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WALL MANIFOLDS

BY

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ABSTRACT. In the calculation of the oriented cobordism ring, it is standard to consider so-called Wall manifolds, for which the first Stiefel-Whitney class is the reduction of an integral class. This paper studies the Wall-type structures in the equivariant case.

1. Introduction. In the calculation of the oriented cobordism ring [7], C. T. C. Wall introduced a class of manifolds which has since come to be referred to as Wall manifolds. These are manifolds for which the first Stiefel-Whitney class is the reduction of an integral class.

In order to understand the cobordism classification of finite group actions on oriented manifolds, several authors have introduced appropriate notions of Wall manifold with G action.

Recently, I had a letter from Frank Capobianco, in which he said, "Also I'd like to look at the various ways of defining Wall structures and Wall bordism to see if they're all equivalent. At this point, I think Rowlett and Komiya's definitions are compatible, while Wheeler's is not." The references are to Rowlett [5], Komiya [4], and Wheeler [8].

Having recently found a nice way to describe Wall manifolds, I began to look at this problem. Eventually, I learned that the most naive definition of Wall structure is just as good.

I also discovered that more care is required in the equivariant case. One must divide the G actions on Wall manifolds into various types in order to have anything meaningful. In particular, Rowlett's approach is not completely satisfying and one should reexamine the case of involutions. At the same time we get the general relation with oriented cobordism.

I am indebted greatly to Frank Capobianco whose question led to this work. I also profited greatly from conversation with John Ewing.

2. Wall structure. There are several different ways to formulate the notion of Wall structure in the literature. While I had assumed the more sophisticated versions would provide greater refinement in structure, that is not the case. It is then sufficient and preferable to take the most naive definition.

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DEFINITION 2.1. A Wall structure on a manifold M is an integral cohomology class $x \in H^1(M; Z)$ for which the mod 2 reduction is the first Stiefel-Whitney class of M, i.e., $\rho_2(x) = w_1(M) \in H^1(M; Z_2)$.

Since integral cohomology classes $x \in H^1(M; Z)$ are in one-to-one correspondence with homotopy classes of maps into S^1, x may be considered as given by a map f_x: $M \to S^1$ with $f_x^*(i) = x$, $i \in H^1(S^1; Z) \cong Z$ being a chosen generator. The condition $\rho_2(x) = w_1(M)$ is equivalent to the assertion that the determinant bundle of the tangent bundle of M, $\det \tau_M$, is isomorphic to the pull back of the nontrivial line bundle over S^1.

Potentially, the choice of the isomorphism of bundles could be part of the structure. Supposing M connected, an isomorphism is specified by a bundle map $\bar{f}$: $\det \tau_M \to \xi$ covering f_x.

$$\begin{array}{ccccc} \det \tau_M & \xrightarrow{\bar{f}} & S^1 \times R/((z, r) \sim (-z, -r)) & & (z, r) \\ \downarrow & & \downarrow \pi & & \downarrow \pi \\ M & \xrightarrow{f_x} & S^1 & & z^2 \end{array}$$

and there are two homotopy classes of isomorphisms (potentially) represented by $\bar{f}$ and $-\bar{f}$, where $-\bar{f}$ is interpreted in cobordism as the "opposite" of the structure $\bar{f}$. However, covering the homotopy F: $M \times [0, 1] \to S^1$ given by $F(m, t) = e^{2\pi i t} \cdot f_x(m)$ is a homotopy $\bar{F}$: $\det \tau_M \times [0, 1] \to \xi$ so that if $\bar{f}(x) = [(z, r)]$ then $\bar{F}(x, t) = [(e^{\pi i t} z, r)]$. This deforms $\bar{f}$ to $-\bar{f}$.

3. Some naive homotopy theory. It is well known that the group of homotopy classes of maps of a space X into the circle S^1 with the group structure being induced by the multiplication on S^1 is isomorphic to $H^1(X; Z)$. This correspondence and its elementary consequences will be used repeatedly in this paper.

Throughout, 1 will denote the constant map 1: $X \to S^1$ sending each x to the unit $1 \in S^1$.

LEMMA 3.1. *If f: $X \to S^1$ is a map homotopic to a constant map (i.e., $f^*(i) \in H^1(X; Z)$ is the zero class) then for each $k \in Z$ there is a map g: $X \to S^1$, also homotopic to a constant so that $g(x)^k = f(x)$ for all $x \in X$. Further, if F: $X \times [0, 1] \to S^1$ with $F(x, 0) = 1$, there is a map G: $X \times [0, 1] \to S^1$ with $G(x, 0) = 1$ and $G(x, t)^k = F(x, t)$, and G is unique.*

PROOF. f, being homotopic to a constant map, will also be homotopic to 1, and one may let F: $X \times [0, 1] \to S^1$ with $F(x, 0) = 1$, $F(x, 1) = f(x)$. Letting e: $R \to S^1$ by $e(t) = \exp(2\pi i t)$, one may lift F by covering homotopy to a map F': $X \times [0, 1] \to R$ with $F'(x, 0) = 0$ and $eF'(x, t) = F(x, t)$. Further, there is a unique such map F' by the unique path lifting property. The map G is defined by $G(x, t) = e((1/k)F'(x, t))$, and g by $g(x) = G(x, 1)$. $\square$

LEMMA 3.2. *Let X be a connected space on which the finite group H acts and $\alpha \in H^1(M, Z)$ a class invariant under this action. Then there is a unique homomorphism $\lambda: H \to S^1$ for which there is a map $\hat{a}: X \to S^1$ satisfying $\hat{a}(hx) = \lambda(h) \cdot \hat{a}(x)$ and $\hat{a}^*(i) = \alpha$. Further, the equivariant homotopy class of the map $\hat{a}$ is unique.*

PROOF. Let α be represented by $a: X \to S^1$. By Theorem 2.4 of Bredon [1], there is a class $\beta' \in H^1(X/H; Q)$ with $\pi^*(\beta') = \rho_Q(\alpha)$ and hence a class $\beta \in H^1(X/H, Z)$ so that $\pi^*(\beta) = k\alpha$ for some $k \in Z$. Letting $b: X/H \to S^1$ represent β, one has a homotopy $F: X \times [0, 1] \to S^1$ with $F(x, 0) = a(x)^k$ and $F(x, 1) = b\pi(x)$. Applying covering homotopy to

$$\begin{array}{ccccc} X & \xrightarrow{a} & S^1 & & z \\ \downarrow \times 0 & & \downarrow & & \downarrow \\ X \times [0,1] & \xrightarrow{F} & S^1 & & z^k \end{array}$$

gives a homotopy $F': X \times [0, 1] \to S^1$ with $F(x, 0) = a(x)$ and $F(x, 1) = \hat{a}(x)$ so that $\hat{a}(x)^k = b\pi(x)$. Thus $\hat{a}$ represents the class α.

For $h \in H$, $\hat{a}(hx)^k = b\pi(hx) = b\pi(x) = \hat{a}(x)^k$ so that

$$\lambda_h: X \to S^1: \lambda_h(x) = \hat{a}(hx)/\hat{a}(x)$$

is a continuous map of the connected space X into the kth roots of unity, and hence a constant $\lambda_h(x) = \lambda(h)$. Now $\lambda(hh')\hat{a}(x) = \hat{a}(hh'x) = \lambda(h)\hat{a}(h'x) = \lambda(h)\lambda(h')\hat{a}(x)$ shows that $\lambda: H \to S^1$ is a homomorphism, and $\hat{a}$ satisfies $\hat{a}(hx) = \lambda(h)\hat{a}(x)$.

If $\hat{a}$ and $\bar{a}$ both represent α and are equivariant with respect to the homomorphisms λ and λ', then $\hat{a}^{-1}\bar{a}: X \to S^1$ is homotopic to 1 and equivariant for $\lambda^{-1} \cdot \lambda' = \mu$. By Lemma 3.1, there is a map $c: X \to S^1$ with $c(x)^j = \hat{a}^{-1}(x) \cdot \bar{a}(x)$ where j is the order of the group H. Now for $h \in H$, $c(hx)^j = \hat{a}^{-1}(hx)\bar{a}(hx) = \mu(h)\hat{a}^{-1}(x)\bar{a}(x) = \mu(h)c(x)^j$ so the map $X \to S^1$ sending x to $c(hx)/c(x)$ maps into the jth roots of $\mu(h)$, so is constant. Thus $c(hx) = \mu'(h)c(x)$ giving $c(x) = c(h^jx) = \mu'(h)^jc(x) = \mu(h)c(x)$, so $\mu(h) = 1$. Hence $\lambda'(h) = \lambda(h)\mu(h) = \lambda(h)$ and so λ is unique.

If $\hat{a}$ and $\bar{a}$ both represent α and are equivariant with respect to λ, then $\hat{a}^{-1}\bar{a}$ is homotopic to a constant and

$$\begin{aligned}(\hat{a}^{-1}\bar{a})(hx) &= \hat{a}^{-1}(hx)\bar{a}(hx) = \lambda(h)^{-1}\hat{a}^{-1}(x)\lambda(h)\bar{a}(x) \\ &= \hat{a}^{-1}(x)\bar{a}(x) = (\hat{a}^{-1}\bar{a})(x)\end{aligned}$$

so $\hat{a}^{-1}\bar{a} = d \circ \pi$ for some $d: X/H \to S^1$. Because $\pi^*(d^*(i)) = 0$ and π^* is monic, d is homotopic to 1, and if $D: X/H \times [0, 1] \to S^1$ with $D(y, 0) = d(y)$, $D(y, 1) = 1$, the map $F: X \times [0, 1] \to S^1$ given by $F(x, t) = \hat{a}(x) \cdot D(\pi(x), t)$ is an equivariant homotopy from $\bar{a}$ to $\hat{a}$. $\square$

4. Equivariant Wall structure. Now turning to our main problem, let G be a finite group and M a compact manifold (with or without boundary) on which G acts. Suppose M has a Wall structure given by $x \in H^1(M; Z)$ and assume that this class is invariant under the action of G, i.e., $g^*(x) = x$ for all $g \in G$.

Because it is more convenient to work with sphere bundles, we assume a G invariant Riemannian metric chosen on M. This induces a G invariant inner product in the fibers of $\det \tau_M$, and so one may consider the unit sphere bundles $S(\tau_M)$ and $S(\det \tau_M)$ with induced G actions $g^*: S(\tau_M) \to S(\tau_M)$ and $\det g^*: S(\det \tau_M) \to S(\det \tau_M)$. Since any two metrics are equivalent, this introduces no extra condition on our situation.

In seeking the form of M there is no loss in assuming M to be G-connected, i.e., M/G is connected. The general case will be a disjoint union of these.

The class x may be assumed given by a map $f: M \to S^1$ with a covering map $\bar{f}: S(\det \tau_M) \to S^1$ so that

$$\begin{array}{ccc} S(\det \tau_M) & \xrightarrow{\bar{f}} & S^1 \\ \pi\downarrow & & \downarrow s \\ M & \xrightarrow{f} & S^1 \end{array}$$

commutes, where $s: S^1 \to S^1$ is the squaring map $s(z) = z^2$.

If $M_0 \subset M$ is a component of M, there is a subgroup $H \subset G$ consisting of those elements of G taking M_0 into itself and M is isomorphic to the extension to G of the H action on M_0, i.e., $M \cong G \times M_0/(gh, h^{-1}m_0) \sim (g, m_0)$ if $h \in H$ and with G action defined by $g'(g, m_0) = (g'g, m_0)$. By Lemma 3.2. there is a unique homomorphism $\lambda: H \to S^1$ for which there is a map $M_0 \to S^1$ homotopic to $f|M_0$ and equivariant with respect to λ, and the equivariant homotopy class of this map is unique. There is then no loss in assuming $f|M_0$ is then actually itself equivariant for λ, for the homotopy can be covered.

If $M_1 \subset M$ is some other component of M, there is a $g \in G$ for which $gM_1 = M_0$. Since x is G invariant, the maps $f|M_1$ and $f|M_0 \circ g$ taking M_1 into S^1 are homotopic. The subgroup preserving the component M_1 is precisely the conjugate $g^{-1}Hg$ of H.

Choose any element $z \in S^1$ and consider the map $\phi: M_1 \to S^1$ defined by $\phi(m) = zf(gm)$. This map is homotopic to $f|M_1$ and

$$\phi(g^{-1}hgm) = zf(gg^{-1}hgm) = zf(hgm) = z\lambda(h)f(gm) = \lambda(h)\phi(m)$$

so ϕ is equivariant for the homomorphism $\lambda_g: g^{-1}Hg \to S^1$ given by $\lambda_g(g^{-1}hg) = \lambda(h)$.

One may then *choose* elements $1, g_2, \ldots, g_n$ where n is $[G : H]$, the index of H in G giving coset representatives and elements $1, z_2, \ldots, z_n$ in S^1 and deform the map f so that on the component g_iM_0 one has $f(m) = z_if(g_i^{-1}m)$.

Any such choice will make f simultaneously equivariant on all components.

We then suppose f has this form, and examine the covering map $\bar{f}$.

Case 1. $S(\det \tau_{M_0})$ is connected.

For $x \in S(\det \tau_{M_0})$ and $h \in H$, one has

$$s\bar{f}(\det h^* x) = f\pi(\det h^* x) = f(h\pi(x)) = \lambda(h) f\pi(x) = \lambda(h) s\bar{f}(x)$$

so the continuous function $S(\det \tau_{M_0}) \to S^1$ sending x to $\bar{f}(\det h^* x)/\bar{f}(x)$ takes values in the set consisting of the two possible square roots of $\lambda(h)$, and so is a constant $\mu(h)$. It is then immediate that $\mu\colon H \to S^1$ is a homomorphism and $\bar{f}$ is equivariant, with λ being the square of μ.

Case 2. $S(\det \tau_{M_0})$ is not connected.

The double cover $S(\det \tau_{M_0}) \to M_0$ is then the trivial double cover and one may let $\sigma_i\colon M_0 \to S(\det \tau_{M_0})$, $i = 1, 2$, be the two sections so that $\sigma_2 = -\sigma_1$. There is then a homomorphism $\psi\colon H \to Z_2 = \{+1, -1\} \subset S^1$ with $\det h^* \sigma_i(m) = \psi(h)\sigma_i(m)$ and a homomorphism $\mu'\colon H \to S^1$ for which $\bar{f} \circ \sigma_i(hm) = \mu'(h)\bar{f} \circ \sigma_i(m)$ with μ' squaring to λ. The map $\bar{f}\colon S(\det \tau_{M_0}) \to S^1$ is then equivariant with respect to the homomorphism $\mu\colon H \to S^1$ given by $\mu(h) = \psi(h)\mu'(h)$, which squares to λ.

In passing to the other components, one sees that

$$s\bar{f}(\det g_i^* x) = f\pi(\det g_i^* x) = f(g_i\pi(x)) = z_i f(\pi(x)) = z_i s\bar{f}(x)$$

for $x \in S(\det \tau_{M_0})$ and using either $S(\det \tau_{M_0})$ or the sections σ_i it follows that $\bar{f}(\det g_i^* x) = w_i \bar{f}(x)$ where $w_i^2 = z_i$. The map $f | g_i M_0$ can, of course, be deformed once around S^1 which can be covered to switch the two roots of $x^2 = z_i$.

Notice that if M were a manifold with boundary then the boundary of M_0 need not be H-connected. In fact, if M_0' is a component of ∂M_0, the subgroup preserving M_0', say H', may be properly smaller than H. The homomorphism μ for H will restrict to that μ' for H' but certain of the corresponding w_i for H' and M_0' are given by $\mu(h_i)$, where the h_i are coset representatives. Thus, putting M_0' inside M_0 actually requires a choice of certain w_i, and hence more structure.

The same sort of additional structure is possible for M. If $H \subset G' \subset G$ and $\mu\colon H \to S^1$ extends to a homomorphism $\mu'\colon G' \to S^1$ then one can choose the w_i corresponding to cosets of H in G' to be given by μ' and then $\bar{f}$ will automatically be equivariant on the extension to G' of M_0. It should be noted that different extensions μ' may exist for μ.

Definition 4.1. Let $H \subset G$ be a subgroup of G and $\mu\colon H \to S^1$ a homomorphism. Then $s\colon (S^1, \mu) \to (S^1, \mu^2)\colon z \to z^2$ is an H-equivariant map, and induces a G-equivariant map

$$s\colon G \times (S^1, \mu)/(gh, h^{-1}z) \sim (g, z) \to G \times (S^1, \mu^2)/(gh, h^{-1}z) \sim (g, z)$$

of the extensions to G.

If M is a Wall manifold with G action, M will be said to have type (H, μ) if there is (chosen) an equivariant map

$$f: (M, G) \to G \times (S^1, \mu^2)/(gh, h^{-1}z) \sim (g, z)$$

inducing the action on det τ_M from s.

Notes. (1) The equivariant homotopy class of f is essentially unique. The Wall structure on M is derived by mapping $G \times_H (S^1, \mu^2)$ to S^1 by a map of degree one on each component.

(2) The choice of the map f selects the components of M to be invariant under H as those mapped by f into the copy $1 \times (S^1, \mu^2) \subset G \times_H (S^1, \mu^2)$. Each component M_0 into this class will be preserved by a subgroup of H and its associated homomorphism will be the restriction of μ.

(3) One cannot meaningfully speak of isomorphism betwen two Wall manifolds with G action except within types. Only part of the type of M is recoverable from M itself except in very special circumstances, i.e., a G connected M has a minimal type derivable from one of its components, and any structure beyond that depends on a *choice* of $w_i \in S^1$.

(4) For $G = Z_2$, there are precisely three types given by $(\{1\}, 1)$ $(Z_2, -1)$, and $(Z_2, 1)$ where 1 and -1 denote trivial and nontrivial homomorphisms into S^1. For the types $(Z_2, \pm 1)$, $\mu^2 = 1$ and a manifold of type $(\{1\}, 1)$ admits structures of both other types.

Comparing with Rowlett **[5]**, one sees that any Wall manifold with involution having the structure class $x \in H^1(M, Z)$ invariant can be mapped into $(S^1, 1)$. The two types $(Z_2, \pm 1)$ can be seen in his Proposition 2.2, and the third can be seen in the copy of W_* which he interprets in both $\mathfrak{N}_*(S^1)$ and $W_*(RP^\infty)$. *Note*: His homomorphism a: $W_* \to \mathfrak{N}_*(S^1)$ is actually incorrectly chosen. $a(W, f)$ should be the class of f: $W \to S^1$ rather than the point map. With his construction the point map gives the orientation cover rather than the trivial cover. Further $ja(W, f) = kb(W, f) = (W \times Z_2, -1, f \circ \pi)$ rather than the point map to S^1 if correctly done.

Looking at Komiya **[4]**, a map $M \xrightarrow{\alpha} (S^1, 1)$ with $\bar{\alpha} \circ (-\det d\mu) = \bar{\alpha}$ is the same thing as having $\bar{\alpha}$ equivariant into $(S^1, -1)$. Thus, his two types are identical with $(Z_2, \pm 1)$.

It is obvious that Wheeler **[8]** has based his analysis on Wall manifolds with action of type $(G, 1)$.

5. Equivariant Wall bordism. There is no difficulty in setting up adequate cobordism theories using equivariant Wall manifolds *provided* one considers G manifolds of a given type (H, μ). Being given (M, x, G) and (N, y, G), manifolds with boundary and with $M_1 \subset \partial M$, $N_1 \subset \partial N$, so that $M_1 \cong N_1$ carrying the restrictions of the classes x and y to one another, the classifying maps f_M and f_N may be equivariantly deformed to coincide along M_1 and N_1

and hence one may form the union of M and N along M_1 and N_1 obtaining a manifold of the same type.

One may also introduce all the usual refinements such as restrictions on isotropy groups via families of subgroups of G as in [6] and equivariant bordism with equivariant maps into G pairs. Notationally $W_*^G(H, \mu)(\mathfrak{F}, \mathfrak{F}')(X, A, \phi)$ should just about suffice.

There are a couple of minor points.

(1) In doing families, the only possible isotropy groups of an action of type (H, μ) are subgroups of the kernel of μ^2: $H \to S^1$ up to conjugacy, and there is no point in considering more general families.

(2) Because an action of type (H, μ) is an extension from H to G, the problem of calculation reduces entirely to H; i.e.

$$W_*^G(H, \mu)(\mathfrak{F}, \mathfrak{F}')(X, A, \phi) \cong W_*^H(H, \mu)(\mathfrak{F}_H, \mathfrak{F}'_H)(X, A, \phi|_{H\times X}).$$

In relating type (H, μ) actions and type (H', μ') actions with $H \subset H'$, $\mu = \mu'|_H$ a bit of care must be excercised, however. The action of H' on (X, A) is material and the families must be kept track of.

(3) The point of doing equivariant Wall bordism is to relate it to G actions on oriented manifolds, of course. One must really classify actions of G on oriented manifolds by types as well. This is, of course, exactly what Komiya did in his paper [4] for free involutions.

It is fairly common in the literature to consider the effect of G on orientations to be described by a homomorphism ϕ: $G \to Z_2$. While this is undoubtedly the most interesting case, it is not really the most general situation. If M is G-connected and $M_0 \subset M$ is a component with H the subgroup preserving M_0, then there is a homomorphism μ: $H \to Z_2$ so that $\mu(h) = +1$ or -1 as h preserves or reverses orientation on M_0. The other components of M can, however, be oriented indiscriminantly, with the entire behavior being described by picking $1, g_2, \ldots, g_n$ coset representatives for G with respect to H and assigning elements $w_i \in Z_2$ to describe whether g_i; $M_0 \to g_i M_0$ preserves or reverses the chosen orientations. The choices may actually give a homomorphism to Z_2 on some subgroup properly containing H, but that depends on a *choice* and is not intrinsic to M.

Thus, one should consider G actions on oriented manifolds as being divided into types (H, μ) where H is a subgroup of G and μ: $H \to Z_2$ is a homomorphism. The choice of a type (H, μ) structure on M is given by an equivariant map f: $M \to (G/H$, multiplication) with the orientation cover induced from $G \times_H (Z_2, \mu)$.

Obviously, given $H \subset G$ and a homomorphism μ: $H \to Z_2$, one can consider μ as a homomorphism into S^1, with equivariant maps

$$\begin{array}{ccc} G \times_H (Z_2, \mu) & \to & G \times_H (S^1, \mu) \\ \downarrow & & \downarrow s \\ G/H & \to & G \times_H (S^1, \mu^2) \end{array}$$

and a resulting induced homomorphism

$$\rho\colon \Omega_*^G(H, \mu)(\mathfrak{F}, \mathfrak{F}')(X, A, \phi) \to W_*^G(H, \mu)(\mathfrak{F}, \mathfrak{F}')(X, A, \phi).$$

Thus, the only Wall types which relate to actions of G on oriented manifolds in any meaningful way are those for which $\mu\colon H \to S^1$ actually maps into Z_2. In particular, Wheeler is concerned in **[8]** primarily with groups of odd order, and there everything reduces to the case $(G, 1)$ in this situation, for $\Omega_*^G(H, 1) = \Omega_*^H(H, 1)$.

6. Relation with oriented bordism. Surprisingly, once one realizes the existence of types, every bit of the relation between Wall and oriented cobordism works beautifully.

PROPOSITION 6.1. *If G is supersolvable, there is a homomorphism*

$$\delta\colon W_*^G(H, \mu)(\mathfrak{F}, \mathfrak{F}')(X, A, \phi) \to \Omega_{*-1}^G(H, \mu)(\mathfrak{F}, \mathfrak{F}')(X, A, \phi)$$

defined in the usual way for which the sequence

$$\begin{tikzcd} \Omega_*^G(H, \mu)(\mathfrak{F}, \mathfrak{F}')(X, A, \phi) \arrow[rr, "2"] & & \Omega_*^G(H, \mu)(\mathfrak{F}, \mathfrak{F}')(X, A, \phi) \arrow[dl, "\rho"] \\ & W_*^G(H, \mu)(\mathfrak{F}, \mathfrak{F}')(X, A, \phi) \arrow[ul, "\delta"] & \end{tikzcd}$$

is exact.

PROOF. Given a Wall manifold M of type (H, μ) with M_0 the H invariant part, one has

$$\bar{f}\colon M_0 \to (S^1, 1).$$

Because G is supersolvable, this map may be made equivariantly transverse regular to $1 \in S^1$, or equivalently transverse to the zero section in $(\xi, 1) \to (S^1, 1)$, or equivalently transverse to the zero section in $\det \tau_{M_0} \to M_0$ where H acts on $\det \tau_{M_0}$ by letting h act as $\mu(h)\det h^*$. Transversality comes from Lemma 1 in Wheeler **[8]**, and this coincides with the constructions of all of the authors. Let $M_0' \subset M_0$ be the submanifold obtained and $M' \subset M$ be its G extension. Then M' is a G-invariant submanifold, is oriented, and the normal bundle in M is obtained from $\det \tau_M$ with the action coming from $\mu(\ )\det(\ )^*$, so M' is an (H, μ) action.

This construction can be performed whether M is closed or not, and M' will be $(\mathfrak{F}, \mathfrak{F}')$ free if M is so.

The proof of exactness, is and should be standard: it has been done often enough in the literature.

$\rho \circ 2 = 0$ for as noted M and $\bar{M}$ with the opposite Wall structure are isomorphic, so $2M = 0$ in Wall cobordism.

$\delta \circ \rho = 0$ for $\bar{f}$ deforms off of $1 \in S^1$ or the appropriate zero sections making M' empty.

$2 \circ \delta = 0$ since 2 annihilates W_*^G and $2 \circ \delta = \delta \circ 2$.

If $2M = \partial V$ one forms W from V by identifying the two copies of M in ∂V. Then $\delta W = M$.

If $\delta M = M'$ with $M' = \partial V$ oriented, one forms $M \times I$ and sews $D \det \tau_V = V \times [-1, 1]$ to it by attaching $M' \times [-1, 1]$ along a tubular neighborhood to cobord M to an oriented manifold.

If $\rho M = \partial V$ with V a Wall manifold, one first deforms M off S^1 and applies the construction as in δ to V, obtaining $V' \subset V$ and not meeting M. If one cuts out a tubular neighborhood of V', the remainder of V is a cobordism from M to the trivial double cover (equals orientation cover) of V', and hence gives the oriented cobordism from M to $2V'$. □

Note. To anyone who has read Rowlett's paper this must seem unbelievable. He notes that $\mathrm{Im}\, 2 \subset \ker \rho$ properly, but the difficulty is by not keeping proper care with types his $\ker \rho$ becomes too large.

There is not a single step in the standard argument which causes any difficulty if one uses types.

7. Involutions. Having indicated problems with Rowlett's paper, it is now time to examine involutions in detail.

For Rowlett, a Wall manifold with involution is a manifold together with a map f: $(M, t) \to (S^1, 1)$ for which $f^*(i)$ reduces to $w_1(M)$. A cobordism is a manifold with boundary with the same sort of structure for which one takes the induced maps on ∂M.

Observation. This sense of cobordism for Wall manifolds with involution is *not* an equivalence relation.

The difficulty would arise if one could find a connected M of type $(Z_2, 1)$ and a connected cobordism to some $N \times Z_2$ with an inherited type $(Z_2, 1)$ structure. One then considers $N \times Z_2$ as of type $(\{1\}, 1)$ and changes it to a type $(Z_2, -1)$ structure finding a connected cobordism to a connected manifold M' of type $(Z_2, -1)$. While the two cobordisms have a common boundary portion, they cannot be joined along that common boundary $N \times Z_2$ to obtain a cobordism from M to M'.

If M and M' were not individually boundaries, say M does not bound, there could be no cobordism from M to M'; i.e. if $M \cup M' = \partial W$ then consider the component of W to which M belongs, say V. Since M does not bound, $\partial V \neq M$ and so $\partial V = M \cup M'$. Thus, there is a connected cobordism from M to M' which cannot have type $(Z_2, 1)$ since M' does not, and cannot have type $(Z_2, -1)$ for M does not.

This situation can actually occur. In the following part of this section there will be sufficient information to demonstrate this fact. A specific example can be constructed as follows. Let $F\colon V \to RP(\infty)$ be a Wall cobordism with V connected, between $f\colon \mathbf{C}P(2) \to \text{point} \in RP(\infty)$ and $f_0\colon \mathbf{C}P(2) \# RP(1)^4 \to RP(\infty)$, where $f_0^*(i) = (0, i \otimes 1 \otimes 1 \otimes 1)$ (H^1 of a connected sum thought of as the direct sum of the two H^1 groups). This can actually be done with V oriented and $F(V) \subset RP(1)$. The induced double cover via the map into $RP(\infty)$ with Wall structure from V is the first cobordism and has type $(Z_2, 1)$. Let $G\colon W \to S^1$ be an unoriented connected cobordism from $f\colon \mathbf{C}P(2) \to \text{point} \in S^1$ to $f_1\colon \mathbf{C}P(2) \# RP(1)^4 \to S^1$ where $f_1^*(i) = (0, i \otimes 1 \otimes 1 \otimes 1)$. This can actually be done with W oriented. The induced double cover from the line bundle $\det \tau_W \otimes G^*\xi$ with Wall structure from $G \circ$ projection is the cobordism of type $(Z_2, -1)$. In the double covers $\mathbf{C}P(2) \times Z_2$ first has orientation preserving involution and is not a boundary, so the cover of f_0 is not a boundary, and secondly has orientation reversing involution so bounds $(\mathbf{C}P(2) \times [-1, 1], 1 \times (-1))$.

What happens is that when one sews the two cobordisms together, one obtains a Wall manifold, and it has an involution, but there is no class invariant under the involution which reduces to the Stiefel-Whitney class.

Observation. Except for minor errors, everything in Rowlett's paper seems correct *if* interpreted with the equivalence relation generated by his cobordism relation.

Note. In an erratum, Rowlett noted that exactness in the sequence of Proposition 2.3 does not hold at the W^{rel}_* term. With types such sequences are exact.

The complete analysis of Wall cobordism of involutions may be obtained from Rowlett's work by keeping track of types. One has:

Proposition 7.1. *The structure of Wall cobordism of involutions is described by*

(a) $W_*^{Z_2}(\{1\}, 1)\,(\mathfrak{F}\mathrm{ree}) \cong W_*^{Z_2}(\{1\}, 1)\,(\mathfrak{A}\mathrm{ll}) \cong W_*$ *assigning to* $M \times Z_2$ *the class of* M.

(b) $W_*^{Z_2}(Z_2, 1)\,(\mathfrak{F}\mathrm{ree}) \cong W_*(RP(\infty))$ *classifying* $M \to M/Z_2$. *The homomorphism from type* $(\{1\}, 1)$ *actions gives the point maps into* $RP(\infty)$.

(c) $W_*^{Z_2}(Z_2, -1)\,(\mathfrak{F}\mathrm{ree}) \cong \mathfrak{N}_*(S^1)$ *assigning to* $f\colon M \to (S^1, 1)$ *the induced map* $\bar{f}\colon M/Z_2 \to S^1$. *M is the double cover associated to* $\det_{M/Z_2} \otimes \bar{f}^*(\xi)$. *The homomorphism from type* $(\{1\}, 1)$ *actions assigns to* $M \times Z_2$ *the map* $M \to S^1$ *giving the Wall structure of* M.

(d) $W_*^{Z_2}(Z_2, 1)\,(\mathfrak{A}\mathrm{ll}, \mathfrak{F}\mathrm{ree}) \cong \bigoplus \tilde{W}_*(MO(2k))$ *assigning to an action the map of the disc bundle of the normal bundle to the codimension* $2k$ *fixed* *components into the disc bundle of the universal bundle over $BO(2k)$ obtained by classifying the normal bundle. The boundary homomorphism to free actions maps onto precisely those classes in $W_*(RP(\infty))$ represented by Wall manifolds with even Euler characteristic.*

(e) $W_*^{Z_2}(Z_2, -1)$ $(\mathcal{A}\mathrm{ll}, \mathcal{F}\mathrm{ree}) \cong \bigoplus \tilde{W}_*(MO(2k+1))$ *assigning to an action the normal data to the fixed components of codimension* $2k+1$. *The boundary homomorphism to free actions* ∂: $\tilde{W}_*(MO(1)) \to \mathfrak{N}_*(S^1)$ *is an isomorphism.*

(f) $W_*^{Z_2}(Z_2, -1)$ $(\mathcal{A}\mathrm{ll})$ *maps monomorphically into* $(\mathcal{A}\mathrm{ll}, \mathcal{F}\mathrm{ree})$ *actions with image isomorphic to* $\bigoplus_{k>0}\tilde{W}_*(MO(2k+1))$. *The image of free actions and hence also type* $(\{1\}, 1)$ *actions is zero in this group.*

(g) *The images of* $W_*^{Z_2}(Z_2, 1)$ $(\mathcal{F}\mathrm{ree})$ *and* $W_*^{Z_2}(\{1\}, 1)$ $(\mathcal{A}\mathrm{ll})$ *in* $W_*^{Z_2}(Z_2, 1)$ $(\mathcal{A}\mathrm{ll})$ *coincide and can be represented by the classes* $(\mathbf{C}P(2)^k \times Z_2, 1 \times (-1))$.

PROOF. Essentially this is in Rowlett, and certainly all of parts (a), (b), (c). For parts (d) and (e), note that if $F^j \subset M^n$ is the j dimensional part of the fixed set, $\det \tau_M|_F$ is $\det \nu \otimes \det \tau_F$ with Z_2 acting as $(-1)^{n-j}$ in the fibers. Thus fixed sets have even or odd codimension for actions of types $(Z_2, 1)$ or $(Z_2, -1)$, respectively. The isomorphisms then follow from Rowlett.

For (e) Rowlett gives the isomorphism $\tilde{W}_*(MO(1)) \cong \mathfrak{N}_*(S^1)$ although his proposed calculation is not the nicest way. For (d) his argument shows that $\tilde{W}_*(MO(2)) \to \tilde{W}_*(RP(\infty))$ is epic and the real projective space bundle of a 2-plane bundle always bounds, being a circle bundle. To see that the image is contained in the classes with even Euler characteristic, note that if $\xi \to F$ is a $2k$ plane bundle, $RP(\xi)$ fibers over F with an odd dimensional projective space as fiber.

In studying manifolds fibered with an odd projective space as fiber, Capobianco [2] has given enough examples to see that you get all classes with even Euler characteristic. For completeness, note that

$$\varepsilon\partial\colon \bigoplus \tilde{W}^*(MO(2k)) \xrightarrow{\partial} W_*(RP(\infty)) \xrightarrow{\varepsilon} W_*$$

is a homomorphism of W_* modules. W_* is a polynomial ring generated by the classes of $\mathbf{C}P(2)$ of odd Euler characteristic and the classes of M^n where

(1) if n is odd, $n \neq 2^s - 1$, $n = 2k + 2j + 1$ for some k and j with $\binom{k+j}{k} = 0$ and $M^n = RP(\xi \oplus (2j+1)) \to RP(2k)$,

(2) if n is even, $n \neq 2^s$, $n = 2k + 2j + 2$ for some k and j with $\binom{k+j}{k} = 0$ and $M^n = RP(\xi_1 \oplus \xi_2 \oplus 2j) \to RP(1) \times RP(2k)$,

(3) if $n = 2^s > 8$, $M^n = RP(\xi_1 \oplus \xi_2 \oplus \xi_3 \oplus (2^s - 5)) \to RP(1)^3$.

These are all $RP(\eta)$'s where $\eta \to F$ is an even dimensional bundle with $D\eta$ a Wall manifold so lie in the image of $\varepsilon\partial$.

The cokernel of $\varepsilon\partial$ then has base the $\mathbf{C}P(2)^k$, giving part (g). $\square$

The comparison with oriented cobordism of involutions then is nicely

$$\begin{array}{ccccccccccc}
0 & \to & \Omega_*/\mathrm{Tor} & \to & \mathcal{O}_*(Z_2) & \to & \oplus\tilde{\Omega}_*(MO(2k)) & \to & \tilde{\Omega}_{*-1}(RP(\infty)) \oplus \mathrm{Tor}\,\Omega_{*-1} & \to & 0 \\
 & & \downarrow \rho' & & \downarrow \rho & & \downarrow & & \downarrow & & \\
0 & \to & W_*/(\chi = 0) & \to & W_*^{Z_2}(Z_2, 1) & \to & \oplus \tilde{W}_*(MO(2k)) & \to & \tilde{W}_{*-1}(RP(\infty)) \oplus (\chi = 0) & \to & 0
\end{array}$$

and

$$\begin{array}{ccccccccc}
0 & \to & \mathfrak{R}_*(Z_2) & \to & \oplus\tilde{\Omega}_*(MO(2k+1)) & \to & \mathfrak{N}_{*-1} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & W_*^{Z_2}(Z_2, -1) & \to & \oplus \tilde{W}_*(MO(2k+1)) & \to & \mathfrak{N}_{*-1}(S^1) & \to & 0
\end{array}$$

and in the latter all the vertical maps are monic.

One should note that $\ker \rho'$ consists of all oriented M with even Euler characteristic or equivalently, even index, which is entirely compatible with Conner's theorem ([3], (4.5)) that $M \times Z_2 \in 2\mathcal{O}_*(Z_2)$ if and only if the index of M is even. This is not to imply that the kernel of ρ' is $2\Omega_*/\mathrm{Tor}$, for it is definitely not.

Note. The final point here is to validate §4 of Rowlett's paper. Because he must go to a different relation to define $\partial: W_*^I \to O_*(Z_2)$, it is not clear that ∂ is well defined. Given $f: M \to S^1$ which is equivariant you can deform f to be transverse to $1 \in S^1$ equivariantly. Unfortunately, $f^{-1}(1) = N \subset M$ is only orientable, and not oriented. The problem is to see that the class in $O_*(Z_2)$ is well defined.

When M is connected, there is no problem. The only problem is when $M = M_0 \times Z_2$, when you change orientation on switched components.

The easiest way to see that ∂ is well defined is to note that $\partial + \partial$: $W_*^{Z_2}(Z_2, 1) \oplus W_*^{Z_2}(Z_2, 1) \to O_*(Z_2)$ factors through W_*^I. Equivalently, $\partial(M_0 \times Z_2) = 0$ with either orientation, for ρ' is epic.

References

1. G. E. Bredon, *Introduction to compact transformation groups*, Academic Press, New York, 1972.

2. F. L. Capobianco, *Cobordism classes represented by fiberings with fiber RP(2k + 1)*, Michigan Math J. **24** (1977), 185–192.

3. P. E. Conner, *Lectures on the action of a finite group*, Springer-Verlag, Berlin, 1968.

4. K. Komiya, *Oriented bordism and involutions*, Osaka J. Math. **9** (1972), 165–181.

5. R. J. Rowlett, *Wall manifolds with involution*, Trans. Amer. Math. Soc. **169** (1972), 153–162.

6. R. E. Stong, *Unoriented bordism and actions of finite groups*, Mem. Amer. Math. Soc. No. 103 (1970).

7. C. T. C. Wall, *Determination of the cobordism ring*, Ann. of Math. (2) **72** (1960), 292–311.

8. E. R. Wheeler, *Equivariant bordism exact sequences*, Duke Math. J. **42** (1975), 451–458.

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