FUNDAMENTAL SOLUTIONS FOR DIFFERENTIAL EQUATIONS ASSOCIATED WITH THE NUMBER OPERATOR

BY

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ABSTRACT. Let (H, B) be an abstract Wiener space. If u is a twice H-differentiable function on B such that $Du(x) \in B^*$ and $D^2u(x)$ is of trace class, then we define $\mathfrak{N}u(x) = -\Delta u(x) + (x, Du(x))$, where $\Delta u(x) = \text{trace}_H D^2u(x)$ is the Laplacian and $(\cdot, \cdot)$ denotes the B-B^* pairing. The closure $\overline{\mathfrak{N}}$ of $\mathfrak{N}$ is known as the number operator. In this paper, we investigate the existence, uniqueness and regularity of solutions for the following two types of equations: (1) $u_t = -\mathfrak{N}u$ (initial value problem) and (2) $\mathfrak{N}^k u = f \ (k > 1)$. We show that the fundamental solutions of (1) and (2) exist in the sense of measures and we represent their solutions by integrals with respect to these measures.

1. Introduction. Let (H, B) be an abstract Wiener space [1]. H is a given real separable Hilbert space with norm $|\cdot| = \sqrt{\langle , \rangle}$ and B is the completion of H with respect to a weaker norm $\| \ \|$ (a measurable norm in the sense of L. Gross [1]). The integration over B is performed with respect to the Wiener measure p_t, which is generated by the Gauss cylinder set measure with variance $t > 0$ on H. It is well known from the constructive quantum field theory that the Fock space (over H) $\mathfrak{F}(H)$ can be identified with $L^2(p_1)$ (see [9], [13]). Under this identification, we can further identify the number (of particles) operator on $\mathfrak{F}(H)$ with the infinitesimal generator $\overline{\mathfrak{N}}$ of the Ornstein-Uhlenbeck semigroup for the Brownian motion on B. When H is finite dimensional, $\overline{\mathfrak{N}}$ is the closure of $\mathfrak{N} = -\Delta + x \cdot \nabla$. However $\mathfrak{F}(H)$ is usually constructed over an infinite dimensional H, so a suitable infinite dimensional expression of $\mathfrak{N}$ as a differential operator must be formulated.

A formulation of $\mathfrak{N}$ which will be used in this paper is defined as follows: For a real-valued function f on B, we may regard f as a function g defined in a neighborhood of the origin of H by defining $g(h) = f(x + h)$. If g is k-times Fréchet differentiable at 0, then we say that f is k-times H-differentiable at x and we denote the k-fold H-derivative at x by $D^kf(x)$. If $D^2f(x)$ is of trace class, we then define the Laplacian $\Delta f(x) = \text{trace}_H[D^2f(x)]$ [2]. Next, we interpret the term "$x \cdot \nabla f(x)$" by $(x, Df(x))$ if $Df(x) \in B^*$, where $(\cdot, \cdot)$ denotes the B-B^* pairing. Then we define $\mathfrak{N}f(x) = -\text{trace}_H[D^2f(x)] + (x, Df(x))$. The domain of $\mathfrak{N}$ contains the C^2-functions on B (when f is C^2, $D^2f(x)$ is necessarily of trace class (see [4, p. 81; Goodman's Theorem])). It is well known that $\mathfrak{N}$ is a selfadjoint operator on $L^2(p_1)$ and the Hermite cylinder polynomials form a complete set of eigenfunctions

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for $\mathfrak{N}$. Moreover, $\mathfrak{N}$ extends to the generator $\overline{\mathfrak{N}}$ of the Ornstein-Uhlenbeck semigroup (which is a contraction semigroup on $L^2(p_1)$ [12]). $\overline{\mathfrak{N}}$ is known as the number operator.

In this paper we shall concern ourselves with the existence, uniqueness and regularity properties of the fundamental solution of the differential operator $\mathfrak{N}^k$ for $k \geqslant 1$. From the work of Gross [2] and Piech [10], [11], we see that the fundamental solutions of certain parabolic and elliptic second order equations are given by measures. For the case of higher order equations, less work has been done. By using the technique of the Fourier-Wiener transform, we have shown that for any exponential type analytic function f with $\int_B f(x)p_1(dx) = 0$, the solution of $\mathfrak{N}^k u(x) = f(x)$ exists uniquely (up to an addition of a constant) and it can also be represented by a series which is always convergent uniformly on bounded subsets of B [7]. (One might wish to solve the equation $\mathfrak{N}^k u = f$ by using the Hermite polynomial expansion. Since the Hermite polynomial expansion is convergent in L^2 and "$\mathfrak{N}f$" exists only in a weak sense [12], we would not obtain any knowledge of whether the solution $u(x)$ is really differentiable, or whether "$\mathfrak{N}u(x)$" really exists pointwise.) Following the approach of Gross [2], we shall show that there is a family of measures $\{Q(x, dy)\}$ (which serve as the "fundamental solution" of $\mathfrak{N}^k$) so that for a smooth function f (possibly nonanalytic) with $\int_B f(x)p_1(dx) = 0$, the function $Qf(x) = \int_B f(y)Q(x, dy)$ is, up to an addition of a constant, the only solution of $\mathfrak{N}^k u = f$. This also extends the results of [6, Theorem 5] and [7, Theorem 5.1].

In the course of proving the above result, we also obtain some regularity properties for the fundamental solution of the initial value problem associated with the equation $u_t(x, t) = -\mathfrak{N}u(x, t)$ and extend the results of Piech [11] to a class of unbounded initial functions.

The following lemma to be used later is an immediate consequence of [12].

LEMMA 1.1. *Assume f and g are in the domain of $\mathfrak{N}$. If f, g, $\|Df\|_{B^*}$, $\|Dg\|_{B^*}$, $\|D^2f\|_{tr}$ and $\|D^2g\|_{tr}$ (where $\| \ \|_{B^*}$ and $\| \ \|_{tr}$ denote the B^*-norm and trace norm respectively) belong to $L^2(p_1)$, then we have*

$$(1) \qquad \int_B (\mathfrak{N}f)(x)g(x)p_1(dx) = \int_B f(x)(\mathfrak{N}g)(x)p_1(dx).$$

COROLLARY 1.2. *Let g be an L^2 function. If $\mathfrak{N}u(x) = g(x)$ has a solution $u(x)$ in L^2 with either $\|D^2f(x)\|_{tr}$ or $\|Df(x)\|_{B^*}$ being in $L^2(p_1)$, then we must have $\int_B g(x)p_1(dx) = 0$.*

REMARK 1.3. In this paper, all the functions are assumed to be real in order to avoid computational complexity. However, one can extend all the results to complex-valued functions in the natural way be considering the real and imaginary parts.

In what follows, we shall use "D" or "D_x" to denote the differentiation with respect to the space variable and use "$\partial/\partial t$" or "$(\cdot)_t$" to denote the derivative with respect to the time variable. $\square$

2. Solution of $u_t = -\Re u$. In this section we shall consider the following Cauchy problem

(2)
$$\partial u(x, t)/\partial t = -\Re u(x, t) \qquad (x \in B, t > 0),$$
$$\lim_{t \searrow 0} u(x, t) = f(x) \quad \text{uniformly for } x \text{ in bounded sets.}$$

Our goal is to extend the results of Piech [**11**, Theorem 1] and Lee [**7**, Theorem 4.7] (both papers dealt with problem (2)) to a wider class of initial functions which includes both the class $\mathcal{C}$ of bounded uniformly Lip-1 functions and the class $\mathcal{E}_a(B)$ of real analytic exponential type functions.

The class of functions which shall be considered in this paper is defined as follows:

A real-valued measurable function f is said to be in the class $\mathcal{L}$ if there are constants $c, c' \geqslant 0$ such that

(3)
$$|f(x) - f(y)| \leqslant c \cdot e^{c'\|x\|} e^{c'\|y\|} \|x - y\|$$

for all x, y in B (see [**8**]).

It follows from Fernique's Theorem (see, for instance, [**4**]) that $\mathcal{C} + \mathcal{E}_a(B) \subset \mathcal{L}$ $\subset L^p(p_1)$ $(p \geqslant 1)$. Let $p_t f(x) = \int_B f(x + y) p_t(dy)$. If f is in $\mathcal{L}$, then it is not hard to see that $p_t f$ is a C_H^1-function in $\mathcal{L}$. (By a C_H^1-function we mean an H-differentiable function f defined on B such that $Df(x)$ is continuous from B into H^* with the norm topology.)

We shall establish the following:

THEOREM 2.1. *Assume that f is a function in $\mathcal{L}$. Let*

$$u(x, t) = o_t f(x) = p_{1 - e^{-2t}} f(e^{-t} x).$$

Then we have

(a) *$Du(x, t) \in B^*$ and $\|Du(x, t)\|_{B^*} \leqslant c_1 e^{-t} e^{2c'\|x\|}$ for some constants c_1, c' depending only on f.*

(b) *For each $t_1, t_2 > 0$ and any bounded set U, the map $(x, t) \to D^2 u(x, t)$ is uniformly continuous on $U \times [t_1, t_2]$ into the Banach space of trace class operators on H.*

(c) *$\|D^2 u(x, t)\|_{tr} \leqslant c_2 t^{-1/2} e^{-t} e^{c'\|x\|}$ for some constants c_2, c' which depend only on f.*

(d) *The function $u(x, t) = o_t f(x)$ is jointly uniformly continuous on $U \times [0, \lambda]$ for each finite $\lambda > 0$ and each bounded subset U of B.*

(e) *For each $t > 0$, $(\partial/\partial t)u(x, t)$ exists and $(\partial/\partial t)u(x, t) = -\Re u(x, t)$. Moreover, $\lim_{t \searrow 0} u(x, t) = f(x)$ uniformly in U.*

The proof of Theorem 2.1 will be accomplished by the following lemmas.

LEMMA 2.2 [**8**]. *Assume that $f \in \mathcal{L}$ and $v(x, t) = p_t f(x)$. Then the function $x \to p_t f(x)$ is infinitely H-differentiable. Moreover, we have*

(a) *$D^2 v(x, t)$ is a trace class operator for each x in B and $t > 0$. Furthermore,*

$$\|D^2 v(x, t)\|_{tr} \leqslant c_f \cdot t^{-1/2} e^{c'\|x\|} \left\{ \int_B e^{2c'\|y\|} p_{t/2}(dy) \right\}^{3/2},$$

where the constants c_f and c' depend only on f.

(b) *For each pair of positive numbers a, a' and each bounded set U, the map $(x, t) \rightarrow D^2 v(x, t)$ is uniformly continuous on $U \times [a, a']$ into the Banach space of trace class operators on H.*

(c) *For each $t > 0$, $\partial v / \partial t$ exists uniformly for x in a bounded set and $(\partial/\partial t)v(x, t) = \frac{1}{2}\text{trace}_H[D^2 v(x, t)]$. Moreover, $\lim_{t \searrow 0} v(x, t) = f(x)$ uniformly for x in each bounded set.*

The next lemma is useful for proving Theorem 2.1. Since its proof is quite similar to that of [11, Lemma 2.5] or [8, Lemma 1], we omit the proof.

LEMMA 2.3 (SEE [11]). *Assume f is a function in $\mathcal{L}$ satisfying the condition (3). Then we have*

(a) *$Dp_t f(x) \in B^*$ and $\|Dp_t f(x)\|_{B^*} \leqslant c \cdot (\int_B e^{2c'\|y\|} p_t(dy)) e^{2c'\|x\|}$.*

(b) *The map $(y, x) \rightarrow (y, Dp_t f(x))$ is continuous.*

(c) *For fixed t and x,*

$$\lim_{s \searrow 0} s^{-1}(p_t f(x + sy) - p_t f(x)) = (y, Dp_t f(x))$$

for all y in B. Moreover, for fixed t and y, the limit exists uniformly for x in a bounded set.

REMARK 2.4. In general, $p_t f(x)$ is not Fréchet differentiable in B. However it follows from Lemma 2.3 that $p_t f(x)$ ($f \in \mathcal{L}$) has directional derivatives in all directions of B. □

PROOF OF THEOREM 2.1. Let $a(t) = 1 - e^{-2t}$ and $b(t) = e^{-t}$. Then $u(x, t) = p_{a(t)} f(b(t)x)$.

(a) Since $f(e^{-t}x)$ is in $\mathcal{L}$, Lemma 2.3 ensures that $Du(x, t)$ belongs to B^* and $Du(x, t) = D_x p_{a(t)} f(b(t)x) = e^{-t} D_y p_{a(t)} f(y)|_{y = b(t)x}$. Also by 2.3(a) we have

$$\|Du(x, t)\|_{B^*} \leqslant c \left(\int_B e^{2c'\|y\|(\sqrt{a(t)})} p_1(dy) \right) e^{-t} e^{2c'\|x\|}$$

$$\leqslant c \left(\int_B e^{2c'\|y\|} p_1(dy) \right) e^{-t} e^{2c'\|x\|}$$

$$= c_1 e^{-t} e^{2c'\|x\|}.$$

(b) Since the map $(x, t) \rightarrow D_y^2 p_{a(t)} f(y)|_{y = e^{-t}x}$ is the composition of the following two uniformly continuous functions $(x, t) \rightarrow (e^{-t}x, a(t))$ and $(x, t) \rightarrow D^2 p_t f(x)$, (b) follows from Lemma 2.2(b) and the fact $D^2 u(x, t) = e^{-2t} D_y^2 p_a f(y)|_{y = bx}$.

(c) By Lemma 2.2(a), we have

$$\|D^2 p_t f(x)\|_{\text{tr}} \leqslant c_f \cdot \left(\int_B e^{2c'\|y\|} p_1(dy) \right)^{3/2} t^{-1/2} e^{c'\|x\|}.$$

It follows that

$$\|D^2 u(x, t)\|_{\text{tr}} \leqslant c_f \cdot \left(\int_B e^{2c'\|y\|} p_1(dy) \right)^{3/2} e^{-2t}(1 - e^{-2t})^{-1/2} e^{c'\|x\|}$$

$$\leqslant c_1 t^{-1/2} e^{-t} e^{c'\|x\|},$$

where $c_1 = 2 \cdot c_f \cdot (\int_B e^{2c'\|y\|} p_1(dy))^{3/2}$ which depends only on f.

(d) Let $F(x, t) = (b(t)x, a(t))$ and $v(x, t) = p_s f(x)$. $F(x, t)$ is uniformly continuous on $U \times [0, \lambda]$ and $u(x, t) = v \cdot F(x, t)$. Observe that if $\|x\| \leq M$ in U the range of F is a subset of $\{\|x\| \leq M\} \times [0, 1]$ on which $v(x, t)$ is also uniformly continuous. Therefore $u(x, t)$, as a composition of two uniformly continuous functions, is uniformly continuous.

(e) It follows by the chain rule that $\partial u / \partial t$ exists and

(4) $\qquad (\partial / \partial t) u(x, t) = (\partial / \partial a) p_a f(bx)(\partial a / \partial t) + (\partial / \partial b) p_a f(bx)(\partial b / \partial t)$.

Next, by (c) of Lemma 2.2, we see that

(5)
$$(\partial / \partial a) p_a f(y)|_{y=bx} = \tfrac{1}{2} \, \text{trace}_H \big[D^2 p_a f(y) \big]\big|_{y=bx}$$
$$= \tfrac{1}{2} e^{2t} \, \text{trace}_H \big[D_x^2 p_a f(bx) \big] = \tfrac{1}{2} e^{2t} \, \text{trace}_H \big[D^2 u(x, t) \big].$$

Similarly,

(6)
$$(\partial / \partial b) p_a f(bx) = (x, D p_a f(y))|_{y=bx} = e^t (x, D_x p_a f(bx))$$
$$= e^t (x, Du(x, t)).$$

Putting (5) and (6) into (4), we get

$$(\partial / \partial t) u(x, t) = \text{trace}_H \big[D^2 u(x, t) \big] - (x, Du(x, t)) = -\mathfrak{N} u(x, t).$$

Finally,

$$|u(x, t) - f(x)| = \left| \int_B f\big(e^{-t}x + (1 - e^{-2t})^{1/2} y\big) p_1(dy) - f(x) \right|$$

$$\leq \int_B |f\big(e^{-t}x + (1 - e^{-2t})^{1/2} y\big) - f(x)| p_1(dy)$$

$$\leq c \cdot e^{2c'\|x\|} \bigg\{ (1 - e^{-2t})^{1/2} \bigg(\int_B e^{c'\|y\|} \|y\| p_1(dy) \bigg)$$

$$+ (1 - e^{-t}) \|x\| \int_B e^{c'\|y\|} p_1(dy) \bigg\}$$

$\rightarrow 0 \quad$ uniformly on bounded subsets of B

as t goes to 0. This completes the proof of Theorem 2.1. $\square$

REMARK 2.5. If we define $o_t(x, A) = p_{1-e^{-2t}}(e^{-t}x, A)$ for each x in B and for each Borel set A, then $o_t(x, \cdot)$ is a probability measure and the family $\{o_t(x, dy)\}$ serves as the fundamental solution for the Cauchy problem (2). $\square$

As a further application of Lemma 1.1, we prove the uniqueness for the Cauchy problem (2) in the following.

THEOREM 2.6 (UNIQUENESS). *Assume that $u(x, t)$ is a function defined on $B \times [0, \infty)$ satisfying the following conditions*:

(1) $u(x, t)$ *is jointly continuous on $B \times [0, \infty)$ and, for each $s > 0$, $u(x, t) \in L^2(B \times [0, s])$.*

(2) $u(\cdot, t)$ *is twice H-differentiable such that $Du(x, t) \in B^*$ and $D^2 u(x, t)$ is of trace class. Moreover, $\|Du(x, t)\|_{B^*}$ and $\|D^2 u(x, t)\|_{tr}$ are in $L^2(p_1)$ and, for each $s > 0$, $u(x, t) \in L^2(B \times [0, s])$.*

(3) $\partial u / \partial t$ *exists and $(\partial / \partial t) u(x, t) = -\mathfrak{N} u(x, t)$.*

(4) $\lim_{t \searrow 0} u(x, t) = 0$.

Then $u(x, t) = 0$ (see also [11]).

PROOF. Note that condition (1) implies that, for almost all $s \geqslant 0$, $u(\cdot, s) \in L^2(p_1)$. Let N be the exceptional set and $s \in (0, \infty) \setminus N$. For any $f \in \mathcal{E}_a(B)$, we have

$$\int_0^s \int_B \mathfrak{N}u(x, t)o_{s-t}f(x)p_1(x) \, dt = \int_0^s \int_B u(x, t)\mathfrak{N}o_{s-t}f(x)p_1(dx) \, dt$$

$$\text{(by 2.6(2), 2.1 and 1.1)}$$

$$= \int_B \int_0^s u(x, t)\frac{\partial}{\partial t} o_{s-t}f(x) \, dt \, p_1(dx)$$

$$= \int_B \lim_{t \to s} u(x, t)o_{s-t}f(x) - \lim_{t \to 0} u(x, t)o_{s-t}f(x)$$

$$- \int_0^s \frac{\partial u}{\partial t}(x, t) \cdot o_{s-t}f(x) \, dt \, p_1(dx)$$

$$= \int_B u(x, s)f(x)p_1(dx) + \int_B \int_0^s \mathfrak{N}u(x, t)o_{s-t}f(x)p_1(dx).$$

This implies that $\int_B u(x, s)f(x)p_1(dx) = 0$ for arbitrary f in $\mathcal{E}_a(B)$. Since $u(\cdot, s) \in L^2(p_1)$ and $\mathcal{E}_a(B)$ is dense in $L^2(p_1)$, hence $u(\cdot, s) = 0$ a.e. (p_1). Since $u(\cdot, s)$ is continuous, $u(\cdot, s) = 0$. Thus we have proved that, for almost all $s > 0$, $u(\cdot, s) = 0$. Finally, by the continuity of $u(x, \cdot)$, we conclude that $u(x, s) = 0$ for all x and s. $\square$

It follows from Theorem 2.1 that for any f in $\mathcal{L}$, $o_t f(x)$ always satisfies conditions (1)–(3) of Theorem 2.6. Thus we have the following

COROLLARY 2.7 (UNIQUENESS). *Assume that* $u(x, t)$ *is a function defined on* $B \times [0, \infty)$ *satisfying conditions* (1)–(3) *of Theorem* 2.6. *Suppose that* f *is in the class* $\mathcal{L}$ *and* $\lim_{t \to 0} u(x, t) = f(x)$. *Then* $u(x, t) = o_t f(x)$.

3. Solution of $\mathfrak{N}^k u = f$. In this section, we shall show that there exists a family of measures $\{Q(x, dy)\}$ such that for $f \in \mathcal{L}$ with mean zero, the function $u(x) = \int_B f(y)Q(x, dy)$ solves the equation $\mathfrak{N}^k u = f$. This extends the result of [7, Theorem 5.1] and when $k = 1$, this also extends [6, Theorem 5]. Furthermore, we shall show that the solution is unique.

Let $o_t(x, A) = p_{1-e^{-2t}}(e^{-t}x, A)$ and $G(x, A) = \int_0^\infty o_t(x, A) \, dt$, where $x \in B$ and A is a Borel set in B.

Obviously, $G(x, \cdot)$ is a Borel measure.

Let $o_t f(x) = \int_B f(y)o_t(x, dy)$ as in the previous section and define

$$Gf(x) = \int_0^\infty o_t f(x) \, dt \text{ (if it exists).}$$

Notation. We shall denote by $\mathcal{L}_0$ the subclass of functions f in $\mathcal{L}$ with $\int_B f(x)p_1(dx) = 0$. $\square$

The following proposition is basic.

PROPOSITION 3.1. *Assume* $f \in \mathcal{L}_0$; *then the integral* $\int_0^\infty o_t f(x) \, dt$ *is convergent and* Gf *also belongs to* $\mathcal{L}_0$. *Thus* G, *as a mapping, maps* $\mathcal{L}_0$ *into* $\mathcal{L}_0$.

PROOF. Since f is of mean zero, we may write

$$o_t f(x) = \int_B f\big(e^{-t}x + (1 - e^{-2t})^{1/2}y\big)p_1(dy) - \int_B f(y)p_1(dy).$$

Suppose that f satisfies the inequality (3) (see the definition of $\mathcal{L}$). We estimate

$$|o_t f(x)| \leqslant \int_B |f\big(e^{-t}x + (1 - e^{-2t})^{1/2}y\big) - f(y)|p_1(dy)$$

(7)
$$\leqslant \int_B c \cdot e^{c'\|x\|}e^{2c'\|y\|}\big[e^{-t}\|x\| + |(1 - e^{-2t})^{1/2} - 1|\,\|y\|\big]p_1(dy)$$

$$\leqslant c_1 e^{c'\|x\|}\|x\|e^{-t} + c_2 e^{c'\|x\|}e^{-2t}$$

$$\leqslant c_3 e^{c''\|x\|}\|x\|(e^{-t} + e^{-2t}),$$

where $c'' = c' + 1$ and $c_3 = c \cdot \int_B e^{2c'\|y\|}(1 + \|y\|)p_1(dy)$. Consequently,

$$\int_0^\infty |o_t f(x)|\,dt \leqslant (3/2) \cdot c_3 e^{c''\|x\|} = c_4 e^{c''\|x\|}.$$

Next, by Fubini's Theorem and the semigroup property of p_t, we see that

$$\int_B Gf(x)p_1(dx) = \int_0^\infty \int_B \int_B f(x + y)p_{e^{-2t}}(dx)p_{1-e^{-2t}}(dy)\,dt$$

$$= \int_0^\infty \Big[\int_B f(x)p_1(dx)\Big]\,dt = 0.$$

Finally, for x, y in B,

$$|Gf(x) - Gf(y)| \leqslant \int_0^\infty |p_{1-e^{-2t}}f(e^{-t}x) - p_{1-e^{-2t}}f(e^{-t}y)|\,dt$$

$$\leqslant c\Big\{\int_B e^{2c'\|z\|}p_1(dz)\Big\}\Big\{\int_0^\infty e^{-t}\,dt\Big\}e^{c'\|x\|}e^{c'\|y\|}\|x - y\|$$

$$\leqslant c_1 e^{c'\|x\|}e^{c'\|y\|}\|x - y\|.$$

Thus $Gf \in \mathcal{L}_0$. □

LEMMA 3.2. *Assume* $f \in \mathcal{L}_0$; *then* $\int_0^\infty |Do_t f(x)|_H\,dt$ *and* $\int_0^\infty |D^2 o_t f(x)|_{H,H}\,dt$ *are convergent. Moreover, for* $h, k \in H$,

(8)
$$\langle h, DGf(x)\rangle = \int_0^\infty \langle h, Do_t f(x)\rangle\,dt$$

and

(9)
$$\langle k, D^2 Gf(x)h\rangle = \int_0^\infty \langle k, D^2 o_t f(x)h\rangle\,dt.$$

PROOF. Since the B^*-norm $\|\cdot\|_{B^*}$ is stronger than the H-norm $|\cdot|_H$ and the trace norm $\|\cdot\|_{\mathrm{tr}}$ is stronger than the operator norm $|\cdot|_{H,H}$, the first assertion follows by (a) and (c) of Theorem 2.1. The second assertion follows from (b) of Theorem 2.1 and Lemma 2.3. □

PROPOSITION 3.3. *Assume f is an $L^2(p_1)$ function.*

(i) *In order that there exists a $u(x) \in \mathcal{L}$ such that $\mathfrak{N}u(x)$ exists and that $\mathfrak{N}u(x) = f(x)$, it is necessary that $\int_B f(x)p_1(dx) = 0$.*

(ii) *If f is in $\mathcal{L}_0$, then there exists a $u(x)$ in $\mathcal{L}_0$ such that $\mathfrak{N}u(x) = f(x)$. In fact, $\mathfrak{N}Gf(x) = f(x)$.*

PROOF. (i) is a consequence of Corollary 1.2 and the fact that $\|Du(x)\|_{B^*} \leqslant c \cdot e^{2c'\|x\|}$ which belongs to $L^2(p_1)$.

It remains to prove (ii). By (8) (Lemma 3.2), we have

$$\left| \int_0^\infty \langle h, Do_t f(x) \rangle \, dt \right| \leqslant \left| \int_0^\infty (h, Do_t f(x)) \, dt \right|$$

$$\leqslant \|h\| \int_0^\infty \|Do_t f(x)\|_{B^*} \, dt \quad \text{(by 2.1)}$$

$$\leqslant \|h\| \cdot c_1 \cdot e^{2c'\|x\|} < \infty.$$

It follows that $|\langle h, DGf(x) \rangle| \leqslant c_1 \cdot e^{2c'\|x\|}\|h\|$ and so $DGf(x) \in B^*$ and $\|DGf(x)\|_{B^*} \leqslant c_1 \cdot e^{2c'\|x\|}$.

Next, by (9) (Lemma 3.2), we have

$$\langle k, D^2 Gf(x)h \rangle = \int_0^\infty \langle k, D^2 o_t f(x)h \rangle \, dt \quad \text{for all } h, k \text{ in } H.$$

Now for any compact operator T on H and for any orthonormal basis $\{e_j\}$ in H, we see that

$$|\text{trace}_H(TD^2 Gf(x))| = \left| \sum_{j=1}^\infty \langle T^* e_j, D^2 Gf(x)e_j \rangle \right|$$

$$\leqslant \sum_{j=1}^\infty \int_0^\infty |\langle T^* e_j, D^2 o_t f(x)e_j \rangle| \, dt$$

$$\leqslant \int_0^\infty \|TD^2 o_t f(x)\|_{\text{tr}} \, dt$$

$$\leqslant |T|_{H,H^2} e^{c'\|x\|} \int_0^\infty t^{-1/2} e^{-t} \, dt \quad \text{(by 2.1)}$$

$$< \infty \quad \text{for each } x \text{ in } B.$$

Thus $D^2 Gf(x)$ is of trace class and $\|D^2 Gf(x)\|_{\text{tr}} \leqslant c_2 e^{c'\|x\|}$. Therefore $Gf(x)$ is in the domain of $\mathfrak{N}$.

Now, by Theorem 2.1, we get

$$\mathfrak{N}Gf(x) = \int_0^\infty \mathfrak{N}o_t f(x) \, dt$$

$$= \lim_{\varepsilon \to 0, r \to \infty} -\int_\varepsilon^r \frac{\partial}{\partial t} o_t f(x) \, dt = \lim_{\varepsilon \to 0, r \to \infty} (o_\varepsilon f(x) - o_r f(x))$$

$$= f(x) - \int_B f(y)p_1(dy) = f(x).$$

This completes the proof of Proposition 3.3.   □

REMARK 3.4. Proposition 3.3 has shown that the family of measures $\{G(x, dy)\}$ forms a "fundamental solution" of the operator $\mathfrak{N}$.   □

Notations. For $k \geqslant 1$, we shall use $\mathcal{L}(k)$ to denote the class of functions g such that $\mathfrak{N}^j g$ exists for $j = 0, \ldots, k - 1$ ($\mathfrak{N}^0 g = g$) and $\mathfrak{N}^j g \in \mathcal{L}$. We shall use $\mathcal{L}(k)_0$ to denote the subclass of functions f in $\mathcal{L}(k)$ with $\int_B f(x) p_1(dx) = 0$.

We let $G^k f(x) = G \circ G \circ \cdots (k \text{ times}) \cdots \circ Gf(x)$ (if it exists) and define

$$(10) \quad Q(x, A) = \underbrace{\int_B \int_B \cdots \int_B}_{k-1 \text{ times}} \int_B G(y_{k-1}, A) G(y_{k-2}, dy_{k-1}) \cdots G(x, dy_1),$$

where x is in B and A is a Borel set in B. $\square$

Now if we consider the following system of equations

$$(11) \quad \begin{cases} \mathfrak{N} u_1(x) & = & f(x), \\ \mathfrak{N} u_2(x) & = & u_1(x), \\ & \vdots & \\ \mathfrak{N} u_k(x) & = & u_{k-1}(x) \end{cases}$$

and then apply Proposition 3.1 and Proposition 3.3 to the system (11) repeatedly, we obtain the following

THEOREM 3.5. (a) *Assume f is an $L^2(p_1)$ function. Suppose that there exists a function $u(x)$ in the class $\mathcal{L}(k)$ such that $\mathfrak{N}^k u(x)$ exists and $\mathfrak{N}^k u(x) = f(x)$. We must have $\int_B f(x) p_1(dx) = 0$.*

(b) *Assume f is in the class $\mathcal{L}_0$ and let $Q(x, A)$ be the measure defined by* (10). *Then $G^k f(x) = \int_B f(y) Q(x, dy)$ exists, $G^k f \in \mathcal{L}(k)_0$ and $\mathfrak{N}^k(G^k f)(x) = f(x)$.*

In particular, we have the following

COROLLARY 3.6 (SEE [6], [7]). *Assume f is a function in the class $\mathrm{Lip}_1 + \mathcal{E}_a(B)$ with $\int_B f(x) p_1(dx) = 0$. Then the function $u(x) = \int_B f(y) Q(x, dy)$, where $Q(x, dy)$ is defined by* (10), *solves the equation $\mathfrak{N}^k u(x) = f(x)$.*

REMARK 3.7. Let $\mathfrak{N}_c u(x) = -\Delta u(x) + c(x, Du(x))$, $c > 0$. Then $\mathfrak{N}_c$ becomes a selfadjoint operator on $L^2(p_{c^{-1}})$. After some suitable changes, for example: use "$\int_B f(x) p_{c^{-1}}(dx) = 0$" to replace "$\int_B f(x) p_1(dx) = 0$" and replace $o_t(x, dy)$ by the measure $r_t(x, dy) = p_{1 - e^{-2ct}}(e^{-ct} c^{1/2} x, dy)$, etc., one can easily verify that the results we have are still true if we replace $\mathfrak{N}$ by $\mathfrak{N}_c$. $\square$

We discuss the uniqueness of solutions in the following

REMARK 3.8. Obviously, the solution of $\mathfrak{N}^k u = f$ is not unique. For example, if u is a solution and C is a constant then $u + C$ is also a solution. However, it follows immediately from the L^2-theory of $\mathfrak{N}$ (see [12]) that if $u \in \mathcal{L}(k)$ and $\mathfrak{N}^k u = 0$, then $u(x) = $ a constant. This implies that the solution of $\mathfrak{N}^k u = f, f \in \mathcal{L}_0$, is of the form $u(x) = \int_B f(x) Q(x, dy) + C$ provided $u(x)$ is in the class $\mathcal{L}(k)$. (It is easy to see that the uniqueness of solutions still holds if we replace $\mathcal{L}(k)$ by the class $\mathcal{S}(k)$, where $\mathcal{S}(k) = \{f \in L^2(p_1): f \in \mathrm{Domain}(\mathfrak{N}^k) \text{ and } \|D(\mathfrak{N}^j f)(x)\|_{B^*} \in L^2, \text{ for } j = 0, 1, \ldots, k - 1, \text{ where } \mathfrak{N}^0 f = f\}$.) $\square$

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