrak{D}_t = H_t \cup \mathfrak{D}$. The unbounded region $\mathfrak{D}_t$ will have as its boundary the compact and convex set Γ_t. Let ω_t be the element of $\mathcal{C}(\Gamma_t)$ such that $\alpha_\lambda^{(t)}$, the free boundary of ω_t, is the solution free boundary for Γ_t. Existence and uniqueness of ω_t is asserted in facts (i) and (ii). We claim that

$$\omega = \bigcup_{t>0} (\omega_t \cap \mathfrak{D})$$

satisfies (a) and (b). We note that (b) follows easily from (iv) and (ii). It remains to show (a).

We see by (iv) that for $t > s$ we have $\alpha_\lambda^{(s)}$ lies inside $\alpha_\lambda^{(t)}$. Furthermore, for $s > t$, (a) if $z \in \omega_s \cap \mathfrak{D}$, then $z \in \omega_t \cap \mathfrak{D}$ and (b) $V_{\omega_t}(z)$ is nonincreasing with t. Therefore, as $t \to \infty$, $V_{\omega_t}(z) \to V_\omega(z)$. Furthermore, if $z \in \alpha$, the free boundary of ω, then every neighborhood of z contains points from all but finitely many of the $\{\alpha_t\}$. Thus $|\operatorname{grad} V_\omega(z)| = \lambda$.

For Γ unbounded and convex, we will call the region $\omega \in \mathcal{C}(\Gamma)$ from Theorem 1 the construction solution strip. We will call its free boundary the constructed free boundary. Any other $\omega \in \mathcal{C}(\Gamma)$ where V_ω satisfies (1), (2) and (3) will be called a solution. In order to prove that the constructed solution is minimal, we will need the following lemma.

LEMMA 1. *Let* Γ *be convex and compact. Suppose there exists* $\omega^* \in \mathcal{C}(\Gamma)$ *whose free boundary* α^* *is the union of finitely many analytic arcs. Suppose for each* z *interior to one of these arcs we have*

$$\lim_{\substack{\zeta \to z \\ \zeta \in \omega^*}} |\operatorname{grad} V_{\omega'}(z)| \leqslant \lambda.$$

Then, ω^* *contains the solution free boundary for* Γ.

PROOF. Let f be a mapping of $|w| > 1$ onto $\mathfrak{D}$ and let $\tilde{\Gamma} = f^{-1}(\Gamma)$. We consider the general problem for the set $\tilde{\Gamma}$ which is the boundary of the region $\tilde{\mathfrak{D}}$, $|w| > 1$, and the positive function $Q(w) = \lambda|f'(w)|$. The region $\tilde{\omega}^* = f^{-1}(\omega^*)$ satisfies condition (4) at all but finitely many points on $\tilde{\alpha}^* = f^{-1}(\alpha^*)$. For $|w| < 1$, define $Q(w) = Q(1/\bar{w})$ and let $\tilde{\Omega}^*$ be the region obtained by reflecting $\tilde{\omega}^*$ about the circle $|w| = 1$. Because of the regularity condition on the boundary of $\tilde{\Omega}^*$, if $H(w)$ solves the Dirichlet problem in the region $\tilde{\Omega}^*$ with boundary values $\log Q(w)$, and if $g(\zeta, w)$ is the Green's function of $\tilde{\Omega}^*$ we have:

$$\log|\text{grad } V_{\tilde{\omega}^*}(w)| = \frac{1}{2\pi}\int_{\partial\tilde{\omega}^*} \log|\text{grad } V_{\tilde{\omega}^*}(\zeta)| \frac{\partial g(w,\zeta)}{\partial\eta}\, ds$$
$$\leqslant \frac{1}{2\pi}\int_{\partial\tilde{\omega}^*} H(\zeta)\frac{\partial g(w,\zeta)}{\partial\eta}\, ds = H(w) \tag{5}$$

where $\partial/\partial\eta$ represents differentiation with respect to the inward pointing normal. From this we obtain (4) so ω^* contains α_λ the unique free boundary solution for Γ.

We now prove that the constructed solution is minimal.

THEOREM 2. *Let Γ be an unbounded convex curve. Suppose $Q(z) \equiv \lambda$, a positive constant and $\omega' \in \mathcal{C}(\Gamma)$ has its stream function $V_{\omega'}$ which satisfies* (1), (2) *and* (3). *Then, ω' contains ω, the constructed solution strip.*

PROOF. It follows from the Schwarz reflection principle that the free boundary of ω' which we denote by α' is an analytic curve. Using the notation of Theorem 1, we show $\omega_t \cap \mathfrak{D}$ is contained inside ω' for all t by supposing there is some t such that $\omega_t \cap \mathfrak{D}$ does not lie inside ω', and getting a contradiction. In the family $\mathcal{C}(\Gamma_t)$ consider the annulus ω^* which consists of all points z which can be connected to Γ_t by an arc which lies in the set

$$[\omega_t \cap \omega'] \cup [\omega_t \cap (\mathbf{C} - \mathfrak{D}) \cap H_t].$$

Since α_t and α' are each analytic curves, it follows that the free boundary of ω^* consists of finitely many analytic arcs. Furthermore since $V_{\omega^*}(z) \geqslant V_{\omega'}(z)$ and $V_{\omega^*}(z) \geqslant V_{\omega_t}(z)$ for $z \in \omega^*$,

$$|\text{grad } V_{\omega^*}(z)| \leqslant \lambda \tag{6}$$

at all regular points on the free boundary of ω^*. This implies that ω^* contains ω_t which in turn implies ω' contains $\omega_t \cap \mathfrak{D}$ for all $t > 0$ which is our desired contradiction.

We now wish to study the asymptotic behavior of the constructed solution in the case where Γ is convex. To do this we assume Γ is continuously parametrized by the equation $z = z(t)$, $-\infty < t < \infty$, in the following manner;

$$z(0) = 0. \tag{7}$$

$$\begin{aligned}&\text{If } \langle z(t)\rangle \text{ is the signed arc length from 0 to } z(t),\\ &\text{then } t < s \text{ implies that } \langle z(t)\rangle \text{ is less than } \langle z(s)\rangle.\end{aligned} \tag{8}$$

$$\lim_{t\to\infty}|z(t)| = \lim_{t\to\infty}|z(t)| = \infty. \tag{9}$$

$$\text{Re } z(t) > 0 \quad \text{for } t > 0. \tag{10}$$

We will also need the following notation. If β is a curve and $z \in \mathbf{C}$, then

$d(z, \beta)$ is the distance from z to β. If the ray r from the point $z \in \mathbf{C}$ intersects the curve β at a furthest point $\zeta \in \beta$, then the distance from z to β along r is defined as:

$$d(z, \beta; r) = |z - \zeta|. \tag{11}$$

We are now ready to give qualitative properties of the constructed solution strip.

THEOREM 3. *Let Γ be unbounded and convex curve which is parametrized by $z = z(t)$ which satisfies* (7), (8), (9) *and* (10) *for $-\infty < t < \infty$. Let l_1 and l_2 be two straight lines such that*

$$\lim_{t\to\infty} d(z(t), l_1) = \lim_{t\to-\infty} d(z(t), l_2) = 0. \tag{12}$$

If α is the free boundary of the constructed solution strip, then the free boundary of any other solution is not asymptotic to α.

PROOF. We suppose Γ is contained in the lower half plane H and r_1 and r_2 be the rays defined by $r_i = l_i \cap H$ for $i = 1, 2$. Since Γ is convex and for $z \in \alpha$, $d(z, \Gamma) < 1/\lambda$, we must have α asymptotic to a pair of rays $\hat{r}_1$ and $\hat{r}_2$ with $\hat{r}_i$ parallel to r_i for $i = 1, 2$. Let ω' be a solution with free boundary α' which is asymptotic to α. Since α is minimal, we know that α separates α' from Γ. For $i = 1, 2$, given $\varepsilon > 0$ let $r_{i,\varepsilon}$ and $R_{i,\varepsilon}$ be the pair of rays from the real axis which are parallel to $\hat{r}_i$ and are at a distance ε from the ray $\hat{r}_i$ with $r_{i,\varepsilon}$ separating α from Γ. Let c and d be two rays in H, originating at 0, which intersect $R_{i,\varepsilon}$ and $r_{i,\varepsilon}$ for $i = 1, 2$. Since $R_{i,\varepsilon}$ and $r_{i,\varepsilon}$ are parallel, we observe that

$$\frac{d(0, r_{i,\varepsilon}; c)}{d(0, R_{i,\varepsilon}; c)} = \frac{d(0, r_{i,\varepsilon}; d)}{d(0, R_{i,\varepsilon}; d)}. \tag{13}$$

If the ray c has a slope sufficiently close to the slope of $\hat{r}_i$, we have

$$d(0, R_{i,\varepsilon}; c) \geqslant d(0, \alpha'; c) \geqslant d(0, \alpha; c) \geqslant d(0, r_{i,\varepsilon}; c). \tag{14}$$

Now if ε' is a sufficiently small positive number and the slope of c is sufficiently close to the slope of $\hat{r}_i$, then we have

$$1 - \varepsilon' < \frac{d(0, r_{i,\varepsilon}; c)}{d(0, R_{i,\varepsilon}; c)} < 1. \tag{15}$$

But then if the slope of c is sufficiently close to the slope of $\hat{r}_i$, we have:

$$1 - \varepsilon' < \frac{d(0, \alpha; c)}{d(0, \alpha'; c)} \leqslant 1. \tag{16}$$

Let $\hat{R}$ be the collection of all rays through $z = 0$. If

$$M = \inf_{c\in\hat{R}} \frac{d(0, \alpha; c)}{d(0, \alpha'; c)}, \tag{17}$$

then we see that $M < 1$ and that $M\alpha' = \{w\colon w = Mz,\ z \in \alpha'\}$ lies inside α. We claim that $M\alpha'$ is tangent to α at some point $z \in \alpha$. Suppose this is not the case. Relation (16) assures us that given $\delta > 0$ there exists $\rho > 0$ such that if $z' \in \alpha'$ and $|z'| > \rho$, then if $z \in \alpha$ lies on the line connecting z' to the origin we have $|z/z'| > M + \delta$. Now suppose for $z' \in \alpha'$ with $|z'| \leqslant \rho$ we have $|Mz' - z| > 0$ where $z \in \alpha$ lies on the line connecting z' to the origin. Then, we have $|z/z'| < M$ for $|z'| \leqslant \rho$. Thus we see there exists $z' \in \alpha'$ where $Mz' \in \alpha$. From this it follows that $V_{\omega'}(M^{-1}z) \geqslant V_{\omega}(z)$ for $z \in \omega \cap M\omega'$. Taking normal derivatives to α at Mz', we get $M^{-1}\lambda \leqslant \lambda$ which is a contradiction.

The next theorem gives information about the asymptotic behavior of the free boundary.

THEOREM 4. *Suppose* Γ *is parametrized by* $z = z(t)$ *which satisfies* (7), (8), (9), (10) *and* (12). *Then*

$$\lim_{t\to\infty} d(z(t), \alpha) = \lim_{t\to-\infty} d(z(t), \alpha) = 1/\lambda.$$

PROOF. We define a monotone function for $-\infty < t < \infty$ by letting $\theta(t) = \theta(z(t))$ be the acute angle between any line of support for Γ through $z(t)$ and the x-axis. Where there is more than one line of support, one may choose one arbitrarily. Define

$$\theta_1 = \sup_{-\infty<t<\infty} \theta(t), \qquad \theta_2 = \inf_{-\infty<t<\infty} \theta(t). \tag{18}$$

The first portion of this proof is the demonstration of the following statement:

FACT A. For any $N > 0$, $\varepsilon > 0$ there exists $t_0 > 0$ such that for all t with $|t| > t_0$ there is a chord L of length N with the distance from $z(t)$ to the midpoint of L less than ε.

In proving Fact A, we first assume $s > 0$; the corresponding statements about $s < 0$ are essentially equivalent. Let $\hat{s}$ be chosen so that

$$\left|\int_{s}^{\infty} d\theta(s)\right| < \delta, \tag{19}$$

where δ is as yet unspecified. The existence of such an $\hat{s}$ depending on δ is clear from the convexity of Γ. Let s' be chosen so that

$$|z(\hat{s}) - z(s')| > N \tag{20}$$

and $s' > \hat{s}$.

Consider a chord L of length N with endpoints $z_1 = z(s_1)$ and $z_2 = z(s_2)$ where $s_2 > s_1 > s'$. Let L_1 and L_2 be lines of support of Γ at z_1 and z_2 respectively and consider the (possibly degenerate) triangle formed by the lines L, L_1 and L_2. Let L_3 be the straight line carrying the segment L. If M_1,

M_2 and M_3 are the respective slopes of L_1, L_2 and L_3, then

$$\delta \geqslant |\theta_1 - \arctan M_1| \geqslant |\theta_1 - \arctan M_3| \geqslant |\theta_1 - \arctan M_2| \geqslant 0$$

because $M_2 \leqslant M_3 \leqslant M_1$. Therefore, $B_1 = |\arctan M_1 - \arctan M_3| < \delta$, $B_2 = |\arctan M_2 - \arctan M_3| < \delta$ where B_1 and B_2 are the angles made by L and L_1 and by L and L_2 respectively. If $L_1 \cap L_2 = \{z^*\}$ is the apex of the triangle formed by L_1, L_2 and L_3 then the altitude to L from z^* has length h where

$$h \leqslant N/2 \tan \delta. \tag{21}$$

Note that in the degenerate case we can assume $h = 0$. Hence, for all $z(s) \in \Gamma$ with $s_1 < s < s_2$, we have $d(z(s), L) \leqslant h$.

For each $z(s)$ with $s > s'$ we can find a chord to Γ denoted L of length N, such that the perpendicular bisector of L passes through $z(s)$, with $z(s)$ between the endpoints of L. To complete the proof of Fact A, we let δ be small enough so that $N/2 \tan \delta < \varepsilon$ and choose s' as in (20). Clearly, similar arguments apply to $s < 0$. So that repeating the above argument, *mutatis mutandis*, we find $s'' < 0$. Finally we conclude the proof of Fact A by letting $s_0 = \mathrm{Max}(s', |s''|)$.

We now claim the following is true:

FACT B. For all $r > 0$, $\varepsilon > 0$, there exists $s^* > 0$ such that for all s with $|s| > s^*$ there exists a circle of radius whose center depends on $z(s)$ and ε such that (a) $d(z(s), C) < \varepsilon$, (b) C is inside the convex domain bounded by Γ.

To prove Fact B, let s_0 be chosen as in Fact A where $N = 2r$. For each s, $|s| > s_0$ let the chord found in Fact A be denoted L_s. Define C_s as the circle of radius r which is tangent to L_s at the midpoint of L_s with the property that C_s lies outside the compact region bounded by L_s and a subarc of Γ. Since Γ is not confined to any vertical strip, there exists $s^* > S_0$ such that C_s is inside Γ for $|s| > s^*$.

For s fixed, let z_0 be the center of C_s and consider the ring region Δ defined by: $r < |z - z_0| < R$ where

$$\frac{1}{R(\log R/r)} = \lambda. \tag{22}$$

Fact (iii) assures us that $\Delta \cap \mathfrak{D} \subset \omega$. It is easy to show (22) is equivalent to

$$R - r = R - R \cdot e^{-1/\lambda R}. \tag{23}$$

From this we see that

$$\lim_{R\to\infty} (R - r) = 1/\lambda. \tag{24}$$

Since by choosing r sufficiently large we can be assured that R and $|s^*|$ are as large as we please, the proof is now complete.

REMARK. The methods give no information about the free boundary when

Γ is confined to a vertical strip. For $\lambda > 0$, we let ω_λ denote the constructed solution strip and let α_λ denote the free boundary of ω_λ. We now discuss the properties of the constructed solution strip as λ varies.

THEOREM 5. *Let Γ be an infinite convex fixed boundary with Γ asymptotic to a pair of nonparallel straight lines. Then,* (a) $\lambda_1 > \lambda_2$ *implies* $\omega_{\lambda_1} \subset \omega_{\lambda_2}$, (b) $\bigcup_{\lambda>0}\alpha_\lambda = \mathfrak{D}$.

PROOF. (a) is an immediate consequence of (iv) and (v). To show part (b) let $z_1 \in \mathfrak{D}$ and define

$$A = A(z_1) = \{\lambda\colon z_1 \in \omega_\lambda\}, \quad B = B(z_1) = \{\lambda\colon z_1 \notin \omega_\lambda\}.$$

We will show that neither A nor B is empty.

Given $z_1 \in \mathfrak{D}$ we show that for all $r > 0$ there is a circle C_r inside Γ with center z_r and radius r. Choose λ so small that $1/|z_1|[\log(|z_1|/r)] > \lambda$. By the minimality of ω_λ we see that $z_1 \in \omega_\lambda$ and $A \neq \varnothing$.

To show B is not empty, we recall that $\omega_\lambda \subset \{z \in \mathfrak{D}\colon d(z, \Gamma) < 1/\lambda\}$. Hence if $z_1 \in \mathfrak{D}$ choose λ such that $1/\lambda < d(z_1, \Gamma)$ and $z_1 \notin \omega_\lambda$.

Now let $\Omega_1 = \bigcap_{\lambda\in A}\omega_\lambda$, $\Omega_2 = \bigcup_{\lambda\in B}\omega_\lambda$. If Ω_1^0 is the interior of Ω_1 then both Ω_1^0 and Ω_2 are solutions for some λ_1 and λ_2 respectively. If $\lambda_1 \neq \lambda_2$, then we apply a similar argument to that used in [5, p. 843] to obtain a contradiction. From this (b) follows.

3. Other regions. We say Γ is an admissible curve if Γ is described by $x + if(x)$ where $f(x)$ is a continuous bounded function which has bounded Dini derivates at every point. In this section we give similar results regarding the free boundaries for admissible curves. We let $\mathfrak{D} = \{z = x + iy\colon y > f(x)\}$ and show that free boundaries do exist in $\mathfrak{D}$ for admissible curves Γ.

THEOREM 6. *Let Γ be an admissible curve. If $Q \equiv \lambda$ a positive constant in* (3), *then there exists $\omega \in \mathcal{C}(\Gamma)$ with the properties that V_ω satisfies* (1), (2) *and* (3). *Furthermore, if α is the free boundary of ω, then every vertical line intersects α at exactly one point.*

PROOF. Let R_n be the closed strip $-n \leqslant x \leqslant +n$ and let $\Gamma_n = \Gamma \cap R_n$. Let

$$\overline{m}_n^* = \sup\left\{\frac{f(s) - f(t)}{s - t} : s, t \in [-n, n]\right\},$$

$$\underline{m}_n^* = \inf\left\{\frac{f(s) - f(t)}{s - t} : s, t \in [-n, n]\right\},$$

which are finite by hypothesis. Let

$$\overline{m}_n = \max\{\overline{m}_n^* + 1, n\}, \qquad \underline{m}_n = \min\{\underline{m}_n^* + 1, n\}.$$

If r_n is the ray with slope $\overline{m}_n$ originating at the point $n + if(n) = A_n$, then r_n does not intersect Γ_n except at the point A_n. Similarly, if r_{-n} is the ray originating at the point $-n + if(-n) = B_n$ with slope $\underline{m}_n$, then r_{-n} does not intersect Γ_n except at the point B_n. If P_n is the point of intersection of lines r_n and r_{-n}, then $P_n \notin \mathfrak{D} \cup \Gamma$. Let Q_{-n} and Q_n denote the points of intersection of the rays r_n and r_{-n} with the lines $x = -n$ and $x = +n$ respectively. Let Γ_n^* denote the curve defined by Γ_n together with the line segments $[A_n, Q_{n+1}]$, $[Q_{-n}, Q_n]$ and $[Q_{-n}, B_n]$. If P lies in the triangle determined by the three points Q_n, Q_{-n} and P_n and Q lies on the curve Γ_n^*, then the segment $[P, Q]$ lies outside of $\mathfrak{D}$.

In the set $\mathcal{C}(\Gamma_n^*)$ there is a unique solution denoted by $\omega_\lambda^{(n)}$ with free boundary $\alpha_\lambda^{(n)}$.

Using ideas quite similar to those used in the proof of Theorem 1, we could show that

$$\omega = \bigcup_n (\omega_\lambda^{(n)} \cap \mathfrak{D})$$

is a solution. Furthermore, since the free boundary of ω is an analytic curve and all points inside the triangle determined by the points Q_n, Q_{-n}, and P_n are star center points for the curves $\alpha_\lambda^{(n)}$ for each n, we see that every vertical line intersects the free boundary of ω in exactly one point.

If Γ is admissible, the region $\omega \in \mathcal{C}(\Gamma)$ whose existence is shown in Theorem 6, will be called the constructed solution strip. The following theorem gives qualitative properties about the free boundary of ω.

THEOREM 7. *Let Γ be admissible and let ω be the constructed solution strip in $\mathcal{C}(\Gamma)$. If Γ is asymptotic to the line $y = 0$, then, α, the free boundary of ω, is asymptotic to the line $y = 1/\lambda$.*

PROOF. Given $\varepsilon > 0$ and $r > 0$ there exists $M > 0$ such that for each z_0 where $|\mathrm{Re}\, z_0| > M$ and $\mathrm{Im}\, z_0 = -(r + \varepsilon)$, the circle

$$C_r(z_0) = \{z\colon |z - z_0| = r\}$$

lies below Γ. Hence for $z \in \alpha$ and $|\mathrm{Re}\, z| > M$, we have $\mathrm{Im}\, z > \rho$ where

$$(\rho + \varepsilon + r)\exp\left[\frac{-1}{\lambda(\rho + \varepsilon + r)}\right] = r.$$

It follows that given $\delta > 0$ for $|\mathrm{Re}\, z|$ sufficiently large, $|\mathrm{Im}\, z| > 1/\lambda - \delta$.

Since Γ is admissible we let $\Gamma = \{x + iy\colon y = f(x)\}$ and $|f(x)| < k$. Furthermore since Γ is asymptotic to the x-axis given $\varepsilon > 0$ there exists $R > 0$ such that $|\mathrm{Im}\, f(z)| < \varepsilon$ for $|x| > R$. Let Γ_n^* be the curve which bounds the region

$$\Delta_n = [\{z\colon |\mathrm{Re}\, z| < r, |\mathrm{Im}\, z| < k\} \cup \{z\colon |\mathrm{Re}\, z| < \varepsilon\}] \cap \{z\colon |\mathrm{Re}\, z| < n\}.$$

Let $\mathfrak{D}_n$ denote the convex hull of Δ_n and let $k_n = \partial\mathfrak{D}_n$. If ω_n is the unique solution in $\mathcal{C}(k_n)$, then a simple application of (iv) shows that if $z \in \alpha$ and

$|\mathrm{Re}\, z| = n$, then

$$|\mathrm{Im}\, z| < \varepsilon + \left[\frac{\sqrt{(n-r)^2 + (k-\varepsilon)^2}}{(n-r)}\right]\frac{1}{\lambda}. \tag{25}$$

Therefore given $\delta > 0$ for $|\mathrm{Re}\, z|$ sufficiently large, $|\mathrm{Im}\, z| < \delta + 1/\lambda$.

We conclude with the remark that one can use any family which is elliptic in the sense of [4] to study unbounded free boundary problems provided there is some control on the free boundary as the fixed boundary becomes unbounded. An example would be the following theorem whose proof we omit.

THEOREM 8. *Let* Γ *be an infinite starlike fixed boundary such that* (1) $\{x\colon x = \mathrm{Re}\, z,\ z \in \Gamma\}$ *is unbounded above and below, and* (2) $\{y\colon y = \mathrm{Im}\, z,\ z \in \Gamma\}$ *is bounded above and below. Then, there exists a region* $\omega \in \mathcal{C}(\Gamma)$ *such that* V_ω *satisfies* (1), (2) *and* (3) *where* $Q \equiv \lambda$ *a fixed positive constant. Furthermore, the free boundary of* ω *is starlike.*

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