TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 267, Number 1, September 1981

# A CHARACTERIZATION OF BEST $\Phi$-APPROXIMANTS

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ABSTRACT. Let $T$ be an operator from an Orlicz space $L_\Phi$ into itself. It is shown in this paper that four algebraic conditions and one integration condition assure that $T$ is the best $\Phi$-approximator, given a suitable $\sigma$-lattice.

**Introduction and notations.** Let $(\Omega, \mathcal{Q}, \mu)$ be a finite measure space and $\Phi$: $\mathbf{R} \to [0, \infty)$ a strictly convex symmetric function with $\Phi(0) = 0$. Denote by $L_\Phi(\Omega, \mathcal{Q}, \mu)$ respectively $L_\Phi^\infty(\Omega, \mathcal{Q}, \mu)$ the system of all $\mu$-equivalence classes of $\mathcal{Q}$-measurable functions $f$ such that $\int \Phi(\alpha f)\, d\mu < \infty$ for some $\alpha > 0$ respectively for all $\alpha > 0$. If $\Phi(x) = |x|^p$ then $L_\Phi = L_\Phi^\infty$ and we obtain the spaces $L_p$ $(p > 1)$.

Let $\mathfrak{L} \subset \mathcal{Q}$ be a $\sigma$-lattice, i.e. a system which is closed under countable unions and intersections and contains $\varnothing$ and $\Omega$. Denote by $L_\Phi(\mathfrak{L})$ the system of all equivalence classes in $L_\Phi(\Omega, \mathcal{Q}, \mu)$, containing a $\mathfrak{L}$-measurable function, i.e. a function $f$: $\Omega \to \overline{\mathbf{R}}$ with $\{f > a\} \in \mathfrak{L}$ for all $a \in \mathbf{R}$. If $L_\Phi = L_\Phi^\infty$ it was shown in [**11**, Corollary 5], that for each $f \in L_\Phi(\Omega, \mathcal{Q}, \mu)$ there exists a unique $\mu_\Phi(f|\mathfrak{L}) \in L_\Phi(\mathfrak{L})$ such that

$$(*) \qquad \int \Phi(f - \mu_\Phi(f|\mathfrak{L}))\, d\mu = \inf\left\{ \int \Phi(f - g)\, d\mu : g \in L_\Phi(\mathfrak{L}) \right\}.$$

We call $\mu_\Phi(f|\mathfrak{L})$ the best $\Phi$-approximant of $f$, given $\mathfrak{L}$. For $p > 1$ and $\Phi(x) = |x|^p$ we write $\mu_p(f|\mathfrak{L})$ instead of $\mu_\Phi(f|\mathfrak{L})$. The concept of best $\Phi$-approximants, given $\mathfrak{L}$, covers and unifies important concepts of probability theory: the classical concept of conditional expectation ($\Phi(x) = |x|^2$, $\mathfrak{L}$ $\sigma$-field), the concept of conditional expectation given a $\sigma$-lattice ($\Phi(x) = x^2$, $\mathfrak{L}$ $\sigma$-lattice, see [**4**], [**5**]), Ando-Amemiya's [**2**] concept of $p$-predictors ($\Phi(x) = |x|^p$, $\mathfrak{L}$ $\sigma$-algebra) or Brunk's [**6**] concept of conditional $p$-means ($\Phi(x) = |x|^p$, $\mathfrak{L}$ $\sigma$-lattice). Best approximants given $\sigma$-lattices play an essential role in several fields of application, e.g. in Bayes-estimation, prediction theory and isotonic regression (see [**4**]).

It is the aim of this paper to give conditions on an operator $T$: $L_\Phi \to L_\Phi$ assuring that $T$ is the best $\Phi$-approximator, given a suitable $\sigma$-lattice $\mathfrak{L} \subset \mathcal{Q}$, i.e. $Tf = \mu_\Phi(f|\mathfrak{L})$ for all $f \in L_\Phi$.

Many authors have investigated such characterization problems for special cases. Characterization results for classical conditional expectations were given in [**1**], [**3**], [**7**], [**12**], [**13**], [**14**]; a characterization for a conditional expectation, given a

Received by the editors October 1, 1980.
*AMS (MOS) subject classifications* (1970). Primary 46E30; Secondary 41A50.
*Key words and phrases*. Best approximants, $\sigma$-lattices, conditional expectations, characterization.

$\sigma$-lattice, can be found in **[8]** and **[12]**, a characterization for $p$-predictors is given in **[10]** and a characterization for conditional $p$-means is contained in **[12]**. Hence much is known for $\Phi(x) = x^2$, less for $\Phi(x) = |x|^p$ $(p > 1)$ and nothing for general $\Phi$. It is the purpose of this paper to close this gap. In our theorem we prove a characterization result for best $\Phi$-approximants, given a $\sigma$-lattice and for best $\Phi$-approximants, given a $\sigma$-field. Four of the five describing conditions are purely algebraic; only the fifth condition introduces the measure $\mu$ and the function $\Phi$. We consider this an advantage since algebraic conditions are nicer and easier to verify than integration conditions.

**The result.** Let $(\Omega, \mathcal{Q}, \mu)$ be a finite measure space, $\Phi: \mathbf{R} \to [0, \infty)$ strictly convex, symmetric with $\Phi(0) = 0$ and assume that $L_\Phi = L_\Phi^\infty$. Let $\mathfrak{L} \subset \mathcal{Q}$ be a $\sigma$-lattice and put $Tf := \mu_\Phi(f|\mathfrak{L})$, $f \in L_\Phi$. Then the operator $T$: $L_\Phi \to L_\Phi$ has the following property.

(i) $T$ is translation invariant, i.e. $T(f + a) = Tf + a$ for all $a \in \mathbf{R}$.

(ii) $T$ is idempotent, i.e. $TTf = Tf$.

(iii) $T$ is monotone, i.e. $f \leqslant g$ implies $Tf \leqslant Tg$.

Properties (i) and (ii) follow directly from the definition of $T$, (iii) follows from [**11**, Theorem 18].

Properties (i)–(iii) were used in most of the cited characterization results. A further property, namely the positive homogeneity of $T$ (i.e. $T(af) = aTf$ for $a \geqslant 0$), has been used in all characterizations up to now. Though $Tf = \mu_\Phi(f|\mathfrak{L})$ is positive homogeneous if $\Phi(x) = |x|^p$, $p > 1$, unfortunately $T$ is not positive homogeneous any more for general $\Phi$. However, even in the general case $T$ has another simple algebraic property, namely

(iv) $T(f \pm \frac{1}{2}Tf) = (1 \pm \frac{1}{2})Tf$.

The case $T(f - \frac{1}{2}Tf) = \frac{1}{2}Tf$ follows directly from the definition of $Tf = \mu_\Phi(f|\mathfrak{L})$. To see $T(f + \frac{1}{2}Tf) = \frac{3}{2}Tf$ let $h \in L_\Phi(\mathfrak{L})$ be given. As $Tf = \mu_\Phi(f|\mathfrak{L})$ and $\Phi$ is convex we have

$$\begin{aligned}\int \Phi\left(\left(f + \frac{1}{2}Tf\right) - \frac{3}{2}Tf\right) d\mu &= \int \Phi(f - Tf)\, d\mu \leqslant \int \Phi\left(f - \frac{2}{3}h\right) d\mu \\ &= \int \Phi\left(\frac{1}{3}(f - Tf) + \frac{2}{3}\left(f + \frac{1}{2}Tf - h\right)\right) d\mu \\ &\leqslant \frac{1}{3}\int \Phi(f - Tf)\, d\mu + \frac{2}{3}\int \Phi\left(f + \frac{1}{2}Tf - h\right) d\mu\end{aligned}$$

which implies $T(f + \frac{1}{2}Tf) = \frac{3}{2}Tf$.

The following theorem shows that if we add to the algebraic conditions (i)–(iv) on $T$ only one integration condition on $T$ (namely that 0 is not a better $\Phi$-approximant of $f$ than $Tf$), then each operator with these properties is necessarily of the form $Tf = \mu_\Phi(f|\mathfrak{L})$ for an appropriate $\sigma$-lattice $\mathfrak{L} \subset \mathcal{Q}$. The added integration condition is a generalization of Dykstra's condition for $\Phi(x) = x^2$ (see **[8]**); for $\Phi(x) = |x|^p$ this condition is also used in **[10]**.

We remark that the following characterization result is the first characterization result for $\sigma$-lattices which uses only *one* integration condition. For the case

$\Phi(x) = x^2$ Dykstra **[8]** needs besides three algebraic conditions (namely (ii), (iii) and positive homogeneity) three integration conditions (namely (v), $\int Tf\,d\mu = \int f\,d\mu$ and $\|Tf - Tg\|_2 \leqslant \|f - g\|_2$). In **[12]** five algebraic conditions and two integration conditions are used for the case $\Phi(x) = |x|^p$, $p > 1$.

THEOREM 1. *Let $(\Omega, \mathcal{Q}, \mu)$ be a finite measure space and $\Phi\colon \mathbf{R} \to \mathbf{R}_+$ be a symmetric, strictly convex, differentiable function with $\Phi(0) = 0$ such that $L_\Phi = L_\Phi^\infty$. Let $T\colon L_\Phi(\Omega, \mathcal{Q}, \mu) \to L_\Phi(\Omega, \mathcal{Q}, \mu)$ be an operator with the following properties.*

(i) *$T$ is translation invariant.*

(ii) *$T$ is idempotent.*

(iii) *$T$ is monotone.*

(iv) $T(f \pm \frac{1}{2}Tf) = (1 \pm \frac{1}{2})Tf$.

(v) $\int \Phi(f - Tf)\,d\mu \leqslant \int \Phi(f)\,d\mu$.

*Then there exists a $\sigma$-lattice $\mathfrak{L} \subset \mathcal{Q}$ such that $Tf = \mu_\Phi(f|\mathfrak{L})$ for all $f \in L_\Phi$. If, moreover, $T(-Tf) = -Tf$, then there exists a $\sigma$-algebra $\mathcal{Q}_0 \subset \mathcal{Q}$ such that $Tf = \mu_\Phi(f|\mathcal{Q}_0)$ for all $f \in L_\Phi$.*

PROOF. From the properties of $\Phi$ we obtain (see e.g. **[9]**)

(1) $\Phi'$ is strictly increasing, continuous, $\Phi'(0) = 0$ and $\Phi'(-x) = -\Phi'(x)$.

Hence $0 \leqslant x\Phi'(x) \leqslant \int_x^{2x} \Phi'(t)\,dt \leqslant \Phi(2x)$, $x \geqslant 0$. As $L_\Phi = L_\Phi^\infty$ this implies $h\Phi'(h) \in L_1$ if $0 \leqslant h \in L_\Phi$. For $h = |f| + |g|$ this yields together with (1)

(2) $$\Phi'(f)g \in L_1 \quad \text{for all } f, g \in L_\Phi.$$

Applying $T(g - \frac{1}{2}Tg) = \frac{1}{2}Tg$ to $g = f - (1 - 1/2^n)Tf$ one obtains inductively

(3) $$T(f - (1 - 1/2^n)Tf) = (1/2^n)Tf, \qquad n \in \mathbf{N}.$$

As by (i) and (3) for all $a \in \mathbf{R}$

$$f - Tf = f - (1 - 1/2^n)Tf + a - T(f - (1 - 1/2^n)Tf + a)$$

we obtain from (v) that

(4) $$\int \Phi(f - Tf)\,d\mu \leqslant \int \Phi(f - (1 - 1/2^n)Tf + a)\,d\mu, \qquad n \in \mathbf{N}.$$

Hence $n \to \infty$ implies by the Theorem of Lebesgue

$$\int \Phi(f - Tf)\,d\mu \leqslant \int \Phi(f - Tf + a)\,d\mu, \qquad \text{for all } a \in \mathbf{R}.$$

Therefore the function $a \to \int \Phi(f - Tf + a)\,d\mu$, $a \in \mathbf{R}$, attains its minimum at $a = 0$. Since $\sup_{|a| \leqslant 1}|\Phi'(f - Tf + a)| \leqslant \Phi'(|f - Tf| + 1) \in L_1$ by (1) and (2), we obtain from the Theorem of Lebesgue that

(5) $$\int \Phi'(f - Tf)\,d\mu = 0.$$

In the next step we show

(6) $$f_n \in L_\Phi, f_n \uparrow f \in L_\Phi \quad \text{or} \quad f_n \downarrow f \in L_\Phi \text{ implies } Tf_n \to Tf.$$

We prove only the increasing case, the decreasing case runs analogously. Since $Tf_n \leqslant Tf_{n+1} \leqslant Tf$ by (iii), we obtain from (5) and the Theorem of Lebesgue (use (1) and (2)) that

$$\int \Phi'(f - Tf)\,d\mu = 0 = \lim_{n\in\mathbf{N}} \int \Phi'(f_n - Tf_n)\,d\mu = \int \Phi'\Big(f - \lim_{n\in\mathbf{N}} Tf_n\Big)\,d\mu.$$

Since $\lim_{n\in\mathbf{N}} Tf_n \leqslant Tf$ and $\Phi'$ is strictly increasing, this implies $Tf = \lim_{n\in\mathbf{N}} Tf_n$, i.e. (6). Put $H = \{f \in L_\Phi\colon Tf = f\}$. From (i), (iv) and (6) we obtain

$$H + b \subset H \quad \text{for all } b \in \mathbf{R}. \tag{7}$$

$$\left(\tfrac{3}{2}\right)^n H \subset H, \qquad n \in \mathbf{N}. \tag{8}$$

$$f_n \in H, \quad f_n\uparrow f \in L_\Phi \quad \text{or} \quad f_n\downarrow f \in L_\Phi \Rightarrow f \in H. \tag{9}$$

We show that furthermore

$$f, g \in H \Rightarrow f \wedge g, \quad f \vee g \in H. \tag{10}$$

We prove only $f \vee g \in H$, the case $f \wedge g \in H$ runs similarly. As $T(f \vee g) \geqslant Tf \vee Tg = f \vee g$ by (iii) and $\int \Phi'(f \vee g - T(f \vee g))\,d\mu = 0$ by (5) we obtain, using $\Phi'(0) = 0$ and $\Phi'(x) < 0$ for $x < 0$, that $f \vee g = T(f \vee g)$, i.e. $f \vee g \in H$.

Let $\mathfrak{L} = \{A \in \mathfrak{A}\colon 1_A \in H\}$. As $T0 = 0$, $T(1) = 1$ by (v) and (i), $\mathfrak{L}$ is a $\sigma$-lattice according to (9) and (10). If additionally $T(-Tf) = -Tf$, then $\mathfrak{L}$ is a $\sigma$-field, since $T1_A = 1_A$ implies by (i) that

$$T(1_{\bar{A}}) = T(1 - 1_A) = 1 + T(-1_A) = 1 - T1_A = 1_{\bar{A}}.$$

Now we show that

$$H = L_\Phi(\mathfrak{L}). \tag{11}$$

Since $((\frac{3}{2})^n(f - b) \vee 0) \wedge 1 \uparrow 1_{\{f>b\}}$, $f \in H$ implies by (7)–(10) that $1_{\{f>b\}} \in H$. Hence $\{f > b\} \in \mathfrak{L}$, whence $f \in L_\Phi(\mathfrak{L})$. It remains to prove $L_\Phi(\mathfrak{L}) \subset H$. Let $f \in L_\Phi(\mathfrak{L})$. According to (7) and (9) we may assume $f \geqslant 0$. Since

$$\sup\left\{\frac{\nu}{2^n} 1_{\{f>\nu/2^n\}}\colon \nu = 0, \ldots, n2^n\right\}\uparrow f$$

it suffices to prove according to (9) and (10) that $C \in \mathfrak{L}$ implies $a1_C \in H$ for all $a \geqslant 0$. According to (8) we may assume $0 \leqslant a \leqslant 1$. Thus we have to prove $T1_C = 1_C$ implies $T(a1_C) = a1_C$, $0 \leqslant a \leqslant 1$. As $T$ is monotone by (iii) we have $T(a1_C) \leqslant T(a) \wedge T(1_C) = a1_C$. Hence (5) implies $T(a1_C) = a1_C$, i.e. $a1_C \in H$. Thus (11) is shown.

In the next step we prove that

$$\int \Phi'(f - Tf)g\,d\mu \leqslant 0 \quad \text{if } f \in L_\Phi, g \in L_\Phi(\mathfrak{L}). \tag{12}$$

Using (i), (6) and (2) we may w.l.g. assume that $0 \leqslant f \leqslant b$. Using (5) and (2) we may w.l.g. assume that $0 \leqslant g \in L_\Phi(\mathfrak{L})$. As each $\mathfrak{L}$-measurable function $g \geqslant 0$ is increasing limit of $\mathfrak{L}$-measurable simple functions (see e.g. [**4**, p. 313]), it suffices to prove (12) for $g = 1_C$ with $C \in \mathfrak{L}$. Since $b1_C \in L_\Phi(\mathfrak{L}) = H$ by (11) we have $T(b1_C) = b1_C$. As $T$ is monotone and $0 \leqslant f \leqslant b$ we obtain $T(f1_C) \leqslant Tf$ and $T(f1_C) \leqslant T(b1_C) = b1_C$. Hence $T(f1_C) \leqslant (Tf)1_C$ and (1) and (5) imply that

$$\int \Phi'(f - Tf)1_C\,d\mu = \int \Phi'(f1_C - (Tf)1_C)\,d\mu \leqslant \int \Phi'(f1_C - T(f1_C))\,d\mu = 0.$$

Thus (12) is shown.

Now we prove that

$$\int \Phi'(f - Tf)Tf\,d\mu \geq 0 \quad \text{if } f \in L_\Phi. \tag{13}$$

Using (1) and (2) we obtain from the Theorem of Lebesgue that the function $\varphi(t) := \int\Phi(f - tTf)\,d\mu$, $t \in \mathbf{R}$, is differentiable. Let $t_n = 1 - 1/2^n$; we obtain from (4), applied to $a = 0$ that

$$\varphi'(1) = \lim_{n\in\mathbf{N}} \frac{\varphi(1) - \varphi(t_n)}{1 - t_n} \leq 0.$$

As $\varphi'(1) = \int\Phi'(f - Tf)(-Tf)\,d\mu$ by (2) and the Theorem of Lebesgue, this implies (13).

By (12) and (13) we have

$$\int \Phi'(f - Tf)(g - Tf)\,d\mu \leq 0 \quad \text{if } f \in L_\Phi, g \in L_\Phi(\mathfrak{L}). \tag{14}$$

Since $T(Tf) = Tf$ by (ii), we have $Tf \in H$. Hence $Tf \in L_\Phi(\mathfrak{L})$ by (11). To prove $Tf = \mu_\Phi(f|\mathfrak{L})$ we have therefore to show that

$$\int \Phi(f - Tf)\,d\mu \leq \int \Phi(f - g)\,d\mu \quad \text{for all } g \in L_\Phi(\mathfrak{L}). \tag{15}$$

Let $f \in L_\Phi$, $g \in L_\Phi(\mathfrak{L})$ be given and put

$$\Psi(\alpha) := \int \Phi(f - (\alpha Tf + (1 - \alpha)g))\,d\mu, \qquad \alpha \in \mathbf{R}.$$

Then $\psi$ is a convex function as $\Phi$ is a convex function. By (2), the Theorem of Lebesgue and (14) we obtain that

$$\psi'(1) = \int \Phi'(f - Tf)(g - Tf)\,d\mu \leq 0. \tag{16}$$

As $\psi$ is convex, $\psi'(\alpha)$, $\alpha \in \mathbf{R}$, is a nondecreasing function. Hence $\psi(0) \geq \psi(1)$ by (16), i.e. (15) holds. This finishes the proof.

The following lemma clarifies the connection of our integration condition (v) and other integration conditions.

**Lemma 2.** *Let $(\Omega, \mathfrak{A}, \mu)$ and $\Phi$ be as in the theorem above. Let $T$: $L_\Phi \to L_\Phi$ be an operator with $T(f - \frac{1}{2}Tf) = \frac{1}{2}Tf$. Then the following three conditions are equivalent.*

(v) $\int\Phi(f - Tf)\,d\mu \leq \int\Phi(f)\,d\mu$ *for* $f \in L_\Phi$,

(v*) $\int\Phi(f - Tf)\,d\mu \leq \int\Phi(f - aTf)\,d\mu$ *for* $f \in L_\Phi$ *and some* $a = a(f) \in [0, 1)$,

(v**) $\int\Phi'(f)Tf\,d\mu \geq 0$ *for* $f \in L_\Phi$.

**Proof.** (v) obviously implies (v*). Let $f \in L_\Phi$ be given and put $\varphi(a) = \int\Phi(f - aTf)\,d\mu$, $a \in \mathbf{R}$. Since $\Phi$ is a convex function, $\varphi$ is a convex function, too.

Using (1) and (2) of the proof of the preceding theorem we obtain from the Theorem of Lebesgue that the function $\varphi$ is differentiable with

$$\varphi'(a) = -\int \Phi'(f - aTf)Tf\,d\mu, \qquad a \in \mathbf{R}. \tag{1}$$

If (v*) is fulfilled, then $\varphi(a) \geq \varphi(1)$ for some $0 \leq a < 1$. As $\varphi$ is convex and differentiable, $\varphi'$ is nondecreasing; hence $\varphi'(0) \leq 0$. Together with (1) this implies (v**). Let (v**) be fulfilled. Since $T(f - \frac{1}{2}Tf) = \frac{1}{2}Tf$ we have $T(f - (1 - 1/2^n)Tf) = (1/2^n)Tf$ (see (3) in the proof of Theorem 1). Hence (1) and (v**) imply

$$\varphi'(1 - 1/2^n) = -\int \Phi'(f - (1 - 1/2^n)Tf)Tf\, d\mu$$
$$= -2^n \int \Phi'(f - (1 - 1/2^n)Tf)T(f - (1 - 1/2^n)Tf)\, d\mu \leqslant 0.$$

Since $\varphi$ is convex this directly implies $\varphi(0) \geqslant \varphi(1)$, i.e. (v).

Lemma 2 shows that the assertion of Theorem 1 holds also true if we replace condition (v) by

(v**) $\int \Phi'(f)Tf\, d\mu \geqslant 0$ for $f \in L_\Phi$.

If $\Phi(x) = x^2$ condition (v**) means $\int fTf\, d\mu \geqslant 0$ for $f \in L_2$. The stronger integration condition $\int fTf\, d\mu > 0$ for $f \in L_2$ with $Tf \neq 0$ was used in **[4]** and **[12]** together with other integration conditions to characterize conditional expectations, given a $\sigma$-lattice.

If $\Phi(x) = |x|^p$, $p > 1$, condition (v**) means $\int \operatorname{sgn} f |f|^{p-1} Tf\, d\mu \geqslant 0$ for $f \in L_p$. A weaker form of this integration condition was used together with $p$-expectation invariance (i.e. $\int |f - Tf|^{p-1} \operatorname{sgn}(f - Tf)\, d\mu = 0$) and five algebraic conditions to characterize conditional $p$-means, given a $\sigma$-lattice.

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