

## UNFLAT CONNECTIONS IN 3-SPHERE BUNDLES OVER $S^4$

BY

ANDRZEJ DERDZIŃSKI<sup>1</sup> AND A. RIGAS<sup>2</sup>

**ABSTRACT.** The paper concerns connections in 3-sphere bundles over 4-manifolds having the property of unflatness, which is a necessary condition in order that a natural construction give a Riemannian metric of positive sectional curvature in the total space. It is shown that, as conjectured by A. Weinstein, the only 3-sphere bundle over  $S^4$  with an unflat connection is the Hopf bundle.

**1. Introduction.** Suppose  $H$  is a closed subgroup of a compact connected Lie group  $G$  and let  $\omega$  be a connection in a principal  $G$ -bundle  $P \rightarrow M$ . Following A. Weinstein [13] one can call  $\omega$   $H$ -unflat if, for any  $p \in P$ , the curvature form  $\Omega_p$  restricted to the horizontal space at  $p$  has the property that its composite with any nonzero functional on the Lie algebra of  $G$ , annihilating the Lie algebra of  $H$ , is nondegenerate (cf. also [2]). (The term "fat" is used instead of "unflat" in [13].)

The motivation for this concept comes from a natural idea of constructing a metric on the total space  $E = P \times_G G/H = P/H$  of the bundle with fibre  $G/H$ , associated to  $P$ , by means of a connection  $\omega$  in  $P$  and a Riemannian metric  $h$  on the base manifold  $M$ . The construction consists in declaring the fibres orthogonal to the horizontal spaces, the former being isometric to  $G/H$  with a fixed normal homogeneous metric, while the latter carry the inner products pulled back from  $h$ . The  $H$ -unflatness of  $\omega$  is then equivalent to the positivity of the sectional curvatures of all planes spanned by one horizontal and one vertical vector in  $E$ . However, the condition of positivity of *all* sectional curvatures in  $E$  is much stronger. In fact, since the fibres of  $E$  are totally geodesic, the normal homogeneous space  $G/H$  must then have positive sectional curvature (unless  $\dim G/H = 1$ ). The same must hold for  $(M, h)$  in view of the O'Neill formulae [10], as  $E \rightarrow M$  is a Riemannian submersion. This imposes natural restrictions on the bases and fibres of bundles for which the construction of unflat connections could be a tool for finding new compact manifolds with positive curvature.

There are also examples of total spaces of principal  $G$ -bundles with  $G$ -invariant positively curved metrics, for which the corresponding connections are not  $\{1\}$ -unflat, in particular, the exceptional positively curved normal homogeneous space  $\mathrm{Sp}(2)/\mathrm{Sp}(1)$  of Berger [3] fibers principally over  $S^4$  with structure group  $S^3$  acting

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by isometries. In this case, the fibers are not totally geodesic and the connection that occurs is not unflat, which follows, e.g., from Corollary 1 below.

The results of this paper show that, in the case of 3-sphere bundles over four-manifolds, unflat connections appear to be scarce. Our argument is based on constructing a conformal structure in the base manifold, intrinsically associated to the given unflat connection. Namely, we prove (Theorem 2) that the characteristic numbers of a bundle with such a connection satisfy a certain inequality, which implies (Theorem 3) that the only principal  $SO(4)$ -bundle with an  $SO(3)$ -unflat connection over  $S^4$  is the one that has the Hopf fibration  $S^7 \rightarrow S^4$  as the associated 3-sphere bundle. This proves a conjecture of Weinstein [13], implying at the same time that the above procedure does not yield positively curved Riemannian metrics on exotic 7-spheres. However, it does, in some cases, yield Riemannian metrics of nonnegative sectional curvature ([11]; see also [4]).

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**2. Unflat forms.** Let  $T$  and  $V$  be real vector spaces of positive dimensions,  $\Omega$  a  $V$ -valued exterior 2-form on  $T$ . The form  $\Omega$  is called *unflat* if, for any nonzero functional  $f \in V^*$ , the composite  $f \circ \Omega: T \wedge T \rightarrow \mathbf{R}$  is nondegenerate, that is,  $\text{rank}(f \circ \Omega) = \dim T$ . This is clearly equivalent to saying that the map  $T \ni v \mapsto \Omega(u, v) \in V$  is surjective for any nonzero  $u \in T$  (cf. [13]).

As observed by Weinstein [13], the property of unflatness is very restrictive: Given an unflat form  $\Omega: T \wedge T \rightarrow V$ , choose an inner product in  $T$  and let  $S$  be the unit sphere. Then the maps  $\Omega(u, \cdot): T_u S \rightarrow V$ ,  $u \in S$ , define a vector bundle epimorphism of  $TS$  onto  $S \times V$ , i.e.,  $TS$  contains a trivial subbundle of dimension  $\dim V$ . In particular, if  $\dim V > 1$ , then  $\dim T \equiv 0 \pmod{4}$ .

EXAMPLES. (1) Unflat forms  $\Omega: T \wedge T \rightarrow \mathbf{R}$  are just symplectic structures in  $T$ .

(2) Let  $V = \text{Im } \mathbf{H}$ , the 3-dimensional space of pure quaternions. For  $T = \mathbf{H}^k$ , the space of  $k$ -tuples of quaternions, an unflat form  $\Omega: T \wedge T \rightarrow V$  can be defined by

$$\Omega((x_1, \dots, x_k), (y_1, \dots, y_k)) = \text{Im}(x_1 \bar{y}_1 + \dots + x_k \bar{y}_k),$$

where  $\text{Im}$  denotes the pure quaternion part.

(3) If  $\Omega: T \wedge T \rightarrow V_1$  is unflat and  $F: V_1 \rightarrow V_2$  is an epimorphism, then  $F \circ \Omega: T \wedge T \rightarrow V_2$  is unflat.

From an obvious dimension argument, we obtain

LEMMA 1. *For real vector spaces,  $T, V$  with  $\dim T = \dim V + 1$ , a form  $\Omega: T \wedge T \rightarrow V$  is unflat if and only if  $\Omega(u, v) \neq 0$  for every pair of independent vectors  $u, v \in T$ .*

For the remainder of this section, we assume that  $T$  and  $V$  are real vector spaces with  $\dim T = 4, \dim V = 3$ .

The formula

$$\det[a_{ij}] = (a_{12}a_{34} + a_{13}a_{42} + a_{14}a_{23})^2, \tag{1}$$

valid for any skew-symmetric  $4 \times 4$ -matrix (cf. [9, p. 309]), together with the definition of unflatness, yields

LEMMA 2. If  $\Omega: T \wedge T \rightarrow V$  is unflat, then the assignment

$$V^* \ni f \mapsto [\det(f \circ \Omega)]^{1/2} \in \mathbf{R} \tag{2}$$

is a positive definite quadratic form in  $V^*$ , determined by  $\Omega$  up to a scaling factor (since  $\det$  is not well defined).

Thus, (2) defines a conformal structure in  $V^*$ , and hence also in  $V$ . Having in mind the latter, we shall call a basis of  $V$   $\Omega$ -conformal if it is orthonormal for some inner product within the conformal structure.

For later convenience, let us introduce some notations. Given a vector space  $W$ , denote by  $LW$  the set of all bases of  $W$ . For  $e = (e_1, \dots, e_n) \in LW$  and  $0 \neq t \in \mathbf{R}$ , set  $te = (te_1, \dots, te_n)$ . Finally, for vector spaces  $T, V$  with  $\dim T = 4, \dim V = 3$ , a form  $\Omega: T \wedge T \rightarrow V$  and bases  $e \in LT, X \in LV$ , we shall write, by abuse of notation,

$$\Omega(e) = X \tag{3}$$

instead of

$$\begin{aligned} \Omega(e_1, e_2) &= \Omega(e_3, e_4) = X_1, \\ \Omega(e_1, e_3) &= \Omega(e_4, e_2) = X_2, \\ \Omega(e_1, e_4) &= \Omega(e_2, e_3) = X_3. \end{aligned} \tag{4}$$

LEMMA 3. If  $\Omega: T \wedge T \rightarrow V$  is unflat and  $X \in LV$  is  $\Omega$ -conformal, then for any nonzero vector  $e_1 \in T$  there exist unique vectors  $e_2, e_3, e_4 \in T$  and a unique real number  $\lambda \neq 0$  such that  $e = (e_1, e_2, e_3, e_4) \in LT$  and  $\Omega(e) = \lambda X$ . Moreover, the orientation of  $V$  determined by  $\lambda X$  depends only on  $\Omega$ , i.e.,  $\Omega$  distinguishes an orientation in  $V$ .

PROOF. By unflatness, there exist  $u_2, u_3, u_4 \in T$  such that  $\Omega(e_1, u_{i+1}) = X_i, i = 1, 2, 3$ , and they complete  $e_1$  to a basis of  $T$ . Let  $\Omega = \sum_i \Omega_i X_i$ . Setting  $v_2 = u_2 - \Omega_2(u_2, u_3)e_1, v_3 = u_3 + \Omega_1(u_2, u_3)e_1, v_4 = u_4 + \Omega_1(u_2, u_4)e_1$ , we obtain

$$\begin{aligned} \Omega(e_1, v_{i+1}) &= X_i, \quad i = 1, 2, 3, \quad \Omega(v_2, v_3) = \delta X_3, \\ \Omega(v_2, v_4) &= \beta X_2 + \epsilon X_3, \quad \Omega(v_3, v_4) = \alpha X_1 + \gamma X_2 + \zeta X_3 \end{aligned} \tag{5}$$

for some  $\alpha, \beta, \gamma, \delta, \epsilon, \zeta$ . On the other hand, since  $X$  is  $\Omega$ -conformal, (1) and (5) yield

$$\begin{aligned} \mu(x^2 + y^2 + z^2)^2 &= \det(x\Omega_1 + y\Omega_2 + z\Omega_3) \\ &= (\alpha x^2 - \beta y^2 + \delta z^2 + \gamma xy + \zeta xz - \epsilon yz)^2 \end{aligned}$$

for some  $\mu > 0$  and arbitrary real  $x, y, z$ , the determinant being calculated in the basis  $e_1, v_2, v_3, v_4$ . Therefore  $\alpha = -\beta = \delta \neq 0$  and  $\gamma = \zeta = \epsilon = 0$ . Our assertion is now satisfied by  $e'_{i+1} = \alpha^{-1}v_{i+1}, i = 1, 2, 3$ , and  $\lambda = \alpha^{-1}$ . To prove the uniqueness statement, assume  $\Omega(e) = \lambda X, \Omega(e') = \lambda' X$  and  $e'_i = e_i$ . Since  $\Omega$  is unflat, the kernel of  $T \ni u \mapsto \Omega(e_1, u) \in V$  is spanned by  $e_1$ , which implies (cf. (4)) that  $e'_{i+1} = \lambda' \lambda^{-1} e_{i+1} + t_i e_1$  for some real  $t_i, i = 1, 2, 3$ . Evaluating now  $\Omega(e'_i, e'_j), 2 \leq i < j \leq 4$ , we obtain  $\lambda' = \lambda$  and  $t_i = 0, i = 1, 2, 3$ , as desired. Finally, to show that the

orientation of  $\lambda X$  depends only on  $\Omega$ , it is sufficient to observe that the map  $(e_1, X) \mapsto \lambda X$  is continuous and sends  $(e_1, -X)$  to  $(-\lambda)(-X) = \lambda X$ . This completes the proof.

For an unflat form  $\Omega: T \wedge T \rightarrow V$ , let us denote by  $C_\Omega V$  the set of all  $\Omega$ -conformal bases  $X \in LV$ , compatible with the orientation determined by  $\Omega$ , i.e., satisfying (3) for some  $e \in LT$ , and by  $C_\Omega T$  the set of all bases  $e \in LT$  such that

$$\Omega(e_i, e_j) = \Omega(e_k, e_l) \tag{6}$$

for any even permutation  $(i, j, k, l)$  of  $(1, 2, 3, 4)$ , i.e., satisfying (3) for some  $X \in LV$ .

REMARK 1. The matrix group  $SO(4)$  acts on  $LT$  from the left in the obvious way, which gives rise to an action of its universal covering  $Spin(4) = S^3 \times S^3$ . The latter can be described as follows:  $S^3 \times S^3 \times LT \ni (q_1, q_2, e) \mapsto q_1 e \bar{q}_2 \in LT$ , the left (resp. right) action of the unit quaternions on  $LT$  being given, for  $e = (e_1, e_2, e_3, e_4) \in LT$ , by  $ie = (-e_2, e_1, -e_4, e_3)$ ,  $je = (-e_3, e_4, e_1, -e_2)$ ,  $ke = (-e_4, -e_3, e_2, e_1)$  (resp. by  $ei = (-e_2, e_1, e_4, -e_3)$ ,  $ej = (-e_3, -e_4, e_1, e_2)$ ,  $ek = (-e_4, e_3, -e_2, e_1)$ ) and then extended linearly. In terms of the obvious left action of  $SO(3)$  on  $LV$  and the covering homomorphism  $\varphi: S^3 \rightarrow SO(3)$ , it is easy to verify that, given an unflat form  $\Omega: T \wedge T \rightarrow V$ , a real number  $t \neq 0$ ,  $q \in S^3$  and  $e \in LT$ ,  $X \in LV$  such that  $\Omega(e) = X$ , we have

$$\Omega(te) = t^2 X, \quad \Omega(eq) = X, \quad \Omega(qe) = \varphi(q)X. \tag{7}$$

LEMMA 4. Let  $\Omega: T \wedge T \rightarrow V$  be unflat. Then, for any fixed  $X \in C_\Omega V$ , the set of all  $e \in LT$  satisfying (3) forms precisely one orbit of the right action of  $S^3$  on  $LT$ .

PROOF. Suppose  $\Omega(e) = X$ . If  $e' = eq$ , then  $\Omega(e') = X$  by (7). Conversely, if  $\Omega(e') = X$ , then we can clearly find  $q \in S^3$  such that  $e'_1 = \mu(eq)_1$  for some  $\mu > 0$ . Since, by (7),  $\Omega(e') = X$  and  $\Omega(\mu eq) = \mu^2 X$ , the uniqueness statement of Lemma 3 implies  $\mu^2 = 1$  and  $e' = \mu eq$ , i.e.,  $e' = eq$ . This completes the proof.

The group  $GL(V)$  of all automorphisms of  $V$  acts from the left on the set of all unflat forms  $T \wedge T \rightarrow V$  by  $(A\Omega)(u, v) = A(\Omega(u, v))$ ,  $A \in GL(V)$ ,  $u, v \in T$ .

Given real vector spaces  $T, W$  with  $\dim T = 4$ , suppose that  $T$  is endowed with an oriented conformal structure, i.e., an orientation together with a homothety class of inner products. The Hodge star operator acting on  $\wedge^2 T = T \wedge T$  is then defined by  $\ast(e_1 \wedge e_2) = e_3 \wedge e_4, e_1, \dots, e_4$  being an arbitrary oriented conformal basis of  $T$ . A 2-form  $\Omega: T \wedge T \rightarrow W$  is called self-dual if  $\Omega \circ \ast = \Omega$ .

LEMMA 5. (i) For any unflat form  $\Omega: T \wedge T \rightarrow V$ , the set  $C_\Omega T$  of all bases  $e \in LT$  satisfying (6) is an orbit of the natural action of  $\text{Conf}^+(4) = \mathbf{R}_+ \times SO(4)$  on  $LT$ . In other words,  $\Omega$  defines an oriented conformal structure in  $T$ .

(ii) Every unflat form  $\Omega: T \wedge T \rightarrow V$  is self-dual with respect to the oriented conformal structure which it determines in  $T$ .

(iii) The oriented conformal structure in  $T$ , determined as above by any unflat form  $T \wedge T \rightarrow V$ , is invariant under the natural action of  $GL(V)$  on unflat forms.

PROOF. We use the conventions introduced in Remark 1.

(i) If  $e \in C_\Omega T$ , say  $\Omega(e) = X$ , and  $e' = tq_1e\bar{q}_2$ ,  $t \in \mathbf{R}_+$ ,  $q_1, q_2 \in S^3$ , then, by (7),  $\Omega(e') = t^2\varphi(q_1)X$ , hence  $e' \in C_\Omega T$ . Conversely, if  $e, e' \in C_\Omega T$ , say,  $\Omega(e) = X$ ,  $\Omega(e') = X'$ , then  $X, X' \in C_\Omega V$ . Therefore  $X' = t^2\varphi(q_1)X$  for some  $t > 0$  and  $q_1 \in S^3$ , which implies  $\Omega(tq_1e) = X' = \Omega(e')$  in view of (7). Lemma 4 now yields  $e' = tq_1e\bar{q}_2$  for some  $q_2 \in S^3$ , as required.

(ii) Our assertion is immediate from the fact that the bases  $e \in LT$  compatible with the oriented conformal structure determined by the unflat form  $\Omega$  are characterized by (6).

(iii) If  $e \in C_\Omega T$  and  $A \in GL(V)$ , say,  $\Omega(e) = X$ , then  $(A\Omega)(e) = AX = (AX_1, AX_2, AX_3)$ , so that  $e \in C_{A\Omega}T$ . This completes the proof.

**3. Relative unflatness.** Let  $T$  and  $W$  be real vector spaces of positive dimensions. Given a subspace  $V$  of  $W$  and a 2-form  $\Omega: T \wedge T \rightarrow W$ , one says that  $\Omega$  is  $V$ -unflat (cf. [13]) if the composite  $T \wedge T \rightarrow W \rightarrow W/V$  of  $\Omega$  with the natural projection is unflat in the sense of §2.

Suppose now that  $\dim T = 4$ ,  $\dim V = 3$ ,  $W = V + V$  and that an inner product has been chosen in  $V$ . Let  $SO(V)$  be the group of orientation preserving linear isometries of  $V$ . We are interested in  $D$ -unflat forms  $T \wedge T \rightarrow V + V$ , where  $D = \{(X, X): X \in V\} \subset V + V$  is the diagonal. The group  $SO(V) \times SO(V)$  acts then on 2-forms  $\Omega: T \wedge T \rightarrow V + V$  by  $((A, B)\Omega)(u, v) = (A\Omega_1(u, v), B\Omega_{-1}(u, v))$ ,  $\Omega_1$  and  $\Omega_{-1}$  being the components of  $\Omega$  in  $V + V$ . This action does not, in general, preserve  $D$ -unflatness, since  $D$  is not invariant under  $SO(V) \times SO(V)$ .

LEMMA 6. Let  $\dim T = 4$ ,  $\dim V = 3$ . Suppose  $V$  is endowed with an inner product. For a 2-form  $\Omega = (\Omega_1, \Omega_{-1}): T \wedge T \rightarrow V + V$ , the following three conditions are equivalent:

(i) Every 2-form in the  $SO(V) \times SO(V)$ -orbit of  $\Omega$  is  $D$ -unflat,  $D \subset V + V$  being the diagonal.

(ii) For every pair of independent vectors  $u, v \in T$ ,  $|\Omega_1(u, v)| \neq |\Omega_{-1}(u, v)|$ .

(iii) For some  $\epsilon \in \{1, -1\}$ ,  $\Omega_{-\epsilon}$  is unflat (as a  $V$ -valued form) and we have

$$|\Omega_\epsilon(u, v)| < |\Omega_{-\epsilon}(u, v)| \tag{8}$$

whenever  $u, v \in T$  are independent.

PROOF. By Lemma 1,  $D$ -unflatness of  $\Omega = (\Omega_1, \Omega_{-1})$  is equivalent to  $\Omega_1(u, v) \neq \Omega_{-1}(u, v)$  for arbitrary independent  $u, v \in T$ . Thus, the orbit of  $\Omega$  consists of  $D$ -unflat forms if and only if  $A\Omega_1(u, v) \neq B\Omega_{-1}(u, v)$  for all  $A, B \in SO(V)$  and arbitrary independent  $u, v \in T$ , which is obviously equivalent to (ii). Assume now (ii). From a connectivity argument we obtain (8) for some  $\epsilon \in \{1, -1\}$  and any independent  $u, v \in T$ . By Lemma 1,  $\Omega_{-\epsilon}$  is unflat, which completes the proof.

**4. Unflatness in principal bundles.** Let  $P \rightarrow M$  be a differentiable principal  $G$ -bundle,  $G$  being a Lie group with Lie algebra  $\mathfrak{g}$ . By a horizontal tensorial 2-form on  $P$  we shall mean a  $\mathfrak{g}$ -valued 2-form  $\Omega$  on  $P$  such that  $\Omega(u, \cdot) = 0$  for any vertical tangent vector  $u$  and  $\Omega(ua, va) = \text{ad } a^{-1} \cdot \Omega(u, v)$  for  $a \in G$  and  $u, v \in TP$ ,  $\text{ad}$  being the adjoint representation. For example, these conditions are satisfied by the

curvature form of any connection in  $P$ . A horizontal tensorial 2-form  $\Omega$  in  $P$  is called *unflat* if, for any  $p \in P$ ,  $\Omega_p$  restricted to a complement of the vertical subspace at  $p$  is unflat.

For a four-manifold  $M$  endowed with an oriented conformal structure, the Hodge star operator is a vector bundle endomorphism  $*$ :  $\wedge^2 M \rightarrow \wedge^2 M$  with  $*^2 = 1$ . Therefore  $\wedge^2 M$  splits as the direct sum  $\wedge_+ M + \wedge_- M$  of the 3-dimensional subbundles of *self-dual* and *anti-self-dual* forms, constituted by the  $\pm 1$ -eigenspaces of  $*$ . Clearly, the bundles  $\wedge_{\pm} M$  do not essentially depend on the conformal structure, i.e., they are determined up to equivalence by the *oriented* 4-manifold  $M$  alone. The principal  $SO(3)$ -bundle associated to  $\wedge_{\pm} M$  will be denoted by  $P \wedge_{\pm} M$ . More generally, replacing the tangent bundle  $TM$  by any oriented 4-plane bundle  $\xi$  over any paracompact space  $N$ , one can similarly define the oriented 3-plane bundles  $\wedge_{\pm} \xi$  over  $N$  by means of an arbitrary fibre metric (or conformal structure) in  $\xi$ . We have the following formulae for characteristic classes, the coefficient field being, respectively,  $\mathbf{R}$  or  $\mathbf{Z}_2$ :

$$\begin{aligned}
 p_1(\wedge_+ \xi) &= p_1(\xi) + 2e(\xi), & (9) \\
 w_2(\wedge_+ \xi) &= w_2(\xi). & (10)
 \end{aligned}$$

In fact, (9) follows immediately from the curvature description of characteristic classes [9, pp. 308, 311], while (10) can be easily obtained with the aid of a splitting map for  $\xi$  [6, p. 235].

REMARK 2. Suppose we are given an unflat horizontal tensorial 2-form  $\Omega$  in a principal  $G$ -bundle  $P \rightarrow M$ , where  $\dim M = 4$  and  $G = SO(3)$  or  $G = S^3$ . For any  $x \in M$  and any  $p \in \pi^{-1}(x)$ , the horizontal form  $\Omega_p$  projects onto an  $so(3)$ -valued unflat form on  $T_x M$  and, since  $\Omega$  is tensorial, the oriented conformal structure in  $T_x M$  defined by  $\Omega_p$  is independent of  $p \in \pi^{-1}(x)$  (cf. Lemma 5). Thus,  $\Omega$  defines an oriented conformal structure on  $M$ . With respect to this structure,  $\Omega_p$  is self-dual when viewed as a form on  $T_{\pi(p)} M$ , i.e.,  $f \circ \Omega_p \in \wedge_+ M_{\pi(p)}$  for any  $f \in so(3)^*$ . In other words, considering  $\Omega$  as a 2-form on  $M$  valued in the adjoint bundle  $\text{ad } P = P \times_{\text{ad}} so(3)$  of Lie algebras, we have  $\Omega \circ * = \Omega$ .

The following theorem can be viewed as a special case of Weinstein’s Theorem 7.2 of [13] (cf. Remark 3).

THEOREM 1. *Let  $\pi: P \rightarrow M$  be a principal  $SO(3)$ -bundle over a four-manifold  $M$ . If  $P$  admits an unflat horizontal tensorial 2-form  $\Omega$ , then  $M$  admits an orientation such that  $P$  is isomorphic to  $P \wedge_+ M$ .*

PROOF. Fix a basis  $X_1, X_2, X_3$  of  $so(3)$  and set  $\Omega = \sum_i \Omega_i X_i$ . Viewing the forms  $\Omega_p$  and  $\Omega_i(p)$  as defined in  $T_{\pi(p)} M$ , it is clear from Remark 2 that  $\Psi(p) = (\Omega_1(p), \Omega_2(p), \Omega_3(p))$  is a basis of  $\wedge_+ M_{\pi(p)}$  for any  $p \in P$ , the oriented conformal structure involved being the one determined by  $\Omega$ . It is now immediate that the map  $\Psi: P \rightarrow B$  is  $SO(3)$ -equivariant,  $B$  being the principal  $GL(3)$ -bundle of  $\wedge_+ M$ . This completes the proof.

REMARK 3. In [13] A. Weinstein proved the following (Theorem 7.2): “Let  $P$  be a principal  $SO(4)$ -bundle over a compact orientable 4-manifold  $M$ . Denote by  $\xi$  the

four-plane bundle associated to  $P$ . If  $P$  admits an  $SO(3)$ -unflat connection, then, for properly chosen orientations in  $\xi$  and  $M$ ,  $\bigwedge_+ \xi$  is isomorphic to  $\bigwedge_+ M$ . Note that, for a fixed base manifold, the functors {oriented 4-plane bundles}  $\ni \xi \mapsto \bigwedge_{\pm} \xi \in$  {oriented 3-plane bundles} correspond, on the principal bundle level, to {principal  $SO(4)$ -bundles}  $\ni P \mapsto P_{\epsilon} = P/S_{\epsilon}^3 \in$  {principal  $SO(3)$ -bundles},  $\epsilon = \pm 1$ . Here  $S_{\epsilon}^3$  denotes the unique connected proper normal subgroups of  $SO(4)$ , both isomorphic to  $S^3$ . In terms of the covering homomorphism  $F: Spin(4) = S^3 \times S^3 \rightarrow SO(4)$ , which assigns to  $(q_1, q_2) \in S^3 \times S^3$  the isometry  $x \mapsto q_1 x \bar{q}_2$  of  $\mathbf{H} = \mathbf{R}^4$ , we have  $S_1^3 = F(\{1\} \times S^3)$  and  $S_{-1}^3 = F(S^3 \times \{1\})$ . The assertion of Weinstein's theorem says precisely that, for some  $\epsilon \in \{1, -1\}$  and a suitable orientation of  $M$ ,  $P \bigwedge_+ M$  is isomorphic to  $P_{\epsilon}$ .

REMARK 4. Let  $P \rightarrow M$  be a principal  $SO(3)$ -bundle over an oriented compact four-manifold  $M$  with a fixed conformal structure. For any connection in  $P$ , the curvature form  $\Omega$  can be viewed as a 2-form on  $M$ , valued in the adjoint bundle  $ad P = P \times_{ad} so(3)$  of Lie algebras. Using any Riemannian metric on  $M$ , compatible with the conformal structure, and the bi-invariant metric on  $SO(3)$ , we have the Chern-Weil formula

$$c_4 \int_M \langle \Omega, \Omega \circ * \rangle = p_1(P)[M],$$

where  $c_4$  is a universal constant and  $p_1(P)$  denotes the first Pontryagin class of  $ad P$ . Moreover, the Schwartz inequality yields

$$|p_1(P)[M]| \leq c_4 \int_M |\Omega|^2$$

with equality if and only if  $\Omega$  is self-dual or anti-self-dual, i.e.,  $\Omega \circ * = \pm \Omega$  (cf. [1]).

For a closed subgroup  $H$  of a Lie group  $G$  and a connection  $\omega$  in a principal  $G$ -bundle  $P \rightarrow M$ , Weinstein's definition of unflatness was given in §1. If  $G = SO(4)$  and  $H = SO(3)$  is embedded in  $SO(4)$  in the obvious way as the set of all orthogonal  $4 \times 4$ -matrices keeping the vector  $(1, 0, 0, 0)$  fixed, then  $\omega$  is  $H$ -unflat if and only if, for any  $p \in P$ , the curvature form  $\Omega_p$ , restricted to the horizontal space at  $p$ , is  $D$ -unflat,  $D$  being the diagonal subspace of  $so(4) = so(3) + so(3)$ .

In the notations of Remark 3, we have

THEOREM 2. Let  $\pi: P \rightarrow M$  be a principal  $SO(4)$ -bundle with an  $SO(3)$ -unflat connection  $\omega$  over a compact four-manifold  $M$ . Then there exist an orientation in  $M$  and  $\epsilon \in \{1, -1\}$  such that

- (i)  $P_{-\epsilon}$  is isomorphic to  $P \bigwedge_+ M$ , and
- (ii)  $0 \leq |p_1(P_{\epsilon})[M]| < 3\tau(M) + 2\chi(M)$ ,  $\tau(M)$  being the signature and  $\chi(M)$  the Euler characteristic of  $M$ .

PROOF. As in Remark 3, we can form the quotient principal  $SO(3)$ -bundles  $P_{\epsilon}$ ,  $\epsilon = \pm 1$ , with equivariant projections  $\pi_{\epsilon}: P \rightarrow P_{\epsilon}$ . Now  $\omega$  projects onto connections  $\bar{\omega}_{\epsilon}$  in  $P_{\epsilon}$  with curvature forms  $\bar{\Omega}_{\epsilon}$  such that  $\pi_{\epsilon}^* \bar{\Omega}_{\epsilon} = \Omega_{\epsilon}$ ,  $\epsilon = \pm 1$ , where  $\Omega_1, \Omega_{-1}$  are the components of  $\Omega$  in  $so(3) + so(3)$  [7, pp. 79–80]. On the other hand, for any  $p \in P$ ,  $\Omega_p$  may be viewed as a form in  $T_{\pi(p)}M$  and it is easy to see that it satisfies

the hypotheses of Lemma 6. Therefore, for some  $\epsilon \in \{1, -1\}$ ,  $\Omega_{-\epsilon}$  is unflat and (8) holds for any independent vectors  $u, v$  tangent to  $M$ . Thus, the curvature form  $\bar{\Omega}_{-\epsilon}$  in  $P_{-\epsilon}$  is unflat and hence self-dual with respect to the oriented conformal structure that it induces in  $M$  (cf. Remark 2), while  $P_{-\epsilon}$  is isomorphic to  $P \wedge_+ M$  by Theorem 1. In view of (9), (8), Remark 4 and Hirzebruch's signature theorem [9, p. 224], we have

$$0 \leq |p_1(P_\epsilon)[M]| \leq c_4 \int_M |\bar{\Omega}_\epsilon|^2 < c_4 \int_M |\bar{\Omega}_{-\epsilon}|^2 \\ = p_1(P_{-\epsilon})[M] = p_1(\wedge_+ M)[M] = 3\tau(M) + 2\chi(M),$$

which completes the proof.

REMARK 5. By Theorem 2, the condition

$$3\tau(M) + 2\chi(M) > 0 \tag{11}$$

for an oriented compact 4-manifold  $M$  is necessary in order that some principal  $SO(4)$ -bundle over  $M$  admit an  $SO(3)$ -unflat connection (inducing the given orientation of  $M$ ). However, (11) follows merely from the fact that  $\wedge_+ M$  carries a nonflat self-dual connection (which leads to  $p_1(\wedge_+ M)[M] > 0$ , cf. [1]). For instance, (11) is satisfied by any oriented, compact, non-Ricci-flat Einstein 4-manifold: for such a manifold, the Riemannian connection of  $\wedge_+ M$  is self-dual and nonflat (see [1]). Since (11) holds now for both orientations of  $M$ , we obtain the Thorpe-Hitchin inequality  $|\tau(M)| < \frac{2}{3}\chi(M)$  (cf. [5]).

We can now use Theorem 2 to prove a conjecture of Weinstein [13].

THEOREM 3. *Let  $P$  be a principal  $SO(4)$ -bundle over  $S^4$  with an  $SO(3)$ -unflat connection. Then  $P$  is isomorphic to the principal  $SO(4)$ -bundle, associated with the Hopf 3-sphere bundle  $S^7 \rightarrow S^4 = \mathbf{H}P^1$ .*

PROOF. By (i) of Theorem 2, one of the quotient  $SO(3)$ -bundles of  $P$ , say,  $P_{-\epsilon}$ , is isomorphic to  $P \wedge_+ S^4$ . This is nothing but the principal Hopf bundle  $\mathbf{R}P^7 \rightarrow S^4$ . In fact, the standard metric of  $\mathbf{R}P^7$  comes from the construction described in §1, so that the Hopf bundle carries a  $\{1\}$ -unflat connection and Theorem 1 works. On the other hand, since principal  $SO(3)$ -bundles over  $S^4$  are pull-backs of the Hopf bundle, we have  $p_1(P_\epsilon)[S^4] \equiv 0 \pmod{4}$  (cf. (9) and [12, p. 256]) and, by (ii) of Theorem 2,  $P_\epsilon$  is trivial. Our assertion follows now immediately from the classification of principal  $SO(4)$ -bundles over  $S^4$ .

REMARK 6. A principal  $SO(n)$ -bundle  $P \rightarrow M$  admits a *spin structure*, i.e., a double equivariant covering by a principal  $\text{Spin}(n)$ -bundle over  $M$  if and only if its second Stiefel-Whitney class  $w_2 = 0$  [8, p. 199]. On the other hand, given a principal  $\text{Spin}(n)$ -bundle  $Q \rightarrow M$ , one can use the normal subgroup  $\mathbf{Z}_2$  of  $\text{Spin}(n)$  to form the quotient principal  $SO(n)$ -bundle  $P = Q/\mathbf{Z}_2 \rightarrow M$  with an equivariant projection  $Q \rightarrow P$ .

For  $n = 3$ , we can apply Theorem 1 to  $P = Q/\mathbf{Z}_2$  and use (10) to obtain

COROLLARY 1. Let  $Q \rightarrow M$  be a principal  $S^3$ -bundle over a four-manifold  $M$ . If  $Q$  admits an unflat horizontal tensorial 2-form, then

- (i)  $M$  is an orientable spin manifold, i.e.,  $w_1(M) = w_2(M) = 0$ , and
- (ii)  $Q$  is a spin structure over  $P \wedge_+ M$  for a suitable orientation of  $M$ .

Similarly, one can prove a statement analogous to Theorem 2 for principal Spin(4) ( $= S^3 \times S^3$ )-bundles with (diagonal  $S^3$ )-unflat connections over a compact four-manifold  $M$ . As in Corollary 1, we have in this case  $w_2(M) = 0$ .

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MATHEMATICAL INSTITUTE, WROCŁAW UNIVERSITY, 50-384 WROCŁAW, POLAND

IMECC-UNICAMP, C. P. 1170, 13100-CAMPINAS, S. P., BRASIL