

MULTIPLICATIVELY INVARIANT SUBSPACES OF BESOV SPACES

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ABSTRACT. We study subspaces of Besov spaces $\dot{B}_p^{s,q}$ which are invariant under pointwise multiplication by characters. The case $s > 0$ is completely described, and for the case $s < 0$ we extend known results.

Multiplicatively invariant subspaces of Besov spaces. The study of subspaces of the homogeneous Besov spaces $\dot{B}_p^{s,q}$ invariant for multiplication by characters was initiated by R. Johnson [4], [5]. In this paper we will simplify and improve some of his results and we will also settle some points left open in [4]. More generally if X is any Banach space of (tempered) distributions we associate with X several multiplicative invariant subspaces $\pi \cdot X$, $\pi_c X$, $\pi_{\omega(h)} X$, $\pi_s X = \pi_{(1+|h|^{2\gamma})/2} X$. In the case of Besov spaces with $s > 0$ we find that $\pi_s \dot{B}_p^{s,q} = \pi \cdot \dot{B}_p^{s,q}$ (which is Theorem 1.1 in [4]) while $\pi_{s-\varepsilon} \dot{B}_p^{s,q} = 0$ if $\varepsilon > 0$. It turns out that the cases $s = 0$ and $s < 0$ behave quite differently. For instance if $s < 0$ we prove $\pi_c \dot{B}_p^{s,q} = 0$ if $s < -n/p'$ or $s = n/p'$, $q = \infty$. In [4] this is done only for the case $1 < p < 2$. We also remark that these techniques have applications to Fourier multipliers between L^p -spaces. See [4], [5].

While in [4] the Besov spaces are defined using finite differences we will use the definition of Besov spaces depending on a general partition on the Fourier side as in [8]. From this definition it is apparent that the origin on the Fourier side plays a particular role. Since multiplication by a character means translation after Fourier transform this puts a heavy restriction on the multipliers.

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0. Conventions. All function or distribution spaces are considered on \mathbf{R}^n . Likewise all integrals without integration limits are taken over all \mathbf{R}^n . In particular, L^p , where $1 < p \leq \infty$, denotes the Lebesgue space of measurable functions f such that the norm $\|f\|_p = (\int |f|^p dx)^{1/p}$ is finite. We let \mathfrak{M} be the space of bounded measures on \mathbf{R}^n and denote its norm likewise by $\|\cdot\|_1$. As usual \mathfrak{S} is the space of rapidly decreasing functions and its dual, the space of tempered distributions, will be denoted by \mathfrak{S}' . Similarly \mathfrak{D}' is the space of all (L. Schwartz) distributions. The relation $X \subset Y$, where X and Y are topological vector spaces, means that we have

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a continuous imbedding. The notation $A \approx B$, where A and B are norms, means that $C_1A \leq B \leq C_2B$ for some positive constants C_1, C_2 .

1. General results. We consider a Banach space X of tempered distributions in \mathbf{R}^n , i.e. $X \subset \mathcal{S}'$.

REMARK. Sometimes we are forced to work modulo polynomials of some fixed degree, but in order not to complicate things we presently disregard this.

If $x \in \mathbf{R}^n$ and $h \in \mathbf{R}^n$ we put

$$\langle x, h \rangle = \sum_{i=1}^n x_i h_i \quad \text{and} \quad \chi_h(x) = e^{i\langle x, h \rangle}.$$

We are interested in subspaces of X invariant under multiplication by χ_h . There is a largest such space which we denote by $\pi \cdot X$, i.e. $f \in \pi \cdot X$ iff $f \in \mathcal{S}'$ and $\chi_h f \in X$ for each $h \in \mathbf{R}^n$. On $\pi \cdot X$ we have a natural topology given by the norms $f \rightarrow \|\chi_h f\|$ where h ranges over \mathbf{R}^n .

We further denote by $\pi_c X$ the space of all $f \in \mathcal{S}'$ such that $\chi_h f$ is a bounded set in X when h belongs to a compact set in \mathbf{R}^n . We give this space the topology defined by the norms $f \rightarrow \sup_{h \in K} \|\chi_h f\|$ where K runs over all compact subsets of \mathbf{R}^n .

Finally we want to put global restriction on $\|\chi_h f\|$. If ω is a given positive function on \mathbf{R}^n we let $\pi_{\omega(h)} X$ be the space of all $f \in \pi \cdot X$ satisfying $\sup_{h \in \mathbf{R}^n} \|\chi_h f\| / \omega(h) < \infty$. With the norm $f \rightarrow \sup_{h \in \mathbf{R}^n} \|\chi_h f\| / \omega(h)$, $\pi_{\omega(h)} X$ becomes a Banach space. With no loss of generality we may assume that ω is submultiplicative, i.e. $\omega(h_1 + h_2) \leq \omega(h_1)\omega(h_2)$. However the only case of real interest for us is $\omega(h) = (1 + |h|^2)^{s/2}$, $s \geq 0$. Therefore we right away introduce the abbreviation $\pi_s X = \pi_{(1+|h|^2)^{s/2}} X$. Clearly we have the following chain of inclusions.

$$X \supset \pi \cdot X \supset \pi_c X \supset \pi_{s'} X \supset \pi_s X \quad (s' \leq s).$$

As we will see in §3, in general we cannot expect equality here. It is also clear that, for instance, $X_1 \subset X_2 \Rightarrow \pi \cdot X_1 \subset \pi \cdot X_2$, and similarly for π_c and $\pi_{\omega(h)}$.

We now establish further properties of these spaces.

LEMMA 1. *Let G be a group of affine transformations on \mathbf{R}^n which acts continuously on X . Then G acts continuously on $\pi \cdot X$. In particular, if X is translation (dilation) invariant, $\pi \cdot X$ is translation (dilation) invariant.*

PROOF. Let $a \in G$. Then we can write $ax = \tau + Ax$ where $\tau \in \mathbf{R}^n$ and A is a nonsingular linear transformation. Let $f \in \pi \cdot X$ and take $h \in \mathbf{R}^n$. Then we have the formula

$$\chi_h a(f) = \exp(-i\langle A^{-1}\tau, h \rangle) a(\chi_{(A^{-1}h)} f).$$

Therefore $\chi_h a(f) \in X$ for each $h \in \mathbf{R}^n$, i.e. $a(f) \in \pi \cdot X$. The continuity is obvious. \square

LEMMA 2. *Assume that X is relatively closed in \mathcal{D}' in the sense of Gagliardo [3] (i.e. if $(\varphi_\nu)_{\nu \in \mathbf{Z}}$ is a bounded sequence in X which converges to φ in \mathcal{D}' then $\varphi \in X$). Let $f \in \pi_{\omega(h)} X$ and $\varphi \in \mathcal{S}$ with $\int \omega(h) |\hat{\varphi}(h)| dh < \infty$. Then $\varphi f \in X$.*

PROOF. For any linear combination of characters χ_{h_i} we have the inequality:

$$\left\| \sum c_i \chi_{h_i} f \right\| \leq \sum |c_i| \omega(h_i) \|f\|_{\pi_{\omega(h_i)} X}. \tag{1}$$

By Fourier's inversion formula,

$$\varphi(x) = (2\pi)^{-n} \int e^{i\langle x, h \rangle} \hat{\varphi}(h) dh.$$

If we approximate the integral with suitable Riemann sums we get a sequence $(\varphi_\nu)_{\nu \in \mathbb{Z}}$ of finite linear combinations of characters which converge to φ in C^∞ . It follows then that $\varphi_\nu f \rightarrow \varphi f$ in \mathcal{D}' . Moreover (1) shows that $(\varphi_\nu f)_{\nu \in \mathbb{Z}}$ is bounded in X . As X is relatively closed in \mathcal{D}' we may conclude that $\varphi f \in X$. \square

REMARK. The assumptions of Lemma 2 are, in particular, fulfilled if X is a dual space. If $f \in \pi_C X$ and $\text{supp } \hat{\varphi}$ is compact the above proof also yields $\varphi f \in X$. (This will be needed in Theorem 3.)

2. Some function spaces. In this section we define some of the spaces which will be needed.

2.1. *Besov spaces* (see [1], [8]). Let $(\varphi_\nu)_{\nu \in \mathbb{Z}}$ be a family of testfunctions such that:
 $\varphi_\nu \in \mathcal{S}$, $\text{supp } \hat{\varphi}_\nu \subset \{2^{\nu-1} < |\xi| \leq 2^{\nu+1}\}$,
 $|\hat{\varphi}_\nu(\xi)| \geq C_\epsilon > 0$ if $2^\nu(2 - \epsilon)^{-1} < |\xi| \leq 2^\nu(2 - \epsilon)$, for each $\epsilon > 0$,
 $|D^\alpha \hat{\varphi}_\nu(\xi)| < c_\alpha |\xi|^{-|\alpha|}$ for every multi-index α .

Without loss of generality we may assume that, for a suitable φ_0 ,

$$\varphi_\nu(x) = 2^{\nu n} \varphi_0(2^\nu x).$$

In what follows s, p, q will always denote numbers such that $s \in \mathbb{R}$, $1 < p, q < \infty$. We then define the homogeneous Besov space $\dot{B}_p^{s,q}$ to be the space of all distributions $f \in \mathcal{S}'$ such that $\|f\|_{\dot{B}_p^{s,q}} < \infty$, where

$$\|f\|_{\dot{B}_p^{s,q}} = \left(\sum_{-\infty}^{\infty} (2^{s\nu} \|\varphi_\nu * f\|_p)^q \right)^{1/q}.$$

Further, let Φ denote a function satisfying:

$\Phi \in \mathcal{S}$, $\text{supp } \hat{\Phi} \subset \{|\xi| \leq 1\}$,
 $|\hat{\Phi}(\xi)| \geq C_\epsilon > 0$ if $|\xi| \leq 1 - \epsilon$, for each $\epsilon > 0$,
 $\hat{\Phi}(\xi) = 1$ if $|\xi| \leq \frac{1}{2}$.

Moreover, we define Φ_ν by $\Phi_\nu(x) = 2^{\nu n} \Phi(2^\nu x)$. The inhomogeneous Besov space $B_p^{s,q}$ is now the space of all $f \in \mathcal{S}'$ such that $\|f\|_{B_p^{s,q}} < \infty$ where

$$\|f\|_{B_p^{s,q}} = \|\Phi * f\|_p + \left(\sum_1^{\infty} (2^{s\nu} \|\varphi_\nu * f\|_p)^q \right)^{1/q}.$$

Finally we introduce, mainly for technical reasons, the space $\mathfrak{B}_p^{s,q}$. Put

$$\|f\|_{\mathfrak{B}_p^{s,q}} = \left(\sum_{-\infty}^{\infty} (2^{s\nu} \|\Phi_\nu * f\|_p)^q \right)^{1/q}.$$

Then $\mathfrak{B}_p^{s,q}$ is the space of all $f \in \mathcal{S}'$ such that $\|f\|_{\mathfrak{B}_p^{s,q}} < \infty$.

The following proposition gives some relations between these spaces.

- PROPOSITION 1.** (i) $\mathfrak{B}_p^{s,q} = \dot{B}_p^{s,q}$ if $s < 0$.
 (ii) $\mathfrak{B}_p^{0,\infty} = L^p$ if $p > 1$, $\mathfrak{B}_1^{0,\infty} = \mathfrak{N}$.
 (iii) $\mathfrak{B}_p^{s,q} = 0$ if $s > 0$ or $s = 0$ and $q < \infty$.
 (iv) $B_p^{s,q} = L_p \cap \dot{B}_p^{s,q}$ if $s > 0$.

PROOF. (i) We begin by proving $\mathfrak{B}_p^{s,q} \subset \dot{B}_p^{s,q}$. We have $\varphi_\nu = \Phi_{\nu+1} * \varphi_\nu$, as is easily seen by taking Fourier transforms. Young's inequality then yields

$$\|\varphi_\nu * f\|_p \leq \|\varphi_\nu\|_1 \|\Phi_{\nu+1} * f\|_p \leq C \|\Phi_{\nu+1} * f\|_p.$$

This immediately implies that

$$\|f\|_{\dot{B}_p^{s,q}} \leq C \|f\|_{\mathfrak{B}_p^{s,q}}.$$

Conversely we now prove that $\dot{B}_p^{s,q} \subset \mathfrak{B}_p^{s,q}$ if $s < 0$. Assuming, in addition, that $\sum_{-\infty}^\infty \hat{\varphi}_\nu(\xi) = 1$, we obtain as above that $\Phi_\nu = \sum_{\mu < \nu+1} \varphi_\mu * \Phi_\nu$. By application of the triangle inequality and Young's inequality we obtain

$$\begin{aligned} 2^{\nu s} \|\Phi_\nu * f\|_p &\leq \sum_{\mu < \nu+1} 2^{\nu s} \|\Phi_\nu\|_1 \|\varphi_\mu * f\|_p \\ &\leq C \sum_{\lambda > -1} 2^{\lambda s} (2^{(\nu-\lambda)s} \|\varphi_{\nu-\lambda} * f\|_p). \end{aligned}$$

Minkowsky's inequality now implies that

$$\|f\|_{\mathfrak{B}_p^{s,q}} \leq C \left(\sum_{\lambda > -1} 2^{\lambda s} \right) \|f\|_{\dot{B}_p^{s,q}},$$

where the geometrical sum converges since $s < 0$.

(ii) If $f \in \mathfrak{B}_p^{0,\infty}$ we have $\sup_\nu \|\Phi_\nu * f\|_p \leq C$. Since $\Phi_\nu * f \rightarrow f$ in \mathfrak{S}' it follows by a classical argument involving weak compactness that $f \in L^p$ if $p > 1$ and $f \in \mathfrak{N}$ if $p = 1$.

(iii) If $s > 0$ we may clearly assume that $q = \infty$. For $f \in \mathfrak{B}_p^{s,\infty}$ it follows that $\|\Phi_\nu * f\|_p \leq C 2^{-\nu s}$. Thus $\Phi_\nu * f \rightarrow 0$ in L^p . But as in (ii), $\Phi_\nu * f \rightarrow f$ in \mathfrak{S}' and therefore $f \equiv 0$ if $s > 0$. The case $s = 0, q < \infty$ is handled similarly.

(iv) See [1, p. 148]. \square

2.2. *Sobolev spaces* (see [8]). Let $k \in \mathbb{Z}_+$ and $1 < p < \infty$. The Sobolev space W_k^p is then the space of all $f \in \mathfrak{S}'$ such that $D^\alpha f \in L^p$ for $|\alpha| < k$.

2.3. *Spaces of Morrey type* (see [6], [7]). In what follows let $\lambda > 0, 1 < p < \infty$. If $f \in L^1_{loc}, x \in \mathbb{R}^n$ and $r > 0$ we put

$$G_\lambda^p(f, x, r) = \left(r^{-\lambda} \int_{|x-y| < r} |f(y)|^p dy \right)^{1/p}.$$

The space $M_q^{p,\lambda}(\cdot)$ is defined to consist of all $f \in L^1_{loc}$ such that for each $x \in \mathbb{R}^n$ the norm

$$\|f\|_{M_q^{p,\lambda}(\cdot),x} = \left(\sum_{-\infty}^\infty (G_\lambda^p(f, x, 2^\nu))^q \right)^{1/q}$$

is finite. We equip $M_q^{p,\lambda}(\cdot)$ with the topology given by the totality of these norms as x ranges over \mathbb{R}^n .

In analogy with §1 we now introduce the space $M_q^{p,\lambda}(C)$. A function $f \in L^1_{loc}$ belongs to $M_q^{p,\lambda}(C)$ iff the norms

$$\|f\|_{M_q^{p,\lambda}(C),K} = \left(\sum_{-\infty}^{\infty} \left(\sup_{x \in K} G_{\lambda}^p(f, x, 2^{\nu}) \right)^q \right)^{1/q}$$

are finite for every compact subset K in \mathbf{R}^n . Using these norms we define a topology on $M_q^{p,\lambda}(C)$.

Let ω be a given positive function on \mathbf{R}^n . The space $M_q^{p,\lambda}(\omega(x))$ then consists of all $f \in L^1_{loc}$ satisfying

$$\|f\|_{M_q^{p,\lambda}(\omega(x))} = \left(\sum_{-\infty}^{\infty} \left(\sup_{x \in \mathbf{R}^n} G_{\lambda}^p(f, x, 2^{\nu}) / \omega(x) \right)^q \right)^{1/q} < \infty.$$

With this norm $M_q^{p,\lambda}(\omega(x))$ will be a Banach space. In particular, if $\omega(x) = (1 + |x|^2)^{s/2}$, $s > 0$, which is the only case of interest for us, we write $M_q^{p,\lambda}(s)$ instead of $M_q^{p,\lambda}((1 + |x|^2)^{s/2})$. The usual Morrey spaces correspond to $M_q^{p,\lambda}(0)$ in our notation. They agree with the Stampacchia spaces $L_q^{p,\lambda}$ if $0 < \lambda < n$, $q < \infty$ or $0 < \lambda < n$, $q = \infty$. See [2] and [7]. We clearly have the following inclusions.

$$M_q^{p,\lambda}(\cdot) \supset M_q^{p,\lambda}(C) \supset M_q^{p,\lambda}(s).$$

PROPOSITION 2. (i) $M_q^{p,\lambda}(\cdot) = 0$ if $\lambda > n$ or $\lambda = n$ and $q < \infty$, or $\lambda = 0$ and $q < \infty$.

(ii) $M_{\infty}^{p,n}(0) = L^{\infty}$.

(iii) $M_{\infty}^{p,0}(\cdot) = M_{\infty}^{p,0}(0) = L^p$.

PROOF. (i) If $\lambda > n$ we may assume that $q = \infty$. Take $f \in M_{\infty}^{p,\lambda}(\cdot)$ and let x be a Lebesgue point for f . For some C we have

$$\frac{1}{r^n} \int_{|x-y| < r} |f(y)|^p dy < Cr^{\lambda-n}. \tag{2}$$

Lebesgue's theorem then implies that the left side approaches $|f(x)|$ as $r \rightarrow 0$. Thus $f(x) \equiv 0$ a.e. if $\lambda > n$.

The case $\lambda = n$, $q < \infty$ is treated similarly. With x a Lebesgue point as before, then for some ν_0 we must have

$$\left(\frac{1}{2^{\nu n}} \int_{|x-y| < 2^{\nu}} |f(y)|^p dy \right)^{q/p} \geq \frac{1}{2} |f(x)|^q$$

if $\nu < \nu_0$. Summing over ν we see that f must vanish a.e. The case $\lambda = 0$, $q < \infty$ is trivial.

(ii) If $f \in M_{\infty}^{p,n}(0)$ we may choose C in (2) independent of x . Letting $\nu \rightarrow -\infty$ we see that $f \in L^{\infty}$, again by Lebesgue's theorem.

(iii) Trivial. \square

3. Determination of $\pi \cdot \dot{B}_p^{s,q}$. We now turn to the main topic of this paper. Since the three cases $s > 0$, $s = 0$ and $s < 0$ behave quite differently, we divide this section into three parts.

3.1. *The case $s > 0$.*

THEOREM 1. *If $s > 0$ and $1 < p, q \leq \infty$ then $\pi \cdot \dot{B}_p^{s,q} = B_p^{s,q}$. Moreover we have for each $h \in \mathbf{R}^n$ and f in this space*

$$\|f\|_{\dot{B}_p^{s,q}} + \|\chi_h f\|_{\dot{B}_p^{s,q}} \approx |h|^s \|f\|_p + \|f\|_{\dot{B}_p^{s,q}}. \tag{3}$$

The main component in the proof will be the observation that multiplication by χ_h corresponds to translation on the Fourier side, which is expressed in the following lemma.

LEMMA 3. *Assume that $\psi \in L^1$ and $\text{supp } \hat{\psi}$ is compact. For some ν_0 we then have the inequality*

$$\|\psi * f\|_p \leq \|\psi\|_1 \|\varphi_\nu * \chi_h f\|_p$$

if $\nu > \nu_0$, $|h| \approx 2^\nu$ and the right-hand side is finite, $1 < p < \infty$.

PROOF OF LEMMA 3. Choose $(\varphi_\nu)_{\nu \in \mathbf{Z}}$, as in the definition of $\dot{B}_p^{s,q}$, satisfying, in addition, $\hat{\varphi}_\nu(\xi) = 1$ if $\xi \in I_\nu = \{\xi: |\xi| - 2^\nu < C_0 2^\nu\}$. As $\text{supp } \hat{\psi}$ is compact, there exists a ν_0 such that $\text{supp } \hat{\psi} \subset \{\xi: |\xi| < C_0 2^{\nu_0}\}$. Thus if $\nu > \nu_0$, $|h| \approx 2^\nu$, we have $\text{supp}(\chi_h \hat{\psi}) = h + \text{supp } \hat{\psi} \subset I_\nu$. Our choice of $(\varphi_\nu)_{\nu \in \mathbf{Z}}$ then implies that $\chi_h \hat{\psi} * \varphi_\nu = \chi_h \hat{\psi}$. Rewriting $\chi_h(\psi * f)$ as $\varphi_\nu * \chi_h \psi * \chi_h f$ and applying Young's inequality, we get

$$\|\psi * f\|_p = \|\chi_h(\psi * f)\|_p \leq \|\psi\|_1 \|\varphi_\nu * \chi_h f\|_p.$$

PROOF OF THEOREM 1. We begin by proving (3) for a fixed h . With $\psi = \Phi$, as in the definition of $B_p^{s,q}$, and h_0 such that $|h_0| \approx 1$ we obtain from Lemma 3,

$$\|\Phi * f\|_p \leq C \|\chi_{h_0} f\|_{\dot{B}_p^{s,q}}.$$

This implies that

$$\|f\|_{B_p^{s,q}} \leq C \|\chi_{h_0} f\|_{\dot{B}_p^{s,q}} + \|f\|_{\dot{B}_p^{s,q}}. \tag{4}$$

In order to obtain the converse inequality we first use the imbedding $B_p^{s,q} \subset \dot{B}_p^{s,q}$, $s > 0$ (Proposition 1(iv)) which gives

$$\begin{aligned} \|f\|_{\dot{B}_p^{s,q}} &\leq C \|f\|_{B_p^{s,q}}, \\ \|\chi_{h_0} f\|_{\dot{B}_p^{s,q}} &\leq C \|\chi_{h_0} f\|_{B_p^{s,q}}. \end{aligned} \tag{5}$$

But χ_h acts continuously on $B_p^{s,q}$. Indeed, in view of the fact that $B_p^{s,q} = (L^p, W_k^p)_{\theta,q}$, $s = \theta k$, $0 < \theta < 1$ (see [8, p. 64]), it suffices to show that χ_h acts continuously on W_k^p , which is obvious. We thus have

$$\|\chi_{h_0} f\|_{\dot{B}_p^{s,q}} \leq C \|f\|_{B_p^{s,q}}. \tag{6}$$

By combining (4)–(6) we get the desired inequalities:

$$\|f\|_{B_p^{s,q}} \approx \|\chi_{h_0} f\|_{\dot{B}_p^{s,q}} + \|f\|_{\dot{B}_p^{s,q}} \quad \text{with } |h_0| \approx 1. \tag{7}$$

An argument with dilations will now establish (3). Indeed take $0 \neq h \in \mathbf{R}^n$. As rotations act continuously on $\dot{B}_p^{s,q}$ (see Lemma 1) we may assume that h and h_0 are collinear, i.e. $h = \lambda h_0$ for some $\lambda > 0$.

Let τ_δ denote the dilation operator defined by

$$(\tau_\delta f)(x) = f(\delta x), \quad \delta > 0.$$

It is well known that τ_δ acts continuously on $\dot{B}_p^{s,q}$ and we have

$$\|\tau_\delta f\|_{\dot{B}_p^{s,q}} \approx \delta^{s-n/p} \|f\|_{\dot{B}_p^{s,q}}. \tag{8}$$

We further notice the formula

$$\chi_h f = \tau_\lambda(\chi_{h_0} \tau_{1/\lambda} f). \tag{9}$$

If we apply (7) with $\tau_{1/\lambda} f$ and multiply by $\lambda^{s-n/p}$ we get, in view of (8) and (9),

$$\|\chi_h f\|_{\dot{B}_p^{s,q}} + \|f\|_{\dot{B}_p^{s,q}} \approx \lambda^{s-n/p} \|\tau_{1/\lambda} f\|_{\dot{B}_p^{s,q}}.$$

Finally, the right side is simplified by once again invoking Proposition 1(iv) thereby obtaining

$$\begin{aligned} \lambda^{s-n/p} \|\tau_{1/\lambda} f\|_{\dot{B}_p^{s,q}} &\approx \lambda^{s-n/p} (\|\tau_{1/\lambda} f\|_p + \|\tau_{1/\lambda} f\|_{\dot{B}_p^{s,q}}) \\ &\approx \lambda^s \|f\|_p + \|f\|_{\dot{B}_p^{s,q}}. \end{aligned}$$

As $\lambda^s \approx |h|^s$ we have now proved (3), and thus $\pi \cdot \dot{B}_p^{s,q} = B_p^{s,q}$. \square

COROLLARY 1. *Let $s > 0, 1 < p, q \leq \infty$. Then $\pi_{s'} \dot{B}_p^{s,q} = B_p^{s,q}$ if $s' > s, \pi_{s'} \dot{B}_p^{s,q} = 0$ if $s' < s$.*

2.2. *The case $s = 0$.*

THEOREM 2. (i) *If $1 < p, q \leq \infty$ then $\pi \cdot \dot{B}_p^{0,q} \subset B_p^{0,q}$.*

(ii) *If $q = \infty$ and $1 < p \leq \infty$ or $q \geq \max(2, p)$ and $1 < p < \infty$, then*

$$\pi_0 \dot{B}_p^{0,q} = L^p \quad \text{if } p > 1, \pi_0 \dot{B}_1^{0,\infty} = \mathfrak{N}.$$

PROOF. (i) As in the proof of Theorem 1 we obtain

$$\|f\|_{\dot{B}_p^{0,q}} \leq C \|\chi_h f\|_{\dot{B}_p^{0,q}} + \|f\|_{\dot{B}_p^{0,q}} \quad \text{if } |h| \approx 1.$$

Thus $\pi \cdot \dot{B}_p^{0,q} \subset B_p^{0,q}$.

(ii) If p and q are as in the hypothesis, we have the imbedding $L^p \subset \dot{B}_p^{0,q}$ (see [8, p. 80]). It follows that $L^p \subset \pi_0 \dot{B}_p^{0,q}$. We now show that, conversely, $\pi_0 \dot{B}_p^{0,q} \subset L^p$ holds. Take $f \in \pi_0 \dot{B}_p^{0,q}$. Then for each $\nu, \Phi_\nu * f \in \pi_0 \dot{B}_p^{0,q}$. Lemma 3 now gives, if $|h| \approx 2^\nu$ and $\psi = \Phi_{\nu+1}$,

$$\begin{aligned} \|\Phi_\nu * f\|_p &\leq C \|\chi_h(\Phi_\nu * f)\|_{\dot{B}_p^{0,q}} \\ &\leq C \|\Phi_\nu * f\|_{\pi_0 \dot{B}_p^{0,q}} \\ &\leq C \|f\|_{\pi_0 \dot{B}_p^{0,q}}, \end{aligned}$$

so by Proposition 1(ii), $f \in L^p$ if $p > 1, f \in \mathfrak{N}$ if $p = 1$. \square

2.3. *The case $s < 0$.* We first observe that Proposition 1(ii) allows us to replace $\dot{B}_p^{s,q}$ with $\mathfrak{B}_p^{s,q}$ as $s < 0$. This will be done in the proofs given below. It will be convenient to introduce the following terminology.

DEFINITION. We say that s, p, q , where $s < 0, 1 < p, q \leq \infty$, are good indices iff $-n/p' \leq s < 0$ if $q = \infty$, or $-n/p' < s < 0$ if $q < \infty$. (p' denotes the conjugate index to p .) Otherwise we say that s, p, q are bad indices.

Using this notation we now state our main result.

THEOREM 3. *If s, p, q are bad indices then*

- (i) $\pi \cdot \dot{B}_p^{s,q} \cap L^1 = 0,$
- (ii) $\pi_c \dot{B}_p^{s,q} = 0.$

PROOF. We reduce to the case $p = \infty$. This may be accomplished by using Besov's imbedding theorem (see [8, p. 63]) implying that $\dot{B}_p^{s,q} \subset \dot{B}_\infty^{s-n/p,q}$.

(i) Take $f \in \pi \cdot \dot{B}_\infty^{s,q} \cap L^1$. We then obtain, using Parseval's formula,

$$\begin{aligned} 2^{\nu s} \|\Phi_\nu * f\|_\infty &\geq 2^{\nu s} |(\Phi_\nu * f)(0)| = 2^{\nu s} \left| \int \Phi_\nu(x) f(-x) dx \right| \\ &= C 2^{\nu s} \left| \int \hat{\Phi}(\eta/2^\nu) \hat{f}(-\eta) d\eta \right|. \end{aligned}$$

We may assume that $\hat{\Phi}(\eta) \geq 0$. If we apply the above inequality to $\chi_h(f * \bar{f})$ and then use Young's inequality we find that

$$2^{\nu s} \int_{|\eta-h| < 2^{\nu-1}} |\hat{f}(\eta)|^2 d\eta \leq C 2^{\nu s} \|\Phi_\nu * \chi_h f\|_\infty \|f\|_1.$$

Therefore it follows that $\hat{f} \in M_{2q}^{2,-s}(\cdot)$. After invoking Proposition 2(ii), we arrive at the result $\hat{f} \equiv 0$ if s, ∞, q are bad indices.

(ii). Take $f \in \pi_c \dot{B}_\infty^{s,q}$. Our aim is to modify f so that we may use (i). Assume first that $q > 1$. Take ψ with $\hat{\psi} \in C_0^\infty$. Then $f * \psi \in \pi_c \dot{B}_\infty^{s,q}$. As $\text{supp}(\psi * f)^\wedge$ is compact, it is easily seen that $f * \psi \in L^\infty$. Indeed we clearly have for some finite N ,

$$f * \psi = \sum_{-\infty}^N \varphi_\nu * f * \psi.$$

As $s < 0$, an application of Minkowsky's inequality yields

$$\|f * \psi\|_\infty \leq \|f * \psi\|_{\dot{B}_\infty^{0,1}} \leq 2^{-Ns} \|f * \psi\|_{\dot{B}_\infty^{s,q}}.$$

Take $\varphi \in L^1 \cap \mathcal{S}$ with $\text{supp } \hat{\varphi}$ compact. Hölder's inequality now implies that $\varphi(f * \psi) \in L^1$. As $\dot{B}_\infty^{s,q}$ is a dual space for $q > 1$ (see [8, p. 74]), we find, by using Lemma 2, that $\varphi(f * \psi) \in \dot{B}_\infty^{s,q}$. Thus $\varphi(f * \psi) \in \pi \cdot \dot{B}_\infty^{s,q} \cap L^1$ if we apply the above argument to $\chi_h \varphi(f * \psi)$. (i) then shows that $\varphi(f * \psi) \equiv 0$ for all φ and ψ . Thus f must vanish.

The case $q = 1$ follows trivially from what we have proved for $q > 1$. Just notice that $\dot{B}_p^{s,1} \subset \dot{B}_p^{s,q}$ for any $q \geq 1$. \square

The main idea in this proof was the pointwise behaviour of f . The Hausdorff-Young theorem allows us to sharpen Theorem 2 if $1 < p < 2$. Let \mathcal{F} denote the Fourier transform.

THEOREM 4. *Let $s < 0, 1 \leq p \leq 2, 1 \leq q \leq \infty$. Then $\mathcal{F}: \pi \cdot \dot{B}_p^{s,q} \rightarrow M_q^{p',-sp'}(\cdot)$. In particular, $\pi \cdot \dot{B}_p^{s,q} = 0$ if s, p, q are bad indices.*

PROOF. An application of the Hausdorff-Young theorem yields for each $h \in \mathbf{R}^n$ and ν ,

$$\begin{aligned} 2^{\nu s} \|\Phi_\nu * \chi_h f\|_p &\geq C 2^{\nu s} \|\hat{\Phi}(\cdot/2^\nu) \hat{f}(\cdot - h)\|_{p'} \\ &\geq C \left(2^{\nu sp'} \int_{|\eta-h| < 2^{\nu-1}} |\hat{f}(\eta)|^{p'} d\eta \right)^{1/p'} \\ &= C G_{-sp'}^{p'}(\hat{f}, h, 2^{\nu-1}). \end{aligned}$$

Raising this to the q th power and summing over ν we obtain

$$\|\hat{f}\|_{M_q^{p', -s'(\cdot), h}} \leq C \|\chi_h f\|_{\dot{B}_p^{s, q}}.$$

This proves the first part of the theorem.

That $\pi \cdot \dot{B}_p^{s, q} = 0$, for s, p, q bad indices, is now a consequence of Proposition 2(i). \square

Conversely, reversing all inequality signs in the above proof, we obtain the following theorem and corollary.

THEOREM 5. *If $s < 0$, $2 \leq p \leq \infty$, $1 \leq q \leq \infty$. Then $\mathfrak{F}: M_q^{p', -s'(\cdot)} \rightarrow \pi \cdot \dot{B}_p^{s, q}$.*

COROLLARY 2. *If $s < 0$, $1 \leq q \leq \infty$ then*

$$\pi \cdot \dot{B}_2^{s, q} = \mathfrak{F} M_q^{2, -2s}(\cdot).$$

REMARKS. (i) Theorems 4 and 5 and Corollary 2 have obvious extensions to the spaces $\pi_{\omega(h)} \dot{B}_p^{s, q}$ and $\pi_c \dot{B}_p^{s, q}$. The conclusions will then involve the spaces $M_q^{p, \lambda}(\omega(x))$ and $M_q^{p, \lambda}(C)$.

(ii) Corollary 2 shows that the spaces $\pi \cdot X$, $\pi_c X$ and $\pi_{\omega(h)} X$ do not coincide in general. For instance it is easily seen that $M_q^{2, -2s}(\cdot)$ and $M_q^{2, -2s}(C)$ behave topologically differently. Further, $f \in M_\infty^{2, n}(\omega(x))$ implies that $|f(x)| < C\omega(x)$. As $M_\infty^{2, n}(0) = L^\infty$ we thus see that $\pi_{\omega(h)} \dot{B}_2^{-n/2, \infty} \neq \pi_0 \dot{B}_2^{-n/2, \infty}$ if, e.g., $\omega(h) \rightarrow 0$ as $h \rightarrow \infty$.

As is seen by Theorem 5, $\pi \cdot \dot{B}_p^{s, q}$ is a "large" space if $2 \leq p \leq \infty$ and s, p, q are good indices. This is true without any restrictions on p , as is seen from the following proposition.

PROPOSITION 3. $\mathfrak{S} \subset \pi_0 \dot{B}_p^{s, q}$ if s, p, q are good indices.

We omit the proof since this simply amounts to a use of Young's inequality.

REMARK. Some of our results are already found in [4]. More precisely, Johnson considers $\pi_s \dot{B}_p^{s, q}$ if $s > 0$ and $\pi_0 \dot{B}_p^{s, \infty}$ if $s \leq 0$. He establishes a weaker version of Theorem 1. He also proves Theorem 2(ii), if $q = \infty$, Theorems 4, 5 and Corollary 2 for $\pi_0 \dot{B}_p^{s, \infty}$.

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