

A REPRESENTATION THEOREM AND APPLICATIONS TO TOPOLOGICAL GROUPS

BY

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ABSTRACT. We show that, given a set S dense in a compact Hausdorff space X and a complex-valued bounded linear functional Λ on the space $C(X)$ of continuous complex-valued functions on X with uniform norm, there exist an algebra \mathcal{A} of sets in S and a unique bounded finitely additive set function $\mu: \mathcal{A} \rightarrow \mathbb{C}$ which is inner and outer regular with respect to the zero and cozero sets respectively and such that $\int_S f|S d\mu$ exists and is equal to $\Lambda(f)$ for all $f \in C(X)$. In the context of topological groups, this theorem permits us to obtain (1) a concrete representation theorem for bounded complex-valued linear functionals on the space of almost periodic functions with uniform norm, (2) a representation theorem for (not necessarily continuous) positive definite functions, (3) a characterization of the space B of finite linear combinations of positive definite functions, and (4) a necessary and sufficient condition to have a linear transformation from B to B .

1. Introduction.

1.1. In a paper of Hewitt [4] an attempt was made at obtaining a concrete representation theorem for bounded complex-valued linear functionals defined on the space of continuous almost periodic functions on the real line. Though the problem was not solved satisfactorily (see the footnote on p. 379 in [2]), we were inspired by the methods of that paper to obtain a representation theorem for bounded linear functionals on subspaces of continuous functions defined on sets more general than locally compact Abelian groups. Though a general approach to representation theorems has been developed in recent years (see [3], [10], and [16]), we have found that, to obtain a complex-valued finitely additive set function representing the linear functional, which is inner and outer regular with respect to the zero and cozero sets respectively, the method of Loomis [9, p. 169] to generate an algebra of sets (on which is defined the set function) by means of the linear functional in question was our best approach. In the special case of a topological group, it is this algebra, which depends on the linear functional defined on the almost periodic functions, that makes the difference between our results and those of Hewitt.

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2. A representation theorem on spaces of continuous functions.

2.1. Let X be a topological space. By a measure on the Borel subsets $\mathfrak{B}(X)$ of X we mean a complex-valued (and so finite) countably additive set function on $\mathfrak{B}(X)$. Let $C(X)$ and $C_b(X)$ denote the continuous and the bounded continuous complex-valued functions on X respectively. Clearly, if μ is a measure on $\mathfrak{B}(X)$, then any $f \in C_b(X)$ is integrable with respect to μ and so the set $\{\omega \in \mathbb{C}: \mu(f^{-1}(\omega)) \neq 0\}$ is at most countable. Let $|\mu|$ denote the total variation measure on $\mathfrak{B}(X)$ associated with μ . Given an algebra of sets $\mathcal{Q} \subset \mathfrak{B}(X)$, let \mathcal{Q}_μ be the algebra of sets $E \in \mathcal{Q}$ such that $|\mu|(\bar{E} \setminus \dot{E}) = 0$. If $A, B \in \mathcal{Q}_\mu$ and if $A \cap S = B \cap S$ for some set S dense in X , then $|\mu|(A \Delta B) = 0$. This is easily shown from the inclusion $A \setminus B \subset (A \setminus \dot{A}) \cup (\dot{A} \setminus \bar{B}) \cup (\bar{B} \setminus B)$. If for any algebra $\mathcal{G} \subset \mathfrak{B}(X)$ we write $\mathcal{G} \cap S$ for $\{E \cap S: E \in \mathcal{G}\}$ then $\mathcal{Q}_\mu \cap S$ is an algebra of subsets of S such that the set function $\mu_S: \mathcal{Q}_\mu \cap S \rightarrow \mathbb{C}$ given by $\mu_S(E \cap S) = \mu(E)$ ($E \in \mathcal{Q}_\mu$) is a well-defined bounded finitely additive set function. If $|\mu_S|$ is the total variation of μ_S then clearly $|\mu|(E) = |\mu_S|(E \cap S) = |\mu_S|(E \cap S)$ for all $E \in \mathcal{Q}_\mu$.

2.2 DEFINITION. Given a set Y and an algebra \mathcal{G} of subsets of Y , a function $f: Y \rightarrow \mathbb{C}$ is said to be \mathcal{G} -continuous if, given $\epsilon > 0$, there exists a finite partition of \mathbb{C} into rectangles (E_1, \dots, E_n) with $f^{-1}(E_i) \in \mathcal{G}$ and $|x - y| < \epsilon$ for all $x, y \in E_i \cap f(Y)$ ($i = 1, \dots, n$). We write $C(Y, \mathcal{G})$ for the class of \mathcal{G} -continuous functions on Y .

2.3 REMARKS. (1) In the context of the definition above, a necessary condition for a function $f: Y \rightarrow \mathbb{C}$ to be \mathcal{G} -continuous is that $\int_Y f \, d\nu$ exists for every bounded finitely additive set function ν on \mathcal{G} [1, p. 293], where the integral is defined by the usual Moore-Smith method ([11, pp. 183–191], [15, pp. 401–404]) or equivalently [7] by the Dunford-Schwartz method [2, pp. 101–125]. Note that our definition of \mathcal{G} -continuity is more restrictive than the one usually given [1, p. 293].

(2) Let $\mathcal{Q} \subset \mathfrak{B}(X)$ be an algebra of subsets of the topological space X such that each $f \in C_b(X)$ is \mathcal{Q} -continuous. For example, $\mathcal{Q} = \mathfrak{B}(X)$ is such an algebra. Given a complex-valued measure μ on \mathcal{Q} and a set S dense in X such that $f|_S$ is $\mathcal{Q}_\mu \cap S$ -continuous for all $f \in C_b(X)$, then $\int_S f|_S \, d\mu_S = \int_X f \, d\mu$.

2.4. Let $MR(\mathfrak{B}(X))$ denote the space of complex-valued measures ν on $\mathfrak{B}(X)$ which are such that, given $\epsilon > 0$ and $E \in \mathfrak{B}(X)$, there is a closed set C and an open set U in X such that $C \subset E \subset U$ and $|\nu|(U \setminus C) < \epsilon$. Given an algebra $\mathcal{G} \subset \mathfrak{B}(X)$, a finitely additive set function $\mu: \mathcal{G} \rightarrow \mathbb{C}$ will be said to be regular on \mathcal{G} if, given $\epsilon > 0$ and $E \in \mathcal{G}$, there exist constants $\alpha, \beta > 0$ and real-valued functions $f, g \in C_b(X)$ such that the sets $K = \{x \in X: f(x) \leq \alpha\}$ and $V = \{x \in X: g(x) < \beta\}$ have the properties (i) $K, V \in \mathcal{G}$, (ii) $K \subset E \subset V$ and (iii) $|\mu|(V \setminus K) < \epsilon$.

2.5 LEMMA. Given a normal space X and $\mu \in MR(\mathfrak{B}(X))$, let $\mathcal{Q} \subset \mathfrak{B}(X)$ be an algebra for which each $f \in C_b(X)$ is \mathcal{Q}_μ -continuous. If S is dense in X , then μ_S is regular on $\mathcal{Q}_\mu \cap S$ and $\mathcal{Q}_\mu \cap S$ consists of all $E \in \mathcal{Q} \cap S$ such that

$$\inf \left\{ \int_S (h|_S - g|_S) \, d|\mu_S|: g|_S \leq \chi_E \leq h|_S; g, h \in C_b(X) \right\} = 0$$

where χ_E is the characteristic function for E .

PROOF. Since $\mu \in MR(\mathfrak{B}(X))$, given $\varepsilon > 0$ and $E \in \mathcal{Q}_\mu$, there exist a closed set C and an open set U in X such that $C \subset \dot{E} \subset E \subset \bar{E} \subset U$, $|\mu|(\dot{E} \setminus C) < \frac{1}{2}\varepsilon$ and $|\mu|(U \setminus \bar{E}) < \frac{1}{2}\varepsilon$. By Urysohn's Lemma, there exist continuous functions $\varphi, \psi: X \rightarrow [0, 1]$ such that $\varphi|_C = 0$, $\varphi|(X \setminus \dot{E}) = 1$, and $\psi|\bar{E} = 0$, $\psi|(X \setminus U) = 1$. Since φ and ψ are \mathcal{Q}_μ -continuous, there exist constants $\alpha, \beta \in (0, 1)$ for which $|\mu|(\varphi^{-1}(\alpha)) = 0$ and $|\mu|(\psi^{-1}(\beta)) = 0$ and if $K = \{x \in X: \varphi(x) < \alpha\}$ and $V = \{x \in X: \psi(x) < \beta\}$, then $K, V \in \mathcal{Q}_\mu$; $C \subset K \subset \dot{E} \subset E \subset \bar{E} \subset V \subset U$. So from the inclusion $V \setminus K \subset (U \setminus \bar{E}) \cup (\bar{E} \setminus \dot{E}) \cup (\dot{E} \setminus C)$ follows that $|\mu|(V \setminus K) < \varepsilon$. This proves that μ is regular on \mathcal{Q}_μ . It now follows that μ_S is regular on $\mathcal{Q}_\mu \cap S$.

Since $\int_S f d|\mu_S| = \int_X f d|\mu|$ for all $f \in C_b(X)$, to identify $\mathcal{Q}_\mu \cap S$ it is sufficient to show that \mathcal{Q}_μ consists of all $F \in \mathcal{Q}$ such that

$$\inf \left\{ \int_X (f_2 - f_1) d|\mu| : f_1 < \chi_F < f_2; f_1, f_2 \in C_b(X) \right\} = 0. \tag{2.5.1}$$

Given $F \in \mathcal{Q}_\mu$, then $\dot{F}, \bar{F} \in \mathcal{Q}_\mu$. By regularity on \mathcal{Q}_μ , given $\varepsilon > 0$ there is a closed set $K \in \mathcal{Q}_\mu$ and an open set $V \in \mathcal{Q}_\mu$ such that $K \subset \dot{F} \subset F \subset \bar{F} \subset V$, $|\mu|(K \setminus \dot{F}) < \frac{1}{2}\varepsilon$, and $|\mu|(V \setminus \bar{F}) < \frac{1}{2}\varepsilon$. By Urysohn's Lemma, there exist continuous functions $\varphi, \psi: X \rightarrow [0, 1]$ such that $\varphi|_K = 1$, $\varphi|(X \setminus \dot{F}) = 0$, and $\psi|\bar{F} = 1$, $\psi|(X \setminus V) = 0$. Hence

$$\begin{aligned} \int_X \varphi d|\mu| &< \sup \left\{ \int_X f d|\mu| : f < \chi_F; f \in C_b(X) \right\} \\ &< \inf \left\{ \int_X f d|\mu| : \chi_F < f; f \in C_b(X) \right\} < \int_X \psi d|\mu| \end{aligned}$$

and

$$\begin{aligned} \int_X \psi d|\mu| - \int_X \varphi d|\mu| &= \int_X (\psi - \varphi) d|\mu| = \int_{V \setminus K} (\psi - \varphi) d|\mu| \\ &< |\mu|(V \setminus K) = |\mu|(V \setminus \bar{F}) + |\mu|(\bar{F} \setminus \dot{F}) + |\mu|(\dot{F} \setminus K) < \varepsilon \end{aligned}$$

imply equation (2.5.1).

Conversely, if $F \notin \mathcal{Q}_\mu$, then $|\mu|(\bar{F} \setminus \dot{F}) > 0$. Choose $\varepsilon > 0$ such that $|\mu|(\bar{F} \setminus \dot{F}) > \varepsilon$. Given any two continuous functions $\varphi, \psi: X \rightarrow (-\infty, \infty)$ such that $\varphi < \chi_F < \psi$, then $\varphi|(\bar{F} \setminus \dot{F}) < 0$, $\psi|(\bar{F} \setminus \dot{F}) > 1$ and so $\int_X \psi d|\mu| - \int_X \varphi d|\mu| > |\mu|(\bar{F} \setminus \dot{F}) > \varepsilon$. Thus, if $F \notin \mathcal{Q}_\mu$, then equation (2.5.1) does not hold. This completes the proof of the lemma.

2.6. Given a topological space Y and a complex-valued linear functional Λ on a subspace B of the Banach space $C_b(Y)$ with uniform norm, we write $\|\Lambda\|$ for the norm of Λ :

$$\|\Lambda\| = \sup\{|\Lambda(f)| : f \in B, |f| < 1\}.$$

If $\|\Lambda\| < \infty$, then we say that Λ is bounded on B . We are now in a position to prove the following crucial result.

2.7 THEOREM. Let S be a dense subset of a compact Hausdorff space X , $\mathcal{Q} \subset \mathfrak{B}(X)$ an algebra such that $f|_S$ is $\mathcal{Q}_\mu \cap S$ -continuous for all $f \in C(X)$ and all $\mu \in MR(\mathfrak{B}(X))$, and Λ a complex-valued bounded linear functional on $C(X)$. The class \mathcal{E}_Λ of sets $E \in \mathcal{Q} \cap S$ such that

$$\inf_{f_1, f_2} \sup_{|g|=1} \{ |\Lambda((f_2 - f_1)g)| : f_1|_S \ll \chi_E \ll f_2|_S; f_1, f_2, g \in C(X) \} = 0$$

is an algebra of sets for which $f|_S$ is \mathcal{E}_Λ -continuous for all $f \in C(X)$, and there exists a unique regular bounded finitely additive set function $\lambda : \mathcal{E}_\Lambda \rightarrow \mathbb{C}$ with

$$\Lambda(f) = \int_S f|_S d\lambda \quad (f \in C(X)).$$

Moreover,

$$\|\Lambda\| = |\lambda|(S).$$

PROOF. *Step 1.* By the Riesz representation theorem, there exists a unique $\mu \in MR(\mathfrak{B}(X))$ for which

$$\Lambda(f) = \int_X f d\mu \quad (f \in C(X)).$$

Let $\lambda = \mu_S$. By the second remark in 2.3, if $f \in C(X)$, then

$$\Lambda(f) = \int_S f|_S d\lambda.$$

Since a compact Hausdorff space is normal, by Lemma 2.5, λ is regular on $\mathcal{Q}_\mu \cap S$ and $\mathcal{Q}_\mu \cap S$ consists of all $E \in \mathcal{Q} \cap S$ such that

$$\inf \left\{ \int_X (f_2 - f_1) d|\mu| : f_1|_S \ll \chi_E \ll f_2|_S; f_1, f_2 \in C(X) \right\} = 0.$$

But $d|\mu| = g d\mu$ for some Borel measurable function g such that $|g| = 1$ [12, p. 126]. Hence $\mathcal{Q}_\mu \cap S$ consists of all $E \in \mathcal{Q} \cap S$ such that

$$\inf \left\{ \int_X (f_2 - f_1)g d\mu : f_1|_S \ll \chi_E \ll f_2|_S; f_1, f_2 \in C(X) \right\} = 0.$$

But, by Lusin's theorem, any complex Borel measurable function on X can be approximated in μ -measure by functions in $C(X)$. Hence, $\mathcal{Q}_\mu \cap S$ consists of all $E \in \mathcal{Q} \cap S$ such that

$$\inf_{f_1, f_2} \sup_{|g|=1} \left\{ \left| \int_X (f_2 - f_1)g d\mu \right| : f_1|_S \ll \chi_E \ll f_2|_S; f_1, f_2, g \in C(X) \right\} = 0$$

i.e. such that

$$\inf_{f_1, f_2} \sup_{|g|=1} \{ |\Lambda((f_2 - f_1)g)| : f_1|_S \ll \chi_E \ll f_2|_S; f_1, f_2, g \in C(X) \} = 0.$$

Let $\mathcal{E}_\Lambda = \mathcal{Q}_\mu \cap S$.

Step 2. In this step we establish the uniqueness of λ on \mathcal{E}_Λ . Clearly, the total variation of the difference $\nu = \lambda - \lambda'$ of two bounded finitely additive set functions regular on \mathcal{E}_Λ is also a bounded finitely additive set function regular on \mathcal{E}_Λ . Thus it is enough to show that, given a nontrivial bounded finitely additive

nonnegative set function ν regular on \mathcal{E}_Λ , then $\int_S h|S \, d\nu \neq 0$ for some $h \in C(X)$. Since $\nu \neq 0$, there exists $E \in \mathcal{E}_\Lambda$ such that $\nu(E) > 0$. By regularity, given $\epsilon < \frac{1}{2}\nu(E)$, there exist continuous functions $f, g: S \rightarrow [0, \infty)$ and $\alpha, \beta > 0$ such that if $K = \{x \in S: f(x) < \alpha\}$ and if $V = \{x \in S: f(x) < \beta\}$ then (i) $K, V \in \mathcal{E}_\Lambda$, (ii) $K \subset E \subset V$, and (iii) $\nu(V \setminus K) < \epsilon$. Now if \bar{K} and \bar{V} are respectively the closed and open subsets of X for which $K = \bar{K} \cap S$ and $V = \bar{V} \cap S$, then by Urysohn's Lemma, there exists a continuous function $h: X \rightarrow [0, 1]$ such that $h|\bar{K} = 1$ and $h|(X \setminus \bar{V}) = 0$. Thus

$$\begin{aligned} \int_S h|S \, d\nu &= \int_K h|S \, d\nu + \int_{V \setminus K} h|S \, d\nu = \nu(K) + \int_{V \setminus K} h|S \, d\nu \\ &= \nu(E) - \nu(E \setminus K) + \int_{V \setminus K} h|S \, d\nu > \nu(E) - \nu(E \setminus K) - \int_{V \setminus K} h|S \, d\nu \\ &> \nu(E) - 2\nu(V \setminus K) > \nu(E) - 2\epsilon > 0. \end{aligned}$$

Step 3. This final step consists of showing that $\|\Lambda\| = |\lambda|(S)$. By the Riesz representation theorem

$$\|\Lambda\| = |\mu|(X) = \sup \left\{ \left| \int_X f \, d\mu \right| : f \in C(X), |f| < 1 \right\}.$$

But if $L(X, \mu)$ is the Banach space of μ -integrable functions with norm $\int_X |f| \, d\mu < \infty$, then $C(X) \subset C(X, \mathcal{Q}_\mu) \subset L(X, \mu)$ and, by Lusin's theorem, $C(X)$ is dense in $L(X, \mu)$. Hence

$$\begin{aligned} \|\Lambda\| &= \sup \left\{ \left| \int_X f \, d\mu \right| : f \in C(X, \mathcal{Q}_\mu), |f| < 1 \right\} \\ &= \sup \left\{ \left| \int_S g \, d\lambda \right| : g \in C(S, \mathcal{E}_\Lambda), |g| < 1 \right\} = |\lambda|(S). \end{aligned}$$

The theorem is proved.

2.8 REMARK. If Λ is a positive linear functional on $C(X)$, then the associated regular set function $\lambda: \mathcal{E}_\Lambda \rightarrow \mathbb{C}$ of Theorem 2.7 is nonnegative. One way to see this is to apply the Riesz representation theorem for positive linear functionals in Step 1 of the proof. Furthermore, \mathcal{E}_Λ can be chosen to consist of all $E \in \mathcal{Q} \cap S$ such that

$$\inf \{ \Lambda(f_2 - f_1) : f_1|S < \chi_E < f_2|S; f_1, f_2 \in C(X) \} = 0.$$

2.9 COROLLARY. Given a completely regular (Hausdorff) space Y and a bounded linear functional L on $C_b(Y)$, let \mathcal{E}_L be the class of sets $E \in \mathcal{B}(Y)$ such that

$$\inf_{f_1, f_2} \sup_{|g|=1} \{ |L((f_2 - f_1)g)| : f_1 < \chi_E < f_2; f_1, f_2, g \in C(X) \} = 0.$$

\mathcal{E}_L is an algebra of sets for which (i) f is \mathcal{E}_L -continuous for all $f \in C_b(Y)$, and (ii) there exists a unique regular bounded finitely additive complex-valued set function λ on \mathcal{E}_L with

$$L(f) = \int_Y f \, d\lambda \quad (f \in C_b(Y)).$$

Moreover,

$$\|L\| = |\lambda|(Y).$$

PROOF. Let X be the Stone-Ćech compactification of Y . Then L admits a unique (norm-preserving) bounded linear extension $\Lambda: C(X) \rightarrow \mathbb{C}$. Taking $S = Y$ and $\mathcal{Q} = \mathfrak{B}(X)$ in Theorem 2.7, and noting that, for any $\mu \in MR(\mathfrak{B}(X))$ and any $f \in C(X)$, $f|S$ is $\mathcal{Q}_\mu \cap S$ -continuous since the set $\{\omega \in \mathbb{C}: \mu(f^{-1}(\omega)) \neq 0\}$ is at most countable, we get the corollary.

2.10 REMARK. In [6, pp. 207, 210–211] sufficient conditions are given for a linear functional on $C(Y)$ (Y completely regular (Hausdorff)) to admit an integral representation by means of a countably additive measure on Y .

2.11. Given a uniform space Y , let $UC_b(Y)$ denote the Banach space of complex-valued uniformly continuous bounded functions on Y with uniform norm. We obtain a compactification for Y by the following well-known construction. Let $\{\varphi_y: y \in Y\}$ be those linear functionals on $UC_b(Y)$ given by $\varphi_y(f) = f(y)$. Then the weak closure X of $\{\varphi_y: y \in Y\}$ in the dual of $UC_b(Y)$ is weakly compact and we can view Y as a dense subset of X such that $UC_b(Y) = \{f|Y: f \in C(X)\}$. Thus, Theorem 2.7 yields the following.

2.12 COROLLARY. *Given a uniform space Y and a bounded linear functional L on $UC_b(Y)$, let \mathfrak{E}_L be the class of sets $E \in \mathfrak{B}(Y)$ such that*

$$\inf_{f_1, f_2} \sup_{|g|=1} \{|L((f_2 - f_1)g)|: f_1 \chi_E \leq f_2, f_1, f_2, g \in UC_b(Y)\} = 0.$$

\mathfrak{E}_L is an algebra of sets for which (i) f is \mathfrak{E}_L -continuous for all $f \in UC_b(Y)$ and (ii) there exists a unique regular bounded finitely additive complex-valued set function λ on \mathfrak{E}_L with

$$L(f) = \int_Y f d\lambda \quad (f \in UC_b(Y)).$$

Moreover,

$$\|L\| = |\lambda|(Y).$$

3. Linear functionals on almost periodic functions.

3.1. Let G be an Abelian locally compact group with dual group G^\wedge and Bohr compactification \bar{G} ([8, p. 137], [5, §24], [13, p. 30]). The value of an element $\hat{z} \in G^\wedge$ at the point $z \in G$ will be denoted by $\langle z, \hat{z} \rangle$ or $\hat{z}(z)$. Let $AP(G)$ be the space of continuous almost periodic functions on G with uniform norm. By a well-known theorem of harmonic analysis [8, p. 168] every element of $AP(G)$ is the restriction on G of a unique element of $C(\bar{G})$. Since $G^\wedge \subset AP(G)$, any given $\hat{z} \in G^\wedge$ can be extended to a unique continuous function on \bar{G} with values in $T = \{\omega \in \mathbb{C}: |\omega| = 1\}$. This extension shall be denoted by the same symbol, i.e. by \hat{z} . Let m denote the normalized Haar measure on $\mathfrak{B}(\bar{G})$.

3.2 LEMMA. *If \mathcal{Q} is the algebra of sets in $\mathfrak{B}(\bar{G})$ generated by sets of the form $\{z \in \bar{G}: \hat{z}(z) \in T_0\}$ for $\hat{z} \in G^\wedge$ and T_0 an arc on T with $m(\{\hat{z}^{-1}(\bar{T}_0 \setminus \dot{T}_0)\}) = 0$, then $f|G$ is $\mathcal{Q}_\mu \cap G$ -continuous for all $f \in C(\bar{G})$ and all $\mu \in MR(\mathfrak{B}(\bar{G}))$.*

PROOF. Given $\epsilon > 0$, by the Stone-Weierstrass theorem there exists a polynomial $p(z)$ on \bar{G} such that $|f(z) - p(z)| < \epsilon/3$ for all $z \in \bar{G}$. Since every finite linear

combination of $\mathcal{Q}_\mu \cap G$ -continuous functions is $\mathcal{Q}_\mu \cap G$ -continuous and since $p(z)$ is a finite linear combination of characters, which are clearly $\mathcal{Q}_\mu \cap G$ -continuous, it follows that there exists a finite $\mathcal{Q}_\mu \cap G$ -measurable partition Π of G such that, for all $E \in \Pi$, we have $|p(z) - p(z')| < \varepsilon/3$ for all $z, z' \in E$. So from the inequality $|f(z) - f(z')| \leq |f(z) - p(z)| + |p(z) - p(z')| + |p(z') - f(z')|$ follows that $|f(z) - f(z')| < \varepsilon$ ($z, z' \in E$). The lemma is proved.

This lemma now permits us to generalize a concrete representation theorem due to Hewitt [4, pp. 307–308] for bounded linear functionals on $AP(G)$. We note that our algebra is smaller than his and depends on the linear functional in question.

3.3 THEOREM. *Given a locally compact Abelian group G with dual group G^\wedge and given a bounded linear functional $L: AP(G) \rightarrow \mathbb{C}$, let \mathbf{A} be the algebra of subsets of G generated by sets of the form $\{z \in G: \hat{z}(z) \in T_0\}$ for $\hat{z} \in G^\wedge$ and T_0 an arc on T with $m(\{z \in \bar{G}: \hat{z}(z) \in \bar{T}_0 \setminus \hat{T}_0\}) = 0$ and let \mathcal{E}_L be the class of sets $E \in \mathbf{A}$ such that*

$$\inf_{f_1, f_2} \sup_{|g|=1} \{|L((f_2 - f_1)g)|: f_1 \leq \chi_E \leq f_2; f_1, f_2, g \in AP(G)\} = 0.$$

\mathcal{E}_L is an algebra for which (i) f is \mathcal{E}_L -continuous for all $f \in AP(G)$ (or equivalently, \hat{z} is \mathcal{E}_L -continuous for all $\hat{z} \in G^\wedge$) and (ii) there exists a unique regular bounded finitely additive complex-valued set function λ on \mathcal{E}_L with

$$L(f) = \int_G f d\lambda \quad (f \in AP(G)).$$

Moreover,

$$\|L\| = |\lambda|(G).$$

PROOF. Recall that a function lies in $AP(G)$ if and only if it is the restriction on G of a (unique) function in $C(\bar{G})$ [8, p. 168]. Hence, there exists a bounded linear functional Λ on $C(\bar{G})$ given by $\Lambda(h) = L(h|G)$. Let \mathcal{Q} be the algebra of sets of Lemma 3.2. Then $\mathbf{A} = \mathcal{Q} \cap G$. To complete the proof, apply Theorem 2.7 for $X = \bar{G}$, $S = G$, and $\mathcal{E}_\Lambda = \mathcal{E}_L$.

3.4. Given $f \in C(\bar{G})$, write $\hat{f}: G^\wedge \rightarrow \mathbb{C}$ for its Fourier transform

$$\hat{f}(\hat{z}) = \int_{\bar{G}} f(z) \overline{\langle z, \hat{z} \rangle} dm(z).$$

Let $F(G)$ denote the family of finite linear combinations of characters. By Bohr's fundamental theorem (or equivalently, the Stone-Weierstrass theorem on the Bohr compactification), $F(G)$ is dense in $AP(G)$ with respect to the uniform norm.

3.5 DEFINITION. A complex-valued function $p(\hat{z})$ defined on G^\wedge is said to be positive definite if

$$\sum_{i,j=1}^n p(\hat{z}_i - \hat{z}_j) c_i \bar{c}_j > 0$$

for all $\hat{z}_1, \dots, \hat{z}_n \in G^\wedge$ and for all $c_1, \dots, c_n \in \mathbb{C}$.

The following is a generalization of Hewitt's representation of positive definite functions [4, pp. 310–311].

3.6 THEOREM. Given a locally compact Abelian group G with dual group G^\wedge and given a complex-valued function $p(\hat{z})$ on G^\wedge , let \mathbf{A} be the algebra of subsets of G generated by sets of the form $\{z \in G: \hat{z}(z) \in T_0\}$ for $\hat{z} \in G^\wedge$ and T_0 an arc on T with $m(\{z \in \bar{G}: \hat{z}(z) \in \bar{T}_0 \setminus \dot{T}_0\}) = 0$, and let \mathfrak{E}_p be the class of sets $E \in \mathbf{A}$ such that

$$\inf \left\{ \left| \sum_{\hat{z} \in G^\wedge} (\hat{f}_2(\hat{z}) - \hat{f}_1(\hat{z}))p(\hat{z}) \right| : f_1 < \chi_E < f_2; f_1, f_2 \in F(G) \right\} = 0. \quad (3.6.1)$$

The function $p(\hat{z})$ is positive definite on G^\wedge if and only if (i) \mathfrak{E}_p is an algebra such that f is \mathfrak{E}_p -continuous for all $f \in AP(G)$ (or equivalently, \hat{z} is \mathfrak{E}_p -continuous for all $\hat{z} \in G^\wedge$) and (ii) there exists a unique regular bounded finitely additive nonnegative set function λ on \mathfrak{E}_p such that

$$p(\hat{z}) = \int_G \langle z, \hat{z} \rangle d\lambda(z)$$

for all $\hat{z} \in G^\wedge$.

PROOF. Sufficiency is easily shown. Necessity can be established by repeating the ideas behind the proof of Bochner's theorem as found in [13, pp. 19–21]. The proof that we now provide is somewhat shorter.

Clearly, a function lies in $F(G)$ if and only if it is the restriction on G of a (unique) function in $F(\bar{G})$. Also, since $|p(z)| < p(0)$ [13, p. 19], $p(z)$ is uniformly bounded. Assume $p(0) = 1$. On $F(G)$, define the linear functional Λ by

$$\Lambda(f|G) = \sum_{\hat{z} \in G^\wedge} \hat{f}(\hat{z})p(\hat{z}) \quad (f \in F(\bar{G})).$$

Let $\hat{F}(\bar{G}) = \{\hat{f}: f \in F(\bar{G})\}$ and given $f, g \in F(\bar{G})$, put

$$(\hat{f}, \hat{g}) = \Lambda(f\bar{g}|G).$$

This defines an inner product on $\hat{F}(\bar{G})$ and so $|\Lambda(f\bar{g}|G)|^2 < \Lambda(|f|^2|G)\Lambda(|g|^2|G)$. Thus,

$$\begin{aligned} |\Lambda(f|G)|^2 &< \Lambda(|f|^2|G) < \Lambda(|f|^2|G)^{1/2} < \dots < \Lambda(|f|^2|G)^{2^{-n}} = \Lambda(|f^{2^n}|^2|G)^{2^{-n}} \\ &= \left(\sum_{\hat{z}} \sum_{\hat{w}} (f^{2^n})^\wedge(\hat{z})(f^{2^n})^\wedge(\hat{w})p(\hat{z} - \hat{w}) \right)^{2^{-n}} \\ &< \left(\sum_{\hat{z}} |(f^{2^n})^\wedge(\hat{z})| \right)^{2^{-n+1}} = \left(\int_{\bar{G}} |f^{2^n-1}|^2 dm \right)^{2^{-n+1}} \\ &\rightarrow \sup\{|f(z)|^2: z \in \bar{G}\} \quad (n \rightarrow \infty). \end{aligned}$$

Thus Λ is a bounded linear functional on $F(G)$ and so can be extended to a unique bounded linear functional Λ' on the closure $AP(G)$ of $F(G)$ with respect to the uniform norm. Note that $\|\Lambda'\| = 1$. Thus, by Theorem 3.3, there exist an algebra $\mathfrak{E} \subset \mathbf{A}$ and a unique regular finitely additive complex-valued set function λ on \mathfrak{E} such that

$$\Lambda'(f) = \int_G f d\lambda \quad (f \in AP(G))$$

and

$$\|\Lambda'\| = |\lambda|(G).$$

In particular, if $f(z) = \langle z, \hat{z} \rangle$, then

$$p(\hat{z}) = \int_G \langle z, \hat{z} \rangle d\lambda(z)$$

for all $\hat{z} \in G^\wedge$. Furthermore, since $1 = p(0) = \int_G d\lambda = \lambda(G) < |\lambda|(G) = \|\Lambda'\| = 1$, then λ is nonnegative on \mathfrak{E} and by Remark 2.8, \mathfrak{E} consists of all $E \in \mathbf{A}$ with

$$\inf\{\Lambda'(f_2 - f_1) : f_1 \leq \chi_E \leq f_2; f_1, f_2 \in AP(G)\} = 0.$$

Hence, since $F(G)$ is dense in $AP(G)$ with respect to the uniform norm, equation (3.6.1) identifies \mathfrak{E} . The theorem is proved.

4. Fourier-Stieltjes transforms.

4.1. Given a locally compact Abelian group G , let $M(G)$ denote the family of tuples (μ, \mathcal{Q}) where \mathcal{Q} is an algebra of sets in $\mathfrak{B}(G)$ such that each $f \in AP(G)$ (or equivalently, each character $\hat{z} \in G^\wedge$) is \mathcal{Q} -continuous and μ is a bounded finitely additive complex-valued set function on \mathcal{Q} . Recall that the Fourier-Stieltjes transform $\hat{\mu}$ of (μ, \mathcal{Q}) is given by $\hat{\mu}(\hat{z}) = \int_G \langle z, \hat{z} \rangle d\mu(z)$. Write $B(G^\wedge)$ for the space of all finite linear combinations of positive definite functions on G^\wedge . By the Pontryagin duality theorem, the group G can be identified with G^\wedge and so, if $z \in G$, the character $\hat{z} \rightarrow \langle z, \hat{z} \rangle$ lies in $B(G^\wedge)$. Also, given $(\mu, \mathcal{Q}) \in M(G)$, then $\mu = (\mu_1^+ - \mu_1^-) + i(\mu_2^+ - \mu_2^-)$ where μ_j^\pm ($j = 1, 2$) are nonnegative bounded set functions on \mathcal{Q} [15, p. 401]. So, it is easy to see that $\hat{\mu} \in B(G^\wedge)$. In fact, we have the following.

4.2 LEMMA. *A complex-valued function on G^\wedge lies in $B(G^\wedge)$ if and only if it is the Fourier-Stieltjes transform of an element of $M(G)$.*

PROOF. It remains to prove necessity. Let $p \in B(G^\wedge)$. Then $p(\hat{z}) = \sum_k p_k(\hat{z})$ where the sum is finite and p_k is a scalar multiple of a positive definite function on G^\wedge . So, by Theorem 3.6, there exist $(\mu_k, \mathcal{Q}_k) \in M(G)$ for all k , such that, $p_k(\hat{z}) = \int_G \langle z, \hat{z} \rangle d\mu_k(z)$. Let Λ_k be those bounded linear functionals on $AP(G)$ given by $\Lambda_k(f) = \int_G f d\mu_k$ and put $\Lambda = \sum_k \Lambda_k$ on $AP(G)$. Then, by Theorem 3.3, there exists $(\mu, \mathfrak{E}) \in M(G)$ such that $\Lambda(f) = \int_G f d\mu$ ($f \in AP(G)$) and so, if $g(z) = \langle z, \hat{z} \rangle$, then

$$\hat{\mu}(\hat{z}) = \Lambda(g) = \sum_k \Lambda_k(g) = \sum_k p_k(\hat{z}) = p(\hat{z}).$$

The lemma is proved.

4.3. Since a function lies in $B(G^\wedge)$ if and only if it is the Fourier-Stieltjes transform $\hat{\mu}$ of a particular $(\mu, \mathcal{Q}) \in M(G)$ given uniquely by Theorem 3.6, we define a norm on $B(G^\wedge)$ by $\|\hat{\mu}\| = |\mu|(G)$. We can now prove the following analog for $B(G^\wedge)$ of Eberlein's characterization of $B(G^\wedge) \cap C(G^\wedge)$ (see [13, pp. 32-34]).

4.4 THEOREM. *A function p lies in $B(G^\wedge)$ if and only if*

$$\left| \sum_G \hat{f}(\hat{z}) p(\hat{z}) \right| < \|p\| \sup\{|f(z)| : z \in G\} \tag{4.4.1}$$

for all $f \in F(G)$.

PROOF. Suppose $p \in B(G^{\wedge})$. By Theorem 3.6, there exists $(\lambda, \mathcal{E}) \in M(G)$ such that $p(\hat{z}) = \int_G \langle z, \hat{z} \rangle d\lambda(z)$. Hence $\|p\| = |\lambda|(G)$. Let Λ be the linear operator on $AP(G)$ given by $\Lambda(f) = \int_G f d\lambda$. Then $\Lambda(f) = \sum_G \hat{f}(\hat{z})p(\hat{z})$ for all $f \in F(G)$. Thus, since $\|\Lambda\| = |\lambda|(G) = \|p\|$ and $|\Lambda(f)| \leq \|\Lambda\| \sup\{|f(z)|: z \in G\}$, then (4.4.1) holds for all $f \in F(G)$.

Conversely, suppose that (4.4.1) holds for all $f \in F(G)$. Let Λ be the linear functional on $F(G)$ defined by $\Lambda(f) = \sum_G \hat{f}(\hat{z})p(\hat{z})$. Then, by (4.4.1), Λ is bounded by $\|p\|$ on $F(G)$. Let Λ' be the (unique) linear functional extending Λ to the closure $AP(G)$ of $F(G)$ with respect to the uniform norm. Then $\|\Lambda'\| \leq \|p\|$. By Theorem 3.3, there exists a tuple $(\lambda, \mathcal{E}) \in M(G)$ such that $\Lambda'(f) = \int_G f d\lambda$ for all $f \in AP(G)$. So, given $\hat{z} \in G^{\wedge}$, if $g(z) = \langle z, \hat{z} \rangle$, then $\hat{\lambda}(\hat{z}) = \int_G g d\lambda = \Lambda'(g) = \Lambda(g) = p(\hat{z})$. Since $\hat{\lambda} \in B(G^{\wedge})$, then $p \in B(G^{\wedge})$ and this completes the proof of the theorem.

4.5 COROLLARY. *Given a pointwise convergent net in $B(G^{\wedge})$ uniformly bounded in norm by some constant, the limit also lies in $B(G^{\wedge})$ and is bounded in norm by the same constant.*

PROOF. This follows immediately from (4.4.1).

4.6. Let X and Y be arbitrary sets and let $f: X \times Y \rightarrow \mathbb{C}$. We say that f satisfies the double limit condition if, whenever (x_i) and (y_j) are sequences in X and Y respectively such that the iterated limits $\alpha = \lim_{i \rightarrow \infty} \lim_{j \rightarrow \infty} f(x_i, y_j)$ and $\beta = \lim_{j \rightarrow \infty} \lim_{i \rightarrow \infty} f(x_i, y_j)$ exist, then $\alpha = \beta$. In particular, for a locally compact Abelian group G , the function $f: G \times G^{\wedge} \rightarrow T$ given by $f(z, \hat{z}) = \langle z, \hat{z} \rangle$ satisfies the double limit condition. To see this, we first note that the restriction to G of an element of $(\overline{G^{\wedge}})^{\wedge}$ (also denoted $G^{-\wedge-\wedge}$) belongs to $\overline{G^{\wedge}}$ (also denoted $G^{\wedge-}$). From this follows that the restriction to $G^{\wedge-}$ of an element of $G^{-\wedge-\wedge-\wedge}$ belongs to $G^{\wedge-}$. Note also that G and G^{\wedge} with discrete topology are topological subgroups of $G^{-\wedge-\wedge}$ and $G^{-\wedge-\wedge-\wedge}$ respectively. Hence, if (z_i) and (\hat{z}_j) are sequences in G and G^{\wedge} respectively, then there are subsequences (z_{im}) and (\hat{z}_{jn}) of (z_i) and (\hat{z}_j) respectively such that $\lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \langle z_{im}, \hat{z}_{jn} \rangle = \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} \langle z_{im}, \hat{z}_{jn} \rangle$. From this follows that the function $f(z, \hat{z}) = \langle z, \hat{z} \rangle$ satisfies the double limit condition.

An algebra \mathcal{Q} of sets in X is said to separate points on X if, whenever $x, y \in X$, $x \neq y$, there are disjoint sets $A, B \in \mathcal{Q}$ such that $x \in A$, and $y \in B$. Clearly, if \mathcal{Q} is a subalgebra of $\mathfrak{B}(G)$ such that each $f \in AP(G)$ is \mathcal{Q} -continuous, then by Urysohn's Lemma and the Stone-Weierstrass theorem on \overline{G} , it follows that \mathcal{Q} separates points on G . Now, by a theorem of Sinclair [14, pp. 363, 364], if \mathcal{Q} and \mathcal{E} are algebras of sets which separate points on X and Y respectively and if $f: X \times Y \rightarrow \mathbb{C}$ is a bounded function for which (i) $f(\cdot, y)$ is \mathcal{Q} -continuous for all $y \in Y$, (ii) $f(x, \cdot)$ is \mathcal{E} -continuous for all $x \in X$, and (iii) f satisfies the double limit condition, then (1) $\int_X f(x, y) d\mu(x)$ and $\int_Y f(x, y) d\lambda(y)$ are integrable with respect to μ and λ respectively and (2) $\int_X \int_Y f d\lambda d\mu = \int_Y \int_X f d\mu d\lambda$, for all finitely additive bounded complex-valued set functions μ and λ on \mathcal{Q} and \mathcal{E} respectively. Thus, if $(\mu, \mathcal{Q}) \in M(G)$ and if $(\lambda, \mathcal{E}) \in M(G^{\wedge})$, then (1) the Fourier-Stieltjes transforms $\hat{\mu}(\hat{z}) = \int_G \langle z, \hat{z} \rangle d\mu(z)$ and $\hat{\lambda}(z) = \int_{G^{\wedge}} \langle z, \hat{z} \rangle d\lambda(\hat{z})$ are integrable with

respect to λ and μ respectively, and (2) $\int_G \int_G \langle z, \hat{z} \rangle d\lambda(\hat{z}) d\mu(z) = \int_G \int_G \langle z, \hat{z} \rangle d\mu(z) d\lambda(\hat{z})$.

4.7 REMARK. Let $L(G^\wedge)$ denote the space of complex-valued functions on G^\wedge which are integrable with respect to the Haar measure $d\hat{z}$ on $\mathfrak{B}(G^\wedge)$. Given $h \in L(G^\wedge)$, write \tilde{h} for the continuous complex-valued function on G given by the Fourier transform $\tilde{h}(z) = \int_{G^\wedge} \langle z, \hat{z} \rangle h(\hat{z}) d\hat{z}$. If $p \in B(G^\wedge)$, then

$$\left| \int_G h(\hat{z}) p(\hat{z}) d\hat{z} \right| \leq \|p\| \sup\{|\tilde{h}(z)| : z \in G\}$$

for all $h \in L(G^\wedge)$. This well-known property of Fourier-Stieltjes transforms (see [5, §33.20]) can be proved as follows. Suppose that $p \in B(G^\wedge)$. Then, there exists $(\lambda, \mathfrak{E}) \in M(G)$ such that $p(\hat{z}) = \int_G \langle z, \hat{z} \rangle d\lambda(z)$. Also, for any $h \in L(G^\wedge)$, $h(\hat{z}) d\hat{z}$ is a finitely additive set function, say μ , on $\mathfrak{B}(G^\wedge)$. So, by Sinclair's theorem,

$$\begin{aligned} \left| \int_G h(\hat{z}) p(\hat{z}) d\hat{z} \right| &= \left| \int_{G^\wedge} p(\hat{z}) d\mu(\hat{z}) \right| = \left| \int_G \int_G \langle z, \hat{z} \rangle d\lambda(z) d\mu(\hat{z}) \right| \\ &= \left| \int_G \int_G \langle z, \hat{z} \rangle d\mu(\hat{z}) d\lambda(z) \right| = \left| \int_G \int_G \langle z, \hat{z} \rangle h(\hat{z}) d\hat{z} d\lambda(z) \right| \\ &= \left| \int_G \tilde{h}(z) d\lambda(z) \right| \leq |\lambda|(G) \sup\{|\tilde{h}(z)| : z \in G\} \\ &= \|p\| \sup\{|\tilde{h}(z)| : z \in G\} \quad \text{for all } h \in L(G^\wedge). \end{aligned}$$

4.8. Recall that if $G = T$, then $G^\wedge = Z$ where $Z = \{0, \pm 1, \pm 2, \dots\}$. Let $\{a_{k,n} : k, n \in Z\}$ be a matrix of complex numbers such that $\sum_n |a_{k,n}| < \infty$ for all $k \in Z$. Then the sums $\sum_n p(n) a_{k,n}$ are well defined for all $p \in B(Z)$. We wish to give a criterion for the matrix to have the property that, for all $p \in B(Z)$, $\sum_n p(n) a_{k,n}$ lies in $B(Z)$ as a function of k . As the following theorem shows, such a condition is given by $\sum_n z^n a_{k,n} \in B(Z)$ for all $z \in T$.

4.9 THEOREM. Let G be a locally compact Abelian group and let $\{L_{\hat{z}} : \hat{z} \in G^\wedge\}$ be a family of complex-valued linear functionals on $B(G^\wedge)$ bounded with respect to the uniform norm. In order that, for all $(\mu, \mathfrak{Q}) \in M(G)$, $\hat{z} \rightarrow L_{\hat{z}}(\hat{\mu})$ be the Fourier-Stieltjes transform of an element of $M(G)$, it is necessary and sufficient that $L_{\hat{z}}(z)$ lie in $B(G^\wedge)$ for each function $z \in G^{\wedge\wedge}$.

PROOF. By Theorem 3.3, for each $\hat{z} \in G^\wedge$, there exists $(\lambda_{\hat{z}}, \mathfrak{Q}_{\hat{z}}) \in M(G^\wedge)$ such that $L_{\hat{z}}(g) = \int_{G^\wedge} g(\hat{w}) d\lambda_{\hat{z}}(\hat{w})$ for all $g \in F(G^\wedge)$. Hence the theorem reduces to showing the following: given $\{(\lambda_{\hat{z}}, \mathfrak{Q}_{\hat{z}}) : \hat{z} \in G^\wedge\} \subset M(G^\wedge)$, then in order that, for all $(\mu, \mathfrak{Q}) \in M(G)$, $\int_{G^\wedge} \hat{\mu}(\hat{w}) d\lambda_{\hat{z}}(\hat{w})$ be the Fourier-Stieltjes transform of an element of $M(G)$, it is necessary and sufficient that $\hat{\lambda}_{\hat{z}}(z) \in B(G^\wedge)$ for each $z \in G^{\wedge\wedge}$.

To prove necessity, suppose that, for any given $w \in G$, $(\mu, \mathfrak{Q}) \in M(G)$ is such that for any $E \in \mathfrak{Q}$, $\mu(E) = 1$ if $w \in E$ and $\mu(E) = 0$ otherwise. Then, by hypothesis, there exists $(\nu, \mathfrak{E}) \in M(G)$ such that

$$\begin{aligned} \hat{\nu}(\hat{z}) &= \int_{G^*} \hat{\mu}(\hat{w}) d\lambda_{\hat{z}}(\hat{w}) = \int_{G^*} \int_G \langle z, \hat{w} \rangle d\mu(z) d\lambda_{\hat{z}}(\hat{w}) \\ &= \int_{G^*} \langle w, \hat{w} \rangle d\lambda_{\hat{z}}(\hat{w}) = \hat{\lambda}_z(w). \end{aligned}$$

But $\nu = (\nu_1^+ - \nu_1^-) + i(\nu_2^+ - \nu_2^-)$ where ν_j^\pm ($j = 1, 2$) are nonnegative bounded finitely additive set functions on \mathcal{G} [15, p. 401]. So, $\hat{\nu} = (\hat{\nu}_1^+ - \hat{\nu}_1^-) + i(\hat{\nu}_2^+ - \hat{\nu}_2^-)$ where, by Theorem 3.6, $\hat{\nu}_j^\pm$ ($j = 1, 2$) are positive definite on G^* . Hence, for any given $w \in G$, $\hat{\lambda}_z(w)$ is a linear combination of four positive definite functions in \hat{z} .

We now prove sufficiency. Given $(\mu, \mathcal{Q}) \in M(G)$, there exist bounded nonnegative finitely additive set functions μ_j^\pm ($j = 1, 2$) on \mathcal{Q} such that $\mu = (\mu_1^+ - \mu_1^-) + i(\mu_2^+ - \mu_2^-)$. Hence, for $j = 1, 2$,

$$\begin{aligned} \int_{G^*} \hat{\mu}_j^\pm(\hat{w}) d\lambda_{\hat{z}}(\hat{w}) &= \int_{G^*} \int_G \langle z, \hat{w} \rangle d\mu_j^\pm(z) d\lambda_{\hat{z}}(\hat{w}) \\ &= \int_G \int_{G^*} \langle z, \hat{w} \rangle d\lambda_{\hat{z}}(\hat{w}) d\mu_j^\pm(z) = \int_G \hat{\lambda}_z(z) d\mu_j^\pm(z). \end{aligned}$$

Thus, by hypothesis on $\hat{\lambda}_z(z)$, $\int_{G^*} \hat{\mu}_j^\pm(\hat{w}) d\lambda_{\hat{z}}(\hat{w})$ lies in $B(G^*)$. So, by Lemma 4.2, there exist elements $(\nu_j^\pm, \mathcal{Q}_j^\pm) \in M(G)$ ($j = 1, 2$) such that $\hat{\nu}_j^\pm(\hat{z}) = \int_{G^*} \hat{\mu}_j^\pm(\hat{w}) d\lambda_{\hat{z}}(\hat{w})$. Let Λ_j^\pm ($j = 1, 2$) be those bounded complex-valued linear functionals on $AP(G)$ given by $\Lambda_j^\pm(f) = \int_G f(z) d\nu_j^\pm(z)$ and, on $AP(G)$, put $\Lambda = (\Lambda_1^+ - \Lambda_1^-) + i(\Lambda_2^+ - \Lambda_2^-)$. Then, by Theorem 3.3, there exists $(\nu, \mathcal{G}) \in M(G)$ such that $\Lambda(f) = \int_G f(z) d\nu(z)$ for all $f \in AP(G)$, and so $\hat{\nu}(\hat{z}) = \Lambda(\hat{z}) = \int_{G^*} \hat{\mu}(\hat{w}) d\lambda_{\hat{z}}(\hat{w})$. The theorem is proved.

4.10 COROLLARY. *If $(\lambda, \mathcal{G}) \in M(G^*)$, then, for all $(\mu, \mathcal{Q}) \in M(G)$,*

$$\int_{G^*} \hat{\mu}(\hat{z} - \hat{w}) d\lambda(\hat{w})$$

is the Fourier-Stieltjes transform of an element of $M(G)$.

PROOF. Let $\{L_z: \hat{z} \in G^*\}$ be that family of bounded complex-valued linear functionals on $B(G^*)$ given by $L_z(f) = \int_{G^*} f(\hat{z} - \hat{w}) d\lambda(\hat{w})$. Then, for $g(\hat{z}) = \langle z, \hat{z} \rangle$ where z is given in G , $L_z(g) = \int_{G^*} \langle z, \hat{z} - \hat{w} \rangle d\lambda(\hat{w}) = \langle z, \hat{z} \rangle \hat{\lambda}(-z) \in B(G^*)$. So, the corollary follows from Theorem 4.9.

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