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SUBGROUPS OF CLASSICAL GROUPS GENERATED BY LONG ROOT ELEMENTS[1]

BY

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ABSTRACT. All conjugacy classes of subgroups G of classical groups of characteristic p are determined, which are generated by a conjugacy class of long root elements and satisfy $O_p(G) < G' \cap Z(G)$.

1. Introduction. The purpose of this paper is to determine all the conjugacy classes of subgroups G of a classical group of characteristic p, which are generated by a conjugacy class $\mathfrak{X}$ of elements of long root groups, and satisfy $O_p(G) \leqslant G' \cap Z(G)$. Here, $\mathfrak{X}$ consists of transvections in the case of special linear, symplectic and unitary groups. In the case of orthogonal groups, if the dimension is at least 5 then each $x \in \mathfrak{X}$ is an element of order 1 or p centralizing $A^{\perp}$ for some totally singular 2-space A.

The resulting classification is presented in §§2 and 11. The lengths of the lists are due to our not having assumed the irreducibility of G.

These results are a first step towards the determination of all subgroups of Chevalley groups generated by elements of long root groups. Comments on this general problem, along with some examples, are presented in §12.

A number of related results have already been proved. McLaughlin **[18]**, **[19]**, Piper **[22]**, **[23]**, Wagner **[34]**, **[35]**, Pollatsek **[26]**, and Key **[16]** have studied irreducible groups generated by transvections, settling all but the case in which each axis arises from exactly one nontrivial transvection. Stark **[27]** studied subgroups of odd characteristic orthogonal groups generated by entire long root groups; however, she missed one class of examples (see (I 4) in §2). The results of Thompson **[32]** and Ho **[13]**, **[14]** on quadratic pairs produce a characterization of the possible groups G, provided $p > 2$ and G is irreducible; however, these results do not provide information concerning which groups can be embedded in which others. Moreover, reducibility and characteristic $p = 2$ allow a number of interesting examples, related to indecomposability and cohomological questions.

Our proof is quite different from those of the above references, in that we start by knowing the structure of G. This is accomplished by quoting very difficult classification theorems due to Fischer **[7]**, Aschbacher **[1]**, **[2]**, and

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Timmesfeld [33]. More generally, these apply immediately to the aforementioned general problem concerning Chevalley groups. While it would perhaps be preferable to obtain a direct approach to such problems, a significant reduction in labor is provided by knowing G, and then studying its embeddings, as opposed to what is almost the reverse point of view. However, these classification theorems are of much less value when $O_\infty(G) > Z(G)$–and in fact we have not used them in this case, relying instead on the given module. Another way in which the module can provide simplifications is through the use of results of Steinberg [29]. However, we have not assumed irreducibility, and not even the degrees of the basic modules of most Chevalley groups are known; moreover, we are interested in embeddings in $\Omega(V)$, not just $GL(V)$.

§§3 and 4 contain preliminary results and notation. The proof of the main theorem occupies §§5–10.

Most group-theoretic notation will be standard. $G^\# = G - \{1\}$; $O_\infty(G)$ is the largest solvable normal subgroup of G; $A \rtimes B$ is the semidirect product of A and B, with A normal; $n \cdot G$ is an extension of G by a group of order n. Let G act on the vector space V. If W is a subspace of V, then G_W is its stabilizer, $G_W^W = G_W / C_G(W)$ is the group induced on W, and $[G, W] = [W, G] = \langle wg - w | w \in W, g \in G \rangle$; similar notation applies even when W is the quotient of two subspaces. Further notation will be found in the next section.

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2. Examples. While notation for special linear groups is standard, that for orthogonal ones is less so. Ours will be as follows. Let $V = V(n, q)$ be an n-dimensional vector space over $GF(q)$. Equip V with a nondegenerate quadratic form Q, along with the associated symmetric form (,). Thus, rad $V = V \cap V^\perp$ is 0, except when q is even, n is odd, dim rad $V = 1$, and $Q(\text{rad } V) \neq 0$. A subspace W of V is *totally singular* if $Q(W) = 0$, *nondegenerate* if Q restricted to W is nondegenerate, and *nonsingular* if it is nondegenerate and either rad $W = 0$ or dim $W = 1$. A vector $v \in V$ is *singular* if $Q(v) = 0$, *nonsingular* if $Q(v) \neq 0$.

$O(V)$ is the group of all linear transformations preserving Q. This is denoted $O^+(n, q)$ or $O^-(n, q)$ when n is even and Q has index $\frac{1}{2}n$ resp. $\frac{1}{2}n - 1$; while $O(n, q) = O^+(n, q) = O^-(n, q)$ when n is odd (a convenient notation for future use). Excluding the cases $O^\pm(2, q)$, $O(3, q)$ and $O^-(4, q)$, a *long root element* is an $x \in O(V)$ of the form $(v)x = v - (v, a)b + (v, b)a$ for a, b in a totally singular 2-space T; if $T = \langle a, b \rangle$ then $x \neq 1$, and T is denoted $A(x)$. The group X of all long root elements corresponding to T is a *long root group*. Set $\langle X^{O(V)} \rangle = \Omega(V)$ (alias $\Omega^\pm(n, q)$). $P\Omega(V)$ and $P\Omega^\pm(n, q)$ are now defined as usual.

$$(v)t_a = v - (v, a)Q(a)^{-1}a.$$

If q is odd, t_a is a reflection; write $\Omega^{\pm,\pi}(V) = \Omega^{\pm}(V)\langle t_a\rangle$ with $\pi = +$ if $Q(a)$ is a square, and $-$ otherwise. $P\Omega^{\pm,\pi}(V)$ is defined similarly; note that $P\Omega^{\pm,+}(2n, q) \cong P\Omega^{\pm,-}(2n, q)$. The notation $P\Omega^{\pm,\pi}(2n+1, q)$, which is especially redundant when $-t_a \in \Omega(2n+1, q)$, will be used to indicate which class of reflections is under consideration.

If q is even, then t_a is a transvection and $O(V)$ is generated by these transvections (except for $O^+(4, 2)$). Also, $t_a \in \Omega(V)$ if and only if n is odd, in which case $\Omega(V) = Sp(V/\mathrm{rad}\ V)$, $\{t_b | b \in \langle a, \mathrm{rad}\ V\rangle, Q(b) \neq 0\}$ induces the group of q transvections of $V/\mathrm{rad}\ V$ with direction $\langle a, \mathrm{rad}\ V\rangle/\mathrm{rad}\ V$, and the long root groups of $\Omega(V)$ become short root groups of $Sp(V/\mathrm{rad}\ V)$.

There are natural isomorphisms $P\Omega(5, q) \cong PSU(4, q)$ sending long root elements to transvections. Also, $PO^{\pm,\pi}(5, 3) \cong PSU(4, 2)\langle\sigma\rangle$ with σ a field automorphism, $P\Omega^{-,\pi}(4, 3) \cong S_6$, and $PO^-(4, q) \cong PGL(2, q^2)\langle\sigma\rangle$ with σ an involutory field automorphism.

We will be concerned with the following examples of subgroups G of $\Omega(V)$ generated by long root elements. In each case, there are no proper G-invariant subspaces V_1, V_2 with $V = V_1 \perp V_2$. In each case, we give the number of conjugacy classes of embeddings of G in $\Omega(V)$ of the stated type, unless there is just one class.

Examples irreducible on $V/\mathrm{rad}\ V$.

(I 1) $G = \Omega^{\pm}(n, q)$.

(I 2) $G = \Omega^-(2n, q) < \Omega^+(2n, q^2)$, obtained by extending $GF(q)$ to $GF(q^2)$. (Alternatively, this is the natural embedding obtained by twisting $\Omega^+(2n, q^2)$.)

(I 3) $G = SU(2n, q) < \Omega^+(4n, q)$ or $G = SU(2n+1, q) < \Omega^-(4n+2, q)$. Here, $Q(v) = \mathrm{trace}[v, v]$, where [,] denotes a nondegenerate hermitian form on $V(2n, q^2)$ and trace refers to the trace map $GF(q^2) \to GF(q)$. For $SU(2n, q)$ there are two such embeddings, conjugate in $O^+(4n, q)$, corresponding to the two possibilities for the class of maximal totally singular subspaces for Q which contains the maximal totally isotropic subspaces for [,].

(I 4) $G/Z(G) = P\Omega(7, q)$, $|Z(G)| = (2, q-1)$, $G < \Omega^+(8, q)$. There are two classes of such embeddings, conjugate in $O^+(8, q)$, arising from the spin representation of $\Omega^+(8, q)$. The embeddings of $G/Z(G)$ are conjugate to the usual embedding $P\Omega(7, q) < P\Omega^+(8, q)$ in Aut $P\Omega^+(8, q)$.

(I 5) $G = O^{\pm}(4, q) < \Omega(5, q)$ for q even. These are obtained by taking the usual embedding of $O^{\pm}(4, q)$ in $\Omega(5, q)$, and then applying the graph automorphism. The involutions of the usual embedding which are short root elements are transformed into long root elements. ($O^+(4, 2)$ must be excluded.)

(I 6) $G = G_2(q)' < \Omega(7, q)$, using the standard embedding of $G_2(q)$.

(I 7) $G = {}^3D_4(q) < \Omega^+(8, q^3)$, using the standard embedding.

(I 8) $G = HJ < G_2(4) < \Omega(7, 4)$. Here, there are two classes of embeddings of the Hall-Janko group HJ in $G_2(4)$, conjugate under the field automorphisms of both $G_2(4)$ and $\Omega(7, 4)$.

(I 9) $G = 3 \cdot P\Omega^{-,\pi}(6, 3) < GU(6, 2) < \Omega^+(12, 2)$. The latter embedding is as in (I 3). There is exactly one possibility for the former embedding (Fischer [7, 16.1.12]).

(I 10) $G = A \rtimes S_n < \Omega^{\pm}(2n, q)$, with q even, $+$ for n even and $-$ for n odd, and A the direct product of $n - 1$ cyclic subgroups of order $a|q + 1$. This is obtained by writing $V = V_1 \perp \cdots \perp V_n$ with each V_i a 2-dimensional space having no nonzero singular vectors. Then let A_0 be the direct product of n cyclic groups of order a fixing the above decomposition, let S_n permute the V_i's, and set $A = [A_0, S_n]$.

Note that S_n splits V as $W_1 \oplus W_2$ with W_i totally singular, and transpositions thus act as long root elements. One example of such a group G arises with $n = a = 3$ and $|G| = 3^3 \cdot 2$, $O_3(G)$ being extraspecial; in fact, $G = SU(3, 2)'$.

(I 11) $G = (A \rtimes E) \rtimes S_n < \Omega^{\pm}(2n, q)$, with q even, A the direct product of n cyclic groups of order $a|q \pm 1$, and E an elementary abelian 2-group of order 2^{n-1}. This is obtained by writing $V = V_1 \perp \cdots \perp V_n$ with the V_i isomorphic nonsingular 2-spaces. Then let A be the direct product of n cyclic groups of order a, one per V_i. Let E_0 be the group generated by n transvections, one per V_i, so that AE_0 induces a dihedral group on each V_i, and set $E = E_0 \cap \Omega(V)$. Finally, S_n permutes the V_i's. A typical example is the subgroup N of BN fame.

Examples having G fixing maximal totally singular subspaces W_1, W_2 with $V = W_1 \oplus W_2$. The actions of G on W_1 and W_2 are contragredient. If $\dim W_i$ is even, then there will be two classes of embeddings of each type, fused in $O(V)$.

(RT 1) $G = SL(n, q)$ or $Sp(n, q)$ in $\Omega^+(2n, q)$, or $G = SU(n, q)$ in $\Omega^+(2n, q^2)$.

(RT 2) $G = O^{\pm}(n, q) < \Omega^+(2n, q)$, q even.

(RT 3) $G = S_n$, $n \geqslant 6$, embedded in $\Omega^+(2n - 2d, 2)$; $d = (2, n)$. The action of G on W_1 can be described as follows. Regard G inside $SL(n, q)$, consisting of permutations of a basis $u_1, \ldots, u_n$ of $V_n = V(n, 2)$. Then transpositions are transvections. There are exactly two proper invariant subspaces: $V_1 = \langle \Sigma u_i \rangle$ and $V_{n-1} = \{\Sigma a_i u_i | \Sigma a_i = 0\}$, where $V_1 < V_{n-1}$ iff n is even. Now $W_1 = V_{n-1}/(V_{n-1} \cap V_1)$.

Note that this even embeds G in an orthogonal group on V_n: define the quadratic form Q' by $Q'(u_i) = 1$ and $(u_i, u_j) = 1$ for $i \neq j$, and note that G

preserves Q'. Hence, $G \leqslant O^{\pm}(V_n)$, the sign depending on the residue class of n (mod 4).

Note also that G acts indecomposably on V_n if n is even. This leads directly to (RT 4):

(RT 4) $G = S_{2n} < \Omega^+(4n, 2)$ and $\Omega^+(4n-2, 2)$, $2n \geqslant 6$.

(RT 5) $G = 3 \cdot A_6 < \Omega^+(6, 4)$.

(RT 6) $G = SL(2, 5) < SL(2, 9) < \Omega^+(4, 9)$.

(RT 7) $G = 3 \cdot P\Omega^{-,\pi}(6, 3) < GU(6, 2) < \Omega^+(12, 4)$ as in (I 9).

(RT 8) $G = SU(4, 2) < GL(5, 4) < \Omega^+(10, 4)$. To construct this, let $x \in L = 3 \cdot P\Omega^{-,\pi}(6, 3) < GU(6, 4)$ as in (I 9), with x a transvection. Then $C_L(x) = \langle x \rangle \times G_0 \times Z(L)$, with $\langle x \rangle \times G_0$ generated by transvections and $G_0 \cong SU(4, 2)$. Now regard G as G_0 acting on the axis W_1 of x. By Fischer [7, 16.1.10], there is just one class of indecomposable embeddings $G < GL(5, 4)$.

Note that this embedding is equivalent to its contragredient: an element of $\Omega^+(10, 4)$ interchanging W_1 and W_2 can be chosen to normalize G and hence induce its graph automorphism.

(RT 9) $G = A \rtimes S_n < SL(n, 2^i) < \Omega^+(2n, 2^i)$, corresponding to (I 10). A typical example is a monomial subgroup of $SL(n, 2^i)$. Note that A has an element inducing a scalar transformation on W_i of order (a, n).

Examples having G reducible on $V/\text{rad } V$ in which G is contained in no Levi factor.

(RL 1) $G = \Omega(2n-1, q) < \Omega^{\pm}(2n, q)$, q even and $n \geqslant 3$.

(RL 2) $G = O^{\pm}(4, q) < \Omega(5, q) < \Omega^{\pm}(6, q)$, q even, the former embedding being (I 5).

(RL 3) $G = G_2(q)' < \Omega^{\pm}(8, q)$, q even.

(RL 4) $HJ < \Omega^{\pm}(8, 4)$. There are two classes, conjugate by a field automorphism.

(RL 5) $G = SU(4, 2) < \Omega^+(10, 2)$. This is obtained by taking a transvection x in $L = 3 \cdot P\Omega^{-,\pi}(6, 3)$, regarding $L < GU(6, 2) < \Omega^+(12, 2)$ as in (I 9), and letting $G = C_L(x)'$ act on $W^{\perp}/W$ for a 1-space W of $A(x)$. Note that $V \otimes_{GF(2)} GF(4)$ yields (RT 8).

(RL 6) $G = SU(4, 2) < \Omega^+(10, 4)$. This is more complicated. There are exactly two totally singular invariant 5-spaces W_1, W_2, with $W_1 \cap W_2 = R$ of dimension 1 and G acting on $R^{\perp}/R$ as in (RT 1). There is a single class of embeddings in $O^+(10, 4)$ (two in $\Omega = \Omega^+(10, 4)$), constructed as follows.

Set $P = O_2(\Omega_R)$. Let $L = SU(4, 2)$ act on $R^{\perp}/R$ as in (RT 8), with L centralizing a nonsingular 2-space of $V = V(10, 4)$. Let L fix the totally singular 4-spaces W_i/R of $R^{\perp}/R$, and let $P_i < P$ correspond to W_i/R as in (4.6). Then $|P_i| = 4^4$ and $P = P_1 \times P_2$. Since the P_i are contragredient for L

(i.e., are obtained from one another by a field automorphism), dim $H^1(L, P_i) = 1$ by Fischer [7, (16.1.10)]. Thus, dim $H^1(L, P) = 2$. There are then 4^2 classes of complements to P in LP. Note that $C = C_{\Omega_R}(L)$ has order 9 and acts on $H^1(L, P)$. (One group of order 3 occurs inducing scalars on $R^{\perp}/R$, and another occurs centralizing R and inducing scalars on W_1/R.) Since C normalizes LP_i, it has just 4 orbits on $H^1(L, P)$, of lengths 1, 3, 3, 9. The first yields L; the next two both yield (RT 8); the last yields the desired example.

In order to demonstrate that a group G obtained in this manner is generated by long root elements of Ω, we only need to explicitly construct one such group. Set $U = V(12, 4)$, and let U_1, U_2 be totally singular 6-spaces for $\Omega^+(U)$ with $U = U_1 \oplus U_2$. Let $K = \langle x \rangle \times SU(4, 2)$ act as a subgroup of $\Omega^+(U)$, fixing U_1, U_2, and acting indecomposably on each as a group generated by transvections, with the representations K^{U_i} contragredient; $D_i = [x, U_i]$ is the direction of the transvection x^{U_i}. (For the existence of these representations, use $3 \cdot P\Omega^{-,\pi}(6, 3)$ again.) Let D denote any 1-space in $D_1 + D_2 = C_U(K)$. Then K acts on the 10-spaces $D^{\perp}/D$ as $SU(4, 2)$, and is generated by long root elements. In each of them, R is $(D_1 + D_2)/D$. The cases $D = D_1$ or D_2 are just (RT 9). Assume $D \neq D_1, D_2$. Then $(D + U_i \cap D^{\perp})/D$, $i = 1, 2$, are the only fixed 5-spaces in $D^{\perp}/D$, and both contain R. This produces the desired example.

Our main result is

THEOREM I. *Let G be a subgroup of $\Omega(V) = \Omega^{\pm}(m, s)$ generated by a conjugacy class of long root elements, such that $O_p(G) \leqslant G' \cap Z(G)$ for $p|s$. Assume that either* dim $V \geqslant 5$ *or* $\Omega(V) = \Omega^+(4, s)$, *and that $V = V_1 \perp V_2$ with V_1, V_2 invariant under G and V_2 centralized by G implies that $V_2 = 0$. Then G acts on V as one of* (I 1–11), (RT 1–9), *or* (RL 1–6), *tensored with $GF(s)$.*

Note that the indicated tensor product merely amounts to extending the field of definition of G, V, and the form on V.

Since $SL(m, s)$ is contained in $\Omega^+(2m, s)$, fixing totally singular m-spaces as in (RT 1–9), we obtain

THEOREM II. *Suppose G is a subgroup of $SL(W)$ generated by a conjugacy class of transvections, such that $O_p(G) \leqslant G' \cap Z(G)$ with p the characteristic of W. Then $W = W_0 \oplus T$ with G trivial on T and indecomposable on W_0, such that G acts on W_0 as one of the following tensored with $GF(s)$.*

(T 1) *$G = SL(n, q)$ or $Sp(n, q)$ in $SL(n, q)$, or $G = SU(n, q)$ in $SL(n, q^2)$.*

(T 2) *$G = O^{\pm}(n, q) < SL(n, q)$, q even.*

(T 3) *$G = S_n < SL(n - d, 2)$, $d = (2, n)$.*

(T 4) *$G = S_{2n}$ in $SL(2n - 1, 2)$ fixing a 1-space or a $2n$-space, or in $SL(2n, 2)$ fixing a 1-space and a $2n - 1$-space.*

(T 5) $G = 3 \cdot A_6 < SL(3, 4)$.
(T 6) $G = SL(2, 5) < SL(2, 9)$.
(T 7) $G = 3 \cdot P\Omega^{-,\pi}(6, 3) < SL(6, 4)$.
(T 8) $G = SU(4, 2) < SL(5, 4)$ *fixing a* 1-*space or a* 4-*space.*
(T 9) $G = A \rtimes S_n$ *belongs to a Borel subgroup of* $SL(n, 2^i)$.

Note that there are six classes of embeddings (T 7), all conjugate in $\Gamma L(6, 4)$. Similarly, there are three classes for (T 5), fused in $GL(3, 4)$.

3. Long root groups of Chevalley groups. The following useful lemma is essentially (12.1) of [**3**].

LEMMA 3.1. *Let G be any Chevalley group of rank at least* 2, *other than* ${}^2F_4(q)$. *Let X and Y be centers of distinct long root groups, of order q. Then one of the following holds*:

(i) $\langle X, Y\rangle$ *is elementary abelian, and* $\langle X, Y\rangle$ *is the union of* $q + 1$ *long root groups*;

(ii) $\langle X, Y\rangle$ *is elementary abelian, and* $X \cup Y$ *is its set of long root elements*;

(iii) $\langle X, Y\rangle$ *is isomorphic to a Sylow subgroup of order* q^3 *in* $SL(3, q)$, $Z = Z(\langle X, Y\rangle)$ *is a conjugate of X, and XZ is the union of conjugates of X*; *or*

(iv) $\langle X, Y\rangle \cong SL(2, q)$ (*or* $PSL(2, q)$ *if G is* $P\Omega^+(4, q)$).

THEOREM 3.2. *Let M be a Chevalley group defined over a field of characteristic p*; *exclude the Suzuki and Ree groups. Suppose G is a subgroup of M generated by a conjugacy class* $\mathfrak{X}$ *of elements central in long root groups. Let* $N = O(G)$ *if* $p > 2$, *and* $N = O_\infty(G)$ *if* $p = 2$, *and assume* $G > N$. *Then* G/N *and* $\bar{\mathfrak{X}} = \mathfrak{X}N/N$ *are one of the following*:

(i) *A Chevalley group of charactreristic p, with* $\bar{\mathfrak{X}}$ *a class of central elements of long root groups* (*or, in the case of* $Sp(4, 2^i)$, $F_4(2^i)$ *or* $G_2(3^i)$, *short root elements*);

(ii) $SL(2, 5)$ *or* $PSL(2, 5)$, *with M defined over* $GF(9^i)$, $\bar{\mathfrak{X}}$ *the class of elements of order* 3;

(iii) $O^{\pm}(n, 2^i)$, *where* $\bar{\mathfrak{X}}$ *consists of all transvections*, $p = 2$;

(iv) $\Omega^+(8, 2^i) \cdot S_3$, *where the* S_3 *is generated by the triality automorphism and a transvection x, and* $\bar{\mathfrak{X}}$ *is the conjugacy class of x*; *or* $P\Omega^+(8, r) \cdot S_3$, $r = 3$ *or* 5, *obtained similarly using a reflection x*;

(v) $P\Omega^{\pm,\pi}(n, r)$, $r = 3$ *or* 5, $p = 2$, *with* $\bar{\mathfrak{X}}$ *the class of reflections corresponding to* π;

(vi) S_n, $n > 6$, $\mathfrak{X}$ *the class of transpositions*;

(vii) F_{22}, F_{23}, F_{24}, $\mathfrak{X}$ *the class of* 3-*transpositions*;

(viii) A_6, $\mathfrak{X}$ *the class of involutions*;

(ix) HJ, $\mathfrak{X}$ *the class of central involutions of the Hall-Janko group HJ*;

(x) $PSL(2, q) \wr S_n$, $\mathfrak{X}$ *consisting of preimages of transpositions in this wreath product.*

PROOF. If $p = 2$, this follows easily from Fischer [7], Aschbacher [1] and Timmesfeld [33]. We therefore assume that $p > 2$, and also that M is universal.

Let S be an $SL(2, s)$ of M generated by central elements of long root groups, with $J = S \cap G$ as large as possible. Then J is $SL(2, q)$, where $q|s$, or else $q = 5$ and $s = 9^i$ [9, p. 44]. Each p-element of J is in a unique center of a long root group, so $N_G(J) = N_G(S)$. It follows that G satisfies Hypothesis Ω of Aschbacher [2]. Moreover, $G = \langle \mathfrak{X} \rangle = \langle J^G \rangle$. Thus, by [2], $\bar{G} = G/N$ is one of the groups on the above list, or else one of the following holds for $\bar{G}$ and $\bar{\mathfrak{X}}$: (α) $\bar{G}$ is a Chevalley group over $GF(5)$, and $\bar{\mathfrak{X}}$ is a class of elements of order $p = 3$; (β) $\bar{G} = M_{11}$, $\bar{\mathfrak{X}}$ is the class of elements of order $p = 3$; or (γ) $\bar{G} = G_2(q)$ or ${}^3D_4(q)$, and $\bar{\mathfrak{X}}$ consists of all short root elements.

(α) Suppose there is $x \in (G - J) \cap \mathfrak{X}$ such that $L = \langle x, J \rangle$ satisfies $\bar{L} = \bar{P} \rtimes \bar{J}$ for a 5-group $\bar{P} \nleq Z(\bar{L})$. Choose $y \in J \cap \mathfrak{X}$ and a 5-element $g \in P$ with $[y, g] \notin N$. Then $\langle y, y^g \rangle$ contradicts (3.1).

Thus, $\bar{G}$ can only be $SL(2, 5^i)$, and (3.1) again fails if $i \geq 1$.

(β) Here there exist two elements of order 3 generating $\bar{G}$.

(γ) The commutator relations contradict (3.1) except for the case $G_2(3^i)$.

COROLLARY 3.3. *With the hypotheses of* (3.2), *if* $p > 2$ *then* $O_\infty(G/O_p(G)) = Z(G/O_\infty(G))$, *except possibly if* $p = 3$ *and* $O_2(G/O_3(G)) \nleq Z(G/O_3(G))$.

PROOF. We continue with the notation of the preceding proof. If $x \in \mathfrak{X}$ and $n \in N$, then $L = \langle x, x^n \rangle \leqslant \langle x \rangle N$ and (3.1) imply that either L is a p-group or else $p = 3$ and L is $SL(2, 3)$.

REMARK. Assume in (3.3) that $p = 3$ and $O_2(\tilde{G}) \nleq Z(\tilde{G})$, where $\sim$ denotes images modulo $O_3(G)$. Further assume that G is minimal subject to $\tilde{G} \ncong SL(2, 3)$, set $R = O_2(\tilde{G})$, and pick $r \in R$, $x \in \tilde{\mathfrak{X}}$ with $\langle x, r \rangle = \langle x, x^r \rangle \cong SL(2, 3)$.

We claim that $\tilde{G} = \langle x, x^r, x^s \rangle$ for some $s \in R$ with $\langle x, x^s \rangle \cong SL(2, 3)$. Suppose first that $\tilde{G} \neq \langle x \rangle R$. Then there is a $y \in \tilde{\mathfrak{X}}$ with $\langle x, y \rangle$ an $SL(2, 3)$ not in $\langle x \rangle R$. Now minimality forces $\tilde{G} = \langle x, y, r \rangle$, so $O_2(\tilde{G}) > R$. Thus, $\tilde{G} = \langle x \rangle R$. Then $s \in R - \langle x, r \rangle$ exists with $\langle x, s \rangle = \langle x, x^s \rangle$ an $SL(2, 3)$, and clearly $\tilde{G} = \langle x, x^r, x^s \rangle$.

We have not checked that such a group G, generated by three long root elements, cannot occur in the exceptional Chevalley groups. Anticipating some very elementary results of the next section, we can, however, deal with all other cases using an argument which will occur frequently:

COROLLARY 3.4. *Assume that G in* (3.2) *is a subgroup of a classical group or a rank* 2 *group. If* $p > 2$ *and* $O_\infty(\tilde{G}) \neq Z(\tilde{G})$ *for* $\tilde{G} = G/O_p(G)$, *then* $\tilde{G}$ *is* $SL(2, 3)$.

PROOF. Choose a minimal counterexample G. Then G can be generated by 3 elements of $\mathfrak{X}$. We may assume, initially, that $G < \Omega(V)$ for some V. Then, in the notation of (4.3), $\dim \mathfrak{A}(G)/\mathrm{rad}\, \mathfrak{A}(G) \leqslant 6$. Here, $A(x) \subseteq \mathfrak{A}(G)$ implies that $\mathrm{rad}\, \mathfrak{A}(G) \subseteq A(x)^{\perp}$ for $x \in \mathfrak{X}$. Consequently, G centralizes both $\mathrm{rad}\, \mathfrak{A}(G)$ and $V/\mathfrak{A}(G)$, inducing on $\mathfrak{A}(G)/\mathrm{rad}\, \mathfrak{A}(G)$ a group having $\tilde{G}$ as a quotient. We may thus assume $\dim V = 6$.

Now $P\Omega(V) = PSL(4, q)$ or $PSU(4, q)$ implies that we could have assumed $G < SL(4, s)$ for some s, with G generated by a class $\mathfrak{X}$ of transvections. By (4.3), we may then assume $G < SL(3, s)$. However, $SL(3, s)$ has no subgroup generated by transvections of the desired type (Mitchell **[20]**).

4. Classical groups. Let $V = V(m, s)$, let p be the prime dividing s, and consider $\Omega(V)$; exclude the cases $m \leqslant 3$ and $\Omega^{-}(4, s)$. The following elementary lemmas will be used very frequently.

LEMMA 4.1. *If $x \neq 1$ is a long root element, and x fixes a subspace W, then either $A(x) \subseteq W$, $W \subseteq A(x)^{\perp}$, or $A(x) \cap \mathrm{rad}\, W \neq 0$.*

LEMMA 4.2. *If X and Y are distinct long root groups, then one of the following holds*:

(i) *$\langle X, Y\rangle$ is abelian and consists of long root elements, and $A(X) + A(Y)$ is a totally singular* 3-*space*;

(ii) *$\langle X, Y\rangle$ is abelian, $X \cup Y$ is its set of long root elements, and either $A(X) + A(Y)$ is a totally singular* 4-*space or a* 3-*space with radical $A(X) \cap A(Y)$*;

(iii) *$\langle X, Y\rangle$ is the Sylow p-subgroup of an $SL(3, s)$, $A(X) \cap A(Y) = 0 \neq A(X)^{\perp} \cap A(Y)$; or*

(iv) *$\langle X, Y\rangle$ is $SL(2, s)$, and $A(X) + A(Y)$ is a nonsingular* 4-*space.*

(4.2) should be compared with (3.1).

DEFINITION. Write $\mathfrak{A}(G) = [G, V]$ for $G \leqslant \Omega(V)$. This notation is motivated by the following

LEMMA 4.3. *If $G \leqslant \Omega(V)$ is generated by long root elements $x_1, \ldots, x_k$, then*

(i) *$\mathfrak{A}(G) = A(x_1) + \cdots + A(x_k)$ and*

(ii) *$\dim \mathfrak{A}(G) \leqslant 2k$.*

PROOF. $A(x_i) = [x_i, V]$.

The corresponding result for transvections is

LEMMA 4.4. *If $G \leqslant GL(W)$ is generated by transvections $x_1, \ldots, x_k$, then the subspace $\mathfrak{D}(G)$ spanned by the directions $D(x)$ of all transvections $x \in G$ satisfies $\mathfrak{D}(G) = [G, W] = D(x_1) + \cdots + D(x_k)$ and $\dim \mathfrak{D}(G) \leqslant k$.*

LEMMA 4.5. *Let $L \leqslant \Omega(V)$ be generated by a set $\mathfrak{X}$ of long root elements, and let $x \in \Omega(V)$ be a long root element.*

(i) *If* $[x, L] = 1$, *then either* $\mathcal{Q}(L) \subseteq A(x)^{\perp}$ *or* $A(x) \cap A(y) \neq 0$ *for some* $y \in \mathfrak{X}$. (*In particular, if* $\mathfrak{X}$ *is a class and contains* y, y' *with* $A(y) \cap A(y') = 0$, *then either* $\mathcal{Q}(L) \subseteq A(x)^{\perp}$ *or* $A(x) \subseteq \mathcal{Q}(L)$.)

(ii) *Suppose* L *fixes a totally singular subspace* W *with* $A(y) \cap W \neq 0$ *for all* $y \in \mathfrak{X}$, *and* $\dim C_W(x) \geqslant \dim W - 1$. *Then there is an* $\langle x, L\rangle$*-invariant totally singular subspace* $W + R$ *for some* 1*-space* R *of* $A(x)$.

PROOF. (i) Assume $\mathcal{Q}(L) \nsubseteq A(x)^{\perp}$ and apply (4.2).

(ii) By hypothesis, $W^{\perp} \cap A(x)$ contains a 1-space R. By (4.1), $W + R$ is fixed by both L and x.

LEMMA 4.6. *Suppose* R *is a singular* 1*-space of* V, *and set* $\Omega = \Omega(V)$ *and* $P = O_p(\Omega_R)$. *Then* P *is elementary abelian of order* s^{m-2}. *The group* $C_{\Omega}(R)$ *acts on* P *precisely as it does on* $R^{\perp}/R$, *with long root elements in* P *corresponding to singular vectors in* $R^{\perp}/R$.

PROOF. [**5**, (3.1)].

COROLLARY 4.7. *Let* L *be a subgroup* $\Omega^{\pm}(m-2, s)$ *of* $\Omega = \Omega^{\pm}(m, s)$, *centralizing the nonsingular* 2*-space* T. *Let* $R < T$ *be a singular* 1*-space*, $G \leqslant L$ *with* $C_V(G) = T$, $P = O_p(\Omega_R)$, *and* $G_1 \leqslant \Omega_R$. *Assume* P_0 *is a* G*-invariant subgroup of* P *such that* $C_{P/P_0}(G) = 1$, *and* G_1 *is a complement to* P_0 *in* P_0G *not conjugate to* G *in* P_0G. *Then* G *and* G_1 *are not conjugate in* Ω.

PROOF. Suppose $G_1^b = G$, $b \in \Omega$. Then R^b is fixed by G, so $R^b = R^{c^{-1}}$ for some $c \in C_{\Omega}(L) \leqslant C_{\Omega}(G)$. Hence, $bc \in \Omega_R = PLH$ (H a diagonal subgroup). Since $(LH)^T$ consists of scalar transformations, it follows that we may assume $bc \in PL$, say $bc = al$ with $a \in P$, $l \in L$. Here $G = G_1^{al}$ implies that $G_1^a \leqslant L \cap PG = G$, so $G_1^a = G$. Now $[G, a^{-1}] \leqslant P \cap GP_0 = P_0$, and hence $a \in P_0$ by hypothesis. This contradiction proves that G and G_1 are not conjugate in Ω.

LEMMA 4.8. *Let* W_1 *and* W_2 *be isomorphic irreducible* G*-modules over* $GF(s)$. *Assume that* $\langle w_1\rangle$ *is the only fixed* 1*-space of* $C_G(w_1)$ *on* W_1 *for some* $w_1 \in W_1$.

(i) $V = W_1 \oplus W_2$ *contains exactly* $s + 1$ *nontrivial submodules*.

(ii) *If* V *is equipped with an orthogonal geometry making* W_1 *and* W_2 *totally singular, then* $C_{\Omega(V)}(G)$ *induces at least* $PSL(2, s)$ *on the set of submodules in* (i).

PROOF. (i) Any two such submodules W clearly meet trivially, and $W \cong W_1$. Since $\dim C_V(C_G(w_1)) = 2$, there are at most $s + 1$ choices for W. On the other hand, if $f: W_1 \to W_2$ is a G-isomorphism and α is a fixed scalar, then G fixes $\{\alpha w + wf \mid w \in W_1\}$.

(ii) Suppose $W \neq W_1$. Then there exists an element $h \in \Omega(V)$ with $W_1^h = W_1$ and $W_2^h = W$. Since Ω_{W_1W} induces $GL(W)$ on W, we can modify h so as to induce a G-isomorphism $W_2 \to W$. Consequently, $C_{\Omega(V)}(G)$ is 2-transitive on the $s + 1$ submodules in (i).

In view of (4.3), the next result will be crucial for us.

THEOREM 4.9. *The following table gives numbers of elements of long root groups which can generate the indicated groups.*

Group	$Sp(2n, q), n \geqslant 1$	$SL(n, q), n \geqslant 3$	$\Omega^+(2n, q), n \geqslant 4$	$\Omega(2n+1, q), n \geqslant 3$
Number	$2n - 1 + (2, q)^*$	n	$n + 1$	$n + 1$

$\Omega^-(2n, q), n \geqslant 4$	$SU(n, q)', n \geqslant 3$	$G_2(q)'$	${}^3D_4(q)$	$Sz(q)$
$n + 1$	n^{**}	4	4	3

* *Exception*: $SL(2, 9)$ *requires* 3 *root elements.*

** *Exception*: $SU(4, 2)$ *requires* 5 *elements.*

PROOF. If $G = SL(2, q) = Sp(2, q)$ with q odd, use [**9**, p. 44]. For q even, G is generated by a dihedral group D of order $2(q + 1)$ along with any involution in $G - D$; the case $Sz(q)$ is very similar.

If $G = SL(3, q)$ or $SU(3, q)'$, then for a suitable long x, $G = \langle SL(2, q), x\rangle$ or $\langle D, x\rangle$ with D as above or $D = SL(2, 5)$ when $q = 9$ (Mitchell [**20**], Hartley [**12**]). Inducting along the Dynkin diagram now settles the cases $SL(n, q)$ and $\Omega^+(2n, q)$.

Since $\Omega(2n + 1, q) = \langle SL(n - 1, q), x\rangle$ and $\Omega^-(2n, q) = \langle \Omega^+(2n - 2, q), y\rangle$ for suitable long root elements x, y, these groups are generated as in the table.

Similarly, $Sp(2n, q) = \langle Sp(2n - 2, q), x, y\rangle$ for transvections x, y with $\langle Sp(2n - 2, q), x\rangle$ of index $q - 1$ in a maximal parabolic.

Also, $G_2(q)$ and ${}^3D_4(q)$ can both be written $\langle SL(3, q), x\rangle$.

Finally, $SU(n, q) = \langle SU(n - 1, q), x\rangle$ handles every situation except for $SU(4, 2)$ and $SU(5, 2)$ (since $SU(3, 2)$ is not generated by transvections). Four transvections in $SU(4, 2)$ can generate the monomial subgroup $L = 3^3 \cdot S_4$ of $SU(4, 2)$, and one further transvection yields $SU(4, 2)$. Similarly, if $G = SU(5, 2)$ then $G = \langle L, x\rangle$ for L (as above) fixing a nonsingular 4-space W of $V(5, 4)$ and a suitable transvection x moving W.

LEMMA 4.10. (i) *S_n can be generated by $n - 1$ transpositions.*
(ii) *For q even, $O^{\pm}(n, q)$ can be generated by $n + 1$ transvections.*

PROOF. (i) This is elementary.

(ii) For n odd, $O(n, q) = Sp(n - 1, q)$ can be generated by n transvections by (4.9). Since $O^{\pm}(n + 1, q) = \langle O(n, q), x\rangle$ for a suitable transvection x, this proves (ii).

LEMMA 4.11. *Let K be a finite extension of the field k, and let G act on the k-space V. Set $W = V \otimes_k K$ and $V_1 = V \otimes 1$. Then*
(i) $\dim_K H^1(G, W) = \dim_k H^1(G, V)$; *and*
(ii) *two complements to V_1 in GV_1 are conjugate in GV_1 if they are conjugate in GW.*

PROOF. (i) Let $a_1 = 1, \ldots, a_n$ be a k-basis of K. Then $W = \bigoplus\sum V \otimes a_i$ with G acting on $V \otimes a_i$ as it does on V. Consequently, $\dim_k H^1(G, W) = \sum \dim_k(G, V \otimes a_i) = [K : k]\dim_k H^1(G, V)$, as required.

(ii) Suppose G_1 and G_2 are complements to V_1 in GV_1, and $G_1^w = G_2$, $w \in W$. Write $w = v + u$ with $v \in V_1$ and $u \in \bigoplus\sum_{i>1} V \otimes a_i = U$. Then $G_1^u \leqslant GV_1$. If $g \in G_1$ then $g^u = g(u^{-1})^g u \in G_1U$. Thus, $G_1^u \leqslant G_1V_1 \cap G_1U = G_1$, and hence $G_2 = G_1^w = G_1^v$.

5. Preliminary reductions. Throughout §§5–10, G and V will satisfy the hypotheses of Theorem I. We will induct on $\dim V + |G|$. Let p denote the characteristic of $V = V(m, s)$. Let $\Omega = \Omega^{\pm}(V)$ preserve the quadratic form Q. Let $\mathfrak{X}$ denote a class of long root elements of G generating G. By (4.1), $A(x) \cap \operatorname{rad} V = 0$ for $x \in \mathfrak{X}$. Throughout the proof, the letters x and y will always stand for suitable elements of $\mathfrak{X}$.

An *$\mathfrak{X}$-subgroup* K of G is a subgroup $K = \langle K \cap \mathfrak{X}\rangle$.

Set $\bar{G} = G/Z(G)$ and $\bar{V} = V/\operatorname{rad} V$. For $L \subseteq G$ we will write $\bar{L} = LZ(G)/Z(G)$, and $\tilde{L} = L/Z(L)$ if L is a group.

Since the various possibilities occurring in (3.2) will frequently be dealt with separately, Table 1 is provided both to fix notation for $\bar{\mathfrak{X}}$ and to indicate where a given case for $\bar{G}$ is studied.

The cases in which G has a noncentral solvable normal subgroup are dealt with in §9. Note that we are using notation distinguishing the cases of the identical groups $\Omega(2k + 1, 2^i)$ and $Sp(2k, 2^i) = O(2k + 1, 2^i)$; of course, no such distinction is needed when $k = 2$, in view of the graph automorphism. We will also use the following additional conventions.

1. If the embedding of G in Ω arises as in (I k), (RT k) or (RL k), by simply extending the base field as in Theorem I, we will say that (I k), (RT k) or (RL k) holds.

2. We may assume that G fixes no nondegenerate hyperplane of V. In view of the $V = V_1 \perp V_2$ condition of Theorem I, such a hyperplane H can occur only if V has both even dimension and characteristic. Such an H arises only in (I 1, 5, 6, 8), accounting for (RL 1, 2, 3, 4), respectively.

The last convention has a very useful consequence:

LEMMA 5.1. *If $\mathfrak{A}(G)$ is nondegenerate, then $V = \mathfrak{A}(G)$.*

TABLE 1

$\bar{G}$	$\bar{\mathfrak{X}}$	
$PSL(n, q)$	transvections	(6.1, 9), (7.1, 2, 7), (10.1, 4)
$PSp(n, q)$	transvections	(6.1), (7.1, 5, 6), (10.1, 2, 3, 4)
$PSU(n, q)'$, $n \geqslant 3$	transvections	(6.1), (7.2, 7, 8), (10.1, 5)
$P\Omega^{\pm}(n, q)$, $n \geqslant 7$	long root elements	(6.8), (10.1)
$G_2(q)'$	long root elements	(8.2), (10.1)
${}^3D_4(q)$	long root elements	(8.2), (10.1)
$F_4(q)$, ${}^2E_6(q)$, $E_n(q)$	long root elements	(6.1)
$O^{\pm}(2n, q)$, q even	transvections	(6.1), (7.4, 9), (10.1, 3)
$P\Omega^+(8, q)\cdot S_3$	conjugates of transvections or reflections	(5.6)
S_n, $n \geqslant 7$	transpositions	(6.1, 10), (7.9), (10.1, 3)
A_6	involutions	(7.2)
HJ	long root elements of $G_2(4)$	(8.2), (10.1)
$P\Omega^{\pm,\pi}(n, q)$, $q = 3$ or 5	reflections	(5.5), (6.1), (10.1)
F_{22}, F_{23}, F_{24}	3-transpositions	(5.4)
$PSL(2, q) \wr S_n$	pre-images of transpositions	(7.4, 10)
$Sz(q)$	involutions	(7.2)

PROOF. Since each $A(x) \subseteq \mathcal{Q}(G)$, necessarily $\mathcal{Q}(G)^{\perp} \subset A(x)^{\perp}$. Then G centralizes $\mathcal{Q}(G)^{\perp}$. If $\dim \mathcal{Q}(G) \geqslant \dim V - 1$, apply convention 2. If $\dim \mathcal{Q}(G) \leqslant \dim V - 2$, then $\mathcal{Q}(G)^{\perp}$ contains a nonsingular 2-space T, and $V = T \perp T^{\perp}$ with $G^{T^{\perp}} = 1$. This contradicts one of the hypotheses of Theorem I.

LEMMA 5.2. *If G fixes a singular* 1-*space R, then $\mathcal{Q}(G) \subseteq R^{\perp} \neq V$ and the only G-invariant decomposition $R^{\perp}/R = W/R \perp T/R$ with $G^{W/R} \neq 1$ has $W = R^{\perp}$, $T = R$. Moreover, G fixes no nondegenerate hyperplane of $R^{\perp}/R$.*

PROOF. Since $R \subseteq A(x)^{\perp}$ for all $x \in \mathfrak{X}$, clearly $\mathcal{Q}(G) \subseteq R^{\perp}$. By (4.1), $G^{T/R} = 1$ and hence $G^T = 1$. Since T/R is nondegenerate, T contains a nondegenerate subspace T_0. By (4.1) and one of the hypotheses of Theorem I, T_0 cannot be nonsingular. Hence, if $T/R \neq 0$ then $T/R = \mathrm{rad}(R^{\perp}/R)$ and W/R is a nondegenerate hyperplane of $R^{\perp}/R$. Hence, it suffices to prove the final assertion.

Let W/R be a G-invariant nondegenerate hyperplane of $R^{\perp}/R$. Since $R \not\subseteq A(x)$, necessarily $A(x) \subseteq W$ by (4.1). Thus, G centralizes V/W. Write $W = W_0 \perp \mathrm{rad}\, W$, so $V = W_0 \perp W_0^{\perp}$. Then $W_0^{\perp}$ has a nondegenerate hyperplane $U \supset \text{rad } W$, and $W + U = W_0 \perp U$ is a nondegenerate hyperplane of V. This contradicts convention 2.

LEMMA 5.3. *If G can be generated by k elements of $\mathfrak{X}$, and if $2k$ is the smallest dimension appearing for $\mathfrak{C}(G)$ in* (I 1–11), (RT 1–10), *then $V = \mathfrak{C}(G)$.*

PROOF. Suppose $V \supset \mathfrak{C}(G) = \mathfrak{C}$. By (5.1), $Q^{-1}(0) \cap \text{rad } \mathfrak{C} = R \neq 0$. Then $R \subseteq A(x)^{\perp}$ for each $x \in \mathfrak{X}$ implies that $G^R = 1$ and $\mathfrak{C} \subseteq R^{\perp}$. Induction applies to $G^{R^{\perp}/R}$ by (4.1) and (5.2). Thus, by (4.3) and our hypothesis, $2k \geqslant \dim \mathfrak{C} = 1 + \dim \mathfrak{C}/R = 1 + \dim[G, R^{\perp}/R] \geqslant 1 + 2k$.

LEMMA 5.4. $\overline{G} \neq F_{22}, F_{23}, F_{24}$.

PROOF. None of the examples in §2 is a central extension of $PSU(6, 2)$ by a group of even order. (While the proof of the latter fact in §6 is not difficult, it is perhaps worth noting that the corresponding assertion for Theorem II is easy. For assume $G = G' \leqslant SL(W)$, $\overline{G} = PSU(6, 2)$, and $Z(G)$ contains an involution z. By (4.4), $\dim \mathfrak{D} \leqslant 6$, where $\mathfrak{D} = [G, W]$. Clearly z fixes the direction of each transvection in G, so $\mathfrak{D} \subseteq C_W(z)$. Since G centralizes $W/\mathfrak{D}$, we may assume that $C_W(z)$ is a hyperplane. Then G fixes the 1-space $D(z) = [z, W] \leqslant [G, W] = \mathfrak{D}$, and hence induces $\overline{G}$ on the space $\mathfrak{D}/D(z)$ of dimension $\leqslant 5$.)

LEMMA 5.5. $\overline{G} \neq P\Omega^{\pm,\pi}(n, 3)$ *for* $n \geqslant 7$, $P\Omega^{\pm,\pi}(n, 5)$ *for* $n \geqslant 4$.

PROOF. If one of these groups can occur, then so can $P\Omega^{\pi}(7, 3)$ or $P\Omega^{-,\pi}(4, 5)$. In either case, $p = 2$. By (4.3), $\dim \mathfrak{C}(G) \leqslant 16$ resp. 8. However, according to Landazuri-Seitz [**17**], each nonlinear irreducible constituent of $\mathfrak{C}(G)$ has dimension at least 72 resp. 12.

LEMMA 5.6. $\overline{G} \neq P\Omega^{+}(8, q) \cdot S_3$, $q = 2^i$, 3 *or* 5.

PROOF. Deny! By induction, G has an $\mathfrak{X}$-subgroup $M = O^{+}(8, 2^i)$, and $\mathfrak{C}(M) = W_1 \oplus W_2$ for totally singular M-invariant 8-spaces W_i. Clearly, $\mathfrak{C}(G'') = \mathfrak{C}(M') = \mathfrak{C}(M)$ is G-invariant, so (4.1) yields $\mathfrak{C}(G) = \mathfrak{C}(M')$. However, the normalizer of M' in $GL(16, q)$ does not induce an S_3 on M'.

6. The inductive step. In this section, all cases of Theorem 1 of high rank will be handled (assuming, inductively, that the smaller rank situations have been dealt with).

THEOREM 6.1. *Assume that* $V = \mathfrak{C}(G)$.

(i) $\overline{G} \neq F_4(q)$, ${}^2E_6(q)$, $E_n(q)$.

(ii) *If* $\overline{G} = PSL(n, q) \neq PSL(5, 2)$, $PSL(5, 4)$, *where* $n \geqslant 5$, *then* (RT 1) *holds*.

(iii) *If* $\overline{G} = PSp(n, q)$ *with* $n \geqslant 6$, *then* (RT 1 *or* 2) *holds.*

(iv) *If* $\overline{G} = PSU(n, q)$ *with* $n \geqslant 5$, *then* (I 3) *or* (RT 1) *holds.*

(v) *If* $\bar{G} = O^{\pm}(2n, q) \neq O^{+}(6, 2)$ *with* $2n \geqslant 6$ *and* q *even, then* (RT 2) *holds.*

(vi) *If* $\bar{G} = S_n$ *with* $n \geqslant 10$ *even, then* (RT 3 *or* 4) *holds.*

(vii) *If* $\bar{G} = P\Omega^{-,\pi}(6, 3)$, *then* (I 9) *or* (RT 7) *holds.*

These groups $\bar{G}$ have the following common properties. There are conjugate commuting $\mathfrak{X}$-subgroups $S_i \cong SL(2, r)$, where $r = 2$ in (v)–(vii). There is an $\mathfrak{X}$-subgroup $L_i \lhd C_G(S_i)$ with $G = \langle L_1, L_2 \rangle$, $L_i = \langle S_{3-i}^{L_i} \rangle$, and $L_1 \cap L_2$ not a p-group.

Clearly, for (6.1 i) we need only consider the case $\bar{G} = F_4(q)$. With this understanding, the possibilities for suitable quotient groups of L_i and $L_1 \cap L_2$ are as follows: (i) $PSp(6, q)$, $PSp(4, q)$; (ii) $PSL(n-2, q)$, $PSL(n-4, q)$; (iii) $PSp(n-2, q)$, $PSp(n-4, q)$; (iv) $PSU(n-2, q)'$, $PSU(n-4, q)$; (v) $O^{\pm}(2n-2, q)$, $O^{\pm}(2n-4, q)$; (vi) S_{n-3}, S_{n-6}; (vii) $3^3 \cdot S_4$, S_3 (where the S_3 is an $\mathfrak{X}$-subgroup meeting $O_3(\tilde{L}_1)$; note that this involves a choice of S_2).

LEMMA 6.2. $\mathcal{Q}(L_1)$ *is nonsingular.*

PROOF. Suppose rad $\mathcal{Q}(L_1)$ contains a singular 1-space R. Then L_1 must be as in (RT 2, 8) or (RL 5, 6).

In (RT 2), R is the radical of both $\mathcal{Q}(L_1)$ and $\mathcal{Q}(L_1 \cap L_2)$, and hence also of $\mathcal{Q}(L_2)$. Then G fixes R, so $\mathcal{Q}(G) \subseteq R^{\perp} \subset V$.

In (RL 5, 6) and (RT 8), L_1 is $SU(4, 2)$ and $\bar{G} = PSU(6, 2)$. There is an $\mathfrak{X}$-subgroup $K = SU(5, 2)$ containing L_1 in the natural manner. By induction, K acts on $\mathcal{Q}(K)$ as in (I 3) or (RT 1). But this contradicts the assumed action of L_1.

Suppose next that rad $\mathcal{Q}(L_1)$ is nonsingular. The only relevant case is (I 5). But there, $\mathcal{Q}(S_2) = \mathcal{Q}(L_1 \cap L_2)$ is a nonsingular 4-space. Then $L_2 = \langle (L_1 \cap L_2)^{L_2} \rangle \leqslant C_G(S_2)$ implies that $\mathcal{Q}(L_2) = \mathcal{Q}(S_2) = \mathcal{Q}(L_1 \cap L_2) = \mathcal{Q}(L_1) = \mathcal{Q}(G)$, which is absurd.

LEMMA 6.3. $\mathcal{Q}(S_2) \subseteq \mathcal{Q}(S_1)^{\perp}$ *for some* $S_2 \in S_1^G \cap C_G(S_1)$.

PROOF. This follows easily from an examination of the possible actions of L_1 on $\mathcal{Q}(L_1)$, except in the following cases: $\tilde{L}_1 = PSL(4, q)$, $PSU(4, q)$, $PSp(4, q)$, $PSL(3, q)$, $PSU(3, q)$, $O^{\pm}(4, q)$, $3^3 \cdot S_4$.

If $\tilde{L}_1 = PSL(4, q)$ or $PSU(4, q)$, then $\bar{G} = PSL(6, q)$ resp. $PSU(6, q)$, so there is an $\mathfrak{X}$-subgroup $SL(5, q)$ resp. $SU(5, q)$ yielding the desired pair of elements of S_1^G by induction. If $\tilde{L}_1 = PSp(4, q)$, then $\bar{G} = PSp(6, q)$, so there is an $\mathfrak{X}$-subgroup $S_1 S_2 S_3$ with the S_i's commuting conjugate $\mathfrak{X}$-subgroups, so $\mathcal{Q}(S_i) \neq \mathcal{Q}(S_j)$ for some $i \neq j$.

Suppose $\tilde{L}_1 = PSL(3, q)$ or $PSU(3, q)$, so $\bar{G} = PSL(5, q)$ resp. $PSU(5, q)$.

There are at least two conjugate $\mathfrak{X}$-subgroups $K > L_1$ of G with $\tilde{K} = PSL(4, q)$ resp. $PSU(5, q)$. If $\dim \mathcal{Q}(K) \geqslant 8$, the lemma holds. Assume $\dim \mathcal{Q}(K) = 6$. Then $\mathcal{Q}(K) = \mathcal{Q}(L_1)$ for two such K's generating G, from which the contradiction $\mathcal{Q}(G) = \mathcal{Q}(L_1)$ follows.

Finally, suppose $\tilde{L}_1 = O^{\pm}(4, q)$ or $3^3 \cdot S_4$, so $\bar{G} = O^{\pm}(6, q)$ resp. $P\Omega^{-,\pi}(6, 3)$. There is an $\mathfrak{X}$-subgroup $K > L_1$ satisfying $\bar{K} \cong \langle \bar{x} \rangle \times Sp(4, q)$ resp. $\langle \bar{x} \rangle \times SU(4, 2)$, $x \in \mathfrak{X}$. Induction applies to the action of K on $A(x)^{\perp}/A(x)$. We may assume that $\dim[K, A(x)^{\perp}/A(x)] = 5$ resp. 6. Then by (6.2), $\dim \mathcal{Q}(L_1) = 6$ and L_1 acts irreducibly on $\mathcal{Q}(L_1)$ as in (I 11). Since S_1 centralizes L_1, it follows that S_1 must act trivially on $\mathcal{Q}(L_1)$, as required. (Note that this proves that any S_2 will work in this case, which is needed in view of the nature of $C_{L_1}(S_2) = L_1 \cap L_2$.)

LEMMA 6.4. *$V = \mathcal{Q}(L_1 \cap L_2) \perp \mathcal{Q}(S_1) \perp \mathcal{Q}(S_2)$ with L_i trivial on $\mathcal{Q}(S_i)$ and $\mathcal{Q}(L_i) = \mathcal{Q}(L_1 \cap L_2) \perp \mathcal{Q}(S_{3-i})$.*

PROOF. Since $L_i = \langle S_{3-i}^{L_i} \rangle$, $\mathcal{Q}(L_i) \subseteq \mathcal{Q}(S_i)^{\perp}$ by (6.3). By (6.2), $\mathcal{Q}(L_i)$ is nonsingular. An easy examination of the cases shows that $\mathcal{Q}(L_i) = \mathcal{Q}(L_1 \cap L_2) \perp \mathcal{Q}(S_{3-i})$. Now

$$V = \mathcal{Q}(G) = \mathcal{Q}(\langle L_1, L_2 \rangle) = \mathcal{Q}(L_1 \cap L_2) \perp \mathcal{Q}(S_1) \perp \mathcal{Q}(S_2).$$

Note that the preceding lemma fails for such excluded cases as $\bar{G} = PSL(5, q)$, $q = 2$ or 4, in which $L_1 \cap L_2 = 1$.

LEMMA 6.5. *If L_i fixes totally singular subspaces W_i^j of $\mathcal{Q}(L_i)$ with $\mathcal{Q}(L_i) = W_i^1 \oplus W_i^2$, then G fixes totally singular subspaces W_1, W_2 with $V = W_1 \oplus W_2$ and $W_i \supset W_i^j$ for each i, j.*

PROOF. Here $\mathcal{Q}(L_i)$ is nonsingular. By (4.8), (6.3), and an examination of the possibilities for L_i, $L_i^{\mathcal{Q}(L_i)}$ and $(L_1 \cap L_2)^{\mathcal{Q}(L_1 \cap L_2)}$ fix the same numbers of maximal totally singular subspaces (namely, 2 or $s + 1$). We may thus assume that $W_1^j \cap \mathcal{Q}(L_1 \cap L_2) = W_2^j \cap \mathcal{Q}(L_1 \cap L_2)$ for $j = 1, 2$. Now (6.4) shows that $W_j = (W_1^j \cap \mathcal{Q}(L_1 \cap L_2)) \perp (W_1^j \cap \mathcal{Q}(S_2)) \perp (W_2^j \cap \mathcal{Q}(S_1))$, $j = 1, 2$, satisfy the desired requirements.

LEMMA 6.6. *If L_1 acts on $\mathcal{Q}(L_1)$ as in* (I 3), *then V can be regarded as a $GF(s^2)$-space on which G acts as $GF(s^2)$-transformations preserving a nondegenerate hermitian form.*

PROOF. By (6.4), there is an element $c \in O(V)$ of order $s^2 - 1$ such that $c' = c^{\mathcal{Q}(L_1 \cap L_2)}$ centralizes $L_1 \cap L_2$, while $c^{\mathcal{Q}(L_i)}$ is the unique element of $O(\mathcal{Q}(L_i))$ centralizing L_i and restricting to c'. This proves the first part. The second is also clear from (6.4): the desired form is uniquely determined by the form on $\mathcal{Q}(L_1 \cap L_2)$.

LEMMA 6.7. *If* $\bar{G} = P\Omega^{-,\pi}(6, 3)$, *then either*
(i) $V = W_1 \oplus W_2$ *for totally singular G-invariant subspaces, or*
(ii) *the conclusions of* (6.6) *hold.*

PROOF. By (6.5), we may assume that L_1 acts on $\mathfrak{C}(L_1)$ as in (I 9). The proof of (6.6) can then be repeated.

At this point in the proof of (6.1), we have reduced to the corresponding situations in Theorem II. Thus, we may now assume that $G \leqslant SL(W)$ is generated by a class $\mathfrak{X}$ of transvections of the $GF(s)$-space W.

The argument in this situation is quite similar to the one used in (6.4). Recall that $\mathfrak{D}(K)$ was defined in (4.4) to be $[K, W]$ for $K \leqslant G$. We have $W = \mathfrak{D}(L_1 \cap L_2) \oplus \mathfrak{D}(S_1) \oplus \mathfrak{D}(S_2)$, where $\dim \mathfrak{D}(S_i) = 2$. (Note that G cannot fix a 1-space of W, and hence L_i cannot fix a 1-space of $\mathfrak{C}(L_i)$ by (6.2).) Forms on $\mathfrak{C}(L_i)$ can then be extended to ones of W.

Fields are also not hard to deal with. Consider, for example, the case $\bar{G} = P\Omega^{-,\pi}(6, 3)$. Here, let $D(x) = \langle w \rangle$ for an $x \in L_1 \cap L_2 \cap \mathfrak{X}$. Then w is in a unique $L_1 \cap L_2$-invariant $GF(4)$-subspace T_0 spanning $\mathfrak{C}(L_i)$. Thus, $T_0 \subset T_1, T_2$, so we may assume that $s = 4$. Now $\bar{G} < PSU(6, 2)$ since forms have already been extended. This case is then completed using Fischer [7, (16.1.12)].

The remaining possibilities for G are handled in a very similar manner, thereby completing the proof of (6.1).

REMARK. The proof of (6.1) also handles $P\Omega^{\pm}(n, q)$ for $n \geqslant 10$. However, we will use a different approach which allows us to deal with $n \geqslant 7$.

PROPOSITION 6.8. *If* $V = \mathfrak{C}(G)$ *and* $\bar{G} = P\Omega^{\pm}(n, q)$ *for* $n \geqslant 7$, *then* (I 1, 2, *or* 4) *holds.*

PROOF. By (4.3, 9), $V = \mathfrak{C}(G)$ has dimension $\leqslant n + 1$. There is an $\mathfrak{X}$-subgroup M with $\tilde{M} = P\Omega^{+}(n - 1, q)$. Set $\mathfrak{C} = \mathfrak{C}(M)$. By Griess **[10]**, $p \nmid |Z(G)|$. Let P be any abelian group of order q^{n-2} as in (4.6), chosen so $|P \cap M| = q^{n-3}$. There exist $x_1, x_2 \in \mathfrak{X}$ with $x_1 \in M$, $G = \langle M, x_2 \rangle$, and $\langle x_1, x_2 \rangle^{\#} \subset \mathfrak{X}$. Thus, $\dim V \leqslant \dim \mathfrak{C} + 1$.

We first show that, without loss of generality,

$$\bigcap \{A(x) | x \in P\} \text{ is a 1-space } P^* . \qquad (*)$$

If $n - 1 \geqslant 8$, $\bigcap \{A(x) | x \in P \cap M\}$ is a 1-space P^*, by induction. Here, we can pass to an $\mathfrak{X}$-subgroup $\Omega^{\pm}(n - 2, q)$, and then up to a conjugate M_1 of M meeting P in q^{n-3} elements, in order to deduce (*) for $P = \langle P \cap M, P \cap M_1 \rangle$. Assume $n - 1 = 6$. Then (*) will again hold for $P \cap M$ replacing P, unless M acts as $SL(4, q)$ on the 8-space $\mathfrak{C}$. Since $\dim V \leqslant n + 1 = 8$, in the latter case $V = \mathfrak{C}$ and we can apply triality in order to arrive at a situation in which (*) holds. Assume $n - 1 = 7$. This time we may also assume $\dim \mathcal{Q} = 8$ (with $M^{\mathcal{Q}}$ as in (I 4)) and $\dim V = 9$. Then M has commuting $\mathfrak{X}$-subgroups S_1, $S_2 \cong SL(2, q)$ with $\mathcal{Q}(S_1S_2) = \mathcal{Q}(S_1) \perp \mathcal{Q}(S_2) = \mathcal{Q}$ and $C = C_G(S_1S_2) = O^{\pm}(4, q)Z(G)$. Since C fixes $\mathcal{Q}(S_i)$ necessarily C fixes $\mathcal{Q}$. Thus C centralizes $\mathcal{Q}$, whereas $\dim \mathcal{Q} = \dim V - 1$.

We may thus assume (*). This singles out an orbit P^{*G} of singular 1-spaces of V, along with our orbit $A(x)^G$ of 2-spaces. Clearly, the maximal parabolic $N_G(P)$ fixes P^*. Thus, $x \in P$ iff $A(x) \supset P^*$. This can be viewed as producing an isomorphism from the geometry of totally singular 1- and 2-spaces of $V(n, q)$ into that of V.

In particular, if $P_i^* \in P^{*G}$, $i = 1, 2, 3$, then some conjugate M_0 of M meets each $P_i \cap \mathfrak{X}$, $i = 1, 2, 3$, and hence each $P_i^* \in \mathcal{Q}(M_0)$.

Fix distinct conjugates M_1, M_2 of M with $M_1 \cap M_2 \geqslant \Omega^{\pm}(n - 2, q)$. Choose M_i-invariant $GF(q)$-subspaces W_i spanning $\mathcal{Q}(M_i)$ such that $W_1 \cap W_2$ spans $\mathcal{Q}(M_1 \cap M_2)$. We claim that $W_1 + W_2$ is invariant under $G = \langle M_1, M_2 \rangle$, and that $Q(W_1 + W_2) = GF(q)$. For, pick $x \in (M_i - M_1 \cap M_2) \cap \mathfrak{X}$ with $A(x_1) \cap \mathcal{Q}(M_1 \cap M_2) = A(x_2) \cap \mathcal{Q}(M_1 \cap M_2) \neq 0$. Then pick $v_i \in A(x_i) \cap (W_i - W_1 \cap W_2)$. It suffices to show that $v_1^{x_2} \in W_1 + W_2$ and $Q(v_1 + v_2) \in GF(q)$. But both of these follow from the preceding paragraph. (For example, $C_{M_i}(x_i)$ has a subgroup $SL(2, q)$ transitive on the set of nonzero elements of $A(x_i) \cap W_i$. Since $A(x_1) + A(x_2) \subseteq \mathcal{Q}(M_3)$ for some conjugate M_3 of M, it follows that $v_1^{x_2}$ lies in the M_3-invariant $GF(q)$-subspace spanning $\mathcal{Q}(M_3)$ which contains v_1.) It follows that (I 1 or 2) holds.

Proposition 6.9. *If* $\overline{G} = PSL(n, q)$, $n \geqslant 5$, *then* (RT 1) *holds.*

Proof. By (5.3), $V = \mathcal{Q}(G)$. There is an $\mathfrak{X}$-subgroup $K = SL(n - 1, q)$ (or possibly $\Omega^+(6, q)$ if $n = 5$), with K acting on $\mathcal{Q} = \mathcal{Q}(K)$ as in (RT 1) or (I 1). There are $x_i \in \mathfrak{X}$ ($i = 1, 2$) with $\langle K, x_i \rangle = KE_i \rhd E_i$ for elementary abelian $E_i \subset \mathfrak{X} \cup \{1\}$ of order q^{n-1}, such that $G = \langle K, x_1, x_2 \rangle$. Here $\dim \mathcal{Q}(KE_i) \leqslant \dim \mathcal{Q} + 1$.

By (4.2), either all $A(x)$, $x \in E_i^{\#}$, contain a 1-space V_i, or all $A(x)$ are contained in a 3-space. The latter case is impossible, in view of the KE_i-composition factors of $\mathcal{Q}(KE_i)$. For the same reason, $V_i = \operatorname{rad} \mathcal{Q}(KE_i)$, so $V = \mathcal{Q} \perp (V_1 \oplus V_2)$.

Suppose $n = 4$ and $\dim \mathcal{Q} = 6$. Then $\Omega(V) = \Omega^+(8, s)$, and we can apply the triality automorphism in order to reduce to the $\dim \mathcal{Q} = 8$ case.

Thus, we may assume $\dim \mathcal{Q} = 2(n - 1)$, so $\dim V = 2n$. Write $\mathcal{Q} = W_1 \oplus W_2$ for totally singular K-invariant $(n - 1)$-spaces W_j. By (4.1), E_i fixes $V_i + W_j$.

Pick any conjugate K^* of K such that $|K^* \cap E_i| = q^{n-2}$ and $x_i \in K^* \cap E_i$ for $i = 1, 2$; $L = \langle K^* \cap K \cap \mathfrak{X} \rangle$ is $SL(n - 2, q)$. Set $M_i = L(K^* \cap E_i)$.

Then

$$\mathcal{C}(M_i) = ((\mathcal{C}(L) \cap W_1) \oplus (\mathcal{C}(L) \cap W_2)) \perp V_i.$$

From $\mathcal{C}(M_i) \subseteq \mathcal{C}(K^*)$ we conclude that $K^* \cap E_1$ centralizes one of the subspaces $\mathcal{C}(L) \cap W_j$ and $K^* \cap E_2$ centralizes the other. Since there are at least three choices for K^*, it follows that (after possibly relabeling W_1, W_2) x_i centralizes two hyperplanes of W_{3-i} for $i = 1, 2$. Thus, x_i centralizes W_{3-i}.

Since x_{3-i} centralizes a hyperplane W_i of $W_i + V_i$, from (4.5 ii) we conclude that KE_{3-i} fixes a totally singular subspace $T_i \supseteq W_i + V_i$. Since x_i already fixes T_i by (4.1), so does $\langle K, x_1, x_2 \rangle = G$. This easily implies the result.

REMARK. Of course, (6.10) is only needed for $PSL(5, 2)$ and $PSL(5, 4)$, by (6.1 ii). Note that $PSU(n, q)$ can also be dealt with by employing $PSU(n-1, q)$ instead of the $PSU(n-2, q)$ used in (6.1).

PROPOSITION 6.10. *If* $\bar{G} = S_{2k+1}$, $k \geqslant 3$, *and* $V = \mathcal{C}(G)$, *then* (RT 3) *holds.*

PROOF. There is an $\mathfrak{X}$-subgroup $K = S_{2k}$. Then $G = \langle K, x \rangle$ with $L = K \cap K^* \leqslant C_G(x)$. Set $\mathcal{C} = \mathcal{C}(K)$, so $V = \mathcal{C} + A(x)$. Since any pair of elements of $\mathfrak{X}$ can be conjugated into K, the standard generators and relations for S_{2k+1} are easily obtained, so $Z(G) = 1$.

Suppose $V = \mathcal{C}$. Then K acts as in (I 1) or (RT 3). The first case leads to the contradiction $S_7 = G < \Omega(5, s)$. If (RT 3) holds, then both K and L fix exactly $s + 1$ totally singular $(2k-2)$-spaces by (4.8). Since x centralizes L, it fixes one of these $(2k-2)$-spaces W. Then G also fixes W. But L is irreducible on W, so x centralizes W, and hence so does G. Thus, $V \neq \mathcal{C}$.

Suppose K acts on $\mathcal{C}$ as $\Omega(5, 2)$. Then $L = S_5$ must act on $\mathcal{C}$ as in (I 5). But this implies that $\mathcal{C} = \mathcal{C}(L)$, and hence that $V = \mathcal{C}(K)$.

Thus, $\mathcal{C} = W_1 \oplus W_2$ for totally singular K-invariant subspaces W_i with dimensions in $\{2k-1, 2k-2\}$. If $\dim W_i = 2k - 2$ for $i = 1, 2$, then $\mathcal{C} = \mathcal{C}(L)$ again yields a contradiction. Thus, we may assume $\dim W_1 = 2k - 1$.

Since x fixes the nonsingular space $\mathcal{C}(L)$, either $A(x) \subset \mathcal{C}(L)$ or $A(L) \subseteq A(x)^{\perp}$. In the former case, x fixes $\mathcal{C}$, and hence so does G. Thus, x centralizes $\mathcal{C}(L)$. In particular, x centralizes hyperplanes of W_1 and W_2. By (4.5 ii), G fixes totally singular subspaces $T_i \supseteq W_i$, where $\dim T_i \leqslant \dim W_i + 1$ and $A(x) \cap T_i \neq 0$ for $i = 1, 2$. Then $V = T_1 \oplus T_2$, and hence $\dim T_1 = \dim T_2$. It follows that either $\dim T_i = 2k - 1$ or $2k$.

If $\dim T_1 = 2k - 1$, then $T_1 = W_1 \subset A(x)^{\perp}$, and hence G centralizes T_1. Consequently, $\dim T_1 = 2k$. Let $0 \neq u \in T_1 \cap \mathcal{C}^{\perp}$. Then $|u^G| = |G : K| = 2k + 1$, and it is easy to complete the proof.

7. Low rank cases. In this section, we will consider most of the initial cases needed in §6.

LEMMA 7. *If* $\bar{G} = PSL(2, q)$ *and* $V = \mathscr{Q}(G)$, *then* (RT 1) *holds.*

PROOF. By (4.2, 9), we may assume that $q = 9$ or $p = 2$, and that a Sylow p-subgroup P of G is contained in no root group of Ω. There then exist x, y, $z \in \mathfrak{X}$ with $G = \langle x, y, z\rangle$, x and y in P, and $A(x) \neq A(y)$. By (4.2), $\dim A(x) + A(y) = 3$. Thus, $\dim \mathscr{Q}(G) \leqslant 5$ by (4.3). But then $V = \mathscr{Q}(G)$ cannot contain a totally singular 3-space $A(x) + A(y)$. This contradiction proves the lemma.

LEMMA 7.2. *If* $\bar{G} = PSL(3, q)$, $PSU(3, q)$, A_6 *or* $Sz(q)$, *then* (I 3), (RT 1) *or* (RT 5) *holds.*

PROOF. Since $\dim \mathscr{Q}(G) \leqslant 6$ by (4.3, 9), $V = \mathscr{Q}(G)$ by (5.3). Thus, $\bar{G}$ is in $P\Omega(5, s) = PSp(4, s)$, $P\Omega^+(6, s) = PSL(4, s)$, or $P\Omega^-(6, s) = PSU(4, s)$, generated by transvections.

Now G acts on $W = V(4, s)$ or $V(4, s^2)$. By (4.4), G fixes the proper subspace $\mathfrak{D}(G)$, of dimension > 1. Then clearly $\dim \mathfrak{D}(G) = 3$ and G acts irreducibly on $\mathfrak{D}(G)$. The dual of (4.4) thus produces a complementary fixed subspace of $\mathfrak{D}(G)$. By Mitchell **[20]** and Hartley **[12]**, this situation corresponds to (I 3), (RT 1) or (RT 5).

COROLLARY 7.3. *If* $\bar{G}$ *is a Chevalley group containing an* $\mathfrak{X}$*-subgroup* $K = PSL(3, q)$, $SL(3, q)$, $PSU(3, q)$ *or* $SU(3, q)$, *then long root groups of* G *are contained in long root groups of* $\Omega(V)$.

LEMMA 7.4. *If* $\bar{G} = O^{\pm}(4, q)$ *with* q *even, then* (I 5) *or* (RT 2) *holds.*

PROOF. Let S be an $\mathfrak{X}$-subgroup of $C_G(x)$ with $|\bar{S}| = 2(q + 1)$. Then $\mathscr{Q} = \mathscr{Q}(S)$ is a nonsingular 4-space. There exist $y' \in S \cap \mathfrak{X}$ and $y \in C_{\mathfrak{X}}(y')$ with $G = \langle x, y, S\rangle$. Note that $\langle x, y'\rangle$ and $\langle y, y'\rangle$ are conjugate.

By (4.5 i), either (i) $A(x) \subset \mathscr{Q}$ or (ii) $\mathscr{Q} \subseteq A(x)^{\perp}$.

Assume (i). Then $A(y) \cap A(y') \neq 0$, so $\mathscr{Q}(G) = \mathscr{Q} + A(y')$ has dimension 5. (Dimension 4 would imply $\mathfrak{X} \subset \Omega^{\pm}(4, q)$.) Suppose $V = \mathscr{Q}(G)$. Apply a graph automorphism to G and obtain a subgroup G^* of $\Omega(5, s)$ generated by short root elements, that is, transvections. By (4.4), $\dim \mathfrak{D}(G^*) \leqslant 4$. It is now easy to obtain (I 5). Suppose $V \neq \mathscr{Q}(G)$. Clearly $\dim \mathscr{Q}(G)/\text{rad}\, \mathscr{Q}(G) \geqslant 4$. Thus, $R = \text{rad}\, \mathscr{Q}(G)$ is a singular 1-space. By (5.2), $\dim R^{\perp}/R = 5$ and $[G, R^{\perp}/R] = R^{\perp}/R$. Then $\dim V = 7$ and $\text{rad}\, V \subseteq [G, R^{\perp}] + R = \mathscr{Q}(G)$, which is not the case.

Assume (ii), so $A(x) \subseteq \mathscr{Q}^{\perp}$ and $A(y') \subseteq A(y)^{\perp}$. Let $0 \neq v \in A(y)$. Then $v^{\perp} \cap \mathscr{Q}$ contains $A(y')$ and meets $A(z)$ nontrivially for each $z \in S \cap \mathfrak{X}$; here each pair of $A(z)$'s meets only at 0. Thus, $\text{rad}\, v^{\perp} \cap \mathscr{Q}$ is a 1-space in

$A(y)$, and $v^\perp \cap \mathcal{Q}$ contains a totally singular 2-space T meeting each $A(z)$. Here, S fixes T. Thus, $W = \langle v, T, v^\perp \cap A(x)\rangle$ is totally singular and G-invariant by (4.1). Since G^W is generated by transvections, by the preceding paragraph $\dim W = 4$ and G^W is uniquely determined. As v was arbitrary, we can find another such W, say W^*, with $0 \neq W^* \cap A(y) \neq \langle v\rangle$. The irreducibility of G^W implies $W \cap W^{*\perp} = 0$, so $W + W^* = W \oplus W^*$ is nonsingular of dimension 8 and contains $A(y)$. Consequently, $V = W \oplus W^*$, and we are led to (RT 2).

Lemma 7.5. *If* $\bar{G} = PSp(4, q)$, *q odd, and* $V = \mathcal{Q}(G)$, *then* (I 1) *or* (RT 1) *holds.*

Proof. Here $V = \mathcal{Q}(G)$ has dimension $\leqslant 8$. Let S_1, S_2 be $\mathfrak{X}$-subgroups isomorphic to $SL(2, q)$, with $[S_1, S_2] = 1$. Then $\dim \mathcal{Q}(S_i) = 4$ by (7.1). Either (i) $\mathcal{Q}(S_1) = \mathcal{Q}(S_2)$, or (ii) $\mathcal{Q}(S_2) \subseteq \mathcal{Q}(S_1)^\perp$. Let $x \in G \cap \mathfrak{X}$ with $G = \langle S_1S_2, x\rangle$.

In (i), commuting $x, y \in \mathfrak{X}$ have $A(x) \cap A(y) \neq 0$. Thus, $\dim V \leqslant 5$ by (4.3), so $G \leqslant \Omega(5, s)$. Then G can be regarded as a subgroup of $Sp(4, s)$ generated by transvections. Consequently, (I 1) holds.

In (ii), $V = \mathcal{Q}(S_1) \perp \mathcal{Q}(S_2)$, so $\Omega(V) = \Omega^+(8, s)$. Regard $\bar{G} \leqslant P\Omega^+(8, s)$, apply triality, and obtain a group G^* containing an $\mathfrak{X}$-subgroup $K = \Omega^+(4, q)$ with $\dim \mathcal{Q}(K) = 4$. By case (i), $\mathcal{Q}(G^*)$ is a nondegenerate 5-space on which G^* induces $\Omega(5, q)$.

Since G^* centralizes $V/\mathcal{Q}(G^*)$, it follows that G^* fixes a nonsingular 6-space $W \supset \mathcal{Q}(G^*)$ with $\Omega(W) = \Omega^+(6, s)$. Thus, G^* is uniquely determined up to conjugacy, and hence so is G.

Lemma 7.6. *If* $\bar{G} = PSp(4, q)$, *q even, and* $V = \mathcal{Q}(G)$, *then* (I 1) *or* (RT 1) *holds.*

Proof. By (7.4), G has an $\mathfrak{X}$-subgroup $M = O^-(4, q)$, and either (i) $\dim \mathcal{Q} = 5$ or (ii) $\dim \mathcal{Q} = 8$, where $\mathcal{Q} = \mathcal{Q}(M)$. Write $G = \langle M, x\rangle$ with $L = C_M(\bar{x})$ an $\mathfrak{X}$-subgroup of order $q^2(q+1)2$. (This is accomplished by regarding $\bar{G}$ as acting on $V(4, q)$, and letting the direction of x be singular for M.)

In (i), $\mathcal{Q}(L) = \mathcal{Q}$. Thus, x fixes $\mathcal{Q}$, and hence so does G. We can then proceed as in (7.5) to obtain (I 1).

Suppose (ii) holds. Here $\mathcal{Q} = W_1 \oplus W_2$ for M-invariant totally singular 4-spaces W_1, W_2. Also, $\dim \mathcal{Q}(L) = 6$ and $\dim \operatorname{rad} \mathcal{Q}(L) = 2$. By (4.5 i), $\mathcal{Q}(L) \subseteq A(x)^\perp$ or $\mathcal{Q}(L) \supset A(x)$. In the latter case, G fixes $\mathcal{Q}$, so $\dim V = 8$ and we can proceed as in (7.5). Assume $A(x) \not\subseteq \mathcal{Q}(L) \subseteq A(x)^\perp$. By (4.5 ii), $G = \langle M, x\rangle$ fixes totally singular subspaces $U_i \supseteq W_i$. Since $V = \mathcal{Q}(G)$, necessarily $V = U_1 \oplus U_2$. Let G_0 be an $\mathfrak{X}$-subgroup $Sp(4, 2) \cong S_6$ of G such that

$M_0 = M \cap G_0$ is S_5, $L_0 = L \cap G_0$ is S_4, and $x \in G_0$. Since $\mathcal{Q}(L_0) \subseteq A(x)^\perp$, we may assume $q = 2$. Now $G = S_6$ acts on U_i, where dim $U_i = 4$ or 5, fixing no nonzero vector. It follows that $U_i = W_i$, and G acts as $Sp(4, 2)$ in the natural manner. But then $A(x) \subset \mathcal{Q}(L)$ yields a contradiction.

LEMMA 7.7. *If* $V = \mathcal{Q}(G)$ *and* $\bar{G}$ *is* $PSL(4, q)$ *or* $PSU(4, q) \neq PSU(4, 2)$, *then one of* (I 1, 2, 3) *or* (RT 1) *holds.*

PROOF. By (4.9), $V = \mathcal{Q}(G)$ has dimension $\leqslant 8$. Let K be an $\mathfrak{X}$-subgroup with $K = SL(3, q)$ or $SU(3, q)$. Then $\mathcal{Q}(K)$ is a nonsingular 6-space. If $V = \mathcal{Q}(K)$, then $G \leqslant \Omega^\pm(6, s)$ implies that G produces a subgroup of $SL(4, s)$ or $SU(4, s^2)$ generated by transvections. These yield (I 1, 2). We may thus assume dim $V = 7$ or 8.

Let S_1 and S_2 be $\mathfrak{X}$-subgroups isomorphic to $SL(2, q)$ with $[S_1, S_2] = 1$. Conceivably, $\mathcal{Q}(S_1) = \mathcal{Q}(S_2)$. But $G = \langle S_1 S_2, x\rangle$ for some x, which then implies that dim $\mathcal{Q}(G) \leqslant 6$.

Thus, $\mathcal{Q}(S_1 S_2) = \mathcal{Q}(S_1) \perp \mathcal{Q}(S_2)$ must be V. Now apply triality and reduce to the case of a 6-dimensional V.

LEMMA 7.8. *If* $\bar{G} = SU(4, 2)$, *and none of* (I 1, 2, 3), (RT 1) *holds, then* $\mathcal{Q} = \mathcal{Q}(G)$ *satisfies the following conditions*:

(i) dim $\mathcal{Q}/$rad $\mathcal{Q} = 8$, dim $\mathcal{Q} = 9$ *or* 10, *and* rad $\mathcal{Q}$ *is totally singular*; *and*

(ii) *there is an* $\mathfrak{X}$-*subgroup* $M = Sp(4, 2)$ *with* $\mathcal{Q} = \mathcal{Q}(M)$.

PROOF. There is an $\mathfrak{X}$-subgroup $M = Sp(4, 2)$. Then $\mathcal{Q}(M)$ has dimension 6, 8, 9, or 10, and $G = \langle M, x\rangle$ for some $x \in \mathfrak{X}$. The proof of (7.7) shows that we may assume that $V \neq \mathcal{Q}(M)$, and that dim $V > 8$ if $V = \mathcal{Q}$.

Suppose dim $\mathcal{Q}(M) = 6$. Then for S_1, S_2 as in (7.7), we obtain $\mathcal{Q}(S_1) = \mathcal{Q}(S_2)$ and hence dim $\mathcal{Q} \leqslant 6$, $\mathcal{Q} = \mathcal{Q}(M)$, and $V = \mathcal{Q}$ by (5.1).

Thus, dim $\mathcal{Q}(M) \geqslant 8$. There is an elementary abelian $\mathfrak{X}$-subgroup E of order 2^4 with $|E \cap \mathfrak{X}| = 5$ and $N_G(E)/C_G(E)$ inducing S_5 on $E \cap \mathfrak{X}$. Every triple of elements of $E \cap \mathfrak{X}$ can be conjugated to a triple x_1, x_2, x_3 in $E \cap M \cap \mathfrak{X}$. If dim $\mathcal{Q}(M) = 8$ or 9 as in (RT 3 or 4), then $A(x_3) \subset A(x_1) + A(x_2)$. Thus, $\mathcal{Q}(E) = A(x_1) + A(x_2)$ and $\mathcal{Q} = \mathcal{Q}(\langle M, E\rangle) = \mathcal{Q}(M)$, which proves (i) and (ii) in this case.

Suppose $\mathcal{Q}(M)$ is as in (RT 4) with dim $\mathcal{Q}(M) = 10$ and dim rad $\mathcal{Q}(M) = 2$. There is an $\mathfrak{X}$-subgroup L of G with $L = 3^3 \cdot S_4$ as in (I 10) and (RT 9). Then dim $\mathcal{Q}(L) = 8$ and $G = \langle L, x\rangle$ for some $x \in \mathfrak{X}$, so dim $\mathcal{Q} \leqslant 10$. It follows that $\mathcal{Q} = \mathcal{Q}(M)$ once again.

LEMMA 7.9. *If* $\bar{G} = S_8 \cong O^+(6, 2)$, *and* $V = \mathcal{Q}(G)$, *then* (RT 3) *holds.*

PROOF. There is an $\mathfrak{X}$-subgroup $K = S_7$. Then $G = \langle K, x\rangle$ with $L = K \cap K^x \leqslant C_G(x)$. Set $\mathcal{Q} = \mathcal{Q}(K)$, so $V = \mathcal{Q} + A(x)$. Then $\mathcal{Q} = W_1 \oplus W_2$ for totally singular K-invariant 6-subspaces W_i. Also, dim $\mathcal{Q}(L) = 10$, and dim rad $\mathcal{C}(L) = 2$. Note that x fixes the nonsingular 8-space $\mathcal{C}(M)$ for each $\mathfrak{X}$-subgroup $M = S_5$ of L. If $A(x) \subset \mathcal{C}(M)$, then $A(x) \subseteq \bigcap \{\mathcal{C}(M^l) | l \in L\} = 0$. Thus, x centralizes hyperplanes of the 6-spaces W_i, so by (4.5 ii) there are G-invariant totally singular subspaces $T_i \supseteq W_i$ with $V = T_1 \oplus T_2$. Here $\dim T_1 = \dim T_2 \in \{6, 7\}$.

If $\dim T_i = 6$, then $A(x)$ meets $T_i = W_i$ nontrivially, and x induces a transvection on W_i with axis $W_i \cap \mathcal{C}(L)$. Then x also centralizes $W_i/((W_{3-i} \cap \mathcal{C}(L))^\perp \cap W_i)$, and hence has direction $(W_{3-i} \cap \mathcal{C}(L))^\perp \cap W_i =$ rad $\mathcal{C}(L) \cap W_i$. It follows that $A(x) =$ rad $\mathcal{C}(L)$ is uniquely determined from K and L. Moreover, a K-invariant $GF(2)$-subspace V_0 spanning V will meet rad $\mathcal{C}(L)$ in 2-space. Thus, the action of G is uniquely determined and (RT 3) holds. It remains to eliminate the case $\dim T_i = 7$. Pick $0 \neq v \in T_1 \cap \mathcal{C}^\perp$. Then $|v^G| = 8$ implies that G^{T_1} is as in (T 4). But then $\mathcal{C}(G) \neq V$.

LEMMA 7.10. *$\bar{G} \neq PSL(2, q) \sim S_k$ with $k > 2$.*

PROOF. Deny! We may assume that $k = 3$ and $V = \mathcal{C}(G)$. Let M be an $\mathfrak{X}$-subgroup with $\tilde{M} = PSL(2, q) \sim S_2$. Then $M = \bar{M}$ behaves as in (7.4): either (i) $\mathcal{C} = \mathcal{C}(M)$ is a nondegenerate 5-space, or (ii) $\mathcal{C}$ is a nondegenerate 8-space. Let $x \in \mathfrak{X}$ with $G = \langle M, x\rangle$. Set $R =$ rad V.

The structure of M forces $Z(G) = 1$. Let $G \rhd L_1 \times L_2 \times L_3$ with $L_i \cong SL(2, q)$, G inducing S_3 on $\{L_1, L_2, L_3\}$, $M \rhd L_1L_2$, and $L_1^x = L_3$. Then L_3 fixes $\mathcal{C}$. The action of M on $\mathcal{C}$ yields $V \neq \mathcal{C}$ and $L_3^{\mathcal{C}} = 1$. (In (ii), $C_{O(\mathcal{C})}(M)$ does contain an $SL(2, q)$, but no S_3 is available to permute the three $SL(2, q)$'s.)

In (i), $\dim V \leqslant 7$, so $\dim \mathcal{C} \cap \mathcal{C}^x \geqslant 3$. Thus, L_3 centralizes a 3-space of $\mathcal{C}^x = \mathcal{C}(L_2L_3)$, which is not the case.

In (ii), $\dim V \leqslant 10$ and $\dim \mathcal{C} \cap \mathcal{C}^x \geqslant 6$. Since M fixes two complementary totally singular 4-spaces W_1, W_2 of $\mathcal{C}$, and $L_3 = L_1^x$ fixes no nonzero vector of W_i^x, this case is also impossible.

8. Generalized hexagons. In this section we will consider the cases $\bar{G} = G_2(q)'$, ${}^3D_4(q)$ and HJ, assuming that $V = \mathcal{C}(G)$. There is an $\mathfrak{X}$-subgroup M, where $M = SL(3, q)$ in the first two cases and $M = 3 \cdot A_6$ in the last one. (For properties of HJ, see **[11]**, **[32]**.) Here, $\mathcal{C} = \mathcal{C}(M)$ is 6-dimensional, and $\mathcal{C} = W_1 \oplus W_2$ for M-invariant totally singular 3-spaces W_1, W_2 on which M acts in the obvious way.

Let $\mathcal{H}$ denote the generalized hexagon associated with $\bar{G}$. Set $\mathcal{L}^* = A(x)^G$ for $x \in \mathfrak{X}$; then $\mathcal{L}^*$ is in 1-1 correspondence with the root groups of $\bar{G}$ (see (7.3)). Let $\mathcal{P}$ denote the set of maximal abelian subgroups $P \subset \mathfrak{X} \cup \{1\}$ of G; here $|P| = q^2$, where $q = 2$ for HJ. By (4.2), $\bigcap \{A(x) | x \in P^\#\}$ is a 1-space P^*. Set $\mathcal{P}^* = P^{*G}$.

LEMMA 8.1. (i) *Define* φ: $(\mathcal{P}^*, \mathcal{L}^*, \subset) \to \mathcal{H}$ *be letting* $A(x)^\varphi$ *be the unique line of* $\mathcal{H}$ *fixed by* $C_{\bar{G}}(x)$, *and* $P^{*\varphi}$ *the unique point fixed by* $N_{\bar{G}}(P)$. *Then* φ *preserves order and is* 1-1; φ *is even an isomorphism if* $\bar{G} \neq HJ$. *If* $\mathcal{H}$ *is regarded as embedded in* $V(7, q)$, $V(8, q)$, *or* $V(7, 4)$, *respectively, then* φ *preserves the relation of orthogonality on* $\mathcal{P}^* \cup \mathcal{L}^*$.

(ii) dim V *is* 7 *for* $G_2(q)$ *and* HJ, *and* 7 *or* 8 *for* ${}^3D_4(q)$.

PROOF. The definitions, along with (4.2), yield (i). For (ii), consider first $\bar{G} = G_2(q)'$ or HJ. Here, $G = \langle M, x\rangle$ with $x \in \mathfrak{X}$, where $C_{M\cap\mathfrak{X}}(x) \neq \varnothing$. Thus, $V = \mathcal{C}(G)$ has dimension $m \leqslant 7$. Using $\mathcal{C}$ and (i), M must be 7. Similarly, if $\bar{G} = {}^3D_4(q)$ then $G = \langle K, x\rangle$ for an $\mathfrak{X}$-subgroup $K = G_2(q)$ and an $x \in \mathfrak{X}$ having $C_{K\cap\mathfrak{X}}(x) \neq \varnothing$.

PROPOSITION 8.2. *If* $\bar{G} = G_2(q)$, ${}^3D_4(q)$ *or* HJ, *then* (I 6, 7 *or* 8) *holds.*

PROOF. For $G_2(q)'$ and ${}^3D_4(q)$, it is not difficult to show that any two embeddings of $\mathcal{H}$ into V as in (8.1) are equivalent under $\Omega(V)$; we omit the proof. (For $G_2(q)'$, this is essentially contained in Schellekens [**27**, pp. 31–33].) Alternatively, the method about to be used for HJ also applies to $G_2(q)'$ and ${}^3D_4(q)$ (using M to get to $G_2(q)'$, and $G_2(q)'$ to handle ${}^3D_4(q)$).

Let M and x be as before for $\bar{G} = HJ$. Let $S = S_3$ be an $\mathfrak{X}$-subgroup of M. Set $\mathcal{B} = \mathcal{C}(S)$ and $C = C_G(S)$. Then $\bar{C} \cong A_4$, $M \cap C = Z(M)$ has order 3, and $C \cap \mathfrak{X} = \varnothing$. We have $V = \mathcal{B} \perp \mathcal{B}^\perp$ and dim $\mathcal{B}^\perp \cap \mathcal{C} = 2$. Since $G = \langle M, O_2(C)\rangle$, it follows that $O_2(C)$ is nontrivial on $\mathcal{B}^\perp$. Hence, C induces A_4 on $\mathcal{B}^\perp$. Pick a $Z(M)$-invariant $GF(4)$-space W spanning $\mathcal{B}^\perp \cap \mathcal{C}$. There are unique M- and C-invariant $GF(4)$-spaces spanning $\mathcal{C}$ resp. $\mathcal{B}^\perp$ and containing W. Thus, $G = \langle M, x\rangle$ fixes a $GF(4)$-space spanning W. We may now assume $s = 4$, and then apply Wales [**36**].

In fact, it is straightforward to uniquely describe $\mathcal{P}^* \cap \mathcal{B}^\perp$ at this stage, thereby avoiding [**36**]. Alternatively, HJ can also be handled by combining (8.1), [**36**], and the 2-modular information in Hall-Wales [**11**].

9. Solvable normal subgroups. Throughout this section, we will assume that G has a normal l-subgroup $L \nleqslant Z(G)$, where $l \neq p$. We may assume that L is minimal subject to this restriction. By (3.4), we may assume $p = 2$.

PROPOSITION 9.1. *If* L *is abelian, then one of* (I 10, 11), (RT 9) *holds.*

PROOF. Here L is even elementary abelian. Using $\langle x\rangle L$ and (4.2), we find that $l | s \pm 1$. Each irreducible constituent of L thus has dimension 1 or 2.

Case 1. *G is irreducible*. By Clifford's theorem [**9**, p. 70], $V = V_1 \oplus \cdots \oplus V_k$ with each V_i a direct sum of isomorphic irreducible L-modules, irreducible L-submodules of V_i and V_j being nonisomorphic for $i \neq j$, and G is transitive on $\Sigma = \{V_1, \ldots, V_k\}$.

If $k = 1$, extend the field in order to make L consist of scalar transformations. Then $x \in \mathfrak{X}$ inverts L, and hence must be acting as an involutory field automorphism on V. But then $\dim C_V(x) = \frac{1}{2} \dim V$ forces $\dim V = 4$. This is just (RT 9).

Consequently we may assume $k \geqslant 2$. Suppose $V_1^x = V_2$. Then $\dim [x, V_1 + V_2] = \dim V_1$, so $\dim V_1 \leqslant 2$, with equality only if $A(x) \subset V_1 + V_2$. Moreover, we may assume $k \geqslant 3$.

If $\dim V_1 = 1$, then (4.1) implies that $(V_1 + V_2) \cap A(x)$ is a 1-space of $\operatorname{rad}(V_1 + V_2)$. Since L fixes only two 1-spaces V_1, V_2 of $V_1 + V_2$, necessarily $\operatorname{rad}(V_1 + V_2) = V_1 + V_2$. Since x does not centralize $V_3 + \cdots + V_k$, we may assume $V_3^x = V_4$, so $L\langle x \rangle$ fixes $W = V_1 + V_2 + V_3 + V_4$, $W \supset A(x)$, and W is not totally singular. Since $L\langle x \rangle$ fixes rad W, it follows that W is nonsingular. Then $W^\perp = V_5 \oplus \cdots \oplus V_k$, and we may assume $V_1^\perp = \Sigma_{i \neq 3} V_i$ and $V_2^\perp = \Sigma_{i \neq 4} V_i$. Since V_3 is then the unique L-complement to $V_1^\perp$, necessarily $G_{V_1} = G_{V_3}$. We can therefore write $V = W_1 \perp \cdots \perp W_n$ with $n = k/2$, $\dim W_i = 2$, $W_1 = V_1 + V_3$, and G acting transitively on $\Sigma' = \{W_1, \ldots, W_n\}$. Since x induces a transposition on Σ', G induces S_n on Σ'. By the transitivity of G, there is $g \in G$ with $V_1^g = V_3$. Using S_n, we can force $g^{\Sigma'} = 1$ to also hold. Thus, if K is the kernel of the action of G on Σ', then G acts on the nontrivial elementary abelian 2-group $J = O_2(K/O(K))$ of order at most 2^{n-1}. If x acts nontrivially on J, then the composition factors of J indicated in example (RT 3) show that $|J| = 2^{n-1}$, and hence that (I 11) holds. Suppose G centralizes J. Then $K = \langle g \rangle O(K)$ with g an involution fixing no V_i, and $G/O(K)$ is a nonsplit central extension of S_n by Z_2. Clearly $G = C_G(g)O(K)$. Since $V_1^g = V_3$, $L \nleqslant C_G(g)$. Consequently, induction applies to $C_G(g)$. However, no such group occurs, so this situation is impossible.

Next suppose $\dim V_1 = 2$. This time G induces S_k on Σ. Since G_{V_1} is transitive on $\Sigma - \{V_1\}$, necessarily V_1 is nonsingular and $V = V_1 \perp \cdots \perp V_k$. It is now easy to obtain (I 10) or (I 11) as above.

Case 2. *G is reducible*. Clearly G permutes the irreducible constituents of L.

Suppose G fixes a singular 1-space R. By (5.2) and induction, $C_{R^\perp/R}(L) = 0$, so rad $V = 0$ and $\dim C_V(L) \leqslant 2$. By (5.1), $C_V(L)$ cannot be a nonsingular 2-space. Thus, $C_V(L) = R$ has a unique L-complement T, necessarily fixed by G. Then G fixes the 1-space rad $T \neq R$, which is a contradiction.

Thus, $C_V(G) = \operatorname{rad} V$. Assume that $C_V(L) = 0$, and let W be a minimal G-invariant subspace. Then $W = \operatorname{rad} W$ has dimension $\geqslant 2$. Note that $A(x) \nsubseteq W^\perp$ since $W \nsubseteq A(x)^\perp$. Thus, G^W is an irreducible group generated by transvections. Proceeding as before, we find that $G = A \rtimes S_n$ with $|A| = a^{n-1}$, $a|s-1$, and $n = \dim W$. Also, $G^{W^\perp}$ is a group generated by transvections, with G centralizing $W^\perp/W$. Since L centralizes $W^\perp/W$, $C_{W^\perp}(L) = 0$ implies that $W^\perp = W$. Write $W = V_1 \oplus \cdots \oplus V_n$ with G acting on

$\{V_1, \ldots, V_n\}$ as in (T 9). Then $n > 2$ implies that no V_i, V_j are contragredient for L. It follows that $V = V_1 \oplus \cdots \oplus V_{2n}$ with G permuting the nonsingular subspaces $V_i + V_{n+i}$, $1 \leqslant i \leqslant n$. This yields (RT 10).

Finally, assume $W = C_V(L) \neq 0$, and let $W' = [L, V]$ be its unique L-complement. Then G fixes W'. By (5.1), rad $W \neq 0 \neq$ rad W', and then dim rad $W' \geqslant 2$ since $L^{W'}$ is faithful. Also, dim $W \geqslant 2$, as otherwise $W = C_V(G)$ would be nonsingular and W' nondegenerate, contradicting (5.1). Then $G^W \neq 1 \neq G^{W'}$ and (4.1) imply that $A(x)$ meets both rad W and rad W' nontrivially. Consequently, G^W and $G^{W'}$ are both groups generated by transvections. Temporarily replacing V by the direct sum of W' and its dual allows us to revert to a previous situation. Thus, G has the form $A \rtimes S_n$ with $|A| = a^{n-1}$ and $n = \dim W'$, and G acts irreducibly on W'. Then W' is totally singular. However, L fixes $H^\perp \cap W'$ for each hyperplane H of W, whereas L fixes exactly n hyperplanes of W', and these are independent. This proves the impossibility of $C_V(L) \neq 0$, and completes the proof of (9.1).

Proposition 9.2. (I 10, 11) *or* (RT 9) *holds.*

Proof. Let G be a minimal counterexample. Then any choice for L is nonabelian. Certainly $Z(L) \leqslant Z(G)$.

The minimality of G implies that whenever $x \in N_G(Q) - C_G(Q)$ for a q-group Q, $q > 2$, necessarily Q is abelian or $q = 3$. (The latter case occurs in (I 10) when $n = 3$ and $a = 3^i$.)

Lemma 9.3. $[x, L]$ *is either cyclic or nonabelian of order* 27 *and exponent* 3.

Proof. Let x invert $g, h \in L$ with $\langle g, h\rangle$ noncyclic and of maximal order. Viewing $\langle x, g, h\rangle$ in $SL(3, s^2)$ as usual, we find that it is dihedral or $SU(3, 2)'$. Consider the latter possibility. It suffices to prove that $\langle g, h\rangle = [x, L]$, so suppose x inverts $k \in L - \langle g, h\rangle$. Then $G = \langle x, xg, xh, xk\rangle$. As usual, we can push G into $SL(4, s^2) = SL(W)$ as a group generated by transvections. The 3-group L cannot act irreducibly on the 4-space W. Since $\langle g, h\rangle$ is nonabelian, L must fix unique complementary i-spaces W_i of W $(i = 1, 3)$. Now the group induced by G on W_3 cannot exist.

Lemma 9.4. $[x, L]$ *is noncyclic.*

Proof. Suppose $[x, L]$ is cyclic. By the minimality of L, we have $L = \langle [x, L] | x \in \mathfrak{X}\rangle$. Thus $x, y \in \mathfrak{X}$ exist having $\langle [x, L], [y, L]\rangle$ nonabelian. By (4.2), $|xy|$ is 2 or odd.

If xy induces an automorphism of even order on L, then $|x'y'| = 2$ for any $x' \in x[x, L]$, $y' \in y[y, L]$, and then $[x, L]$ and $[y, L]$ commute. This situation cannot occur.

Suppose $|xy|$ is odd. Then $F = \langle x, [x, L], y\rangle$ is, as usual, a subgroup of

$SL(3, s^2)$ generated by transvections and having a normal nonabelian l-subgroup $F \cap L$. Then $F = SU(3, 2)'$ has $[x, F \cap L]$ noncyclic. This proves (9.4).

In order to complete the proof of (9.2), we now assume $[x, L]$ is noncyclic. Suppose there is a $y \in \mathfrak{X}$ with $|xy|$ odd and $\langle x, y\rangle \nleqslant \langle x\rangle L$. Then $G = \langle x, [x, L], y\rangle$. As usual, this is a subgroup of $SL(4, s^2)$, generated by a class of transvections, and having a nonabelian normal 3-subgroup L. This leads to the same contradiction as in (9.3).

Thus, $y \in \mathfrak{X} - \langle x\rangle L$ implies that $|xy| = 2$. (Since $G > \langle x, [x, L]\rangle$, such a y exists.) Then also y centralizes $\langle x, [x, L]\rangle \cap \mathfrak{X}$, and hence even $[x, L]$. Consequently, distinct subgroups of the form $\langle x, [x, L]\rangle$ commute. Since $\mathfrak{X}$ is a class, some $y \in \mathfrak{X}$ must move $\langle x, [x, L]\rangle$, which we have seen cannot occur. This proves (9.2).

10. Singular 1-spaces. The only case remaining in the proof of Theorem I is that in which rad $\mathcal{Q}(G)$ contains a singular 1-space R (cf. (5.1)). By (5.2), G centralizes R.

Set $K = C_G(R^\perp/R)$. If $K \neq 1$ then $([K, V] + R)/R$ is a G-invariant 1-space of $R^\perp/R$.

Write $G^* = G/K = G^{R^\perp/R}$.

By (4.2) and (5.2), induction applies to G^*. By §9, we may assume that $O_\infty(G) = Z(G)$. Note that a few possibilities for $\bar{G}$ have been handled in §7.

Set $P = O_p(\Omega_R)$. If $K \neq 1$, let $B < P$ correspond to $([K, V] + R)/R$ as in (4.6). Note that $K = B \cap G$.

Since $\Omega_R = (\Omega_R \cap \Omega_T)P$ for a nonsingular 2-space $T \supset R$, necessarily $GP = LP$ for some $L < \Omega_T$. By (5.1), G and L are not conjugate in Ω, and hence not in LP. Note, however, that GP is not a semidirect product unless $K = 1$. In any event, $C_{O(V)_R}(T^\perp) \cong GF(s)^*$ acts on $H^1(L, P)$.

While this leads to cohomological questions, a number of cases can be handled directly (a necessity, since Jones **[15]** did not deal with twisted groups of small characteristic).

LEMMA 10.1. *$\bar{G}$ is not $PSL(n, q)$ for $n \geqslant 4$, $PSp(n, q)$ for q odd and $n \neq 4$, $PSU(n, q)$ for $n > 4$, $PSU(4, q)$ for $q > 2$, $P\Omega^\pm(n, q)$ for $n > 6$, ${}^3D_4(q)$, $O^\pm(2n, q)$ for $n \geqslant 3$ (excluding $O^+(6, 2)$), S_{2n+1}, $P\Omega^{-,\pi}(6, 3)$, $G_2(q)'$, or HJ.*

PROOF. By (4.9, 10), we can apply (5.3) except in the cases $P\Omega^\pm(n, q)$ for $n \geqslant 6$ (excluding $P\Omega^-(6, 2) = PSU(4, 2)$), $O^\pm(2n, q)$, $G_2(q)'$, HJ, or $PSp(2, q)$. (Note that $P\Omega^{-,\pi}(6, 3)$ can be generated by 6 elements of $\mathfrak{X}$.)

Suppose $\bar{G} = P\Omega^\pm(n, q)$. Then G^* acts as in (I 1, 2 or 4). There is an $\mathfrak{X}$-subgroup M with $M/Z(M) = P\Omega^+(n-1, q)$; moreover, $\mathcal{Q}(M)$ is nondegenerate, so $R \cap \mathcal{Q}(M) = 0$. There exist $x, y \in \mathfrak{X}$ with $y \in M$, $G = \langle M, x\rangle$ and $\langle x, y\rangle^\# \subset \mathfrak{X}$. Thus, $\dim \mathcal{Q}(G) = \dim(\mathcal{Q}(M) + A(x)) \leqslant$

dim $\mathscr{C}(M) + 1$, from which $\mathscr{C}(G)/R = (\mathscr{C}(M) + R)/R$ follows. This implies that neither (I 1) nor (I 2) can occur on $R^{\perp}/R$, so G^* must act as in (I 4). However in that case dim $\mathscr{C}(G) \leqslant 8 =$ dim $R^{\perp}/R$ by (4.3, 9), and this is a contradiction.

Suppose $\bar{G} = O^{\pm}(2n, q)$ with $n \geqslant 3$. There is an $\mathfrak{X}$-subgroup $M = O^{\pm}(2n-2, q)$ such that $G = \langle M, x, y\rangle$ for some $x, y \in \mathfrak{X}$. This time, dim $\mathscr{C}(G) \leqslant 2(2n-2) + 4 =$ dim $\mathscr{C}(G)/R$ produces the desired contradiction. (Note that the case $n = 2$ was handled in (7.4).)

If $\bar{G} = G_2(q)'$ or HJ, then dim $\mathscr{C}(G) \leqslant 7$ exactly as in (8.1), whereas dim $\mathscr{C}(G)/R = 7$ by induction.

Finally, if $\bar{G} = PSp(2, q)$, q odd, then dim $R^{\perp}/R = 4$, $K = 1$ and $G^* = SL(2, q)$. Consequently, $G = C_{GP}(Z(G))$ implies that $H^1(G, P) = 0$.

LEMMA 10.2. (i) *G^* does not act on $R^{\perp}/R$ as $\Omega(5, q)$.* (ii) *$\bar{G}$ is not $PSp(4, q)$ with q odd.*

PROOF. (i) Here we would have dim $V = 7$. Embed $G < \Omega(V) < \Omega^+(8, s) = \Omega(V^+)$, apply triality, and obtain a group $G^+ < \Omega(V^+)$ fixing a 4-space. Then $\mathscr{C}(G^+) = V^+$ implies that G^+ is as in (RT 1). It follows that $[G, V^+]$ is a nondegenerate 5-space in V. This contradicts (5.1).

(ii) If $\bar{G}$ is $PSp(4, q)$, then dim $\mathscr{C}(G) \leqslant 8$ by (4.9). But then dim $\mathscr{C}(G)/R \leqslant 7$, so (i) applies.

REMARK. Note that we have avoided dealing with such cohomological oddities as $\Omega(5, 3)$ and $\Omega^+(6, 2)$ (cf. Jones [**15**]).

LEMMA 10.3. *If $\bar{G} = S_{2n}$, then* (RT 3 *or* 4) *holds.*

PROOF. Let $R^{\perp} = V_1 + V_2$ for G-invariant totally isotropic subspaces V_i with $V_1 \cap V_2 = R$. Set $W_i = [G, V_i]$. Then $\mathfrak{X}$ induces transvections on V_i, so dim $W_i \leqslant 2n - 1$ by (4.4).

Suppose (RT 4) holds for G^*. Then dim $V_i = 2n + 1$ or $2n$ and dim $\mathscr{C}(G^*) = 4n - 1$ or $4n - 2$. We may assume dim$[G, V_1/R] = 2n - 1$, and then $R \not\subseteq W_1$ and $\mathscr{C}(G) = W_1 \oplus W_2$. Now

$$\begin{aligned}\dim(W_1 \oplus W_2)/R &= \dim \mathscr{C}(G^*)\\ &= \dim(W_1 + R)/R + \dim(W_2 + R)/R\\ &= \dim W_1 + \dim(W_2 + R)/R\end{aligned}$$

implies that $R \subset W_2$. Then dim$[G, V_2/R] = 2n - 2$, dim $R^{\perp}/R = 4n - 2$, and dim $W_1 \oplus W_2 = 4n - 2$. Since W_1 is contained in one totally singular $2n$-space $W_1 + R$, it is contained in a second one U_1, necessarily G-invariant. Similarly, W_2 is in a G-invariant totally singular n-space U_2 with $U_2 \cap W_1 = 0$. Since $U_1 + U_2 \supset R^{\perp}$, necessarily $V = U_1 \oplus U_2$. The action of G on U_2 uniquely determines the action on U_1. There is an $\mathfrak{X}$-subgroup $M = S_{2n-1}$, and $U_2 = (U_2 \cap \mathcal{Q}(M)) \oplus (U_2 \cap \mathcal{Q}(M)^\perp)$ by induction. Note that $M \cap K = 1$, and that every pair of elements of $\mathfrak{X}$ is conjugate to a pair of elements of M. The usual generators and relations for S_{2n} can then be obtained, so $G = S_{2n}$. Now pick $u \in U_2 \cap \mathcal{Q}(M)^\perp$ with $u \notin R$. Then $|u^G| = |G : M| = n$, and it follows easily that (RT 4) holds for G.

Now suppose (RT 3) holds for G^*, so dim $V_i = 2n - 1$ and $K = 1$. Since V_1/R and V_2/R are L-isomorphic (by the construction in (RT 3)), by (4.8) there is a subgroup C of $C_{\Omega_R}(L)$ inducing $SL(2, s)$ on the set of proper L-invariant subspaces of $R^\perp/R$. Moreover, dim $H^1(L, P) =$ dim $H^1(L, P_1)$ + dim $H^1(L, P_2) = 1 + 1$ by (4.11) and Pollatsek [24]. It follows that C is transitive on $H^1(L, P) - \{0\}$, and hence that (RT 4) holds.

LEMMA 10.4. *If* $\overline{G} = Sp(2n, q) \neq Sp(4, 2)$ *with* q *even, then* (RT 2) *holds.*

PROOF. By (10.2), we have $R^\perp = V_1 + V_2$ for G-invariant totally singular subspaces V_i with $V_1 \cap V_2 = R$.

First suppose (RT 1) holds for G^*, so dim $V_i/R = 2n$ and $K = 1$. Let $P_i < P$ correspond to V_i/R as in (4.6). Then $P = P_1 \oplus P_2$ with P_1 and P_2 isomorphic L-modules. Precisely as in the last part of (10.3), we obtain both dim $H^1(L, P) = 1 + 1$ (Pollatsek [25]) and the existence of a subgroup of Ω_R centralizing L and transitive on $H^1(L, P) - \{0\}$. Hence, (RT 2) holds for G.

Now suppose (RT 2) holds for G^*, so dim $V_i/R = 2n + 1$. With P_i as above, P_1 and P_2 are contragredient, so dim $H^1(L, P) = 1 + 0 =$ dim $H^1(L, P/B)$. Hence, $GB/B = MB/B$ for a complement M to P in LP. Then $G = (GB)' = M$ and $K = 1$. Consequently, there is (up to conjugacy) just one possible candidate for G. Such a candidate is obtained by embedding $O(2n + 1, q) < O^+(2n + 2, q) < \Omega^+(4n + 4, q)$. However, this $O(2n + 1, q)$ induces no transvections on its invariant $2n + 2$-spaces. Consequently, this case cannot occur.

LEMMA 10.15. *If* $\overline{G} = SU(4, 2)$, *then* (RT 8), (RL 5) *or* (RL 6) *holds.*

PROOF. This time $R^\perp$ has G-invariant totally singular subspaces $V_i \supset R$ with dim $V_i/V_1 \cap V_2 = 4$ for $i = 1, 2$. Set $W_i = [G, V_i]$, so dim $W_i \leqslant 5$ by (4.4, 9).

Assume (RT 8) or (RL 6) holds for G^*. In each case, $\mathcal{Q}(G) = W_1 \oplus W_2$ as in (10.3), with dim $W_i = 5$ and G fixing a 1-space of W_i. Also as in (10.3), W_i is contained in a G-invariant 6-space U_i, with $V = U_1 \oplus U_2$. It suffices to prove that G cannot act on U_1 as an indecomposable group generated by transvections. We may assume $R \subset W_1$. Then the action of G^* is known on U_1/R, and hence on $E = O_2(SL(U_1)_R)$. Note that $K \leqslant C = C_E(G)$. By (4.11) and Fischer [7, (16.1.10)], dim $H^1(L, E/C) = 1$. Then dim $H^1(L, E) = 1$. (If $LC = MC$ with $M \cap C = 1$, then $L = (LC)' = M$.) It follows that

$G^* = MK/K$ for some complement M of E in LE. Now $GE = LE$ and $\dim H^1(L, E) = 1$ imply that there is at most one choice for G (up to Ω_R-conjugacy). We will describe a realization of a complement G_1 to E in LE which contains no transvections of U_1. Regard $3 \cdot P\Omega^{-,\pi}(6, 3)$ as a subgroup of $SL(U_1)$ generated by transvections. The derived group G_1 of the centralizer of a transvection is then an $SU(4, 2)$ containing no transvections and acting indecomposably on U_1. (See the discussion in (RT 8).) Thus, G cannot be G_1, and neither (RT 8) nor (RL 6) can occur for G^*.

If (RL 5) holds for G^*, extend the field and find that (RT 8) holds for G^*.

If (RT 1) holds for G^*, then $V_1 + V_2 = R^\perp$ with $V_1 \cap V_2 = R$. Let $P_i < P$ correspond to V_i as in (4.6). By (4.11) and Fischer [7], $\dim H^1(L, P) = 1 + 1$. (Note that $GF(4) \subseteq GF(s)$ by induction, so the L-modules P_i are isomorphic via a field automorphism since they are contragredient for $L = SU(4, 2)$.) Then LP contains s^2 classes of complements to P. Precisely as in the discussion in (RL 6), these classes fall into orbits of length $1, s-1, s-1, (s-1)^2$ under $C_{\Omega_R}(L)$, and hence (RT 8) or (RL 6) holds.

Finally, if (I 3) holds for G^*, then $\dim V = 10$, $\dim H^1(L, P) = 1$, and (RL 4) is easily obtained.

Remark. A few parts of the proofs of the preceding lemmas could have been slightly simplified using properties of covering groups of $\bar{G}$ (Griess [**10**]). On the other hand, heavier use of $H^1(L, P)$ (Cline-Parshall-Scott [**4**], Jones [**15**]) seems to produce more complicated proofs.

11. Subgroups of $Sp(2n, s)$, $SU(m, s)$, $G_2(s)$, ${}^3D_4(s)$. We have now completed the proof of Theorem I, and hence also of Theorem II. From these results, and their proofs, it is easy to determine all conjugacy classes of subgroups G of the above groups, generated by a class $\mathfrak{X}$ of elements of long root groups, and having $O_p(G) \leqslant G' \cap Z(G)$. We shall merely list all such subgroups, leaving the proofs to the reader. In the symplectic and unitary cases, we may assume that G centralizes no nonsingular subspace of V.

Symplectic groups.

1. $Sp(2n, q) \leqslant Sp(2n, q^i)$.
2. $O^{\pm}(2n, q) < Sp(2n, q^i)$ for q even.
3. $S_{2n} \leqslant Sp(2n, 2^i)$, $Sp(2n-2, 2^i)$.
4. $S_{2n+1} < Sp(2n, 2^i)$.
5. $SL(2, 5) < Sp(2, 9^i)$.
6. Dihedral subgroups of $Sp(2, 2^i)$.

Unitary groups.

1. $SU(m, q) \leqslant SU(m, q^i)$.
2. $Sp(m, q) < SU(m, q^i)$.
3. $O^{\pm}(m, q) < SU(m, q^i)$, for m and q even.

4. $S_{2n} < SU(2n, 2^i), SU(2n-2, 2^i)$.
5. $S_{2n+1} < SU(2n, 2^i)$.
6. $SL(2, 5) < SU(2, 9^i)$.
7. $3 \cdot P\Omega^{-,\pi}(6, 3) < SU(6, 2^i)$.
8. $A \rtimes S_n < SU(n, s)$, $|A| = a^{n-1}$, $a|s+1$, s even.
9. Dihedral subgroups of $SU(2, 2^i)$.

$G_2(s)'$ and ${}^3D_4(s)$.

1. ${}^3D_4(q) \leqslant {}^3D_4(q^i)$.
2. $G_2(q)' \leqslant G_2(q^i)' < {}^3D_4(q^i)$.
3. $SL(2, q) < G_2(q^i)$.
4. $SL(2, 5) < G_2(9^i)$.
5. $O(3, q) < G_2(q^i)$, q even.
6. $SL(3, q) < G_2(q^i)$.
7. $SU(3, q) < G_2(q^i)$.
8. $SU(3, q) < SL(3, q^2) < G_2(q^{2i})$.
9. $3 \cdot A_6 < G_2(4^i)$.
10. $HJ < G_2(4^i)$.
11. $A \rtimes S_n < G_2(s)$, $n = 2$ or 3, $|A| = a^{n-1}$, $a|s \pm 1$, s even.

12. Concluding remarks. (A) Let Ω denote any finite Chevalley group, not a Suzuki or Ree group. As noted in the introduction, the problem we have considered is a special case of the situation in §3. We may assume Ω has rank $\geqslant 3$, and G is as in (3.2) with $O_p(G) \leqslant G' \cap Z(G)$.

Let J_1 be a subgroup of G maximal with respect to being contained in a long $SL(2, q)$ of Ω, as in the proof of (3.2). Set $M_1 = C_G(J_1)$, let $J_2 < M_1$ be conjugate to J_1, and set $M_2 = C_G(J_2)$. The embeddings of M_i in the appropriate Levi factors L_i of Ω are known, by induction. The commutator relations between M_1 and M_2 are known; so are those between L_1 and L_2 (because of (3.2)). In general, $G = \langle M_1, M_2 \rangle$. Thus, an exhaustive check of commutator relations should be able to verify whether or not a given possibility for G can occur in a given Ω. This is essentially what was done in §6, where linear algebra made these commutator relations particularly easy to check.

Since groups of type E_8 contain all possible Ω's, only $E_8(q)$ needs to be handled. However, for induction, smaller ones will be needed.

It is perhaps of interest to mention some examples of such embeddings $G < \Omega$ not arising from parabolic subgroups. Examples 1–5 are in Stensholt **[30]**, **[31]**,, while 6 is in Fischer **[8]**.

1. $B_4(q)$, ${}^3D_4(q) < F_4(q)$.
2. ${}^2D_5(q)$, $F_4(q) < {}^2E_6(q)$.
3. $F_4(q)$, $C_4(q)$ for q odd $< E_6(q)$.
4. $A_7(q)$, ${}^2A_7(q)$, ${}^2E_6(q) < E_7(q)$.

5. $A_8(q)$, $D_8(q)$, ${}^2A_8(q) < E_8(q)$.

6. $2 \cdot PSU(6, 2) < F_{22} < {}^2E_6(2)$.

(B) *Finiteness*. In Theorems I and II, both G and V were assumed finite. The finiteness of G is certainly essential: just consider subgroups of $SL(2, Z) < SL(2, Q)$ generated by transvections. However, if G is finite but V is not, then the obvious extensions of Theorems I and II (and the the lists in §11) remain valid.

To see this, consider first Theorem II. By Curtis-Reiner [**6**, (70.24)], if $W = V(m, K)$ then there is a finite extension L of K, and a finite subfield k of L, such that $W \otimes_K L$ has a spanning k-subspace U invariant under G. Now Theorem II applies to G^U. An easy induction yields $k \subseteq K$. (Note that $SL(2, k) \leqslant SL(2, K)$ implies that $k \subseteq K$; similarly, $SU(3, q) \leqslant SU(3, K)$ implies that $GF(q^2) \subseteq K$.) Now [**6**, (29.7)] implies that W is already G-isomorphic to $U \otimes_k K$ over K.

The corresponding sort of argument for Theorem I seems to break down. However, given Theorem II, the proof of Theorem I did not use finiteness in any essential way, and in fact goes through almost verbatim.

(C) *Normalizers*. Let $G \leqslant \Omega(V')$ be generated by a class of long root elements, and have $O_p(G) \leqslant G' \cap Z(G)$, for p the characteristic of V'. According to Theorem I, $V' = V \perp T$ with V and T invariant under G, $G^T = 1$, and G^V as in Theorem I unless dim V is small (in which case G^V is easy to determine). In this situation, it is straightforward to determine $N_{O(V')}(G)$, and hence this is left to the reader.

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