

THE PL GRASSMANNIAN AND PL CURVATURE

BY

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ABSTRACT. A space $\mathcal{G}_{n,k}$ is constructed, together with a block bundle over it, which is analogous to the Grassmannian $G_{n,k}$ in that, given a PL manifold M^n as a subcomplex of an affine triangulation of R^{n+k} , there is a natural "Gauss map" $M^n \rightarrow \mathcal{G}_{n,k}$ covered by a block-bundle map of the PL tubular neighborhood of M^n to the block bundle over $\mathcal{G}_{n,k}$. Certain subcomplexes of $\mathcal{G}_{n,k}$ are then studied in connection with immersion problems, the chief result being that a connected manifold M^n (nonclosed) PL immerses in R^{n+k} satisfying certain "local" conditions if and only if its stable normal bundle is represented by a map to the subcomplex of $\mathcal{G}_{n,k}$ corresponding to the condition. An important example of such a condition is a restriction on PL curvature, e.g., nonnegative or nonpositive, PL curvature having been defined by D. Stone.

0. The purpose of this paper is to construct and examine a certain space, which we shall call $\mathcal{G}_{n,k}$, which is related to BPL and to the classification of normal block bundles of locally-flat PL embeddings of n -manifolds in R^{n+k} much as the finite Grassmannian $G_{n,k}$ is related to BO and to the classification of normal vector bundles of smooth embeddings of n -manifolds in R^{n+k} . That is:

(a) The family of spaces $\{\mathcal{G}_{n,k}\}$ forms a double sequence

$$\begin{array}{ccc} \mathcal{G}_{n,k} & \rightarrow & \mathcal{G}_{n+1,k} \\ \downarrow & & \downarrow \\ \mathcal{G}_{n,k+1} & \rightarrow & \mathcal{G}_{n+1,k+1} \end{array}$$

(b) There is a canonical k -block bundle $\gamma_{n,k}$ over $\mathcal{G}_{n,k}$.

(c) Given a triangulated combinatorial manifold M^n embedded as a subcomplex of a piecewise-linear triangulation of R^{n+k} , there is a natural "Gauss" map $\nu: M^n \rightarrow \mathcal{G}_{n,k}$ so that $\nu^*(\gamma_{n,k})$ is the normal block bundle of the embedding.

Specifying further what is meant by "natural" in (c) above, we mean that the Gauss map ν is immediately determined pointwise by the geometric data of the situation. One does not need Brown's Theorem [B], or even Rourke and Sanderson's construction of universal k -block bundles [RS], which, in

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any case, determine classifying maps only up to homotopy. The situation is thus analogous to the case where a smooth manifold M^n embedded in R^n (with R^n having its usual linear structure) automatically acquires a map, i.e., the classical Gauss map, to $G_{n,k}$. This justifies, in part at least, our calling $\mathcal{G}_{n,k}$ a polyhedral analogue of the finite Grassmannian and our reference to ν as the Gauss map.

In §3 we shall use the Grassmannian, and its subcomplexes, to study the problem of polyhedrally immersing a manifold M^n in R^{n+n} so that its curvature (in the sense of D. Stone [S_1], [S_2]), is everywhere nonnegative or nonpositive. We reduce this problem to a homotopy problem (i.e., the usual kind of lifting problem), in the case of manifolds each component of which is open or has nonvoid boundary.

We also take note of the fact that in a preliminary version of this paper, the Grassmannian and Gauss map were used to prove a close approximation to the statement that there exist "local combinational formulae" for cocycles representing rational characteristic classes of PL manifolds. A suggestion of Rourke improved both the result (which is now no longer an approximation, but the statement itself), and the method of proof. A separate, joint paper will appear with this result [LR].

We shall need some preliminary definitions. "Manifold" shall always mean PL manifold and "triangulated manifold" shall always mean combinatorially triangulated. By a j -plane in R^{n+k} we mean a j -dimensional linear subspace. An affine j -plane is the translate of a j -plane by some fixed vector. If U_1 is a j_1 -plane, U_2 a j_2 -plane, with $j_1 < j_2$ we say that U_1 is in U_2 when we mean that U_1 is a linear subspace of U_2 . If U_1 is an affine j_1 -plane, we say that U_1 is in U_2 when it is an affine subspace of U_2 . If U is a j -plane we use D_U to designate the unit disc of U and S_U to denote the unit $(j - 1)$ -sphere in U .

A piecewise linear triangulation of R^{n+k} means a triangulation such that every j -simplex σ is a subset of some affine j -plane.

If M^n is a triangulated manifold and σ is a closed simplex, then $\text{st}(\sigma, M^n)$ is the union of all closed simplices having σ as a face; $\text{lk}(\sigma, M^n)$ is the union of all simplices τ such that $\tau \subseteq \text{st}(\sigma, M^n)$, $\tau \cap \sigma = \emptyset$. The dual cell σ^* is the union of all simplices τ of the first barycentric subdivision of the given triangulation such that $\tau \cap \sigma = \{\text{barycenter of } \sigma\}$.

1. Formal links. We need to define the notion of "formal link" which will play a central role in the subsequent constructions.

To begin with, let U be a $(j + k)$ -plane of R^{n+k} . A triangulation T of S_U shall be called "nice" if and only if:

(a) For each r -simplex σ of T there is a unique $(r + 1)$ -plane in U which contains σ .

(b) If $c(\sigma)$ is the convex hull in U of the vertices of σ , then $c(\sigma)$ is contained

in the union of all those segments from the origin to points in σ .

(c) For any simplex σ^i of T , the convex structure is the same as that obtained by considering σ^i as a subset of S_U and defining, for $x, y \in \sigma^i$, $0 \leq t \leq 1$, the convex combination $t \cdot x + (1 - t) \cdot y$ as $tx + (1 - t)y / \|tx + (1 - t)y\|$ where multiplication by scalars in the latter expression has its usual meaning for the vector space U .

1.1 DEFINITION. A formal link L of dimension (n, k, j) is a triple (U_L, T_L, Σ_L) where U_L is a $(j + k)$ -plane of R^{n+k} , T_L is a nice triangulation of S_{U_L} , Σ_L is a subcomplex of T_L which is a combinatorial $(j - 1)$ -sphere.

If $k = 2$ we shall further assume that the sphere pair (T_L, Σ_L) is unknotted.

Let $L = (U_L, T_L, \Sigma_L)$ be a formal link of dimension (n, k, j) and let v be a vertex of Σ_L . We define a new formal link L_v of dimension $(n, k, j - 1)$ as follows: Let R denote the segment from 0 to v in U_L ; then U_v is the $(j + k - 1)$ -plane of U_L orthogonal to R . Let U' be an affine $(j + k - 1)$ -plane of U_L parallel to U_v and passing through the midpoint m of R . Let S' be a small $(j + k - 2)$ -sphere of radius λ in U' centered at m . If σ is a simplex of $\text{lk}(v, T_L)$ then let $\tau(\sigma) = \sigma * v$ be the corresponding simplex of $\text{st}(v, T_L)$. Let P_σ be the union of all segments in U from the origin to points in $\tau(\sigma)$. We claim that if we set $\sigma_1 = S' \cap P_\sigma$, then σ_1 is homeomorphic to σ , and thus letting σ range over all the simplices of $\text{lk}(v, T_L)$ we obtain a triangulation of S' isomorphic to that of $\text{lk}(v, T_L)$. Now consider the similarity transformation on U given by $u \rightarrow \lambda^{-1} \cdot (u - m)$ which carries U' onto U_v and S' onto S_{U_v} . This induces a triangulation of S_{U_v} with one simplex $\bar{\sigma}$ for each simplex σ of $\text{lk}(v, T_L)$. Call this triangulation T_{L_v} . Clearly T_{L_v} is a nice triangulation of S_{U_v} . Let $\Sigma_{L_v} = \bigcup_{\sigma \in \text{lk}(v, \Sigma_L)} \bar{\sigma}$. Σ_{L_v} is clearly a subcomplex of T combinatorially equivalent to S^{j-2} . We thus set $L_v = (U_v, T_v, \Sigma_v)$.

Furthermore, we may consider a formal link $L = (U_L, T_L, \Sigma_L)$ of dimension (n, k, j) and an arbitrary r -simplex σ of Σ_L . Let v_0, \dots, v_r be its vertices, arbitrarily ordered. Let $L_0 = L_{v_0}$. Clearly there are vertices $v_1^1, v_2^1, \dots, v_r^1$ of Σ_{L_0} corresponding to v_1, \dots, v_r . Then set $L_1 = L_{v_1^1}$ and obtain vertices v_2^2, \dots, v_r^2 of Σ_{L_1} corresponding to v_2^1, \dots, v_r^1 . Continuing in this fashion, we may define $L_{i+1} = (L_i)_{v_{i+1}^{i+1}}$ for $i < r$, finally obtaining L_r , a formal link of dimension $(n, k, j - r - 1)$.

1.2 LEMMA. L_r is independent of the ordering v_0, \dots, v_r .

We sketch the proof. Let b be the barycenter of σ , and let m be the midpoint of the ray in U_L from the origin to b . Let X_σ denote the unique $(r + 1)$ -plane of U_L in which σ lies and let U_σ be the $k + j - r - 1$ plane of U_L orthogonally complementary to X_σ . Let U' denote the affine $(k + j - r - 1)$ -plane of U_L parallel to U_σ and passing through m . Given a simplex τ of $\text{lk}(\sigma, T_L)$, let $\rho(\tau)$ be the simplex $\tau * \sigma$ of $\text{st}(\sigma, T_L)$. Let P_τ be the union of all

rays in U_L from the origin to points in $\rho(\tau)$. If S' is a small sphere of radius λ in U' , centered at m , let $\tau_1 = S' \cap P_\tau$. τ_1 is homeomorphic to τ , and letting τ range over all the simplices of $\text{lk}(\sigma, T_L)$, we obtain a triangulation of S' . We note that the similarity transformation $u \rightarrow \lambda^{-1}(u - m)$ carries S' onto S_{U_σ} thereby providing S_{U_σ} with a nice triangulation T_σ with one simplex $\bar{\tau}$ for each simplex τ of $\text{lk}(\sigma, T_L)$. We let $\Sigma_\sigma = \bigcup_{\tau \subset \text{lk}(\sigma, \Sigma_L)} \bar{\tau}$, and thereby obtain a formal link $L_\sigma = (U_\sigma, T_\sigma, \Sigma_\sigma)$ of dimension $(n, k, j - r - 1)$. We now claim that L_σ is the same as the L_r constructed above from the ordering v_0, \dots, v_r of the vertices of σ , irrespective of which ordering was used. Q.E.D.

Consider once more an arbitrary formal link $L = (U_L, T_L, \Sigma_L)$ of dimension (n, k, j) and let v^* denote the dual $(j - 1)$ -cell of v in Σ_L (as a subcomplex of the first barycentric subdivision of Σ_L). Let \hat{v}^* denote the dual $(j + k - 1)$ -cell of v in T_L (as a subcomplex of the first barycentric subdivision of T_L). We claim that there is an obvious homeomorphism $(\text{st}(v, T_L), \text{st}(v, \Sigma_L)) \rightarrow \hat{v}^*, v^*$. This is obtained by “radially projecting” $\text{lk}(v, T_L)$ onto $\text{bdy}(\hat{v}^*, v^*)$ and then linearly extending to convex combinations $\lambda v + (1 - \lambda)u$, $u \in \text{lk}(v, T_L)$ (which is the generic form of a point in $\text{st}(v, T_L)$).

We now adopt the convention that, if L is a formal link of dimension $(n, k, 0)$, making $\Sigma_L = \emptyset$, we shall interpret $c\Sigma_L$ as the set consisting of a single “cone” point; otherwise c has its usual meaning, i.e., unreduced cone. (Alternatively, we may read cX as the reduced cone on $X^+ = X \cup \{*\}$ where $*$ is a disjoint base point.)

Now consider some formal link L , v a vertex of Σ_L , and $L_v = (U_v, T_v, \Sigma_v)$. We may identify D_{U_v} with cT_v , and, since $\text{st}(v, T_L) = c \text{lk}(v, T_L)$, we obtain a homeomorphism $h_{(L,v)}: cT_v, c\Sigma_v \rightarrow \hat{v}^*, v^*$.

If σ is a simplex of Σ_L and v_0, \dots, v_r an ordering of its vertices, then, by constructing $L_i, i = 0, 1, \dots, r$, as before we have a composite of 1-1 maps

$$\begin{array}{rcl}
 cT_{L_r} & \rightarrow & T_{L_{r-1}} \\
 & & | \cap \\
 & & cT_{L_{r-1}} \rightarrow T_{L_{r-2}} \\
 & & \vdots \\
 & & T_{L_0} \\
 & & | \cap \\
 & & cT_{L_0} \rightarrow T_L
 \end{array}$$

which restricts to

$$\begin{array}{ccc}
 c\Sigma_{L_r} & \rightarrow & \Sigma_{L_{r-1}} \\
 & & | \cap \\
 & & c\Sigma_{L_{r-1}} \rightarrow \Sigma_{L_{r-2}} \\
 & & \vdots \\
 & & \Sigma_{L_0} \\
 & & | \cap \\
 & & c\Sigma_{L_0} \rightarrow \Sigma_L
 \end{array}$$

Here each of the horizontal maps is of the form $h_{(L_{r-1}, v_i)}$ for some vertex v_i of Σ_{L_r} corresponding to v_i .

Since $L_r = L_\sigma$, we have a homeomorphism $cT_{L_\sigma}, c\Sigma_{L_\sigma}$ into T_L, Σ_L and it is easy to see that the image is $\hat{\sigma}^*, \sigma^*$ where $\hat{\sigma}^*$ is the dual cell of σ in the first derived of T_L and σ^* is the dual cell of σ in the first derived of Σ_L .

1.3 LEMMA. *This homeomorphism is independent of the ordering v_0, \dots, v_r of the vertices of σ .*

We leave the proof to the reader. In the light of this lemma we are justified in calling this homeomorphism $h_{(L, \sigma)}$.

Note that if σ is a simplex of Σ_L , and of the form $\sigma = \tau * \rho$, then there is a simplex $\bar{\rho}$ of Σ_L corresponding to ρ and, furthermore, $(L_r)_\rho = L_\sigma$. We claim, in extension of Lemma 1.3 above, that $h_{(L, \sigma)} = h_{(L, \tau)} \circ h_{(L, \bar{\rho})}$.

Given a formal link $L = (U_L, T_L, \Sigma_L)$ of dimension (n, k, j) , let X_L be the plane of dimension $(n - j)$ in R^{n+k} orthogonal to U_L , thus decomposing R^{n+k} as $U_L \oplus X_L$. Given a simplex σ in T_L let Q_σ be the union of all those infinite rays in U_L from the origin through points in σ . Let V_σ denote $Q_\sigma \times X_L$ (as a subset of $R^{n+k} (= U_L \oplus X_L)$.) Let $V_L = \cup_{\sigma \subseteq \Sigma_L} V_\sigma$. V_L is thus an n -dimensional piecewise-linear submanifold of R^{n+k} . Note that if $\dim \sigma = r$, V_σ is a manifold of dimension $(n + r - j + 1)$, which is contained in some $(n + r - j + 1)$ -plane of R^{n+k} . Henceforth, we shall call X_L the axis of V_L .

It is now appropriate to take note of the relationship between formal links and submanifolds of R^{n+k} . Let M^n be a combinatorial submanifold which is a locally flat subcomplex of a piecewise-linear triangulation of R^{n+k} . Let σ be an r -simplex of M^n , $\sigma \not\subseteq \partial M^n$. The formal link $L\langle \sigma, M^n \rangle$ is defined as follows:

Let Y_σ be the affine r -plane containing σ and let U_σ be the $(n + k - r)$ -plane orthogonal to Y_σ . Let U' be an affine $(n + k - r)$ -plane parallel to U_σ and passing through the barycenter b_σ of σ . Let S' be a small $(n + k - r - 1)$ -sphere of radius λ in U' centered at b_σ . Given a simplex τ of $\text{lk}(\sigma, R^{n+k})$,

let $\rho(\tau) = \tau * \sigma \subseteq \text{st}(\sigma, R^{n+k})$. Let $\tau' = S' \cap \rho(\tau)$. Thus S' acquires a triangulation having one simplex τ' for each τ of $\text{lk}(\sigma, R^{n+k})$. Consider $u \rightarrow 1/\lambda \cdot (u - b_\sigma)$ taking U' onto U_σ and S' onto S_{U_σ} . Let T_σ be the induced triangulation of S_{U_σ} under this homeomorphism. T_σ has one simplex $\bar{\tau}$ for each simplex τ of $\text{lk}(\sigma, R^{n+k})$. Moreover, if $\bar{\tau}$ is of dimension i it lies in an $(i + 1)$ -plane of U_σ and, furthermore, $\bar{\tau}$ lies in the union of all rays through $c(\bar{\tau})$. Thus T_σ is a nice triangulation of S_{U_σ} .

Now let $\Sigma_\sigma = \bigcup_{\tau \subseteq \text{lk}(\sigma, M^n)} \bar{\tau}$. Σ_σ is a triangulated sphere of dimension $n - r - 1$. We now define the formal link of σ to be the $(n, k, n - r)$ -dimensional formal link $L\langle\sigma, M^n\rangle = (U_\sigma, T_\sigma, \Sigma_\sigma)$.

We note the following property:

Given $M^n \subseteq R^{n+k}$ as above, let $\tau \subseteq \sigma$ be simplices of M^n so that $\sigma = \tau * \rho$. Thus, corresponding to $\rho \subseteq \text{lk}(\tau, M^n) \subseteq \text{lk}(\tau, R^{n+k})$ there is a simplex $\bar{\rho}$ of Σ_τ . We claim that $(L\langle\tau, M^n\rangle)_{\bar{\rho}} = L\langle\sigma, M^n\rangle$. The reader is invited to verify this.

2. The complex $\mathcal{G}_{n,k}$ and the Gauss map. We wish to define a certain space $\mathcal{G}_{n,k}$, a k -block bundle $\gamma_{n,k}$ over $\mathcal{G}_{n,k}$, and a “natural” Gauss map $\nu: M^n \rightarrow \mathcal{G}_{n,k}$ for every combinatorial n -manifold given as a subcomplex of a piecewise-linear triangulation of R^{n+k} . This map ν will be seen to be covered in a natural way by a block bundle map $E \rightarrow E_{n,k}$ where E is the normal block bundle of the embedding $M^n \subseteq R^{n+k}$ and $E_{n,k}$ is the total space of $\gamma_{n,k}$.

For the remainder of this definitional material, fix n and k . Consider the set of formal links L of dimension (n, k, j) . Consider further a set $\{e_L\}$ of j -dimensional cells indexed by these formal links. Intuitively, it is convenient to think of e_L as the topological j -cell $c\Sigma_L$. We now recall that for a simplex σ of Σ_L we have a homeomorphism $h_{(L,\sigma)}$ from cT_{L_σ} onto the dual cell $\hat{\sigma}^*$ of σ in T_L , and that $h_{(L,\sigma)}$ takes $c\Sigma_{L_\sigma}$ homeomorphically onto the dual cell σ^* of σ in Σ_L . We denote by $h_{(L,\sigma)}^0$ the restriction $h_{(L,\sigma)}|_{c\Sigma_{L_\sigma}} \rightarrow \sigma^*$. If we continue to think of e_L as $c\Sigma_L$, we may define $\mathcal{G}_{n,k}$ as the cell complex obtained by taking $\bigcup_L e_L$ (L a formal link of dimension (n, k, j) , $0 \leq j \leq n$) and then identifying e_{L_σ} with a subspace of e_L via $h_{(L,\sigma)}^0$. $\mathcal{G}_{n,k}$ is thus seen to be a C-W complex with one j -cell for every formal link of dimension (n, k, j) .

To build the block bundle $\gamma_{n,k}$ over $\mathcal{G}_{n,k}$, we think first of the block bundle over the disjoint union $\bigcup_L e_L$, where the “block” over the j -cell $e_L \cong c\Sigma_L$ is the $j + k$ -cell $b_L \cong cT_L$. If we then divide out $\bigcup_L b_L$ by the identifications obtained from the maps $h_{(L,\sigma)}: b_{L_\sigma} \rightarrow b_L$, we obtain a space $E_{n,k}$ which is the total space of a block bundle $\gamma_{n,k}$ over $\mathcal{G}_{n,k}$.

Now let M^n be a locally flat submanifold of R^{n+k} , given as a subcomplex in some piecewise-linear triangulation. We wish to define the Gauss map $\nu: M^n \rightarrow \mathcal{G}_{n,k}$. Let σ be a simplex of M^n ($\sigma \not\subseteq \partial M^n$ if M^n has boundary), and let

$\hat{\sigma}^*$, σ^* denote, respectively, the dual cell of σ in the triangulation of R^{n+k} and in the triangulation of M^n . Specifically, $\hat{\sigma}^*$ and σ^* are subcomplexes of the first barycentric subdivision. If M^n has no boundary the dual cells are, of course, the cells of the dual cell decomposition of M^n . If M^n has a nonvoid boundary, then $\bigcup_{\sigma \not\subseteq \partial M^n} \sigma^*$ is itself an n -manifold which is a deformation retract of M^n .

We define ν explicitly on $M_0^n = \bigcup_{\sigma \not\subseteq \partial M^n} \sigma^*$. We may quickly describe ν as a map of this cell structure to the given cell structure on $\mathcal{G}_{n,k}$ as follows. First, we get a map $\nu: M_0^n \rightarrow \mathcal{G}_{n,k}$ by assigning σ^* to the cell $e_{L\langle\sigma, M^n\rangle}$. It is easily seen that this is compatible with incidence relations.

If fact we may be even more specific; that is, the map ν is easily specified pointwise. In particular, if σ is an $(n - j)$ -simplex then the dual cell σ^* of σ as a subcomplex of the first barycentric subdivision of M^n , is isomorphic in an obvious way to the cone on the first subdivision of $\Sigma_{L\langle\sigma, M^n\rangle}$. This gives an obvious homeomorphism $\nu_\sigma: \sigma^* \rightarrow c\Sigma_{L\langle\sigma, M^n\rangle} = e_{L\langle\sigma, M^n\rangle}$. Moreover, if τ is a face of σ , σ^* is a face of τ^* and we then have a strictly commutative diagram

$$\begin{array}{ccc} \sigma^* & \xrightarrow{\subseteq} & \tau^* \\ \downarrow \nu_\sigma & & \downarrow \nu_\tau \\ e_{L\langle\sigma, M^n\rangle} & \xrightarrow{i} & e_{L\langle\tau, M^n\rangle} \end{array}$$

where i is the map defined by first noting that $L\langle\sigma, M^n\rangle$ is $[L\langle\tau, M^n\rangle]_\rho$ for some simplex ρ of $\Sigma_{L\langle\tau, M^n\rangle}$ thus allowing i to be defined as the composition

$$e_{L\langle\sigma, M^n\rangle} = c\Sigma_{L\langle\sigma, M^n\rangle} \xrightarrow{h(L\langle\sigma, M^n\rangle, \rho)} \Sigma_{L\langle\tau, M^n\rangle} \subseteq c\Sigma_{L\langle\tau, M^n\rangle} = e_{L\langle\tau, M^n\rangle}.$$

The family $\{\nu_\sigma\}$ thus defines a map $\nu: M_0^n \rightarrow \mathcal{G}_{n,k}$.

Now by Rourke and Sanderson [RS], the normal block bundle of M^n in R^{n+k} may be specified as follows (at least over M_0^n). The total space E is the union of $\hat{\sigma}^*$ over all the simplices σ of M^n with $\sigma \not\subseteq \partial M^n$. Here $\hat{\sigma}^*$ is the "block" over σ^* . We shall define a map N from E to $E_{n,k}$. This is done by noting that $\hat{\sigma}^*$ is, as a subcomplex of the first subdivision of R^{n+k} , isomorphic to $cT_{L\langle\sigma, M^n\rangle} = b_{L\langle\sigma, M^n\rangle}$ in a natural way. Let N_σ denote this homeomorphism. Again, if τ is a face of σ , we have a commutative diagram

$$\begin{array}{ccc} \hat{\sigma}^* & \xrightarrow{\subseteq} & \hat{\tau}^* \\ \downarrow N_\sigma & & \downarrow N_\tau \\ b_{L\langle\sigma, M^n\rangle} & \xrightarrow{j} & b_{L\langle\tau, M^n\rangle} \end{array}$$

where j is defined similarly to i above. The family $\{N_\sigma\}$ defines a map $N: E \rightarrow E_{n,k}$, with $N|M_0^n = \nu$.

Thus it is evident that the map ν , which depends only on the triangulation

of M^n and the ambient R^{n+k} , is entitled to be called the Gauss map. (There is no essential difficulty resulting from M_0^n being strictly smaller than M^n when $\partial M^n \neq 0$.)

We also point out that, by an easy extension of this observation, if M^n is immersed in a triangulated R^{n+k} so that the immersion is an embedding on all subspaces of the form $\text{st}(\sigma, M^n)$, σ a simplex of M^n (that is, M^n is immersed as a subcomplex of a triangulated R^{n+k}), then the notion of a Gauss map, classifying the normal block bundle of the immersion is likewise naturally defined.

We observe that there are natural maps $\alpha: \mathcal{G}_{n,k} \rightarrow \mathcal{G}_{n+1,k}$ $\beta: \mathcal{G}_{n,k} \rightarrow \mathcal{G}_{n,k+1}$ thus yielding a double sequence

$$\begin{array}{ccccc}
 & & \downarrow \beta & & \downarrow \beta \\
 \alpha & \rightarrow & \mathcal{G}_{n,k} & \xrightarrow{\alpha} & \mathcal{G}_{n,k+1} & \xrightarrow{\alpha} \\
 & & \downarrow \beta & & \downarrow \beta \\
 \alpha & \rightarrow & \mathcal{G}_{n+1,k} & \xrightarrow{\alpha} & \mathcal{G}_{n+1,k+1} & \xrightarrow{\alpha} \\
 & & \downarrow \beta & & \downarrow \beta
 \end{array}$$

The definition of α and β is as follows:

Let L be a formal link of dimension (n, k, j) , $L = (U_L, T_L, \Sigma_L)$. Let P denote the line in R^{n+k+1} orthogonal to R^{n+k} . Set $U_K = U_L \oplus P$. Then $S_{U_K} \cap P$ consists of two points n and s . We choose a triangulation T_K of S_{U_K} by letting the simplices be those of T_L , together with additional simplices σ^+ and σ^- for each simplex σ of T_L . σ^+ is the union of all 90° arcs from n to the points of σ , and σ^- is the union of all 90° arcs from s to points of σ . We also include n and s as vertices. Thus, T_K is isomorphic as a complex to the unreduced suspension of T_L . We set $\Sigma_K = \Sigma_L$. We let $K(L)$ be the formal link of dimension $(n, k + 1, j)$ given by $K = (U_K, T_K, \Sigma_K)$. The set map $L \rightarrow K(L)$ from (n, k, j) -links to $(n, k + 1, j)$ -links is consistent with incidence relations among the cells of $\mathcal{G}_{n,k}, \mathcal{G}_{n,k+1}$ because $K(L_\sigma) = [K(L)]_\sigma$ for σ a simplex of $\Sigma_L = \Sigma_K$. Thus we obtain a cellular map $\alpha: \mathcal{G}_{n,k} \rightarrow \mathcal{G}_{n,k+1}$.

On the other hand, given the (n, k, j) -link L , we may regard U_L as a $(j + k)$ -plane of R^{n+k+1} , and a fortiori, L may be regarded as a formal link of dimension $(n + 1, k, j)$. This allows us to view $\mathcal{G}_{n,k}$ as a subcomplex of $\mathcal{G}_{n+1,k}$ and it is seen, again without difficulty that this induces an inclusion $\beta: \mathcal{G}_{n,k} \xrightarrow{\subseteq} \mathcal{G}_{n+1,k}$.

It is clear that $\alpha^* \gamma_{n,k+1} = \gamma_{n,k} \oplus \varepsilon^1$ and that $\beta^* \gamma_{n+1,k} = \gamma_{n,k}$.

3. Subcomplexes of $\mathcal{G}_{n,k}$ -applications to D. Stone's theory of polyhedral curvature. We shall use our construction of the polyhedral Grassmannian to study some immersion problems. In particular, we shall study the question of deciding when a manifold M^n may be immersed piecewise linearly in R^{n+k}

subject to constraints on the induced polyhedral sectional curvature in the sense of D. Stone [S₁], [S₂].

Recall that the Hirsch immersion theorem, as applied in the PL case, [H], [HP] tells us that the classifying map $M^n \rightarrow BPL$ for the stable normal bundle of M^n lifts to $BPL(k)$ if and only if M^n immerses in R^{n+k} (provided either $k \neq 0$ or M^n has a handlebody decomposition with no n -handles). The same, of course, holds true if $BPL(k)$ be replaced by $\mathcal{G}_{n,k}$.

We seek to generalize this result.

We shall say that a subcomplex B of $\mathcal{G}_{n,k}$ is “geometric” if it has the property

(i) If $e_L \subseteq B$ and $V_J = V_L$ then $e_J \subseteq B$.

For the remainder of this discussion, let M^n be a manifold admitting a handle decomposition with no n -handles, i.e., every component of M^n is either open or has nonvoid boundary.

3.1 THEOREM. *Let B be a geometric subcomplex of $\mathcal{G}_{n,k}$. M^n immerses in R^{n+k} with a Gauss map $\nu: M^n \rightarrow B \subseteq \mathcal{G}_{n,k}$ if and only if the stable normal classifying map $M^n \rightarrow BPL$ lifts to B .*

Before proceeding to the proof of 3.1, we state a corollary which is the prime motivating example. David Stone has defined an analog of “sectional curvature” for polyhedral manifolds embedded, or immersed, in Euclidean space. (See [S₁], [S₂] for details.) In particular, one may talk of two parameters $K_-(x, d)$, $K_+(x, d)$ for each point $x \in M^n$ and each “tangent direction” d . One may call M “nonnegatively” curved at x if $K_-(x, d) \geq 0$ for all choices d and “nonpositively” curved at x if $K_+(x, d) \leq 0$ for all d . (The meaning of strictly positive or negative curvature is somewhat elusive, since $K_+(x, d) = K_-(x, d) = 0$ for any x in the interior of an n -simplex or $(n - 1)$ -simplex.) However, if M^n is polyhedrally immersed in R^{n+k} , and $x \in \text{int } \sigma$, with σ embedded, it is immediately clear from Stone’s definition that the curvature properties of the immersion at x depend solely on the formal link of σ , in our sense. To put it another way, if L is a formal link of dimension (n, k, j) , then Stone curvatures $K_-(L, d)$, $K_+(L, d)$ are defined for all “directions” d , i.e., all points $d \in \Sigma_L$. We may define $K_-(L) = \min K_-(L, d)$, $K_+(L) = \max K_+(L, d)$.

Let $B_{n,k}^+$ denote the subcomplex of $\mathcal{G}_{n,k}$ consisting of the union of all e_L such that $K_-(L) \geq 0$ and $K_-(J) \geq 0$ for all faces J of L . Similarly, let $B_{n,k}^-$ be the union of e_L such that $K_+(L) \leq 0$, $K_+(J) \leq 0$ for faces J of L . We thus have, as an immediate corollary of 3.1 for manifolds M^n having, as usual, no top-dimensional handle.

3.2 COROLLARY. *M^n immerses in R^{n+k} with everywhere nonnegative (resp. nonpositive) Stone curvature if and only if the stable normal-bundle classifying map $M^n \rightarrow BPL$ lifts to $B_{n,k}^+$ (resp., $B_{n,k}^-$).*

We now proceed to the proof of 3.1. I should like to thank the referee for some suggestions which make some of the constructions to be used simpler and more elegant.

We supplement the given cell decomposition of $\mathcal{G}_{n,k}$ via the cells e_L by a new decomposition into contractible subspaces \bar{e}_L , one for each formal link L . \bar{e}_L will neither contain nor be contained in e_L , in general. However, if C is any subcomplex of $\mathcal{G}_{n,k}$, and for some indexing set \mathcal{G} , $\{e_i\}_{i \in \mathcal{G}}$ is the set of all cells of C , then $\bigcup_{i \in \mathcal{G}} \bar{e}_{L_i} = \bar{C}$ will contain C as a deformation retract.

To define \bar{e}_L , we shall first define spaces $e_{L,J} \subseteq e_J$ where either $L = J$ or $J = L_\sigma$, for some simplex σ of Σ_L . Consider, therefore $c\Sigma_L$, which is the pre-image of e_L before indentifications. We may barycentrically subdivide e_L once to get a complex isomorphic to the cone $c\Sigma'_L$ on the first barycentric subdivision of Σ_L . We subdivide once more obtaining a complex which we call c_L . We let $c_{L,\sigma}$ be the simplicial regular neighborhood in c_L of the vertex β_σ of $c\Sigma'_L$ where β_σ is the barycenter of the simplex σ of Σ_L . We let $c_{L,L}$ be the regular neighborhood of the cone point. Now we let $e_{L,L} \subseteq e_L$ be the (homeomorphic) image of $c_{L,L}$ in e_L . We let $e_{L,J} = \bigcup_{J=L_\sigma} \text{image}(c_{L,\sigma}) \subseteq e_L$.

Finally, we let $\bar{e}_L = e_{K,L}$ where the union is taken over all K having $L = K_\tau$, τ a simplex of Σ_K , together with $K = L$.

We claim that the decomposition $\{\bar{e}_L\}$ has the property required of it, i.e. $\bar{C} = \bigcup_{e_L \subseteq C} \bar{e}_L$ contains C as a deformation retract for any subcomplex C of $\mathcal{G}_{n,k}$.

We wish to define a certain map $F: \mathcal{G}_{n,k} \rightarrow \mathbf{R}^{n+k}$ having the property that $F(\bar{e}_L) \subseteq V_L$ for all formal links L . Moreover, if we set $\gamma_L = \gamma_{n,k}|_{\bar{e}_L}$ there will be block-bundle maps $\phi_L: \gamma_L \rightarrow \nu(V_L)$, where $\nu(V_L)$ denotes the normal block bundle of V_L . The set $\{\phi_L\}$ is to be consistent in the sense that for any two links L, J , $\phi_L|_{\gamma_L \cap \gamma_J} = \phi_J|_{\gamma_L \cap \gamma_J}$.

To facilitate the construction, it is convenient to work with a slightly smaller "copy" of $E(\gamma_{n,k})$ embedded within $E(\gamma_{n,k})$. Consider once more a link L , and the cell pair $(cT_L, c\Sigma_L)$. We take the second subdivision of this pair, i.e. the barycentric subdivision of the simplicial pair $(cT'_L, c\Sigma'_L)$ to obtain the triangulated cell pair (d_L, c_L) (where c_L obviously is the triangulation used earlier in the construction of \bar{e}_L). We let $d_{L,L}$ be the simplicial regular neighborhood of $c_{L,L}$, i.e. the union of all closed simplices of d_L incident to $c_{L,L}$. We let $d_{L,\sigma}$ be the simplicial regular neighborhood of $c_{L,\sigma}$.

Now, we let

$$A_{L,L} = \text{image } d_{L,L}, \quad \text{and} \quad A_{L,J} = \bigcup_{J=L_\sigma} (\text{image } d_{L,\sigma}) \subseteq E(\gamma_{n,k}).$$

Finally, we let $A_L = \bigcup_K A_{K,L}$ where the union is over all K with $L = K_\tau$, together with $K = L$.

It is obvious that $\bar{E} = \cup_L A_L \subseteq E(\gamma_{n,k})$ is a block bundle over $\mathcal{G}_{n,k}$, i.e. it is a regular neighborhood of $\mathcal{G}_{n,k}$ in $E(\gamma_{n,k})$ and thus isomorphic, as a block bundle to $\gamma_{n,k}$ itself. Moreover, A_L is the “block” over \bar{e}_L . Thus in order to define F , and the maps $\phi_L: \gamma_L \rightarrow \nu(V_L)$, we claim that it will suffice to specify a certain map $G: \bar{E} \rightarrow \mathbf{R}^{n+k}$, where $F = G|\mathcal{G}_{n,k}$.

Now consider once more the cell cT_L triangulated as d_L . Let $\bar{d}_L = d_{L,L} \cup \cup_\sigma d_{L,\sigma}$. To define a map $\Gamma_L: \bar{d}_L \rightarrow \mathbf{R}^{n+k}$ it will obviously suffice to take an affine extension of a map on vertices. So we note first that any vertex of \bar{d}_L must either be a barycenter β_σ of some simplex σ of Σ_L , or else the cone point x of $c\Sigma_L$, or must be contiguous to x or some β_σ (i.e. contiguous in the sense of being connected by a 1-simplex). Now for such a vertex v , we denote $l(v) = L$ if v is contiguous to x . Let $l(v) = L_\sigma$ if $v = \beta_\sigma$ or if v is contiguous to β_σ , but not to x , nor to b_τ for any proper face τ of σ . Let $P_K = \{v | l(v) = K\}$ where $K = L$ or L_σ , thus partitioning the vertices of \bar{d}_L into disjoint families. Note that the largest subcomplex of \bar{d}_L containing the vertices P_L (resp. P_{L_σ}) is naturally isomorphic to the cone on the second barycentric subdivision of T_L (resp. T_{L_σ}). So let $\Gamma_L(v)$ be the natural image of v in $|T_L| = S_{U_L}$ for $v \in P_L$ (resp., $\Gamma_{L_\sigma}(v) =$ natural image of v in $|T_L| = S_{U_{L_\sigma}}$, $v \in P_{L_\sigma}$). By linear extension $\Gamma_L: \bar{d}_L \rightarrow \mathbf{R}^{n+k}$ is defined. It is easily seen that this is compatible with identifications in $E(\gamma_{n,k})$, $\mathcal{G}_{n,k}$, so that we obtain a map $G: \bar{E} \rightarrow \mathbf{R}^{n+k}$, which we may think of as the “tautologous” map on \bar{E} . (Note: $\bar{E} = \cup_L$ image \bar{d}_L .)

We now assert that, if $G_L = G|A_L$, $G_L^{-1}(V_L) \supseteq \bar{e}_L$. Moreover, the reader may easily check that G_L is PL transverse regular to V_L . Thus G_L may be regarded as a block-bundle map $\gamma_L \rightarrow \nu(V_L)$. That is, we have a bundle locally defined over \bar{e}_L as $F^*(\nu(V_L))$, and it is easily seen that a tube representing this bundle embeds in $\bar{E} \subseteq E(\gamma_{n,k})$. Thus this “local” block bundle pieces together to form a bundle isomorphic to $\gamma_{n,k}$ and the set $\{\phi_L\}$ is defined in the obvious way.

We now consider a manifold M^n with a reduction of its stable normal block-bundle to a k -block bundle ν_k , such that there is a bundle map $\theta: \nu_k \rightarrow \gamma_{n,k}|B$ where B is a geometric subcomplex. We replace B by \hat{B} , the union of all \bar{e}_L for e_L a cell of B . Let $h: M^n \rightarrow \hat{B}$ be the map covered by θ . By general position considerations (i.e., a series of codimension-one transversality arguments) it is trivial to show that, for $e_L \subset B$, i.e., $\bar{e}_L \subseteq \hat{B}$, $h^{-1}(\bar{e}_L)$ may be assumed to be a codimension-0 submanifold of M^n . Denote this manifold by M_L . For $e_L \not\subset B$, $M^{-1}(\bar{e}_L) = \emptyset$. It is also easy to see that $M_L \cap (\cup_{e_j \subset e_L} M_j)$ is a codimension-0 submanifold of ∂M_L .

Now we consider $g = F \cdot h: M \rightarrow \mathbf{R}^{n+k}$. Note that $g(M_L) \subseteq V_L$, $g(M_L \cap M_j) \subseteq V_{(j,L)}$ for e_L a face of e_j . Clearly, by composing ϕ_L with $\theta|M_L$ we obtain maps $G_L: \nu_k/M_L \rightarrow \nu(V_L)/V_L$ covering $g|M_L$. Moreover, on $M_L \cap M_j \neq \emptyset$, G_L and G_j coincide as maps to $V_j \cap V_L$. But, by stability

considerations for bundles, we must also then have maps $G'_L: \tau(M_L^\circ) \rightarrow \tau(V_L)$ where τ now denotes the PL tangent bundle, and where M_L° denotes M_L minus a disc in the interior of each component. Now we apply the PL version of Hirsch's immersion theorem [HP], stepwise, to deform g so that each $g|M_L^\circ$ is an immersion (codimension 0) of M_L in V_L .

Inductively, one does this by starting with M_L° for $\dim e_L = 0$, and deforming g to an immersion, keeping $M_L^\circ \cap M_J^\circ$ in $V_{(J,L)}$ for each J such that $e_J \supset e_L$. Then, one immerses M_L° in V_L , for $\dim e_L = 1$, extending the given one on $M_L^\circ \cap (\bigcup_{\dim e_K=0} M_K^\circ)$, again using a relative version of the immersion theorem. We proceed, inductively, to immerse M_L° corresponding to successively higher-dimension e_L , in their respective V_L , deforming the underlying map $g|(M^n\text{-discs})$ each time as necessary. In the end, we have made g a codimension- k immersion of M^n -(discs) into R^{n+k} , with each M_L° being immersed in V_L .

Moreover, it is easy to arrange that for $M_L \cap M_J \neq \emptyset$ (i.e., L a face of J or vice versa) the immersion takes $M_L^\circ \cap M_J^\circ$ into the interior of $V_L \cap V_J$.

Now we have immersed M_0^n , i.e. a multiply-punctured M^n but, since M^n admitted a handle decomposition with no n -handle, M_0^n contains a smaller copy of M^n as a codimension-0 submanifold; thus this copy is immersed. Let M'_L be the intersection of this smaller copy with M_L° .

Now triangulate R^{n+k} so that the immersion is a simplicial map on M^n and all the M'_L . We claim that the Gauss map thus engendered, $M^n \rightarrow \mathcal{G}_{n,k}$, must have its image in B . For let S be an arbitrary simplex σ of M^n , and let $P = \{L|\sigma \subseteq M'_L\}$. This set is linearly ordered by the face relation, and thus, if L_1 is the maximal element, then a neighborhood of σ must be embedded in V_{L_1} since a neighborhood of σ in M'_L is embedded in $V_L \cap V_{L_1}$ for all L in P .

Therefore the formal link J of σ (defined by the triangulation of R^{n+k}) must have the property mentioned in property (i) above, i.e., V_J must coincide with $V_{J'}$ for some face J' of L . Thus, since B is geometric, $e_J \subseteq B$.

This completes the proof that if the normal bundle of M^n is induced by a map to B , then M^n immerses in R^{n+k} with a Gauss map going into B .

The converse is, of course, trivial.

4. Appendix. (I) We briefly discuss here the smooth analogue of the problem addressed by Theorem 3.1. That is, given some condition on immersions that is "locally" defined, when may M^n be immersed in R^{n+k} satisfying this condition. (It may be helpful to think of a typical condition of interest, e.g. positive (or negative) curvature.) We will briefly indicate why, for n -dimensional non-closed manifolds, this problem is, in some (not very useful) sense, a homotopy problem.

With some local condition for smooth immersions of n -manifolds in R^{n+k} in mind, consider the set consisting of all pairs (V^n, k) where V^n is some

manifold and $h: V^n \rightarrow \mathbf{R}^{n+k}$ is some immersion of V^n satisfying the condition. We consider the union U of all such V^n as a space and let γ be the tangent bundle of this “nonparacompact” manifold. Clearly if M^n is a nonclosed manifold and there is a bundle map $\tau(M^n) \rightarrow \gamma$, then there is a tangential map of M^n to some V^n where V^n immerses via h , to satisfy the given condition. But then, by Hirsch’s theorem M^n immerses in V^h via some map g , and thus M^n immerses in \mathbf{R}^{n+k} , via $h \circ g$ so as to satisfy the condition. The foregoing falls under the heading of facts which are true, but not very interesting.

On the other hand the “geometric” subcomplexes of $\mathcal{G}_{n,k}$ studied in §3 are interesting precisely because they are given as C-W complexes determined purely by “local” data. That is, it seems entirely feasible that, for interesting examples B of such subcomplexes ($B_{n,k}^+$ and $B_{n,k}^-$, for instance), the algebraic topological questions that naturally arise may be answerable. (That is, it might well be both possible and useful to compute the homotopy groups of the fiber of $B \rightarrow \text{BPL}$ and to determine relations on the characteristic classes of $\gamma_{n,k}(B)$.)

On the other hand, for smooth manifolds and immersions, I know of no such analogous possibilities. The U, γ constructed above (or variants thereof) does not seem to hold out much promise of being understandable in terms of algebraic topology.

(II) Other versions of $\mathcal{G}_{n,k}$: Stability.

We first observe that there are natural mappings

$$\alpha: \mathcal{G}_{n,k} \rightarrow \mathcal{G}_{n+1,k}, \quad \beta: \mathcal{G}_{n,k} \rightarrow \mathcal{G}_{n,k+1}.$$

The map α is completely obvious; a formal link L of dimension (n, k, j) is, by inspection of the definition, also a formal link of dimension $(n + 1, k, j)$. This induces the inclusion of complexes α . As for β , consider a formal link $L = (U_L, T_L, \Sigma_L)$ having dimension (n, k, j) . To L we associate the $(n, k + 1, j)$ -dimensional link J constructed as follows. If $\mathbf{R}^{n+k+1} = \mathbf{R}^{n+k} \oplus \mathbf{R}^1$ in a standard way let $U_J = U_L \oplus \mathbf{R}^1$; let T_J be isomorphic with the suspension of T_L . That is, there is a natural homeomorphism of suspension $|T_L|$ to S_{U_J} which is the identity on the “equator” S_{U_L} and which takes the suspension points to the “north” and “south” poles of S_{U_J} . Let $\Sigma_J = \Sigma_L$. Again the assignment $L \rightarrow J$ defines a map of complexes $\mathcal{G}_{n,k} \rightarrow \mathcal{G}_{n,k+1}$.

Thus it is natural to ask whether the limit of the double sequence

$$\begin{array}{ccccccc} \cdots & \rightarrow & \mathcal{G}_{n,k} & \xrightarrow{\alpha} & \mathcal{G}_{n+1,k} & \rightarrow & \\ & & \downarrow \beta & & \downarrow \beta & & \\ \cdots & \rightarrow & \mathcal{G}_{n,k+1} & \xrightarrow{\alpha} & \mathcal{G}_{n+1,k+1} & & \\ & & \downarrow & & \downarrow & & \end{array}$$

is $\widetilde{\text{BPL}} = \text{BPL}$.

The answer does not seem to be an unqualified “yes”.

There are still other complexes which may be thought of as PL Grassmannians for certain purposes. For instance, we might consider what happens, if we were to define a “formal link” of dimension (n, k, j) by a pair (U_L, Σ_L) where U_L is as before, a $j + k$ -plane of \mathbf{R}^{n+k} and where Σ_L is, as before, a curvilinear triangulation of some $(j - 1)$ -dimensional subsphere of S_{U_L} . That is, consider two formal links (in the original sense) $L = (U_L, T_L, \Sigma_L)$, $L' = (U_{L'}, T_{L'}, \Sigma_{L'})$ to be identical if $U_L = U_{L'}$, $\Sigma_L = \Sigma_{L'}$ (i.e. discount T_L on the complement of Σ_L). Let a formal link (in the new sense) be an equivalence class of formal links (in the old sense). We then may construct $\mathcal{H}_{n,k}$ from these new formal links, with one j -cell for each (h, k, j) -dimensional link. There is a natural map $\mathcal{G}_{n,k} \rightarrow \mathcal{H}_{n,k}$ coming from the equivalence relation on old-style formal links. However, it is no longer natural to consider block bundles over $\mathcal{H}_{n,k}$. On the other hand we claim that there is a natural n -dimensional PL microbundle $\mathcal{O}_{n,k}$ over $\mathcal{H}_{n,k}$ and that the Gauss map $M^n \rightarrow \mathcal{H}_{n,n}$ is naturally covered by a map from the tangent microbundle of M^n to $\mathcal{O}_{n,k}$. Moreover, a version of Theorem 3.1 may be proved for subcomplexes of $\mathcal{H}_{n,k}$.

Finally, we may want to consider what happens when we retopologize $\mathcal{H}_{n,k}$ in the following way: Consider two points x, y as being ϵ -close if x is the image of \bar{x} in $c\Sigma_L$, y is the image of \bar{y} in $c\Sigma_J$ where L, J are of the same dimension, and where \bar{x} is ϵ -close to \bar{y} in \mathbf{R}^{n+k} , and Σ_L is ϵ -close to Σ_J , as closed subsets of \mathbf{R}^{n+k} , as are S_{U_L} and S_{U_J} . Another way of putting this is as follows. Consider the first barycentric subdivision of $\mathcal{H}_{n,k}$ (which is a simplicial complex). Consider the set of j -simplices.

Each such σ is the image of $\bar{\sigma}$, a simplex of the cone on the first derived subdivision of Σ_L for some L . Put a metric on this set which makes σ close to τ when $\bar{\sigma}$ is close to $\bar{\tau}$ in \mathbf{R}^{n+k} , and the corresponding (Σ_L, Σ_J) , (S_{U_L}, S_{U_J}) are also close. The metric converts the abstract simplicial complex to a simplicial space, and its geometric realization $\mathcal{G}_{n,k}$ is a new topology on $\mathcal{H}_{n,k}$.

$\mathcal{G}_{n,k}$ is a Grassmannian for certain kinds of piecewise-differentiable immersions. A theorem analogous to 3.1 holds for it as well. Moreover, there are equivariant versions of 3.1 for finite orthogonal actions on \mathbf{R}^{n+k} , (which induce actions on $\mathcal{G}_{n,k}$ (resp., $\mathcal{H}_{n,k}$, $\mathcal{G}_{n,k}$) in an obvious way). These results concern equivariant immersions of manifolds in \mathbf{R}^{n+k} . Finally, there are analogues to the notion of the $G_{n,k}$ bundle associated to the tangent bundle of a smooth manifold. In the case of a triangulated manifold W^{n+k} , it is possible to construct a complex $\mathcal{H}_{n,k}(W)$ which receives the Gauss map of a PL immersion $M^n \rightarrow W^{n+k}$ having certain properties. Furthermore, for a smooth Riemannian manifold W^{n+k} , one may speak of an associated $\mathcal{G}_{n,k}$ -bundle which receives the Gauss map from a manifold M^n piecewise-

differentiably immersed in W^{n+k} . These generalizations will be dealt with in future papers.

With these other "Grassmannians" in mind, I shall finally note that the $\mathcal{G}_{n,k}$ construction was emphasized in §1–3 since it is the simplest, most elegant, and most directly geometric.

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