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HULLS OF DEFORMATIONS IN $\mathbf{C}^n$

BY

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ABSTRACT. A problem of E. Bishop on the polynomially convex hulls of deformations of the torus is considered. Let the torus T^2 be the distinguished boundary of the unit polydisc in $\mathbf{C}^2$. If $t \mapsto T_t^2$ is a smooth deformation of T^2 in $\mathbf{C}^2$ and g_0 is an analytic disc in $\mathbf{C}^2$ with boundary in T^2, a smooth family of analytic discs $t \mapsto g_t$ is constructed with the property that the boundary of g_t lies in T_t^2. This construction has implications for the polynomially convex hulls of the tori T_t^2. An analogous problem for a 2-sphere in $\mathbf{C}^2$ is also considered.

Introduction. By a result of A. Browder [3], a compact real orientable manifold X of dimension $\geqslant n$ in $\mathbf{C}^n$ is never polynomially convex. Beginning with the work of E. Bishop [1], a great deal of effort has gone into giving a constructive explanation of this phenomenon. Bishop showed that there exist analytic discs in $\mathbf{C}^n$ with boundaries in X near certain "exceptional" points of X. His method was amplified by a number of others; a report on these developments is given in [6].

We shall consider a related problem which has been attributed to E. Bishop in [2, p. 234, Problem 17]. Let $\{F_t\}$ be a deformation of the torus T^2 in $\mathbf{C}^2$; i.e., $F_0 =$ identity and $\{F_t\}$ is a family of diffeomorphisms of T^2 into $\mathbf{C}^2$ which vary smoothly with t. Put $T_t^2 = F_t(T^2)$; T_t^2 is a torus in $\mathbf{C}^2$ which is "close" to T^2 for small t. By Browder's result T_t^2 is not polynomially convex. Also, every point of the closed unit polydisc $\overline{U}^2$, the polynomially convex hull of $T_0^2 = T^2$, lies on some analytic disc g_0 with boundary in T^2. The problem is to show that associated to the smooth deformation $\{F_t\}$ there exists a smooth family of analytic discs $\{g_t\}$ such that the boundary of g_t lies in T_t^2. We shall construct such a family $\{g_t\}$, at least for t sufficiently small. This will account for the nonpolynomial convexity of T_t^2 in the most direct manner. As T^2 is deformed to T_t^2, its hull $\overline{U}^2$ is deformed to a set D_t which is close to $\overline{U}^2$ $(= D_0)$ and which is related to T_t^2 in a way that parallels the relationship of $\overline{U}^2$ to T^2; namely, D_t has a topological boundary composed of analytic discs whose own boundaries lie in T_t^2 and thus D_t has T_t^2 as a distinguished boundary. Eric Bedford [7] has proved that D_t is in fact the polynomially convex hull of T_t^2.

We shall also obtain an analogous result for a particular 2-sphere in $\mathbf{C}^2$:

$$S^2 = \left\{(z_1, z_2) \in \mathbf{C}^2\colon \operatorname{Im} z_2 = 0, |z_1|^2 + (\operatorname{Re} z_2)^2 = 1\right\}.$$

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Specifically, if g_0 is an analytic disc in $\mathbf{C}^2$ with boundary in S^2 and if $t \to S_t^2$ is a smooth deformation in $\mathbf{C}^2$, then there exists a smooth family of analytic discs $t \to g_t$ in C^2 with the boundary of g_t in S_t^2. This result is in some sense complementary to Bishop's construction of analytic discs near the exceptional points of S_t^2. Bedford and Gaveau **[8]** have considered this problem from a different point of view. They obtain the complete hull by a process of continuation applied to the Bishop discs.

In order to produce analytic discs whose boundaries lie in prescribed sets, we must solve certain functional equations for unknown holomorphic functions on the unit disc. Bishop attacked this problem in **[1]** by iterating a contradiction mapping. We shall find it more convenient to use the implicit function theorem for Banach spaces **[4]**; in fact, the same approach can be applied to Bishop's case in order to avoid an explicit iteration. Of course, iteration still lurks in the proof of the implicit function theorem. Banach spaces enter from the fact that rather than deal with a deformation consisting of a 1-parameter family of diffeomorphisms of T^2 we shall consider all diffeomorphisms of T^2 into $\mathbf{C}^2$ which are close to the identity. This infinite dimensional space can be associated with a Banach space such that a small deformation of T^2 is then viewed as a curve through the origin in the Banach space.

Finally I want to thank Eric Bedford for an observation which simplifies Theorem 3 which was overly complicated in its original version.

1. Preliminaries. We shall denote the open unit disc in $\mathbf{C}$ by U and the unit circle by T. Thus U^n and T^n are the open unit polydisc and torus in $\mathbf{C}^n$, respectively. If f is a coninuous complex valued function on a set S, we denote $\sup\{|f(x)|: x \in S\}$ by $|f|_S$. For K a compact set in $\mathbf{C}^n$, $\hat{K}$, the polynomially convex hull of K, is the set $\{z \in \mathbf{C}^n: |f(z)| \leq |f|_K$, for all polynomials f in $\mathbf{C}^n\}$. An analytic disc in $\mathbf{C}^n$ is a map $g: \bar{U} \to \mathbf{C}^n$ such that g is continuous and nonconstant on $\bar{U}$ and holomorphic on U; we do not require that g be injective. We shall also refer to the image of g in $\mathbf{C}^n$ as an analytic disc and shall call the set $g(T)$ the boundary of g. The maximum principle implies that the polynomially convex hull of the boundary of g contains the full analytic disc g. If $g = (g_1, g_2, \dots, g_n)$ is an analytic disc in $\mathbf{C}^n$ with boundary in T^n, then each g_k has unit modulus on T and therefore is a finite Blaschke product, where we admit unimodular constants as Blaschke products of order zero. For K compact in $\mathbf{C}^n$, we distinguish a number of subalgebras of the algebra $C(K)$ of all continuous complex valued functions on K: $A(K)$ consists of those functions in $C(K)$ which are holomorphic on the interior of K; $P(K)$, $R(K)$ and $H(K)$ are the closures in $C(K)$ of the polynomials, the rational functions holomorphic on K, and the functions holomorphic on K, respectively. We shall say that a map is $\mathcal{C}^k$ if it has continuous derivatives of order $\leq k < \infty$. If X is a compact $\mathcal{C}^\infty$ manifold with or without boundary, then $\mathcal{C}^k(X)$ with the topology of uniform convergence of derivatives of order $\leq k$ is a Banach algebra for an appropriate norm.

For functions φ on the unit circle T we have some special notation: $\|\varphi\|_\infty = |\varphi|_T = \sup\{|\varphi(e^{i\theta})|: 0 \leq \theta \leq 2\pi\}$ and $\|\varphi\|_2 = (\int_0^{2\pi} |\varphi(e^{i\theta})|^2\, d\theta)^{1/2}$; $L^2(T)$ is the usual Hilbert space associated with the second norm. Let $\mathcal{H}_2$ be the set of all complex valued functions on T which have distributional derivatives of order ≤ 2 in $L^2(T)$. Here derivatives are with respect to θ where $T = \{e^{i\theta}: 0 \leq \theta \leq 2\pi\}$. One can also describe $\mathcal{H}_2$ as the space of complex valued $\mathcal{C}^1$ functions on T whose first derivatives are absolutely continuous with square integrable derivatives. From this it is easy to see that $\mathcal{H}_2$ is a ring. Although $\mathcal{H}_2$ can be normed as a Hilbert space, we shall use an equivalent norm which is more convenient for working with the multiplicative structure; namely, we define $\|f\| = \|f\|_\infty + \|f'\|_\infty + \frac{1}{2}\|f''\|_2$ for $f \in \mathcal{H}_2$. It is easy to check that $\|fg\| \leq \|f\| \cdot \|g\|$ and that $\mathcal{H}_2$ is a Banach algebra with this norm. We shall find it convenient to take the scalar field of $\mathcal{H}_2$ to be the real numbers.

Every $f \in \mathcal{H}_2$ has a Fourier series $f = \sum_{-\infty}^{\infty} c_n e^{in\theta}$ with $\sum |n^2 c_n|^2 < \infty$ and, conversely, any integrable function whose Fourier coefficients satisfy this bound is in $\mathcal{H}_2$. We shall utilize two closed subspaces of $\mathcal{H}_2$: $\mathcal{Q}_2$ is the set of all functions in $\mathcal{H}_2$ which extend to be holomorphic in the unit disc, $f \in \mathcal{H}_2$ is in $\mathcal{Q}_2$ if and only if $c_n = 0$ for $n < 0$, i.e., $f = \sum_{n=0}^{\infty} c_n e^{in\theta}$; $\mathcal{H}_2(\mathbf{R})$ is the subspace of real valued functions in $\mathcal{H}_2$. Each $u \in \mathcal{H}_2(\mathbf{R})$ has a Fourier series of the form

$$u = a_0 + \sum_{n=1}^{\infty} a_n e^{in\theta} + \bar{a}_n e^{-in\theta}$$

with a_0 real. The conjugate function $\tilde{u}$ of u is defined by

$$\tilde{u} = \sum_{n=1}^{\infty} - i a_n e^{in\theta} + i \bar{a}_n e^{-in\theta}.$$

Then $u + i\tilde{u} \in \mathcal{Q}_2$ and the map J: $\mathcal{H}_2(\mathbf{R}) \to \mathcal{Q}_2$, $Ju = u + i\tilde{u}$ is a bounded linear transformation. This is a theorem of M. Riesz and is easily seen when $\mathcal{H}_2$ is given its equivalent Hilbert space norm.

2. A special case. We begin by treating a rather special class of deformations of T^2. The problem of finding analytic discs with boundaries in the associated deformed tori involves the solution of a certain functional equation (2.3). This functional equation plays a key role in the subsequent discussion of the general deformations of T^n.

Consider the image T^2_φ in $\mathbf{C}^2$ of T^2 under the map

$$(z_1, z_2) \mapsto (z_1, z_2 + \varphi(z_1)) = (w_1, w_2) \tag{2.1}$$

where φ is a smooth function on T which is close to zero in $\mathcal{H}_2$-norm. If g is a finite Blaschke product, then $\lambda \to (\lambda, g(\lambda))$ is an analytic disc in $\mathbf{C}^2$ with boundary in T^2. We seek an analytic disc in $\mathbf{C}^2$ of the form $\lambda \mapsto (\lambda, f_\varphi(\lambda))$ with boundary in T^2_φ. The inverse of (2.1) is given explicitly by

$$z_1 = w_1, \quad z_2 = w_2 - \varphi(w_1). \tag{2.2}$$

Thus $(w_1, w_2) \in T^2_\varphi$ if and only if $|w_1| = 1$ and $|w_2 - \varphi(w_1)| = 1$. What we require of f_φ then it is that it satisfy the functional equation

$$|f_\varphi(\lambda) - \varphi(\lambda)| = 1 \quad \text{for } |\lambda| = 1, \tag{2.3}$$

and that $f_0 = g$ when $\varphi \equiv 0$. We shall now solve (2.3) for all sufficiently small φ with functions f_φ which are analytic in the unit disc and which vary smoothly with φ.

THEOREM 1. *Let g be a finite Blaschke product. There exist an open neighborhood Ω of 0 in $\mathcal{H}_2$ and a $\mathcal{C}^1$ nonlinear operator $E\colon \Omega \to \mathcal{A}_2$ such that $E(0) = g$ and, for $\varphi \in \Omega$, (2.3) holds for $f_\varphi = E(\varphi)$.*

REMARK. To put this into the setting of the Introduction, let $t \mapsto F_t$ with $F_t(z_1, z_2) = (z_1, z_2 + \varphi_t(z_1))$ be a deformation of T^2 of the special type under consideration and fix $g_0(\lambda) = (\lambda, g(\lambda))$, an analytic disc in $\mathbf{C}^2$ with boundary in T^2. Then for t sufficiently small, $\varphi_t \in \Omega$ and so we may define $g_t = (\lambda, E(\varphi_t)) \in \mathcal{A}_2 \oplus \mathcal{A}_2$. Viewing elements of $\mathcal{A}_2$ as holomorphic functions, we see that $t \mapsto g_t$ is a smooth family of analytic discs in $\mathbf{C}^2$ with the boundary of g_t in T_t^2.

PROOF. Let $W = \{\psi \in \mathcal{H}_2\colon \|\psi\|_\infty < 1\}$, an open set in $\mathcal{H}_2$. Define $S\colon W \to \mathcal{H}_2(\mathbf{R})$ by $S(\psi) = u$ where $u(\lambda) = \log|g(\lambda) - \psi(\lambda)|$ for $\lambda \in T$. Since $|g - \psi| > 0$, one can compute the derivatives of u to check that $u \in \mathcal{H}_2(\mathbf{R})$. Put $h = Ju = u + i\tilde{u} \in \mathcal{A}_2$ and define $A\colon W \to \mathcal{A}_2$ by $A(\psi) = e^{-h} = e^{-J \circ S(\psi)}$.

Now set

$$f = ge^{-h} = g \cdot A(\psi) \tag{2.4}$$

and note that $f \in \mathcal{A}_2$ since $h \in \mathcal{A}_2$. From (2.4), $g = fe^h$ and we get

$$\begin{aligned} 1 = |g| = |f|e^u &= |f|\,|g - \psi| = |f| \cdot |1 - \bar{g}\psi| \\ &= |f - \bar{g}f\psi| = |f - e^{-h}\psi| \end{aligned} \tag{2.5}$$

holding on T. Now define

$$Q\colon W(\subset \mathcal{H}_2) \to \mathcal{H}_2 \text{ by } Q(\psi) = \psi \cdot e^{-h} = \psi \cdot A(\psi).$$

We claim

$$\begin{aligned} &\text{(a)} \quad A\colon W \to \mathcal{A}_2 \text{ is a } \mathcal{C}^1 \text{ function with } A(0) = 1. \\ &\text{(b)} \quad Q\colon W \to \mathcal{H}_2 \text{ is } \mathcal{C}^1. \\ &\text{(c)} \quad Q(0) = 0 \text{ and } Q'(0) = \text{identity}. \end{aligned} \tag{2.6}$$

We shall check these conditions below. Assuming them for now, we appeal to the inverse function theorem for Banach spaces [4] to conclude that there exist open neighborhoods W_1 and Ω of 0 in $\mathcal{H}_2$ with $W_1 \subset W$ such that Q is a $\mathcal{C}^1$ diffeomorphism of W_1 onto Ω. Now define $E\colon \Omega \to \mathcal{A}_2$ by $E = M_g \circ A \circ Q^{-1}$ where M_g is the bounded linear transformation of $\mathcal{A}_2$ defined by $M_g(f) = g \cdot f$. Then E is clearly a $\mathcal{C}^1$ map and $E(0) = g$ because $Q(0) = 0$ and $A(0) = 1$.

Given $\varphi \in \Omega$, put $\psi = Q^{-1}(\varphi)$. Then $\varphi = Q(\psi) = \psi \cdot A(\psi) = \psi e^{-h}$ and $E(\varphi) = g \cdot A(\psi) = ge^{-h} = f$ and so (2.5) can be restated as $1 = |E(\varphi)(\lambda) - \varphi(\lambda)|$ for $|\lambda| = 1$. Thus E has the desired properties.

It remains to show that (2.6)(a), (b) and (c) are valid. This will follow from the following observations. The fact that $\mathcal{H}_2$ has been normed as a Banach algebra will simplify some of the indicated computations which have been left to the reader.

(i) S: $W \to \mathcal{H}_2(\mathbf{R})$ is a $\mathcal{C}^1$ map. To see this fix $\psi \in W$ and write, for small $\sigma \in \mathcal{H}_2$,

$$S(\psi + \sigma) = \log|g - \psi - \sigma| = \log|g - \psi| + \log\left|1 - \frac{\sigma}{g - \psi}\right|$$

$$= \log|g - \psi| + \frac{1}{2}\log\left(1 - \frac{\sigma}{g - \psi}\right) + \frac{1}{2}\log\left(1 - \frac{\bar{\sigma}}{\bar{g} - \bar{\psi}}\right).$$

Using the Taylor series for $\log(1 - z)$ at $z = 0$ one can expand each of the last two terms in a series. Now easy estimates of these series in the $\mathcal{H}_2$-norm show that S is differentiable at ψ and that $S'(\psi)(\sigma) = -\mathrm{Re}(\sigma/(g - \psi))$. From this expression it follows that the map $\psi \to S'(\psi)$ of W into $\mathrm{Hom}(\mathcal{H}_2, \mathcal{H}_2(\mathbf{R}))$ is continuous when the latter space of bounded linear transformations is given the norm topology.

(ii) J: $\mathcal{H}_2(R) \to \mathcal{Q}_2$ is a bounded linear transformation and consequently is $\mathcal{C}^1$. Similarly M_g: $\mathcal{H}_2 \to \mathcal{H}_2$ and M_g: $\mathcal{Q}_2 \to \mathcal{Q}_2$ are $\mathcal{C}^1$.

(iii) The map $h \to e^{-h}$ of $\mathcal{Q}_2$ to $\mathcal{Q}_2$ is easily seen to be $\mathcal{C}^1$.

(iv) Now (a) follows because $A(\psi) = e^{-J \circ S(\psi)}$ is a composition of $\mathcal{C}^1$ maps. Also $S(0) = 0$ and $J(0) = 0$ imply $A(0) = 1$.

(v) Using $Q(\psi) = A(\psi) \cdot \psi$, a short computation shows that Q is differentiable at each $\psi \in W$ and that

$$Q'(\psi)(\sigma) = \psi \cdot A'(\psi)(\sigma) + A(\psi) \cdot \sigma. \tag{2.7}$$

It follows that $\psi \mapsto Q'(\psi)$ is continuous. This gives (b).

(vi) Finally $Q(0) = A(0) \cdot 0 = 0$ and (2.7) implies $Q'(0)(\sigma) = A(0) \cdot \sigma = \sigma$, as claimed in (c).

3. The general case for T^n. Let F be a $\mathcal{C}^3$ diffeomorphism of the torus T^n into $\mathbf{C}^n$ which is close to the identity, T_F^n the image of T^n under F. Fix $g = (g_1, g_2, \ldots, g_n)$, an analytic disc in T^n with boundary in T^n. Our problem is to find an analytic disc f_F in $\mathbf{C}^n$ with boundary in T_F^n such that f_F varies smoothly with F when F lies in some neighborhood of the identity I and such that $f_I = g$.

We can write $F(z) = z + \beta(z)$ where $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$ is an n-tuple of complex valued $\mathcal{C}^3$ functions on T^n which are close to zero in $\mathcal{C}^3(T^n)$-norm. Put $\mathbf{C}^* = \mathbf{C} \setminus \{0\}$ and define the retraction ρ: $\mathbf{C}^{*n} \to T^n$ by $\rho(z_1, z_2, \ldots, z_n) = (z_1/|z_1|, \ldots, z_n/|z_n|)$. Then F can be extended to a map $\tilde{F}$: $\mathbf{C}^{*n} \to \mathbf{C}^n$ by setting $\tilde{F}(z) = z + \beta(\rho(z))$. Fix $0 < s < r < 1$ and define $K_t = \{z \in \mathbf{C}^n$: $t < |z_k| < 1/t$ for $1 \leqslant k \leqslant n\}$; $T^n \subseteq K_r \subseteq K_s$.

LEMMA. *If β is sufficiently small in $\mathcal{C}^3(T^n)$, then $\tilde{F}$ is a diffeomorphism of the interior of K_s onto a domain Σ in $\mathbf{C}^n$ containing K_r.*

PROOF. If β is sufficiently small then $\tilde{F}$ is easily seen to be injective on K_s with nonvanishing Jacobian determinant. Hence $\tilde{F}$ maps $\mathrm{int}(K_s)$ diffeomorphically onto a domain Σ in $\mathbf{C}^n$. If β is small, then $\tilde{F}$ is close to the identity and so $\partial\Sigma$ is close to ∂K_s and consequently $\partial\Sigma$ is disjoint from K_r. As $K_r \cap \Sigma$ is nonempty, it follows from the connectedness of K_r that $K_r \subseteq \Sigma$.

We can write the inverse G of $\tilde{F}$ as $G(z) = z - \alpha(z)$ where $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ is an n-tuple of $\mathcal{C}^3$ complex valued functions on Σ which are small in $\mathcal{C}^3(K_r)$ norm when β is small in $\mathcal{C}^3(T^n)$, since the map $\beta \to \alpha$ of a neighborhood of 0 in $\mathcal{C}^3(T^n)$ into $\mathcal{C}^3(K_r)$ is norm continuous. When β is sufficiently small, T_F^n is contained in K_r and then, for $z \in K_r$, $z \in T_F^n$ if and only if $G(z) = z - \alpha(z) \in T^n$; i.e., $|z_k - \alpha_k(z)| = 1$ for $1 \leqslant k \leqslant n$. Hence for any analytic disc $f = (f_1, f_2, \dots, f_n)$ in $\mathbf{C}^n$ with boundary in K_r, the boundary of f lies in T_F^n if and only if

$$|f_k(\lambda) - \alpha_k(f_1(\lambda), \dots, f_n(\lambda))| = 1, \qquad 1 \leqslant k \leqslant n, \tag{3.1}$$

for each $\lambda \in T$. Our problem is thus to solve the system (3.1) for f as a smooth function of α.

THEOREM 2. *Let $g = (g_1, g_2, \dots, g_n)$ be an analytic disc in $\mathbf{C}^n$ with boundary in T^n. There exist a neighborhood N of 0 in $(\mathcal{C}^3(K_r))^n$ and a $\mathcal{C}^1$ map u: $N \to (\mathcal{Q}_2)^n$ with the following properties*:

(a) *$u(0) = g$, and*

(b) *if $u(\alpha) = f = (f_1, f_2, \dots, f_n)$ where $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$, then $|f_k - \alpha_k(f)| = 1$ on T for $1 \leqslant k \leqslant n$.*

REMARK 1. To again relate this to our original formulation of the problem, let $\{F_t\}$, $-\varepsilon < t < \varepsilon$, be a smooth 1-parameter family of $\mathcal{C}^3$ diffeomorphisms of T^n into $\mathbf{C}^n$ with $F_0 =$ identity, $T_t^n = F_t(T^n)$ and let g be an analytic disc in C^n with boundary in T^n. Write $F_t(z) = z + \beta_t(z)$. If α_t is associated to F_t as above, then $t \to \alpha_t$ is a smooth curve through 0 in $N \subset (\mathcal{C}^3(K_r))^n$, for t sufficiently small. Then, for fixed t, $u(\alpha_t)$ is an analytic disc in $\mathbf{C}^n$ with boundary in T_t^n and $u(\alpha_0) = u(0) = g$. Thus the map $t \mapsto u(\alpha_t) \equiv g_t$ yields the desired smooth family of analytic discs.

REMARK 2. The differentiability assumptions are somewhat arbitrary. With the assumption of more (or less) smoothness on the α's the same method yields a family of analytic discs which is more (or less) smooth. Likewise the choices of $0 < s < r < 1$ are arbitrary.

We shall solve (3.1) by applying the implicit function theorem in Banach space. We begin with some simple facts on compositions.

LEMMA 1. *Let X and Y be Banach spaces with Ω an open set in X. Let E_k: $\Omega \to Y$ be a $\mathcal{C}^1$ map for $1 \leqslant k \leqslant n$. Define E: $\Omega^n (\subseteq X^n) \to Y^n$ by $E(x_1, x_2, \dots, x_n) = (E_1(x_1), E_2(x_2), \dots, E_n(x_n))$. Then E is a $\mathcal{C}^1$ map.*

PROOF. It is straightforward to check that E is differentiable at $(x_1, x_2, \dots, x_n) \in \Omega^n$ and that

$$E'(x_1, \dots, x_n)(u_1, \dots, u_n) = (E_1'(x_1)(u_1), \dots, E_n'(x_n)(u_n)).$$

LEMMA 2. *Let $W = \{f = (f_1, f_2, \dots, f_n)$: $f_k \in \mathcal{Q}_2$ and $f(T) \subseteq$ the interior of $K_r\}$, an open set in $\mathcal{Q}_2^n$. The map B: $\mathcal{C}^3(K_r) \times W \to \mathcal{H}_2$ given by $B(\alpha; f_1, f_2, \dots, f_n) = \alpha(f_1, f_2, \dots, f_n)$ is $\mathcal{C}^1$.*

PROOF. That B maps into $\mathcal{H}_2$ is easily seen from a computation of the first two derivatives of $\alpha(f_1, \ldots, f_n)$. Fix $\alpha \in \mathcal{C}^3(K_r)$ and $f \in W$. Write, for small $\sigma \in \mathcal{C}^3(K_r)$ and $h \in \mathcal{Q}_2^n$,

$$B(\alpha + \sigma, f + h) = \alpha(f + h) + \sigma(f + h)$$
$$= \alpha(f) + \nabla\alpha(f) \cdot h + \gamma + \sigma(f) + \delta$$

where $\gamma = \alpha(f + h) - \alpha(f) - \nabla\alpha(f) \cdot h$ and $\delta = \sigma(f + h) - \sigma(f)$; here $\nabla\alpha$ is the real gradient of α and so $\nabla\alpha(f) \cdot h$ is a linear combination of h_k and $\bar{h}_k$. By tedious, but direct computations of γ, γ' and γ'' one sees that $\|\gamma\|/\|h\| \to 0$ as $\|h\| \to 0$ in $\mathcal{H}_2$-norms; likewise, with $\mathcal{C}^3(K_r)$ norm on σ, $\|\delta\|/(\|\sigma\| + \|h\|) \to 0$ as $\|h\| + \|\sigma\| \to 0$. It follows that B is differentiable at (α, f) and that $B'(\alpha, f)(\sigma, h) = \nabla\alpha(f) \cdot h + \sigma(f)$. Another straightforward computation shows that $(\alpha, f) \to B'(\alpha, f)$ is a continuous map into $\mathrm{Hom}(\mathcal{C}^3(K_r) \oplus \mathcal{Q}_2^n, \mathcal{H}_2)$ with operator norm.

PROOF OF THEOREM 2. Apply Theorem 1 n times to each of the Blaschke functions $g_1, g_2, \ldots, g_n$ to get n $\mathcal{C}^1$ functions $E_1, E_2, \ldots, E_n$, all defined on a single open neighborhood Ω of 0 in $\mathcal{H}_2$. The map B of Lemma 2 is continuous at $(0, g)$ and $B(0, g) = 0$. Hence there is a neighborhood N_1 of 0 in $\mathcal{C}^3(K_r)$ and a neighborhood D of g in $\mathcal{Q}_2^n$ such that $B(N_1 \times D) \subseteq \Omega$. Therefore we can define a map

$$R: N_1^n(\subseteq \mathcal{C}^3(K_r)^n) \times D(\subseteq \mathcal{Q}_2^n) \to \mathcal{Q}_2^n$$

by

$$R(\alpha, f) = f - (E_1(\alpha_1(f)), E_2(\alpha_2(f)), \ldots, E_n(\alpha_n(f))).$$

It follows from Lemmas 1 and 2 that R is a $\mathcal{C}^1$ map. We have $R(0, f) = f - (E_1(0), \ldots, E_n(0)) = f - g$ and therefore (a) $R(0, g) = 0$ and (b) $(D_2R)(0, g) =$ identity map: $\mathcal{Q}_2 \to \mathcal{Q}_2$, where D_2R is the "partial derivative" with respect to the second variable f. We can now invoke the implicit function theorem for Banach spaces [4] to conclude from (a) and (b) that there exist a neighborhood $N \subseteq N_1^n$ of 0 in $\mathcal{C}^3(K_r)^n$ and a $\mathcal{C}^1$ map u: $N \to \mathcal{Q}_2$ such that $u(N) \subseteq D$, $u(0) = g$ and $R(\alpha, u(\alpha)) \equiv 0$ for all $\alpha \in N$. The latter implies that if $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in N$ and $u(\alpha) = (f_1, f_2, \ldots, f_n) \in \mathcal{Q}_2^n$, then $f_k = E_k(\alpha_k(f_1, f_2, \ldots, f_n))$ for $1 \leqslant k \leqslant n$. Hence $|f_k - \alpha_k(f_1, f_2, \ldots, f_n)| \equiv 1$ on T for $1 \leqslant k \leqslant n$. Thus u has the desired properties.

4. Varying the initial analytic disc. In the last section we dealt with a fixed analytic disc g with boundary in T^n. Now we allow the initial analytic disc to vary simultaneously with the deformations. For simplicity, we shall limit the discussion to the two dimensional case.

We first need to extend Theorem 1: we let g vary and show that the solution f_φ of (2.3) can be chosen to depend smoothly on both g and φ. To this end, take Q, A, E to be the nonlinear operators and Ω and W_1 the neighborhoods of 0 in $\mathcal{H}_2$ given in the proof of Theorem 1 for the case $g(\lambda) \equiv \lambda$; then Q^{-1}: $\Omega \to W_1$ is $\mathcal{C}^1$ and $E(\varphi)(\lambda) \equiv \lambda \circ A \circ Q^{-1}(\varphi)$ for $\varphi \in \Omega$ satisfies $|E(\varphi) - \varphi| \equiv 1$ on T and $E(0)(\lambda) \equiv \lambda$. We shall let λ denote the variable in $\mathbf{C}$ as well as the identity function on $\mathbf{C}$.

Fix a Blaschke product g and choose a neighborhood Ω_1 of 0 in $\mathcal{H}_2$ such that $\lambda \cdot \bar{g}\varphi \in \Omega$ whenever $\varphi \in \Omega_1$. Then $\psi \equiv Q^{-1}(\lambda \cdot \bar{g} \cdot \varphi) \in W_1$. Put $f = g \cdot A(\psi) \in \mathcal{Q}_2$; then $A(\psi) = \bar{g}f$. We have $|E(\lambda\bar{g}\varphi) - \lambda\bar{g}\varphi| = 1$ on T, where $E(\lambda\bar{g}\varphi) = \lambda A \circ Q^{-1}(\lambda\bar{g}\varphi) = \lambda A(\psi)$. Thus $|\lambda\bar{g}f - \lambda\bar{g}\varphi| = 1$ and so $|f - \varphi| \equiv 1$ on T where $f = g \cdot A \circ Q^{-1}(\lambda\bar{g}\varphi)$. Also, if $\phi = 0$, then clearly $f = g$. Since A and Q are independent of g, this shows that f is a jointly $\mathcal{C}^1$ function of g and φ in the following sense.

Proposition 1. *Let $t \to g_t$ be a $\mathcal{C}^1$ map into $\mathcal{Q}_2$ such that each g_t is a finite Blaschke product where t varies over some open set S in $\mathbf{R}^q$. Suppose that Ω_1 is a neighborhood of 0 in $\mathcal{H}_2$ such that $\lambda\bar{g}_t\varphi \in \Omega$ for each t and each $\varphi \in \Omega_1$. Then the correspondence $(t, \varphi) \to f \equiv g_t \cdot A \circ Q^{-1}(\lambda\bar{g}_t\varphi)$ is a $\mathcal{C}^1$ map of $S \times \Omega_1$ into $\mathcal{Q}_2$ such that $|f - \varphi| = 1$ on T and $f = g_t$ when $\varphi = 0$.*

One would like to use this map together with the method of §3 to show that for a small but fixed α, each member of a family of analytic discs with boundaries in T^2 which completely fills up U^2 can be deformed to a disc with boundary in T^2_α (where we write $T^2_\alpha = F(T^2)$ when $z \mapsto z - \alpha(z)$ is the inverse of $z \to F(z) = z + \beta(z)$). Proposition 1 requires, in order that f be defined, that $\lambda\bar{g}_t\varphi \in \Omega =$ the domain of Q^{-1}. This means that for a fixed φ (corresponding to a fixed α), only g_t's whose norms are not too large are admissible. The following result shows however that in order to exhaust U^2 by analytic discs with boundaries in T^2, one must use Blaschke products of arbitrarily large $\mathcal{H}_2$-norm and thus for no fixed $\alpha \neq 0$ can all of these analytic discs be deformed by this method to discs with boundaries in T^2_α.

Proposition 2. *Let $\{p_n = (0, \lambda_n)\}$, where $|\lambda_n| \nearrow 1$, be a sequence of points in U^2. Let (h_n, k_n) be an analytic disc in $\mathbf{C}^2$ passing through p_n and with boundary in T^2. Then, in $\mathcal{H}_2$-norm, as $n \to \infty$,*

$$\|(h_n, k_n)\| \equiv \|h_n\| + \|k_n\| \to \infty.$$

Proof. Both h_n and k_n are nonconstant Blaschke products. Let $z_n \in U$ be such that $h_n(z_n) = 0$ and $k_n(z_n) = \lambda_n$. Arguing by contradiction we may suppose that $\|h_n\| + \|k_n\| \leq M < \infty$ and that $z_n \to z_0 \in \bar{U}$. We consider two cases.

Case 1. $|z_0| = 1$. Then

$$\frac{h_n(z_0) - h_n(z_n)}{z_0 - z_n} = \frac{1}{z_0 - z_n}\int_{z_n}^{z_0} h_n'(\zeta)\,d\zeta$$

and so we get

$$\frac{1}{|z_0 - z_n|} \leq \|h_n'\|_\infty \leq \|h_n\|.$$

Then as $n \to \infty$, $\|h_n\| \to \infty$, a contradiction

Case 2. $|z_0| < 1$. Because $\{k_n\}$ is a normal family we may assume, without loss of generality, that k_n converges to a holomorphic function k uniformly on compact subsets of U where $|k(z)| \leq 1$. Since $|k(z_0)| = \lim|k_n(z_n)| = 1$, k is a unimodular constant; i.e., $|k_n| \to 1$ uniformly on compact subsets of U. Therefore if z_n' is a zero of k_n in U (it has at least one), we conclude that $|z_n'| \to 1$. We can now apply Case 1 to conclude that $\|k_n\| \to \infty$.

Although we cannot deform all discs in U^2 with boundary in T^2 for a fixed α, we shall be able to do this for all discs in ∂U^2. The topological boundary ∂U^2 of U^2 is the union of "faces" M_1 and M_2 where $M_1 = \{(z_1, z_2) \in \mathbf{C}^2\colon |z_1| < 1 \text{ and } |z_2| = 1\}$ and M_2 is obtained by reversing the roles of z_1 and z_2 in M_1. The M_j are real 3-dimensional compact manifolds with $\partial M_j = T^2$ and $M_1 \cap M_2 = T^2$. M_1 is a disjoint union of analytic discs $g(e^{it})$ where $g(e^{it})(\lambda) = (\lambda, e^{it})$ for $e^{it} \in T$, where the second component is a constant function. Since $\varphi \to e^{-it}\varphi \in \Omega$ is an isometry of $\mathcal{H}_2$, it follows that there exists a neighborhood $\Omega_1 \subseteq \Omega$ of 0 in $\mathcal{H}_2$ such that $\lambda e^{-it}\varphi \in \Omega$ for all $\varphi \in \Omega_1$ and all t. This will allow us to deform all of the discs in M_1 simultaneously.

For $f \in \mathcal{Q}_2^2$ such that $f(T^2) \subseteq K_r$ we have seen that for $\alpha \in (\mathcal{C}^3(K_r))^2$, $(\alpha, f) \mapsto \alpha \circ f$ is a $\mathcal{C}^1$ map B into $\mathcal{H}_2^2$. Therefore for each t, there exist neighborhoods of 0 in $(\mathcal{C}^3(K_r))^2$ and of $g(e^{it})$ in $\mathcal{Q}_2^2$ such that B takes their product into $\Omega_1 \times \Omega_1$. By compactness we get a single neighborhood N of 0 in $\mathcal{C}^3(K_r)^2$ and an open set W in $\mathcal{Q}_2^2$ such that $B(N \times W) \subseteq \Omega_1 \times \Omega_1$ and W contains each $g(e^{it})$. Define a map R: $T \times N \times W \to \mathcal{Q}_2^2$ by

$$R(e^{it}; \alpha_1, \alpha_2; f_1, f_2) = (f_1, f_2) - \big(E(\alpha_1(f)), e^{it} \circ A \circ Q^{-1}(\lambda e^{-it}\alpha_2(f))\big)$$

where $f = (f_1, f_2)$. It follows from our choice of N and W that R is well defined and from previous arguments R is a $\mathcal{C}^1$ map. We have $R(e^{it}; 0; f) = f - (\lambda, e^{it}) = f - g(e^{it})$. Hence $R(e^{it}; 0; g(e^{it})) = 0$ and $D_3R(e^{it}; 0; g(e^{it})) =$ the identity map on $\mathcal{Q}_2^2$. Therefore, by the implicit function theorem, for each fixed t, there exist a neighborhood V_t of e^{it} in T, a neighborhood $N_t \subseteq N$ of 0 in $(\mathcal{C}^3(K_r))^2$ and a $\mathcal{C}^1$ map u_t: $V_t \times N_t \to W$ such that $u_t(e^{it}, 0) = g(e^{it})$ and $R(e^{is}; \alpha; u_t(e^{is}, \alpha)) \equiv 0$ for $(e^{is}, \alpha) \in V_t \times N_t$. Moreover $u_t(e^{is}, 0) = g(e^{is})$ for $e^{is} \in V_t$ by uniqueness. By the compactness of T we can choose a finite subcover $\{V_j\}$ of $\{V_t\}$; let $\{N_j\}$ and $\{u_j\}$ be the corresponding finite subfamilies of $\{N_t\}$ and $\{u_t\}$ respectively. By the uniqueness in the implicit function theorem, it follows that $u_i = u_j$ on $(V_i \cap V_j) \times (N_i \cap N_j)$, provided that $V_i \cap V_j$ is nonempty. Thus by "patching" we get a $\mathcal{C}^1$ map u: $T \times N_0 \to W$ where $N_0 = \bigcap N_j$ is a neighborhood of 0 in $(\mathcal{C}^3(K_r))^2$, such that $u(e^{it}, 0) = g(e^{it})$ and $R(e^{it}; \alpha; u(e^{it}, \alpha)) \equiv 0$. Now write $u(e^{it}, \alpha) = (f_1, f_2)$. It follows from the definition of R that $f_1 = E(\alpha_1(f))$ and $f_2 = e^{it}A \circ Q^{-1}(\lambda e^{-it}\alpha_2(f))$. Taking g_t in Proposition 1 to be e^{it} we conclude that

$$|f_1 - \alpha_1(f)| \equiv 1, \qquad |f_2 - \alpha_2(f)| \equiv 1$$

on T; i.e., $u(e^{it}, \alpha)$ is an analytic disc in $\mathbf{C}^2$ with boundary in T_α^2 for each e^{it}.

For fixed $\alpha \in N_0$ define a map P_α: $\overline{U} \times T (= M_1) \to \mathbf{C}^2$ by $P_\alpha(\lambda, e^{it}) = u(e^{it}, \alpha)(\lambda)$ where $u(e^{it}, \alpha) \in \mathcal{Q}_2^2$ and we view elements of $\mathcal{Q}_2$ as functions on $\overline{U}$, holomorphic on U. Let $M_1(\alpha) = P_\alpha(M_1) \subseteq \mathbf{C}^2$. Note that $P_0 =$ identity and so $M_1(0) = M_1 \subseteq \partial U^2$.

Lemma. (a) *The map* e: $\mathcal{Q}_2 \times \overline{U}(\subseteq \mathbf{R}^2) \to \mathbf{C}$, $e(f, \lambda) = f(\lambda)$ *is* $\mathcal{C}^1$.

(b) P_α: $M_1 \to \mathbf{C}^2$ *is* $\mathcal{C}^1$.

(c) $\alpha \mapsto P_\alpha$ *is a continuous map of* $N_0 \subseteq (\mathcal{C}^3(K_r))^2$ *into* $\mathcal{C}^1(M_1, \mathbf{C}^2)$.

PROOF. (a) A straightforward computation shows that e is differentiable and that

$$e'(f, \lambda)(g, \zeta) = f'(\lambda) \cdot \zeta + g(\lambda).$$

Using the easily verified fact that the derivatives of functions $f \in \mathcal{Q}_2$ satisfy a Hölder condition of the form

$$|f'(\lambda_1) - f'(\lambda_2)| \leq C\|f\| \, |\lambda_1 - \lambda_2|^{1/2}$$

for λ_1 and λ_2 in $\overline{U}$ where C is a universal constant, one sees that $(f, \lambda) \to e'(f, \lambda)$ is continuous.

(b) Consider the map μ: $\overline{U} \times T \times N_0 \to \mathbf{C}^2$ given by $\mu(\lambda, e^{it}, \alpha) = u(e^{it}, \alpha)(\lambda)$. We know that $(\lambda, e^{it}, \alpha) \to (u(e^{it}, \alpha), \lambda)$ is a $\mathcal{C}^1$ map into $\mathcal{Q}_2^2 \times \overline{U}$. It follows from (a) that μ is $\mathcal{C}^1$. Hence for fixed α, $P_\alpha(\lambda, e^{it}) = \mu(\lambda, e^{it}, \alpha)$ is $\mathcal{C}^1$ on M_1.

(c) We can regard μ as a function of two variables p and α where $p = (\lambda, e^{it}) \in M_1$ and $\alpha \in N_0$. Then μ is $\mathcal{C}^1$ on $M_1 \times N_0$ and $DP_\alpha(\lambda, e^{it}) = D_1\mu(p, \alpha)$ is continuous on $M_1 \times N_0$, as is $P_\alpha(p) = \mu(p, \alpha)$. For fixed $\alpha_0 \in N_0$ a simple compactness argument shows that

$$\|D_1\mu(p, \alpha) - D_1\mu(p, \alpha_0)\|_\infty + \|\mu(p, \alpha) - \mu(p, \alpha_0)\|_\infty \to 0$$

as $\alpha \to \alpha_0$ where the sup norms are taken over M_1. This gives (c).

Since the set of imbeddings of M_1 into $\mathbf{C}^2$ is an open set in $\mathcal{C}^1(M_1, \mathbf{C}^2)$ and since $P_0 =$ the identity, it follows from (c) of the lemma that P_α is an imbedding (injective with differential of maximal rank 3 at each point of M_1) for α sufficiently small. Thus $M_1(\alpha)$ is an imbedded compact 3 dimensional manifold in $\mathbf{C}^2$ with $\partial M_1(\alpha) = T_\alpha^2$; moreover $M_1(\alpha)$ is a disjoint union of analytic discs $\lambda \mapsto P_\alpha(\lambda, e^{it})$ for fixed e^{it}. Of course, the same is true for $M_2(\alpha)$ with the obvious definitions of Q_α: $M_2 \to \mathbf{C}^2$ with $M_2(\alpha) = Q_\alpha(M_2)$ and $Q_0 =$ the identity on M_2. Clearly $M_1(\alpha) \cap M_2(\alpha)$ contains T_α^2; moreover $M_1(\alpha)$ and $M_2(\alpha)$ meet transversally at T_t^2. It follows that $M_1(\alpha) \cap M_2(\alpha) = T_t^2$, for α sufficiently small. Hence $\mathbf{C}^2 \setminus M_1(\alpha) \cup M_2(\alpha)$ is the union of two open sets, the bounded component being homeomorphic to U^2. Let D_α be the closure of this bounded component. Then D_α is a compact subset of $\mathbf{C}^2$. If f is holomorphic on D_α, then $|f|_{D_\alpha} \leq |f|_{\partial D_\alpha}$ by the maximum principle and $|f|_{\partial D_\alpha} \leq |f|_{T_\alpha^2}$ because $\partial D_\alpha = M_1(\alpha) \cup M_2(\alpha)$ and the $M_j(\alpha)$ are fibered by analytic discs with boundaries in T_α^2. Thus T_α^2 is a "distinguished" boundary for D_α and, in particular, the polynomially convex hull of T_α^2 contains D_α. The following theorem summarizes the above results. It says that as T^2 is smoothly deformed to T_α^2, the hull $\hat{T}^2 = \overline{U}^2$ is, in some sense, smoothly deformed to the set $D_\alpha \subseteq \hat{T}_\alpha^2$.

THEOREM 3. *There exists a ball B about* 0 *in $(\mathcal{C}^3(K_r))^2$ such that if $\alpha \in B$ and if T_α^2 is the associated deformation of T^2, then the following holds. There exist $\mathcal{C}^1$ imbeddings P_α: $M_1 \to \mathbf{C}^2$, $P_\alpha(M_1) = M_1(\alpha)$ and Q_α: $M_2 \to \mathbf{C}^2$, $Q_\alpha(M_2) = M_2(\alpha)$ with $T_\alpha^2 = \partial M_j(\alpha)$, $j = 1, 2$. The manifolds $M_j(\alpha)$ depend smoothly on α and each is a disjoint union of analytic discs with boundaries in T_α^2. $M_1(\alpha) \cap M_2(\alpha) = T_\alpha^2$. There exists a compact set D_α in $\mathbf{C}^2$ with $\partial D_\alpha = M_1(\alpha) \cup M_2(\alpha)$ such that $D_\alpha \subseteq \hat{T}_\alpha^2$. Moreover the set D_α is close to $\overline{U}^2$ in the following sense: if $\varepsilon > 0$ is given, then $U_{1-\varepsilon}^2 \subseteq D_\alpha \subseteq U_{1+\varepsilon}^2$ (where $U_t = \{\lambda \in \mathbf{C}: |\lambda| < t\}$) provided that α is sufficiently small.*

REMARK. Bedford [7] has shown that D_α is in fact the polynomially convex hull of T_α^2.

PROOF. We have verified all but the last assertion. This follows from the fact that $M_1(\alpha) \cup M_2(\alpha) \subseteq U_{1+\varepsilon}^2 \setminus U_{1-\varepsilon}^2$ for α sufficiently small, because then P_α and Q_α are uniformly close to identity maps.

The construction of D_α raises a number of questions about the hull of T_α^2:

(a) Does every point of D_α (or $\hat{T}_\alpha^2$) lie in an analytic disc with boundary in T_α^2?

(b) Is T_α^2 the Shilov boundary of $A(D_\alpha)$ (or $H(D_\alpha)$ or $P(D_\alpha)$)? Is every point of T_α^2 a peak point for these algebras?

(c) When is T_α^2 the image of T^2 under a biholomorphism of $\overline{U}^2$ into $\mathbf{C}^2$? When is D_α biholomorphic to $\overline{U}^2$?

(d) Let f be a continuous (or smooth) function on T_α^2. Suppose that f extends to be holomorphic on each of the analytic discs in $M_1(\alpha)$ and $M_2(\alpha)$. Does f extend to be holomorphic on the interior of D_α? For f on T^2, the hypothesis amounts to the vanishing of the appropriate Fourier coefficients.

We shall conclude our discussion of perturbations of the torus by considering the algebra $R(T_\alpha^2)$. It follows from the Stone-Weierstrass Theorem that z_1, z_2, $1/z_1$ and $1/z_2$ generate $C(T^2)$, because $\bar{z}_k = 1/z_k$ on T^2. The same is true for $C(T_\alpha^2)$.

PROPOSITION 3. *For α sufficiently small in $\mathcal{C}^3(K_r)^2$, the functions z_1, z_2, $1/z_1$ and $1/z_2$ generate $C(T_\alpha^2)$. In particular $R(T_\alpha^2) = C(T_\alpha^2)$.*

PROOF. Write $1/z_k = \bar{z}_k + R_k(z_1, z_2)$ for $z = (z_1, z_2) \in T_\alpha^2$. By a theorem of Hörmander and Wermer [5], it suffices to check that $R = (R_1, R_2)$ extends to a neighborhood of T_α^2 where it satisfies a Lipschitz condition $\|R(z) - R(z')\| < k\|z - z'\|$ for some $k < 1$ where $\|\cdot\|$ is the Euclidean norm in $\mathbf{C}^2$.

Recall that T_α^2 is the image of T^2 under the map $F(w) = w + \beta(w)$ where $z \to z - \alpha(z)$ is the inverse of F. It follows that if $z = (z_1, z_2) \in T_\alpha^2$ corresponds to $w = (w_1, w_2) \in T^2$, then $\alpha(z) = \beta(w)$ and $|w_k| = 1$ for $k = 1, 2$. Hence, for $z \in T_\alpha^2$,

$$\begin{aligned} R_k(z) &= \frac{1}{z_k} - \bar{z}_k = \frac{1}{w_k + \beta_k(w)} - \bar{w}_k - \overline{\beta_k(w)} \\ &= \frac{1}{w_k + \beta_k(w)} - \frac{1}{w_k} - \overline{\alpha_k(z)} \\ &= \frac{-\beta_k(w)}{(w_k + \beta_k(w))w_k} - \overline{\alpha_k(z)}. \end{aligned}$$

Thus

$$R_k = \frac{-\alpha_k}{z_k(z_k - \alpha_k)} - \bar{\alpha}_k.$$

Clearly the Lipschitz condition on R holds on K_r if α is small.

5. Deformations of a two-sphere. We consider small deformations of the 2-sphere $S^2 = \{(z_1, z_2) \in \mathbf{C}^2\colon \operatorname{Im} z_2 = 0 \text{ and } |z_1|^2 + (\operatorname{Re} z_2)^2 = 1\}$, again restricting to $\mathbf{C}^2$ for simplicity. The polynomially convex hull of S^2 is easily described: $\hat{S}^2 \setminus S^2$ is the disjoint union of analytic discs $g_t(U)$ where $g_t(\lambda) = (\sqrt{1-t^2}\,\lambda, t)$ for $-1 < t < 1$. In fact, the g_t's are, except for a reparameterization, the only analytic discs in $\mathbf{C}^2$ with boundary in S^2. We shall show that given a g_t, every sufficiently small deformation of S^2 in $\mathbf{C}^2$ induces a deformation of g_t to analytic discs with boundaries in the deformed 2-spheres.

For $s > 0$, let $K_s = \{z \in \mathbf{C}^2\colon \text{distance}(z, S^2) < s\}$, a compact neighborhood of S^2 in $\mathbf{C}^2$, where s is fixed small enough so that there exists a smooth retraction ρ of K_s to S^2. Let F be a $\mathcal{C}^3$ diffeomorphism of S^2 into $\mathbf{C}^2$ which is close to the identity. Write $F(z) = z + \beta(z)$ where $\beta \in (\mathcal{C}^3(S^2))^2$ is close to 0. Extend F to K_s by $\tilde{F}(z) = z + \beta \circ \rho(z)$. Then, for $0 < r < s$, if β is sufficiently small, just as in §2, $\tilde{F}$ is a diffeomorphism of the interior of K_s onto a neighborhood of K_r and the inverse G of $\tilde{F}$ can be written $G(z) = z - \alpha(z)$ where $\alpha \in (\mathcal{C}^3(K_r))^2$. Let $S_\alpha^2 = F(S^2)$ be the deformed 2-sphere; S_α^2 is contained in K_r when α is sufficiently small. Then, when $z \in K_r$, $z \in S_\alpha^2$ if and only if $G(z) = z - \alpha(z) \in S^2$; i.e., $|z_1 - \alpha_1(z)|^2 + (\operatorname{Re}(z_2 - \alpha_2(z)))^2 = 1$ and $\operatorname{Im}(z_2 - \alpha_2(z)) = 0$ where $\alpha = (\alpha_1, \alpha_2)$. Hence, if $\lambda \mapsto (f_1(\lambda), f_2(\lambda))$ is an analytic disc f in $\mathbf{C}^2$, then the boundary of f lies in S_α^2 if and only if the boundary of f lies in K_r and

$$\begin{aligned} |f_1 - \alpha_1(f)|^2 + (\operatorname{Re}(f_2 - \alpha_2(f)))^2 &= 1, \\ \operatorname{Im}(f_2 - \alpha_2(f)) &= 0 \end{aligned} \tag{4.1}$$

on T. Thus our problem is to solve the system (4.1) for $f = (f_1, f_2) \in \mathcal{Q}_2^2$ given small $\alpha \in \mathcal{C}^3(K_r)^2$ such that f varies smoothly with α and with the initial condition $f = g_t$ when $\alpha = 0$. In fact, we shall solve (4.1) with the t of the initial condition as an additional parameter. We shall need a refinement of Theorem 1.

THEOREM 4. *There exists an open set Ω in $\mathcal{H}_2 \oplus \mathcal{H}_2(\mathbf{R})$ such that Ω contains each pair $(0, \sigma)$ if $\sigma > 0$ and conversely if $(\varphi, \sigma) \in \Omega$, then $\sigma > 0$ and there exists a $\mathcal{C}^1$ map $E\colon \Omega \to \mathcal{Q}_2$ such that*

(i) *if $(\varphi, \sigma) \in \Omega$ and $E(\varphi, \sigma) = f$, then $|f - \varphi| = \sigma$ on T,*

(ii) *$E(0, c)(\lambda) \equiv c\lambda$ when c is a positive constant function.*

PROOF Let $E_0\colon \Omega_0 \to \mathcal{Q}_2$ be the $\mathcal{C}^1$ map given in Theorem 1 for $g(\lambda) \equiv \lambda$; we have used a zero subscript to avoid confusion with the current notation. Then Ω_0 is a neighborhood of 0 in $\mathcal{H}_2$, $E(0)(\lambda) \equiv \lambda$ and, if $\varphi \in \Omega_0$ and $f = E_0(\varphi)$, then $|f - \varphi| \equiv 1$ on T.

Let $W = \{\sigma \in \mathcal{H}_2(\mathbf{R})\colon \sigma > 0\}$ and define for $\sigma \in W$, $E_1(\sigma) = f \in \mathcal{Q}_2$ by $f = e^{J(\log \sigma)}$ where $J(u) = u + i\tilde{u}$ (with $\tilde{u}$ the conjugate of u) is the map given in §1 and $|f| = \sigma$. Previous arguments show that $E_1\colon W \to \mathcal{Q}_2$ is $\mathcal{C}^1$; the same is true for $E_2\colon W \to \mathcal{Q}_2$ given by $E_2(\sigma) = e^{-J(\log \sigma)}$.

Define $\Omega = \{(\varphi, \sigma) \in \mathcal{H}_2 \oplus \mathcal{H}_2(\mathbf{R})\colon \sigma > 0 \text{ and } \varphi \cdot E_2(\sigma) \in \Omega_0\}$; Ω is an open set in $\mathcal{H}_2 \oplus \mathcal{H}_2(\mathbf{R})$. Now define $E\colon \Omega \to \mathcal{Q}_2$ by $E(\varphi, \sigma) = E_1(\sigma) \cdot E_0(E_2(\sigma) \cdot \varphi)$. Then E is clearly a $\mathcal{C}^1$ map with $E(0, c) = E_1(c)E_0(0) = c\lambda$. To check (i), let $f_1 = E_1(\sigma)$ and let $f_2 = E_0(\varphi/f_1)$, then $f = E(\varphi, \sigma) = f_1 \cdot f_2$ and $|f_2 - \varphi/f_1| = 1$ on T implies $|f - \varphi| = |f_1| = \sigma$.

REMARK. To have $(\varphi, \sigma) \in \Omega$ we required that $\varphi \cdot E_2(\sigma) = \varphi \cdot h \in \Omega_0$ where $h = E_2(\sigma)$ and $|h(\lambda)| = 1/\sigma(\lambda)$ for $\lambda \in T$. Since $\|h\| \to \infty$ as $\inf \sigma \to 0$, we see that for fixed φ, in order that $(\varphi, \sigma) \in \Omega$, the value of $\inf \sigma$ cannot be too small.

Define $\hat{S}^2(t) = \{z \in \hat{S}^2: -t < \operatorname{Re} z_2 < t\}$ for $0 < t < 1$. Then $\hat{S}^2(t) \subseteq \mathbf{C}^2$ is a noncompact 3-dimensional manifold with boundary; $\partial \hat{S}^2(t)$ is an open subset of S^2. We shall now show that if S^2 is smoothly deformed to S^2_α, then, if α is small, $\hat{S}^2(t)$ is smoothly deformed to a 3-manifold whose boundary is contained in S^2_α and which consists of a disjoint union of analytic discs also with boundaries in S^2_α.

THEOREM 5. *There exists an open neighborhood V of $(-1, 1) \times \{0\}$ in $(-1, 1) \times \mathcal{C}^3(K_r)^2$ and there exists a $\mathcal{C}^1$ map u: $V \to \mathcal{Q}_2^2$ such that for $(t, \alpha) \in V$, if $u(t, \alpha) = f = (f_1, f_2)$, then f is an analytic disc in $\mathbf{C}^2$, $f = g_t$ when $\alpha = 0$, and the boundary of f lies in S^2_α; i.e., (4.1) holds on T.*

Moreover for fixed t, $0 < t < 1$, there is a neighborhood $N = N(t)$ of 0 in $\mathcal{C}^3(K_r)^2$ such that $[-t, t] \times N \subseteq V$ and, for $\alpha \in N$, the map P_α: $\hat{S}^2(t) \to \mathbf{C}^2$ defined by

$$P_\alpha(z_1, s) = u(s, \alpha)\left(\frac{z_1}{\sqrt{1 - s^2}}\right)$$

is a $\mathcal{C}^1$ imbedding into $\mathbf{C}^2$. The image set $M(\alpha) = P_\alpha(\hat{S}^2(t))$ is a disjoint union of analytic discs with boundaries in S^2_α and $M(\alpha) \subseteq \hat{S}^2_\alpha$. $M(\alpha)$ is a $\mathcal{C}^1$ 3-dimensional manifold whose boundary $\partial M(\alpha)$ is an open subset of S^2_α. The $M(\alpha)$ depend smoothly on α and $M(0) = \hat{S}^2(t)$.

PROOF. Define a bounded linear transformation H: $\mathcal{H}_2(\mathbf{R}) \to \mathcal{H}_2(\mathbf{R})$ by $H(u) = -\tilde{u}$, where $\tilde{u}$ is the conjugate function of u discussed in §1. For $u \in \mathcal{H}_2(\mathbf{R})$,

$$H(u) + iu = i(u + i\tilde{u}) \in \mathcal{Q}_2.$$

For (t, α, f) in $(-1, 1) \times \mathcal{C}^3(K_r)^2 \times \mathcal{Q}_2^2$ such that $f(T) \subset K_r$ and α small we can define

$$\Phi(t, \alpha, f) = \left(\alpha_1(f), \sqrt{1 - \left[t + H(\alpha_2(f)) - \operatorname{Re}(\alpha_2(f))\right]^2}\,\right),$$

an element of $\mathcal{H}_2 \oplus \mathcal{H}_2(\mathbf{R})$. In order that Φ be defined at (t, α, f), the expression in square brackets must be less than one in absolute value on T. Then Φ is a $\mathcal{C}^1$ map on its (open) domain which includes all points $(t, 0, g_t)$ for $-1 < t < 1$ and $\Phi(t, 0, g_t) = (0, \sqrt{1 - t^2}\,) \in \Omega \subseteq \mathcal{H}_2 \oplus \mathcal{H}_2(\mathbf{R})$. Hence there exists an open set W in $\mathbf{R} \oplus \mathcal{C}^3(K_r)^2 \oplus \mathcal{Q}_2^2$ such that for $(t, \alpha, f) \in W$, we have $-1 < t < 1$, $W \subseteq$ domain of Φ, $\Phi(W) \subseteq \Omega$ and $(t, 0, g_t) \in W$ for all t, $-1 < t < 1$.

Now we can define a map R: $W \to \mathcal{Q}_2^2$ by

$$R(t, \alpha, f) = f - (E \circ \Phi(t, \alpha, f), t + H(\operatorname{Im} \alpha_2(f)) + i \operatorname{Im} \alpha_2(f)).$$

R is a $\mathcal{C}^1$ map. We have $R(t, 0, f) = f - (E(0, \sqrt{1 - t^2}\,), t) = f - g_t$. Hence $R(t, 0, g_t) = 0$ and $D_3 R(t, 0, g_t) =$ the identity map on $\mathcal{Q}_2^2$ for $-1 < t < 1$.

By the implicit function theorem [4], for each t, $-1 < t < 1$, there exist a neighborhood J_t of t in $(-1, +1)$ and a neighborhood N_t of 0 in $\mathcal{C}^3(K_r)^2$ and a $\mathcal{C}^1$ function u_t: $J_t \times N_t \to \mathcal{Q}_2^2$ such that $u_t(t, 0) = g_t$ and $R(s, \alpha, u(s, \alpha)) \equiv 0$ for $s \in J_t$, $\alpha \in N_t$. By uniqueness $u_{t_1} = u_{t_2}$ whenever the domains overlap. It follows that there exists an open set V in $(-1, 1) \times \mathcal{C}^3(K_r)^2$ such that V contains each point $(t, 0)$ and there exists a $\mathcal{C}^1$ map u: $V \to \mathcal{C}_2^2$ such that $u(t, 0) = g_t$ and $R(s, \alpha, u(s, \alpha)) = 0$ for all $(s, \alpha) \in V$.

Let $f = (f_1, f_2) = u(t, \alpha)$. We shall show that f is an analytic disc with boundary in S_α^2. From $R(t, \alpha, f) = 0$ we get

$$f_1 = E\left(\alpha_1(f), \sqrt{1 - \left[t + H(\operatorname{Im} \alpha_2(f)) - \operatorname{Re} \alpha_2(f)\right]^2}\right) \tag{4.2}$$

and

$$f_2 = t + H(\operatorname{Im} \alpha_2(f)) + i \operatorname{Im} \alpha_2(f). \tag{4.3}$$

Now (4.3) implies $\operatorname{Re} f_2 = t + H(\operatorname{Im} \alpha_2(f))$. Using this in (4.2) and then taking the imaginary parts in (4.3) yields

$$\begin{aligned} |f_1 - \alpha_1(f)| &= \sqrt{1 - \left[\operatorname{Re}(f_2 - \alpha_2(f))\right]^2}, \\ \operatorname{Im} f_2 &= \operatorname{Im} \alpha_2(f). \end{aligned} \tag{4.4}$$

The system (4.4) is clearly equivalent to the system (4.1). Hence the boundary of f lies in S_α^2.

For a fixed t, $0 < t < 1$, a compactness argument shows that there is a neighborhood $N = N(t)$ of 0 in $\mathcal{C}^3(K_r)^2$ such that $[-t, t] \times N \subseteq V$. Now define P_α for $\alpha \in N$ as above. Using the fact that u is $\mathcal{C}^1$ we can argue as in §5 to see that P_α is a $\mathcal{C}^1$ imbedding for α sufficiently small and that $M(\alpha)$ depends smoothly on α. Since $\partial M(\alpha) = P_\alpha(S^2 \cap \hat{S}^2(t))$ and since $P_\alpha | S^2$ is an open map into S_α^2 it follows that $\partial M(\alpha)$ is open in S_α^2. P_α is clearly holomorphic in z_1 for fixed s and so $M(\alpha)$ is a union of discs with boundaries in S_α^2. This implies that $M(\alpha) \subset \hat{S}_\alpha^2$.

Remark. In general the open set V of this theorem is not a product set in $(-1, 1) \times \mathcal{C}^3(K_r)^2$; in fact, we would expect that as $t \to \pm 1$, the set of α for which $(t, \alpha) \in V$ will shrink to 0. This is because, by the remark after Theorem 4, for a fixed φ corresponding to a fixed α, the quantity $\sqrt{1 - t^2}$ cannot be too small if the operator E is to be applied; i.e., for fixed α, $|t|$ must be bounded away from 1 if we want to deform g_t to S_α^2. Thus our method is in some sense complementary to that of Bishop [1] who produces analytic discs near the "exceptional" points of S_α^2; our construction yields discs away from the exceptional points. Bedford and Gaveau [8] have shown by other methods that the Bishop discs can be continued from the exceptional points to fill up the full hull. A uniqueness result for discs plays an important role in their approach.

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