A FAMILY OF LINKS AND THE CONWAY CALCULUS

BY

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ABSTRACT. In 1969, J. H. Conway gave efficient methods of calculating abelian invariants of classical knots and links. The present paper includes a detailed exposition (with new proofs) of these methods and extensions in several directions.

The main application given here is as follows. A link L of two unknotted components in S^3 has the *distinct lifting property* for p if the lifts of each component to the p-fold cover of S^3 branched along the other are distinct. The p-fold covers of these lifts are homeomorphic, and so L gives an example of two distinct knots with the same p-fold cover. The above machinery is then used to construct an infinite family of links, each with the distinct lifting property for all $p \geq 2$.

I. Introduction. Let $L \subseteq S^3$ be a link of two unknotted components L_1 and L_2, and denote the p-fold cover of S^3 branched along a knot K by $B_p(K)$; note that $B_p(L_i)$ is homeomorphic to S^3. We say that L has the distinct lifting property (DLP) for p if the lifts of L_1 and L_2 to $B_p(L_2)$ and $B_p(L_1)$ respectively represent distinct links types in S^3. Such an L is automatically asymmetric. As we show later, the p-fold cover of S^3 branched along each of these lifts is the same p^2-fold cover of S^3 branched along L, and so we are provided with a pair of distinct links with homeomorphic p-fold covers. It is unknown whether a pair of distinct knots can have homeomorphic p-fold covers for all p (see Problem 1.27 in [K]), and we do not know of a pair of knots with homeomorphic p-fold covers for two distinct values of p.

The purpose of this paper is 2-fold. We first give an example of an infinite family of links L_k, $k \geq 2$, each with the DLP for $p \geq 2$. The associated lifts in each case have identical signature [M2] but are distinguished by their Alexander polynomials. Thus we obtain for each p an infinite family of pairs of distinct links (in fact, knots) with homeomorphic p-fold covers.

Our second purpose is to follow J. H. Conway [C1, C2] in outlining a fast and efficient method of calculating abelian invariants of links. In particular, we hope to point to these methods not only as convenient computational tools, but also as a means to prove general facts about links and as being of interest in their own right.

In §II.1 we identify an Alexander type polynomial of an oriented link L as an invariant of a certain infinite cyclic cover of L. Specifically, choose an oriented trivial meridian $\mu_i \in H_1(S^3 - L)$ using the right-hand rule for each component of L, and let M_L be the cover corresponding to the map to $\mathbf{Z}$ which sends each μ_i to a fixed generator t. Then $H_1(M_L)$ is a $\mathbf{Z}[t, t^{-1}]$-module in the usual way, and our

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polynomial is the determinant of a presentation matrix for this module. This polynomial is seen to be an isotopy invariant of L but not an invariant of link complement.

In §II.2 we give a new proof of the consistency of a method of Conway for the recursive assignment of a polynomial to L, and identify this Conway polynomial with the polynomial discussed above. Thus the groundwork is laid for a major theme of this paper: that this recursive machinery can be used to gain information about links without direct recourse to covering spaces of complements. §§II.3 and II.4 begin this theme by giving some computational lemmas and examples useful in the recursion, and by interpreting and reproving some well-known facts about links in our recursive framework. §II.7 continues by showing how signatures may be computed recursively and by giving a simple algorithm for finding the polynomial of a 2-bridge knot. Our theme finds even more justification in the following manner:

One way to draw the lift of one component L_1 of a link L to the p-fold cover of S^3 branched along another unknotted component L_2 is to cut along a disk bounded by L_2, straighten the strands belonging to L_1, and juxtapose p of the resulting blocks end-to-end (see Figure 1).
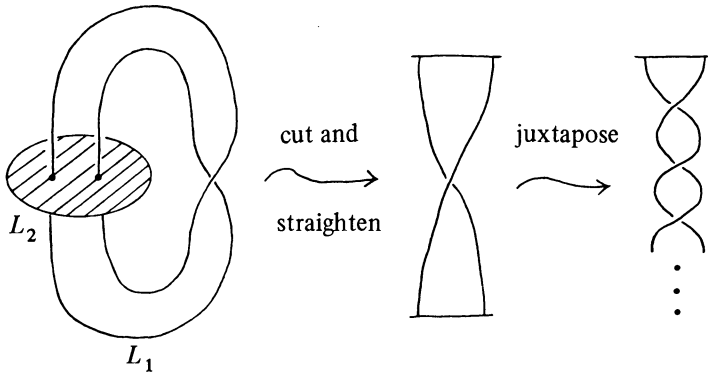

FIGURE 1

In fact, many links can be decomposed into such a block form, and it is desirable to be able to compute polynomials of links in terms of their block components. Following Conway, we define in §II.5 an algebra spanned by certain blocks (which Conway calls skein elements) subject to our previous recursion relations. We investigate the structure of this algebra and gain powerful formulas enabling us to perform the above calculation.

After giving examples in §II.6 both for the reader's practice and for our use later in this paper, we define two links to be skein equivalent if they have the same recursive structure, and we define an operation on links called mutation. Again following Conway, we show that mutants are skein equivalent and in this way give an example of a pair of distinct skein equivalent knots; the 2-fold covers of any two mutants are homeomorphic, and we give pictures of the p-fold covers of these examples and ask when they are homeomorphic.

Finally, in §III we give our example of a family of links with the DLP and prove that the distinct lifts have homeomorphic p-fold covers. The degree of the polynomial of one of the lifts is calculated recursively by the above methods, although Conway's formula cannot be used directly. The other lift is seen to have genus $\leqslant p$, a fact which prevents the degree of its polynomial from agreeing with that of the other. Thus the two lifts are distinguished by their Conway polynomials.

We use the following conventions. All links are assumed oriented, and arrows indicate the chosen orientation. ⊡ stands for n half-twists, right-handed for positive n and left-handed for negative n (for example, ⊟ = ⧚).

With these conventions, the family of links mentioned above is

$$L_k = \qquad , \qquad \text{where } \begin{cases} a + b = 1, \\ b = 2k \geqslant 4. \end{cases}$$
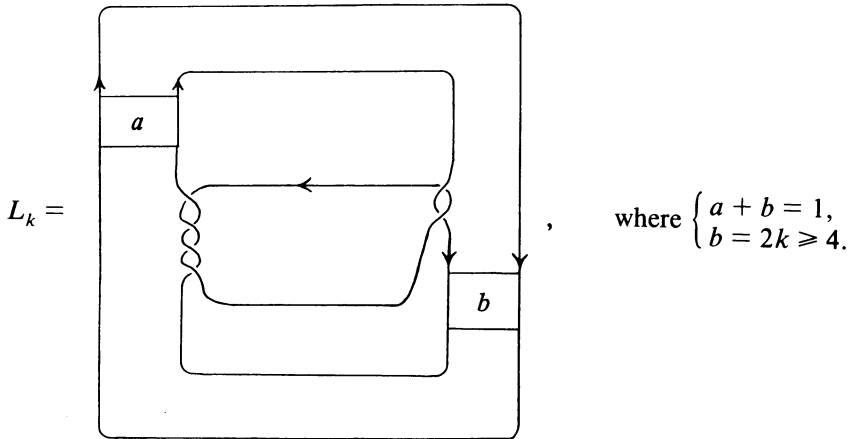
FIGURE 2

Note that the linking number of the two components is 1. A similar family with the same properties exists with linking number 0.

I would like to thank W. B. R. Lickorish for several enlightening discussions and Robion Kirby for providing endless encouragement as well as suggesting the original problem. Thanks go also to both the referee and John Hughes for many helpful suggestions.

II. Alexander polynomials of links.

1. *Definition of $\Delta_L(t)$.* Given an oriented link L, we construct an infinite cyclic cover of $S^3 - L$ in the following standard manner [R]. Let F be a connected Seifert surface for L compatible with the given orientation (say, so that an observer upright in the normal direction to F and walking along L sees F to the left), and equipped with a bicollar $\mathring{F} \times [-1, 1] \hookrightarrow S^3$. Let $\{Y_i = S^3 - F \mid i \in \mathbf{Z}\}$ be infinitely many copies of $S^3 - F$, and if $x \in \mathring{F}$, write $(x, t) \in \mathring{F} \times [-1, 1]$. Then form

$$M_L = \bigcup Y_i / (x_i, t) \sim (x_{i+1}, t - 1)$$

where $x_i \in Y_i$ and $x_{i+1} \in Y_{i+1}$ represent the same point in $\mathring{F}$ and where $t \in (0, 1)$ (i.e., the positive side of $\mathring{F}$ in Y_i is glued to the negative side of $\mathring{F}$ in Y_{i+1}).
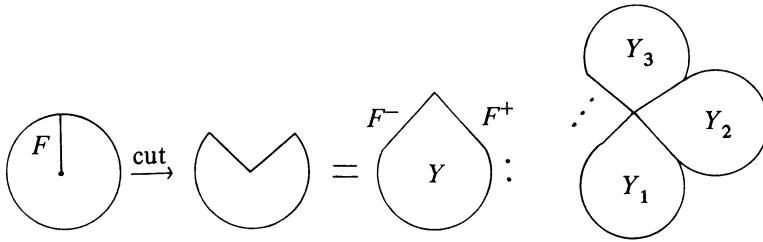
Schematically,

<div align="center">FIGURE 3</div>

M_L along with the obvious projection map is easily seen to be an infinite cyclic cover of $S^3 - L$ (i.e., a covering space of $S^3 - L$ with deck transformation group $\mathbf{Z}$). In general there are many infinite cyclic covers of the (unoriented) $S^3 - L$, and we make the following

PROPOSITION 1. *The choice of infinite cyclic cover of $S^3 - L$ depends only on the orientation of L and not upon the choice of F.*

PROOF. This fact is well known and the proof will be omitted.   □

REMARK. In case L is a knot there is only one infinite cyclic cover.

Recall now that $H_1(M_L)$ is a $\mathbf{Z}[t, t^{-1}]$-module, where t acts by the induced action of the deck transformation $1 \in \mathbf{Z}$. As in [**R**], a Mayer-Vietoris argument shows that

$$H_1(M_L) = \mathbf{Z}^k / (tV - V^T)\mathbf{Z}^k$$

where V is a Seifert matrix for L arising from F. Furthermore, modulo multiplication by $\pm t^n$, $\det(tV - V^T)$ depends only on M_L and not on the choice of V. Thus we have

PROPOSITION 2. *Let L be an oriented link, and V a Seifert matrix for L arising from a compatibly oriented surface F. Then the Alexander polynomial*

$$\Delta_L(t) = \det(tV - V^T)$$

is an isotopy invariant and modulo multiplication by $\pm t^n$ does not depend on the choice of V or F.   □

REMARK. Although $\Delta_L(t)$ is an isotopy invariant of the oriented link L, it is not an invariant of the link complement. Since a given automorphism of $S^3 - L$ may not preserve the preferred meridians corresponding to a choice of Seifert surface, the covers corresponding to these meridians need not be the same.

For example, twisting about the component labelled A in Figure 4 describes a homeomorphism between the complements of the two links, each equipped with preferred orientations and preferred meridians μ_1, μ_2, μ_3 and μ'_1, μ'_2, μ'_3. But μ_1 is carried to $\mu = \mu_1 + \mu_2 + \mu_3$ by this homeomorphism, and so the infinite cyclic cover of the complement obtained by setting $\mu_1 = \mu_2 = \mu_3 = 1$ is not the same as that obtained by setting $\mu'_1 = \mu'_2 = \mu'_3 = 1$. In fact, the polynomials of the above oriented links are $-(1 - t)^2$ and $(1 - t)^4 + 3t(1 - t)^2$ respectively.
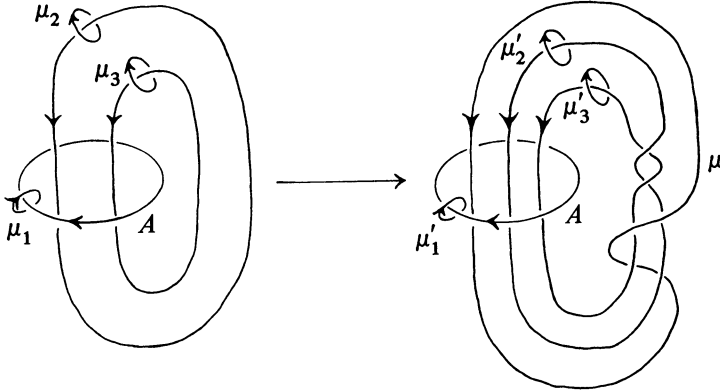
FIGURE 4

2. *Conway's method for computing Alexander polynomials.* Conway's idea is to calculate $\Delta_L(t)$ recursively using the following fact:

Fact. Any knot (or link) can be changed to the unknot (or unlink) by changing a finite number of overcrossings to undercrossings in some projection of the knot (or link).

To this end, we examine $\Delta_L(t)$ as a crossing is changed. To be definite, let L_l be a link in which a left-handed crossing occurs, let L_r be the same link with the crossing changed to a right-handed crossing, and let L_s be L_l with the crossing "smoothed" (see Figure 5). Each shaded surface is part of the Seifert surface given by a standard algorithm [R], and each Δ_L is to be calculated from these chosen surfaces.
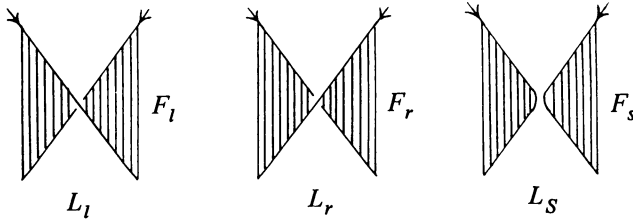
FIGURE 5
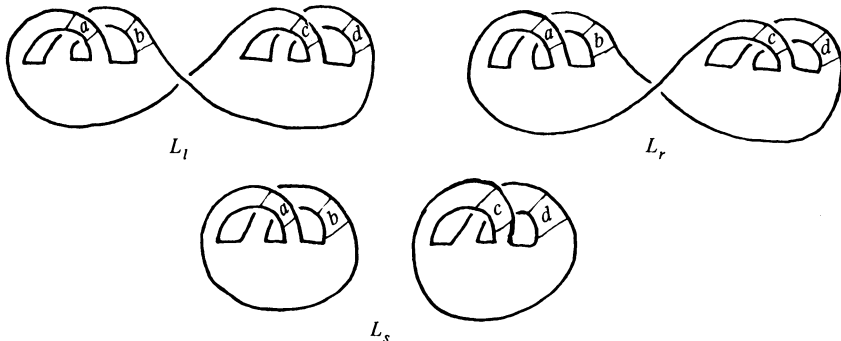
FIGURE 6

LEMMA 3. $\Delta_{L_l}(t) = \Delta_{L_r}(t) + (t-1)\Delta_{L_s}(t)$.

PROOF. If there is no basis element of $H_1(F_l)$ through the given crossing, then L_l must split as a direct sum with $L_l = L_r$ and L_s disconnected (so that $\Delta_{L_l}(t) = \Delta_{L_r}(t)$) (see Figure 6). A connected surface F for L_s is given by tubing together the previous two disjoint surfaces (see Figure 7).
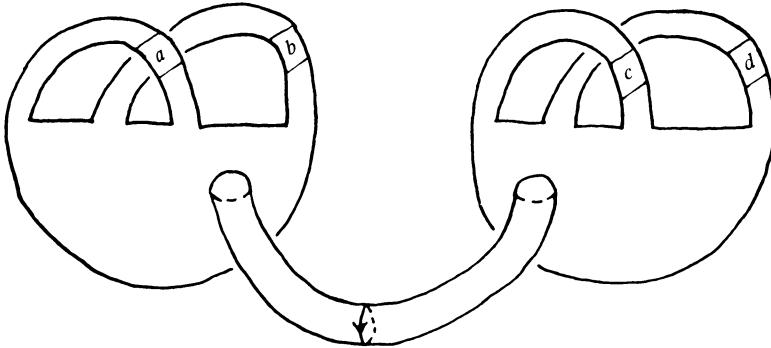

FIGURE 7

Using the meridian α of the tube as a basis element for $H_1(F)$, we see that the row and column corresponding to α in the Seifert matrix V is all 0's, and so $\Delta_{L_s}(t) = \det(tV - V^T) = 0$. Then since $\Delta_{L_l}(t) = \Delta_{L_r}(t)$, the lemma is true in this case.

If there is a basis of $H_1(F_l)$ containing an element through the indicated crossing, then we can rechoose the basis so that only one element α passes through the crossing. Then the Seifert matrices V_l, V_r, and V_s of L_l, L_r, and L_s respectively satisfy

$$V_r = \begin{array}{c} \alpha \\ \alpha \left(\begin{array}{c|c} a & * \\ \hline * & V_s \end{array} \right) \end{array}, \qquad V_l = \left(\begin{array}{c|c} a+1 & * \\ \hline * & V_s \end{array} \right),$$

so that

$$\Delta_{L_l}(t) = \det(tV_l - V_l^T) = \det \left(\begin{array}{c|c} (t-1)(a+1) & * \\ \hline * & tV_s - V_s^T \end{array} \right),$$

and expanding about the first row,

$$= \det \left(\begin{array}{c|c} (t-1)a & * \\ \hline * & tV_s - V_s^T \end{array} \right) + (t-1)\det(tV_s - V_s^T).$$

$$(1) \qquad \Delta_{L_l}(t) = \Delta_{L_r}(t) + (t-1)\Delta_{L_s}(t) \quad \text{as claimed.} \quad \square$$

At first glance, this seems to define $\Delta_L(t)$ recursively; simply change L to the unlink by changing crossings and smoothing (an easy induction on the number of crossings ensures that this can be done), and compute $\Delta_L(t)$ at each stage by Lemma 3. For concreteness, start with $\Delta(\text{unknot}) = 1$. However, this scheme does not quite work; the Alexander polynomial is defined only up to multiplication by $\pm t^n$, and so is only a representative of the set $S = \{\pm t^s \det(tV - V^T) \mid s \in \mathbf{Q}\}$. So, the recursion gives a well-defined choice of $\Delta_L(t)$ only if we can choose for all L a specific

canonical representative for $\Delta_L(t)$, and only if this choice is preserved by the recursion formula. A natural choice is to assume $\Delta_L(t)$ is symmetrized, i.e., of the form

(2) $\qquad a_\alpha t^\alpha + a_{\alpha-1} t^{\alpha-1} + \cdots + a_{-\alpha+1} t^{-\alpha+1} + a_{-\alpha} t^{-\alpha}$, with $a_\alpha, a_{-\alpha} \neq 0$.

However, (1) does not preserve this choice. To fix matters up, we redefine $\Delta_L(t)$ as follows. Let F be the surface from which V arises and WLOG suppose F is built from a disk by adding b bands (it is not hard to show that $b = 2g + k - 1$, where $g = $ genus F and k is the number of components of L). Then V is a $b \times b$ matrix, and we redefine

$$\Delta_L(t) = t^{-b/2} \det(tV - V^T)$$
$$= \det(t^{-1/2}(tV - V^T)) = \det(t^{1/2}V - t^{-1/2}V^T).$$

With this definition, $\Delta_L(t)$ is a representative of S and is clearly symmetric as in (2), although α need not be an integer. Furthermore, repeating the calculation leading to (1) yields

(3) $\qquad\qquad\qquad \Delta_{L_l}(t) = \Delta_{L_r}(t) + (t^{1/2} - t^{-1/2})\Delta_{L_s}$

so that our recursion now preserves symmetry (i.e., if any two of Δ_{L_l}, Δ_{L_r}, and Δ_{L_s} are symmetric, then so is the third). Thus the recursion gives us one of the two symmetrized representatives $\pm\Delta_L(t)$. In fact, we get a unique (even up to $\pm$) representative of S, as shown by the following three lemmas and corollary.

LEMMA 4. *Let K be a knot. Then $\Delta_K(1) = +1$ for any choice of Seifert surface for K.*

PROOF. $\Delta_K(t) = \det(t^{1/2}V - t^{-1/2}V^T)$, so $\Delta_K(1) = \det(V - V^T)$. But $V - V^T$ is the intersection form of F and is a direct sum of copies of $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ for some basis of $H_1(F)$ (see [**R**, p. 202]). $\square$

LEMMA 5. *$\Delta_K(t)$ as defined above gives a unique representative in S.*

PROOF. The condition $\Delta_K(1) = 1$ determines which of $\Delta_K(t)$ is chosen. $\square$

LEMMA 6. *Let F_1 and F_2 be two Seifert surfaces for a link L of two components, and let $\Delta_i(t)$ be the symmetrized form of $\det(t^{1/2}V_i - t^{-1/2}V_i^T)$, where V_i is the Seifert matrix arising from F_i. Then $\Delta_1(t) = \Delta_2(t)$.*

FIGURE 8

PROOF. Let b_n be a band containing n half-twists added to L so that $L \cup \partial b_{2n}$ is a knot and so that $F_1 \cup b_{2n}$ is a Seifert surface for $L \cup \partial b_{2n}$ (see Figure 8). Then (3) gives

$$\Delta(L \cup \partial b_2) = \Delta(L \cup \partial b_0) + (t^{1/2} - t^{-1/2})\Delta_1(L)$$
$$= \left(\Delta(L \cup \partial b_{-2}) + (t^{1/2} - t^{-1/2})\Delta_1(L)\right) + (t^{1/2} - t^{-1/2})\Delta_1(L).$$

So

$$\Delta_1(L) = \frac{\Delta(L \cup \partial b_2) - \Delta(L \cup \partial b_{-2})}{2(t^{1/2} - t^{-1/2})}.$$

If b_n does not intersect F_2, we can form a new surface $F_2 \cup b_{2n}$ for $L \cup \partial b_{2n}$, calculate as above, and find

$$(4) \qquad\qquad \Delta_2(L) = \frac{\Delta(L \cup \partial b_2) - \Delta(L \cup \partial b_{-2})}{2(t^{1/2} - t^{-1/2})}.$$

Since, by Lemma 5, Δ of a knot is independent of the surface chosen, $\Delta_1(L) = \Delta_2(L)$ in this case as claimed.
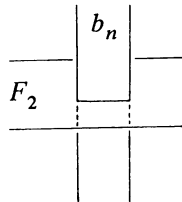
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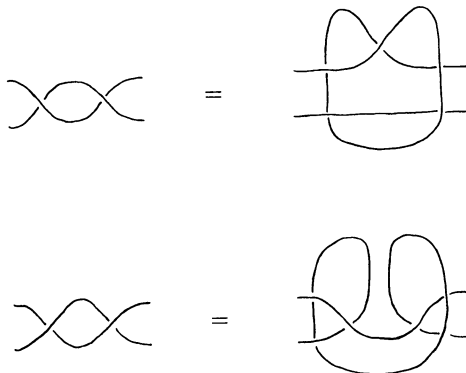
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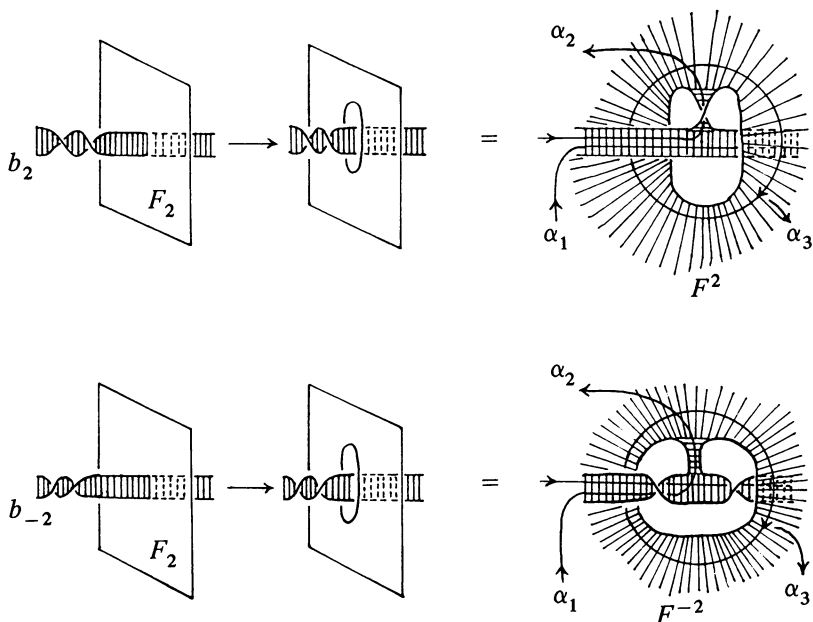
FIGURE 11

Suppose however that b_n intersects F_2 in (for simplicity's sake) one ribbon intersection (see Figure 9). First note the isotopies in Figure 10. Then form new Seifert surfaces F^2 and F^{-2} for $L \cup \partial b_2$ and $L \cup \partial b_{-2}$ respectively by first puncturing F_2 and filling in with the surfaces in Figure 11. Note that if necessary we can replace Figure 12 by Figure 13 so that the surfaces F^2 and F^{-2} are indeed orientable.
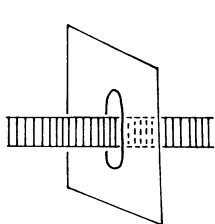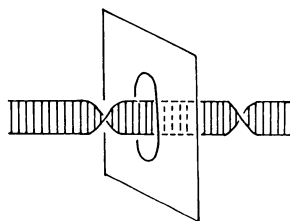

FIGURE 12

FIGURE 13

Then using the elements α_1, α_2, α_3 of $H_1(F^i)$ indicated, the corresponding Seifert matrices are

$$
V^2 = \begin{array}{c} \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \\ \end{array}
\begin{array}{c} \alpha_1 \quad \alpha_2 \quad \alpha_3 \\ \left(\begin{array}{ccc|c} n_1 & n_{12} & 1 & * \\ n_{21} & n_2 & 0 & * \\ 1 & 1 & 0 & 0 \\ \hline * & * & 0 & V_2 \end{array} \right) \end{array} ,
\qquad
V^{-2} = \left(\begin{array}{ccc|c} n_1 & n_{12}+1 & 1 & * \\ n_{21} & n_2 & 0 & * \\ 1 & 1 & 0 & 0 \\ \hline * & * & 0 & V_2 \end{array} \right) ,
$$

where the n_i's are integers and V_2 is the Seifert matrix arising from F_2.

Then expanding about the rows indicated by arrows,

$\det\left(t^{1/2}V^{-2} - t^{-1/2}V^{-2^T}\right)$

$$\rightarrow \atop = \det \left(\begin{array}{ccc|c} n_1(t^{1/2} - t^{-1/2}) & n_{12}t^{1/2} - n_{21}t^{-1/2} + t^{1/2} & t^{1/2} - t^{-1/2} & * \\ n_{21}t^{1/2} - n_{12}t^{-1/2} - t^{1/2} & n_2(t^{1/2} - t^{-1/2}) & -t^{-1/2} & * \\ t^{1/2} - t^{-1/2} & t^{1/2} & 0 & 0 \\ \hline * & * & 0 & t^{1/2}V_2 - t^{-1/2}V_2^T \end{array} \right)$$

$$= \det \left(\begin{array}{ccc|c} n_1(t^{1/2} - t^{-1/2}) & n_{12}t^{1/2} - n_{21}t^{-1/2} & t^{1/2} - t^{-1/2} & * \\ n_{21}t^{1/2} - n_{12}t^{-1/2} - t^{-1/2} & n_2(t^{1/2} - t^{-1/2}) & t^{-1/2} & * \\ t^{1/2} - t^{-1/2} & t^{1/2} & 0 & 0 \\ \hline * & * & 0 & t^{1/2}V_2 - t^{-1/2}V_2^T \end{array} \right) \leftarrow$$

$$-t^{1/2}\det \left(\begin{array}{cc|c} n_{21}t^{1/2} - n_{12}t^{-1/2} - t^{-1/2} & -t^{-1/2} & * \\ t^{1/2} - t^{-1/2} & 0 & 0 \\ \hline * & 0 & t^{1/2}V_2 - t^{-1/2}V_2^T \end{array} \right) \leftarrow$$

$$= \det\left(t^{1/2}V^2 - t^{-1/2}V^{2^T}\right) + t^{-1/2}\det \left(\begin{array}{cc|c} n_{12}t^{1/2} - n_{21}t^{-1/2} & t^{1/2} - t^{-1/2} & * \\ t^{1/2} & 0 & 0 \\ \hline * & 0 & t^{1/2}V_2 - t^{-1/2}V_2^T \end{array} \right)$$

$$-(t^{1/2} - t^{-1/2})\det\left(t^{1/2}V_2 - t^{-1/2}V_2^T\right)$$

$$= \det\left(t^{1/2}V^2 - t^{-1/2}V^{2^T}\right) - 2(t^{1/2} - t^{-1/2})\det\left(t^{1/2}V_2 - t^{-1/2}V_2^T\right).$$

Thus, using the surfaces F^2, F^{-2}, and F_2 we have again formula (4). Then as for the case where b_n does not intersect F_2, we see that $\Delta_2(L)$ and $\Delta_1(L)$ can be written as identical expressions in the polynomials of the knots $L \cup \partial b_2$ and $L \cup \partial b_{-2}$; since such polynomials are independent of which surface is chosen for the computation, $\Delta_2(L) = \Delta_1(L)$ as claimed. $\square$

COROLLARY 7. *The symmetrized* $\Delta_L(t) = \det(t^{1/2}V - t^{-1/2}V^T)$ *is independent (even up to $\pm$) of V.*

PROOF. Induct on the number of components of L, banding together two different components and using the above argument at each stage of the induction. $\square$

For convenience, we set $z = t^{1/2} - t^{-1/2}$. Note that by (3) any $\Delta_L(t)$ may be written as a unique polynomial $\nabla_L(z)$, as long as we set $\nabla(\text{unknot}) = 1$. So finally, we have a recursive scheme

$$(5) \qquad \nabla_{L_l}(z) = \nabla_{L_r}(z) + z\nabla_{L_s}(z), \qquad \nabla_{\text{unknot}}(z) = 1.$$

$\nabla_L(z)$ is called the Conway potential function of L.

REMARK. Kauffman [K] has proven the independence of $\nabla_L(z)$ from the choice of Seifert surface by noting that it is an invariant of S-equivalence, a notion not used in Lemma 6.

3. *Examples*. We will use the following conventions. An asterisk ($*$) by a crossing indicates that the crossing is to be changed and smoothed in the next step as in the previous sections. We will indicate the status of three links as being L_l, L_r, and L_s by

$$L_l = L_r \oplus L_s \quad \text{or} \quad L_r = L_l \ominus L_s.$$

Note that $\oplus$ and $\ominus$ are not commutative or associative. We will write $\nabla_L(z)$ as $\nabla(L)$ or ∇(a picture of L) according to convenience, and we will brutally abuse notation by writing a picture of L to mean either L or $\nabla_L(z)$ depending on the context.

(a)
$$1 = \nabla(\ \bigcirc\ ) = \nabla(\ \infty\ ) = \nabla(\ \infty\ ) + z\,\nabla(\ \bigcirc\ \bigcirc\ )$$

by (5), and so $\nabla(\bigcirc\ \bigcirc) = 0$.

Similarly, for any links L_1 and L_2 separated in S^3 by an S^2, $\nabla(L_1 \cup L_2) = 0$. This was proved geometrically in Lemma 3.

(b)
$$\nabla(\ \text{⌬}\ ) = \nabla(\ \bigcirc\ \bigcirc\ ) + z\,\nabla(\ \bigcirc\ ) = z,$$

by (5) and §3a; we write $\text{⌬} = (\bigcirc\ \bigcirc) \oplus \bigcirc$.

(c)
$$\nabla(\ \text{⌬}\ ) = \nabla(\ \bigcirc\ ) + z\,\nabla(\ \text{⌬}\ ) = 1 + z^2,$$

and we write $\text{⌬} = \bigcirc \oplus \text{⌬}$. Note that $1 + z^2 = 1 + (t^{1/2} - t^{-1/2})^2 = t - 1 - t^{-1}$ as expected.

(d)
$$\nabla(\ \text{⌬}\ ) = \nabla(\ \bigcirc\ \bigcirc\ ) - z\,\nabla(\ \bigcirc\ ) = -z.$$

$$\nabla(\ \text{⌬}\ ) = \nabla(\ \bigcirc\ ) - z\,\nabla(\ \text{⌬}\ ) = 1 + z^2.$$

Note that interchanging L_l and L_r in (5) has the effect of changing the $+$ to a $-$.

(e) Define
$$\nabla_n(z) = \nabla(\ \text{⌬}_n\ ).$$

Then $\nabla_n = \nabla_{n+2} + z\nabla_{n+1}$.

REMARKS. (1) When calculating $\nabla_L(z)$ in this fashion it is not necessary to draw a Seifert surface for L or to write down a Seifert matrix. This represents an enormous savings of time and effort in practical applications.

(2) By construction, when $\nabla_L(z)$ is of degree d in z and is converted to $\nabla_L(t)$ by the substitution $z = t^{1/2} - t^{-1/2}$, the result is a symmetrized polynomial in t of degree $\frac{1}{2}d$: $a_{d/2}t^{d/2} + a_{d/2-1}t^{d/2-1} + \cdots + a_{-d/2+1}t^{-d/2+1} + a_{-d/2}t^{-d/2}$. Multiplying by $t^{d/2}$, we see that the degree of $\Delta_L(t)$ as a polynomial in $\mathbf{Z}[t]$ with constant coefficient $a_{-d/2}$ is the degree d of $\nabla_L(z)$ in z. But an Euler characteristic argument shows that the number of bands of F (and hence the size of the matrix V) is $2g + k - 1$, where k is the number of components of L and g is the genus of F. Thus if g_L is the genus of the link L, then $2g_L + k - 1 \geqslant \deg \nabla_L(z)$. In the case of a knot K,

(6)
$$2g_K \geqslant \deg \nabla_K(z) = \deg \Delta_K(t).$$

This inequality will be used later to distinguish the two lifts.

4. *Computational lemmas.*

LEMMA 8. *Let K be a knot and L be a link. Then*
(a) $\Delta_K(1) = \nabla_K(0) = 1$.
(b) *If L has more than one component, then* $\nabla_L(0) = 0$.
(c) $\nabla_K(z)$ *is a polynomial in* z^2.

(d) $\Delta_K(t)$ and $\nabla_K(z)$ *are independent of the orientation of* K; *in fact,* $\nabla_L(z)$ *does not change when the orientations are simultaneously reversed on all components of* L (*recall that changing the orientation of a strand corresponds to a change in choice of infinite cyclic cover*).

(e) *Substituting* $t^{1/2} - t^{-1/2}$ *for z in* $\nabla_K(z)$ *gives the symmetrized* $\Delta_K(t)$; *furthermore, in the notation of* (2), $a_\alpha = a_{-\alpha}$ (*i.e.*, $\Delta_K(t) = \Delta_K(t^{-1})$).

REMARK. (a) and (b) are equivalent to the fact that if F is a Seifert surface for L, then the determinant of the intersection matrix of F is

$$\begin{cases} 1, & L \text{ a knot}, \\ 0, & L \text{ has more than one component}. \end{cases}$$

PROOF. (a) When $t = 1$, we have $z = 0$; therefore this statement is Lemma 4.

(b) L can be changed to the unlink of greater than one component by changing crossings. Setting $z = 0$ and using (5) shows $\nabla_L(0) = 0$.

(c) If L has more than one component, (b) shows that $\nabla_L(z)$ is a multiple of z. An induction on the crossing number using this fact and (5) finishes the proof.

(d) Changing the orientation on all strands of a link does not change the sign of each crossing, and so the recursion defining $\nabla_L(z)$ remains the same.

(e) By (c), $\nabla_K(z) = \nabla_K(-z)$, i.e., $\Delta_K(t) = \Delta_K(t^{-1})$. □

LEMMA 9. *Recall that*

$$\nabla_n = (\;\text{⌔}\boxed{n}\text{⌕}\;), \qquad n \in \mathbb{Z}.$$

Then

(a) $\nabla_{2n} = -\nabla_{-2n}$,
(b) $\nabla_{2n+1} = \nabla_{-(2n+1)}$,
(c) $\deg \nabla_n(z) = |n| - 1$.

PROOF. (a) and (b). Easy induction, using (5). The examples in §3 show the lemma is true for $k = 1$. Suppose it is true for $k \leqslant n$; then

$$\nabla_{2(n+1)} = \nabla_{2n+2} = \nabla_{2n} - z\nabla_{2n+1} = -\nabla_{-2n} - z\nabla_{-(2n+1)}$$

$$= -(\nabla_{-2n} + z\nabla_{-2n-1}) = -\nabla_{-2(n+1)}.$$

Similarly, $\nabla_{2(n+1)+1} = \nabla_{-(2(n+1)+1)}$. Thus (a) and (b) are proven for $n \geqslant 0$; a similar argument holds for $n \leqslant 0$.

(c) Straightforward induction using Example 3(e). □

The following lemma gives formulas for ∇ of a link in which $\boxed{n}$ occurs. Although useful in specific examples, it will not be used later and so will not be proven here (the proof, of course, is by induction and (5)).

LEMMA 10. (a) *Let* $\text{⍓}\boxed{n}$ *or* $\text{⍓}\boxed{n}$ *occur in a link* L. *Let* L_j *be the link obtained from* L *by replacing n by j. Then*

$$\nabla_L = \nabla_n\nabla_{L_{-1}} + \nabla_{n+1}\nabla_{L_0},$$

with a + *or* − *as n is positive or negative.*

(b) *Similarly, suppose* ⊞ *or* ⊞ *appear in L, and let L_s be the link obtained from L by smoothing the crossing. Then*

$$\nabla_L = \nabla_{L_0} + [n/2] z \nabla_{L_s}. \quad \square$$

COROLLARY 11. *Let L be a link, and let ♯ denote connected sum. Then*

$$\nabla (L \ \sharp \ \langle \boxed{n} \rangle \ ) = \nabla_n \nabla_L.$$

PROOF. By the lemma,

$$\nabla (L \sharp \langle \boxed{n} \rangle \ ) = \nabla_n \nabla_L + \nabla_{n\mp 1} \nabla (L \cup \bigcirc).$$

But the second term is zero by example 3(a). □

REMARK. This is a special case of $\nabla(L_1 \sharp L_2) = \nabla_{L_1} \nabla_{L_2}$ which is well known for knots. It will follow later from Corollary 16.

5. *Conway's room theory.* Following and generalizing Conway [C2], we develop a powerful method to calculate ∇_L.

DEFINITION. A *room R* is a connected domain (usually a rectangle) in $\mathbf{R}^2$ equipped with an equal number of oriented ingoing and outgoing strands:
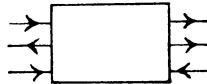
The *skein* of a room, $S(R)$, is the set of all collections of strands in the room which connect ingoing to outgoing strands. For example, Figure 14 is an element of the skein of the above room.
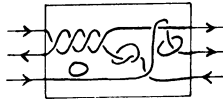
FIGURE 14

In this discussion we will always take R to be the following room with n ingoing and outgoing strands:
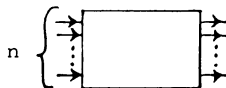
Our arguments generalize in obvious ways to other rooms.

DEFINITION. Let S_n denote the full symmetric group on n letters. Given a skein element s, we assign a permutation $\pi(s) \in S_n$ by numbering the strands of R:

Now pick an ordering $\{\pi_i\}_{i=1,n!}$ of S_n and fix a choice of skein element s_{π_i} for each $\pi_i \in S_n$ such that $\pi(s_{\pi_i}) = \pi_i$, and such that s_{π_i} contains no free component.

EXAMPLE. $n = 2$.

$n = 3$.
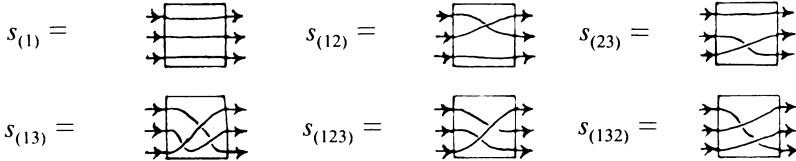
DEFINITION. Define a binary operation on $S = S(R)$ by juxtaposition:
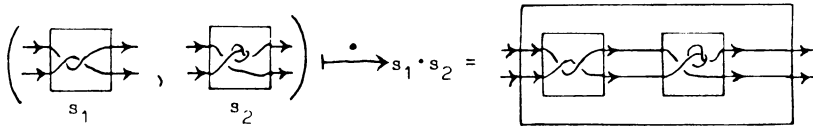
Note that $\pi(s_1 \cdot s_2) = \pi(s_1)\pi(s_2) \in S_n$.

Now let s_l be a skein element in which a left-handed crossing appears, and let s_r (respectively s_s) be the same element with the crossing changed to a right (respectively smoothed) crossing. We call (s_l, s_r, s_s) a *skein triple*.

DEFINITION. Let F be the quotient field of $\mathbf{Z}[z]$, let $V(S)$ be the vector space generated by S over F, and let $N(S)$ be the vector space generated by $\{s_l - s_r - zs_s \mid (s_l, s_r, s_s)$ is a skein triple$\}$ over F. Then define the linearization $L(S)$ by $L(S) = V(S)/N(S)$ (i.e., we impose the previous recursion relations on $V(S)$).

$L(S)$ inherits a vector space structure from $V(S)$, and is in fact an algebra under the extension of $\cdot$ to $L(S)$, since $\cdot$ preserves the relations in $N(S)$:

$$s_1 \cdot (s_r + zs_s) = s_1 \cdot s_r + zs_1 \cdot s_s = s_1 \cdot s_l.$$

DEFINITION. Define $L_0(S)$, $N_0(S)$, and $V_0(S)$ the same as for $L(S)$, $N(S)$, and $V(S)$ respectively, using $\mathbf{Z}[z]$ instead of F.

LEMMA 12. *Setting $z \mapsto 0$ and $s \mapsto \pi(s)$ defines an algebra homomorphism*

$$L_0(S) \xrightarrow{e} \mathbf{Z}[S_n],$$

where $\mathbf{Z}[S_n]$ is the group ring of S_n.

PROOF. An element α of $V_0(S)$ is a linear combination of elements of S with coefficients in $\mathbf{Z}[z]$. The above map takes α into $\mathbf{Z}[S_n]$, and we must show $e(N) = 0$. But $e(s_l - s_r - zs_s) = e(s_l) - e(s_r) = \pi(s_l) - \pi(s_r) = 0$.

Furthermore, for skein elements s_1 and s_2, $e(s_1 \cdot s_2) = \pi(s_1 \cdot s_2) = \pi(s_1)\pi(s_2) = e(s_1)e(s_2)$ so that e may be extended to an algebra homomorphism.   □

PROPOSITION 13. $\{s_\pi \mid \pi \in S_n\}$ *defined previously is a basis for $L(S)$.*

PROOF. By a *skein move* we mean either the equating of s_l with $s_r + zs_s$ (or of s_r with $s_l - zs_s$) as is done on dividing $V(S)$ by $N(S)$, or the geometric analogue of changing and smoothing a crossing. A moment's thought shows that any skein element may be decomposed by skein moves (i.e., rewritten in $L(S)$) to a linear combination over $\mathbf{Z}[z]$ of elements of $\{s_\pi \mid \pi \in S_n\}$ or skein elements containing a free component separated by an S^2.

EXAMPLE.
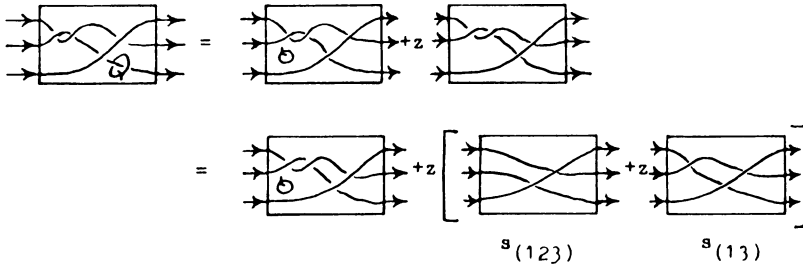
But as in §3a, skein elements with separated free components are zero in $L(S)$. Thus, $\{s_{\pi_i} \mid \pi_i \in S_n\}$ at least spans $L(S)$.

Suppose, however, that $\Sigma a_i(z)s_{\pi_i} = 0$, for $a_i(z) \in F$. Then by multiplying by polynomials if necessary, we may assume $a_i(z) \in \mathbf{Z}[z]$ and that some $a_i(0) \neq 0$. So $\Sigma a_i(z)s_{\pi_i} \in L_0(S)$, and $e(\Sigma a_i(z)s_{\pi_i}) = \Sigma a_i(0)\pi_i = 0$, a contradiction as $\{\pi_i \mid \pi_i \in S_n\}$ is a basis for $\mathbf{Z}[S_n]$. $\square$

In particular, we now have

COROLLARY 14. *Any skein element s can be expressed uniquely as a linear combination of the s_π, so that $\dim_F L(S) = n!$.* $\square$

DEFINITION. Given skein elements s and x, define the dual s^* by
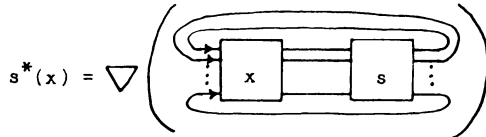
s^* may be extended linearly to all of $L(S)$; note that the so-defined s^* preserves skein moves:

So we get a dual element $s^*: L(S) \to \mathbf{Z}[z] \subseteq F$.

Let M_n be the $n! \times n!$ matrix defined by $(M_n)_{ij} = s_i^*(s_j)$. Then $\det(M_n) \neq 0$ iff $\{s_i^*\}$ is a basis of $L(S)^*$. Direct computation shows that $\det(M_n) \neq 0$ for $n = 2$ and

3; using the above basis, M_n is respectively

$$M_2 = \begin{pmatrix} 0 & 1 \\ 1 & z \end{pmatrix}, \qquad M_2^{-1} = \begin{pmatrix} -z & 1 \\ 1 & 0 \end{pmatrix}, \qquad \det(M_2) = -1,$$

$$M_3 = \begin{vmatrix} 0 & 0 & 0 & z & 1 & 1 \\ 0 & 0 & 1 & 1+z^2 & z & z \\ 0 & 1 & 0 & 1+z^2 & z & z \\ z & 1+z^2 & 1+z^2 & 3z^2+z^4 & 2z+z^3 & 2z+z^3 \\ 1 & z & z & 2z+z^3 & 1+z^2 & z^2 \\ 1 & z & z & 2z+z^3 & z^2 & 1+z^2 \end{vmatrix},$$

$$M_3^{-1} = \frac{-1}{z^2+4} \begin{vmatrix} -z^4-3z^2+2 & z^3+3z & z^3+3z & z & -z^2-2 & -z^2-2 \\ z^3+3z & 2 & -z^2-2 & -2 & z & z \\ z^3+3z & -z^2-2 & 2 & -2 & z & z \\ z & -2 & -2 & 2 & -z & -z \\ -z^2-2 & z & z & -z & -2 & z^2+2 \\ -z^2-2 & z & z & -z & z^2+2 & -2 \end{vmatrix},$$

$$\det(M_3) = -(z^2+4).$$

Thus $\{s_i^*\}$ is a dual base for $L(S)$ when $n = 2$ or 3.

CONJECTURE. $\det(M_n) \neq 0$, all n.

REMARKS. Since $s_i^*(s_j) \in \mathbf{Z}[z]$, one may similarly define dual elements to $e(L_0(S)) = \mathbf{Z}[S_n]$, by setting $\bar{s}_i^*(s_j) = e(s_i^*(s_j))$ (recall that e has the effect of setting $z = 0$). Then the linear independence of the $\bar{s}_i^*$ is equivalent to the nonvanishing of the determinant of the matrix $\overline{M}_n$ obtained by setting $z = 0$ in the entries of M_n; this in turn implies that $\det(M_n) \neq 0$. However, recalling the fact that

$$\nabla_L(0) = \begin{cases} 0, & L \text{ has more than one component}, \\ 1, & L \text{ a knot}, \end{cases}$$

and noting that
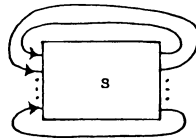
is a knot iff $\pi(s)$ is an n-cycle, we see that

$$M_{ij} = \begin{cases} 1, & \pi_i\pi_j \text{ an } n\text{-cycle}, \\ 0, & \text{otherwise}. \end{cases}$$

Then M is similar to $c = \Sigma(n\text{-cycles})$ under the regular representation in $\mathbf{C}[S_n]$. Using elementary representation theory as in [I] one may in fact show that $\det(M_n) = 0$ for $n \geq 4$. This means that the s_i^* do not form a basis for $\mathbf{Z}[S_n]^*$, although we conjecture that $\{s_i^*\}$ is a basis for $L(S)^*$.

Accordingly, the following discussion holds when $\det(M_n) \neq 0$, and in particular when $n = 2$ or 3.

Define a bilinear form on the vector space $L(S)$ by

$$(s_1, s_2) \overset{\varphi}{\longmapsto} \nabla \left(\text{} \right).$$

φ clearly preserves skein moves and so extends over $L(S)$. Since $\{s_i^*\}$ is a basis of $L(S)^*$, there are elements $a_{k,l}$ of F such that $\varphi = \Sigma a_{k,l} s_k^* \otimes s_l^*$. So

$$\varphi(s_i, s_j) = \sum a_{k,l} s_k^*(s_i) s_l^*(s_j) = \sum a_{k,l} M_{k,i} M_{l,j}$$

$$= \sum_l \left(\sum_k a_{k,l} M_{k,i} \right) M_{l,j}$$

$$= \left((A^T M)^T M \right)_{i,j} = (M^T A M)_{i,j},$$

where $M = M_n$ and $(A)_{k,l} = a_{k,l}$. But

$$\varphi(s_i, s_j) = \nabla \left(\text{} \right) = M_{i,j}$$

and so $M = M^T A M$, i.e., $\varphi = A = (M^T)^{-1}$. So if $\alpha = \Sigma a_i s_i$ and $\beta = \Sigma b_i s_i$, we have

(7)
$$\nabla \left(\text{} \right) = \varphi(\alpha, \beta) = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n!} \end{pmatrix} (M^T)^{-1} (b_1, \ldots, b_{n!}).$$

Special case. $n = 2$. As computed above, using the basis

$$s_{(1)} = \text{} , \qquad s_{(12)} = \text{}$$

we find $M_2 = \begin{pmatrix} 0 & 1 \\ 1 & z \end{pmatrix}$ so that

$$A = \left(M_2^T \right)^{-1} = \begin{pmatrix} -z & 1 \\ 1 & 0 \end{pmatrix}.$$

Thus $\varphi(a, b) = (-z s_{(1)}^* \otimes s_{(1)}^* + s_{(1)}^* \otimes s_{(12)}^* + s_{(12)}^* \otimes s_{(1)}^*)(a, b)$. We claimed before that similar formulas could be derived for any room. As an example, let us make similar computations for the room
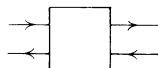
Let s_1 and s_2 be two skein elements of this room; we calculate the bilinear form ψ given by

$$\psi(s_1, s_2) = \nabla\left(\boxed{s_1} \; \boxed{s_2} \right).$$

First define skein elements of our previous room:

$$\sigma_1 = \boxed{s_1} \;, \qquad \sigma_2 = \boxed{s_2} \;.$$

One sees easily that $\psi(s_1, s_2) = \varphi(\sigma_1, \sigma_2)$. Following Conway, we define

$$s^N = \nabla\left(\boxed{s} \right), \qquad s^D = \nabla\left(\boxed{s} \right).$$

As before, N and D define dual elements for this room (N and D stand for numerator and denominator respectively. This notation will become less mysterious shortly).

By drawing links one sees that

$$s^*_{(1)}(\sigma_1) = \nabla\left(\boxed{s_1} \right) = s_1^N,$$

$$s^*_{(12)}(\sigma_1) = \nabla\left(\boxed{s_1} \right) = s_1^D,$$

$$s^*_{(1)}(\sigma_2) = \nabla\left(\boxed{s_2} \right) = s_2^N,$$

$$s^*_{(12)}(\sigma_2) = \nabla\left(\boxed{s_2} \right)$$

$$= \nabla\left(\boxed{s_2} \right) + z\,\nabla\left(\boxed{s_2} \right)$$

$$= s_2^D + z s_2^N.$$

Then by the formula for φ above,

$$\psi(s_1, s_2) = \varphi(\sigma_1, \sigma_2) = -zs^*_{(1)}(\sigma_1)s^*_{(1)}(\sigma_2) + s^*_{(1)}(\sigma_1)s^*_{(12)}(\sigma_2) + s^*_{(12)}(\sigma_1)s^*_{(1)}(\sigma_2)$$
$$= -zs_1^N s_2^N + s_1^N(s_2^D + zs_2^N) + s_1^D s_2^N$$
$$= s_1^N s_2^D + s_1^D s_2^N.$$

Similarly, using the bilinear form defined by

$$(s_1, s_2) \longmapsto \nabla\left(\boxed{s_1} \boxed{s_2} \right)$$
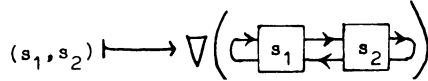
we get

$$\nabla\left(\boxed{s_1} \boxed{s_2} \right) = s_1^D s_2^D .$$

Furthermore, note that a basis for the linearization of this room can be taken to be two elements which Conway calls 0 and ∞:

$$0 = \quad , \qquad \infty = \quad .$$

Note that

$$0^N = 0, \quad \infty^N = 1, \quad 0^D = 1, \quad \infty^D = 0.$$

We continue to abuse notation and write s^N and s^D for the links

$$\boxed{s} \qquad \text{and} \qquad \boxed{s}$$
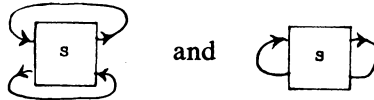
respectively. We also write $s_1 + s_2$ for the skein element

$$s_1 + s_2 = \boxed{s_1} \boxed{s_2} .$$

Summarizing,

PROPOSITION 15.

$$(s_1 + s_2)^N = \nabla\left(\boxed{s_1} \boxed{s_2} \right) = s_1^N s_2^D + s_1^D s_2^N,$$
$$(s_1 + s_2)^D = \nabla\left(\boxed{s_1} \boxed{s_2} \right) = s_1^D s_2^D,$$
$$0^N = 0, \quad \infty^N = 1, \quad 0^D = 1, \quad \infty^D = 0. \quad \square$$

The motivation for Conway's notation is now evident.

REMARK. We may summarize the polynomials of a skein element s by writing the fraction s^N/s^D, so that Proposition 15 allows us to calculate polynomials using arithmetic for fractions. However, we may *not* cancel common terms from numerator and denominator.

DEFINITION. Let L_1 and L_2 be links in S^3 separated by an embedded S^2. A *connected sum* $L_1 \natural L_2$ is a link obtained from $L_1 \cup L_2$ by connect summing a component of L_1 with a component of L_2. Of course, there may be more than one choice of $L_1 \natural L_2$.

COROLLARY 16. *Let L_1 and L_2 be links. Then for any connected sum $L_1 \natural L_2$,*

$$\nabla(L_1 \natural L_2) = \nabla_{L_1} \nabla_{L_2}.$$

(Compare Corollary 11.)

PROOF. Let K_i be the component of L_i used to form the connected sum, let a_i be an arc on K_i, and let b_i be any arc of L_i. Let s_1 and s_2 be the skein elements obtained from L_1 and L_2 by deleting the arcs a_i and b_i so that the endpoints of a_1 appear to the right and those of a_2 appear to the left (see Figure 15). (If necessary, replace the arcs ⌇ by ⌇ to make the orientations match as shown.)
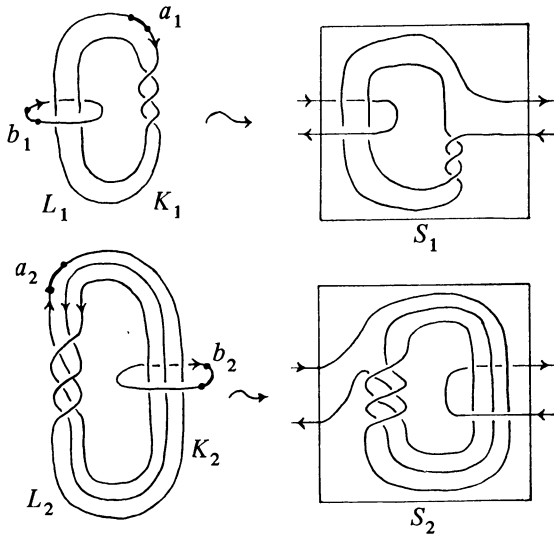
FIGURE 15

Then $s_i^D = L_i$, and

$$\nabla(L_1 \natural L_2) = (s_1 + s_2)^D = s_1^D s_2^D \quad \text{(by Proposition 15)}$$
$$= \nabla_{L_1} \nabla_{L_2}. \quad \square$$

REMARK. Viro [V] points out that the various connected sums of two links are all mutations of one another in the sense of §8 of this paper, and so have identical polynomials.

Application. Let s be a skein element of the room indicated in Figure 16, and let L be the lift of L_0 below to the p-fold cover of S^3 branched along the unknotted component. We can write L as a series of p blocks as in Figure 17 and so $\nabla_L = (s + s + \ldots + s)^N = ps^N(s^D)^{p-1}$ by Proposition 15 and induction.

FIGURE 16

p-times
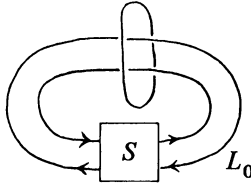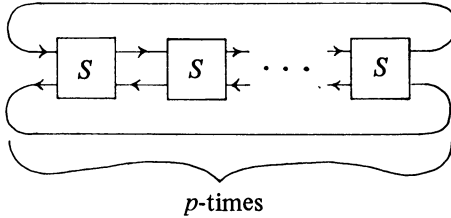
FIGURE 17

6. *More examples.* In the following examples, a and b are integers such that

(8) $$a + b = 1, \qquad b = 2k \geqslant 4.$$

Recall that $\boxed{n}$ stands for n half-twists, and that we will abuse notation as in §3. An asterisk (∗) by a crossing will indicate that a skein move will be performed at that crossing. Note that $\underline{\quad n \quad}$ is equivalent to $\underline{\quad n \quad}$. In all cases the orientation chosen will be such that arrows lead into the top of any box labelled b and out of that labelled a; the consistency of this scheme is ensured in these examples by b being even and a being odd.

(a)

(b)

(c)

by Corollary 16.

(by Proposition 15 and Corollary 16)

(by Example 6(b)) $= 2z \nabla_a \nabla_b + 1$ (by Example 3(b)).

    (d)

by Corollary 16 and the above examples.

    (e)

$$= 2\nabla_a \nabla_b + z(1 + 2z \nabla_a \nabla_b) = 2(z^2 + 1)\nabla_a \nabla_b + z,$$

by Proposition 15 and the above examples; note that the second link is the unknot and that the third link is the first.

    (f)

7. *Another formula, 2-bridge knots, and signatures.* This section is composed of three remarks.

REMARK 1. A similar argument as in §5 using the room
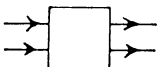
gives the formula $\nabla_{n+m} = \nabla_{n+1}\nabla_m + \nabla_n \nabla_{m+1} + z \nabla_n \nabla_m$.
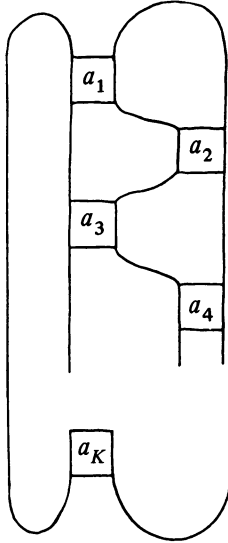
FIGURE 18

REMARK 2. A 2-*bridge knot* is a knot of the form shown in Figure 18. Form the continued fraction

$$p/q = [-a_k, a_{k-1}, -a_{k-2}, \dots, -a_1]$$

$$\left(= -a_k + \cfrac{1}{a_{k-1} + \cfrac{1}{-a_{k-2} + \cdots}} \right).$$

Then it is well known [S] that two such knots giving either p/q or $p/(p-q)$ are isotopic in S^3. The following is not hard to prove by induction.

Fact. Replacing q by $p - q$ if necessary so that we may assume that q is even, p/q has a unique continued fraction expansion with even coefficients, $p/q = [2b_1, 2b_2, \dots, 2b_{2n}]$.

The polynomials of these knots are given by

THEOREM 17 (CONWAY). ∇ *of the above knot is the numerator of* $[-zb_1, zb_2, -zb_3, \dots, zb_{2n}]$ *expressed as a simple fraction, where the* b_i's *are as above.*

SKETCH OF PROOF. By the fact above, we may assume q is even, so

$$p/q = [2b_1, 2b_2, \dots, 2b_{2n}] = [2b_1, 2b_2, \dots, 2b_{2n-1} - 1, 1, - (2b_{2n} + 1)].$$

This last expansion of p/q has an odd number of coefficients and so corresponds to a 2-bridge knot, all of whose a_i's are even except the last three. One may now prove the theorem for these types of knots using Proposition 15 and induction. □

REMARK 3. The *signature of* K, $\sigma(K)$, may be calculated recursively with the following formulas:

(9a) $\sigma(K_l) \leqslant \sigma(K_r) \leqslant \sigma(K_l) + 2,$

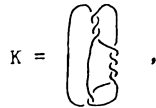(9b) $\text{sign}(\nabla_K(2i)) = i^{\sigma(K)},$

that is,

$$\text{sign}(\nabla_K(2i)) = \text{sign}(\Delta_K(-1)) = \begin{cases} 1, & \text{if } \sigma(K) \equiv 0 \,(\text{mod } 4), \\ -1, & \text{otherwise,} \end{cases}$$

where K_l and K_r are as in previous sections.

Formula (9a) follows from a simple consideration of the Seifert matrices of K_l and K_r as in §1; (9b) follows from [**M3**, p. 400].

To calculate $\sigma(K)$, we use (9b) to determine $\sigma(K)$ mod 4, and (9a) along with a record of the ∇'s to finish the calculation.

EXAMPLE. Let

so that $p/q = [2, -4, -3] = 23/13$. Replacing 13 by $23 - 13 = 10$, we write

$$23/10 = [2, 4, -2, 2] = [2(1), 2(2), 2(-1), 2(1)]$$

so, by Theorem 17, $\nabla_K =$ the numerator of $[-z, 2z, z, z] = -2z^4 - 2z^2 + 1$. Alternatively, we can do skein moves to calculate ∇_K:

So, changing a crossing of K gives the knot
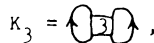
and changing a crossing of this knot gives the unknot; the corresponding ∇'s are $-2z^4 - 2z^2 + 1$, $z^2 + 1$, and 1. We use (9) to compute signatures: $\text{sign}(\nabla_K(2i)) = -1$, and so by (9b), $\sigma(K_3) \equiv 2 \,(\text{mod } 4)$. Similarly, $\sigma(K) \equiv 2 \,(\text{mod } 4)$. But by (9a), the signature can go up by at most 2 at each stage, so that $\sigma(K_3) = 2 = \sigma(K)$.

8. *Mutations, skein equivalence, and the Conway-Kinoshita-Terasaka knots.* Given a link L, we have seen that we can change L to unlinks by a series of skein moves; we write L as $L_1 \oplus L_2$ or $L_1 \ominus L_2$ and continue the process. We can associate a binary tree $T(L)$ to L by drawing a node for each link in the skein decomposition, drawing an edge between the nodes corresponding to L and L_1 and to L and L_2, and writing a $+$ or $-$ at each node depending if a $\oplus$ or $\ominus$ occurred in the decomposition.
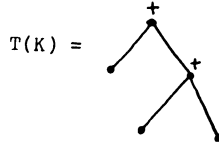
EXAMPLE. Let

So $T(K)$ is formed from
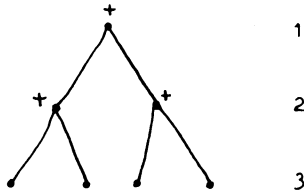

i.e.,

$$T(K) = \quad \text{}$$
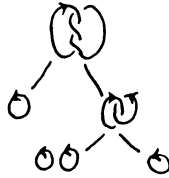
Note that $T(L)$ is by no means unique. For example, since

$$L = L \oplus (L \cup \text{⟲})$$

(see §3a), we can add a branch to any bottom-most node of the tree:

Hence we assume that all trees for the same link L have an identical number $n(T)$ of node levels along any branch; the above tree for the trefoil has 3:


Labelling the nodes at the bottom level of a tree for L as $n_1(L), \ldots, n_k(L)$, we write $\bar{n}_i(L)$ for the link represented by the node $n_i(L)$.

DEFINITION. The two links L_1 and L_2 are said to be *skein equivalent* if there are isomorphic trees for L_1 and L_2 such that, for all i,
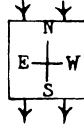
$$\bar{n}_i(L_1) = \bar{n}_i(L_2)$$

where $=$ means isotopic. As we have seen, skein equivalent links have the same Conway polynomials and signatures.

DEFINITION. Suppose the following room appears in a link L:
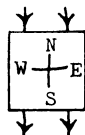
Define a mutation $m(L)$ of L to be the link obtained from L by rotating this room by 180°:

(so the orientation of the strands inside the room are preserved).

REMARK. We could carry out a similar program by rotating in the other direction.

PROPOSITION 18 (CONWAY). *L and $m(L)$ are skein equivalent.*

PROOF. As proved in §5, any skein element can be written as a formula in terms of $s_{(1)}$, $s_{(12)}$, $\oplus$ and $\ominus$. But mutation preserves the orientation of each strand, and hence preserves $\oplus$ and $\ominus$; mutation also clearly preserves $s_{(1)}$ and $s_{(12)}$. Thus by first writing the interiors of the rooms for L and $m(L)$ in terms of $s_{(1)}$ and $s_{(12)}$ and then decomposing the outside of the room by further skein moves, we see that L and $m(L)$ have trees with identical bottom row nodes. □

The agreement of ∇ and σ for skein equivalent links also has a geometric reason:

PROPOSITION 19 (MONTESINOS AND VIRO). *The 2-fold covers M_L and $M_{m(L)}$ of S^3 branched along L and $m(L)$ respectively are homeomorphic.*
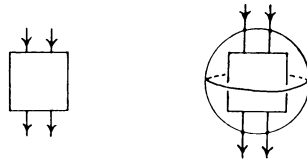

FIGURE 19

PROOF. Let B^3 be a small ball containing the room as shown in Figure 19. Then $S^3 = B \cup (S^3 - B)$, and letting $\hat{A}$ be the 2-fold branched cover of $A \subseteq S^3$ branched along $A \cap L$, we have

$$M_L = \hat{B} \underset{\partial \hat{B} = \hat{S}^2}{\cup} \widehat{S^3 - B}.$$

But $S^2 \cap L$ consists of four points, and we can construct $\hat{S}^2$ as follows. The 2-fold branched cover of a disk along two points a and b is an annulus obtained by gluing together two copies of the disk cut along an arc connecting a and b (see Figure 20). Gluing two of these annuli together along their boundaries gives a torus as the 2-fold cover of S^2 branched on four points, i.e., $\hat{S}^2$.

But now the effect of the mutation is to rotate B by 180°, so that the lift of this rotation to S^2 is a homeomorphism of the torus isotopic to the identity. Thus

$$M_{m(L)} = \hat{B} \underset{h(S^2)}{\cup} \widehat{(S^3 - B)}$$
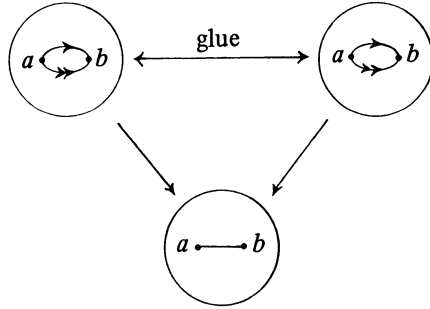
and so is homeomorphic to M_L. □

FIGURE 20

EXAMPLE. The following distinct knots (C for Conway and KT for Kinoshita-Terasaka [KT]) are skein equivalent; in fact they are mutants of each other via the indicated room (see Figure 21).
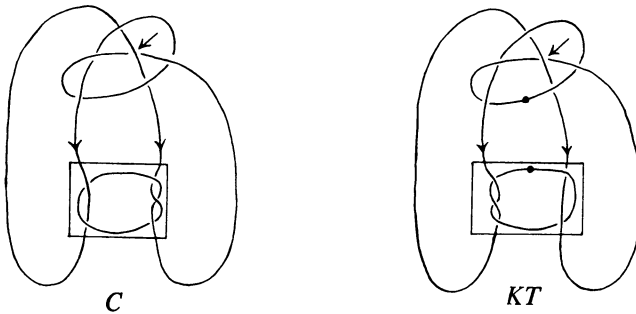

FIGURE 21

REMARKS. These knots can be seen to be distinct by examining representations of their fundamental groups [Ri]. They both have Conway polynomial 1 (in fact, they are the only 11 crossing knots with trivial Alexander polynomial) and both have unknotting number 1. KT is known to be ribbon (the one ribbon move necessary is made by attaching a trivial band between the points on the knot indicated by dots), but C is not known to be slice. The proposition above shows that the 2-fold branched covers of C and KT are homeomorphic (see also [V]), while [M] shows that these covers have nontrivial π_1.

Using the notation in [R], the p-fold branched covers of C and KT can be drawn as in Figure 20 where the framings on all components are -1. For $p = 2$, these pictures are identical as expected (see also [M]). I do not know if they represent the same 3-manifolds for $p > 2$; in fact, since $\nabla(C) = \nabla(KT) = 1$, they are all homology S^3's. Note however that a rotation about the indicated S^2 in the drawing for the cover of C gives that for KT; this S^2 is simply the lift of the original S^2 describing the mutation from C to KT.
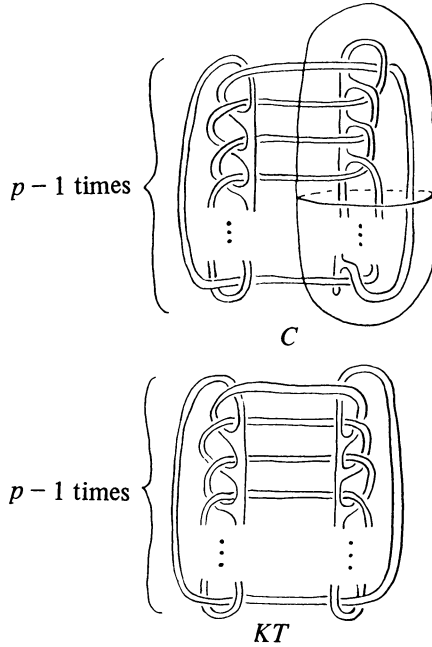
C

KT

FIGURE 22

REMARK. These pictures were obtained using methods in [**R**]. The knots were unknotted at the crossings indicated by arrows by blowing up an unknotted component P of framing -1, drawing the cover of the newly formed unknot, and keeping track of P.

III. The example. The following asymmetric link of two unknotted components K_1 and K_2 with linking number $+1$ satisfies the distinct lifting property for all $p \geqslant 2$ (see Figure 23).
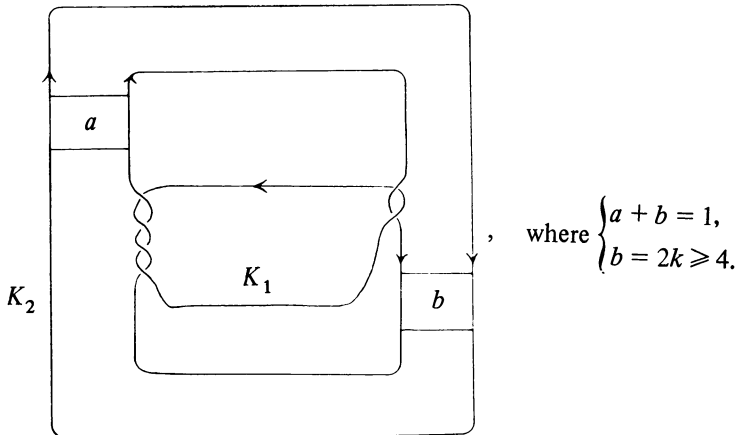

$$\text{where } \begin{cases} a + b = 1, \\ b = 2k \geqslant 4. \end{cases}$$

FIGURE 23

REMARKS. (1) The link shown in Figure 24 is an example of such a family with linking number zero.
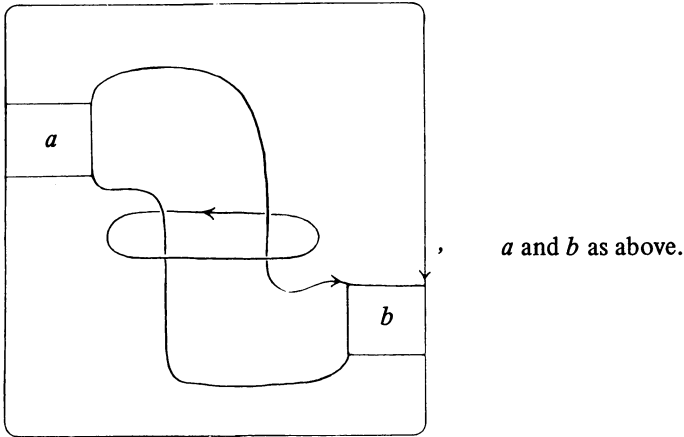

, a and b as above.

FIGURE 24

(2) The case of linking number one is of more interest here because the associated lifts are in fact knots.

Let $\hat{K}_i$ be the lift of K_i to the p-fold cover of K_j, $j \neq i$. We shall distinguish $\hat{K}_1$ from $\hat{K}_2$ by their potential functions, and in particular by showing degree $\nabla \hat{K}_1 <$ degree $\nabla \hat{K}_2$.

PROPOSITION 20. *degree* $\nabla_{K_1} \leqslant 2p$.

PROOF. Consider the punctured torus T which K_1 bounds. Note that K_2 intersects T in one point (see Figure 25).
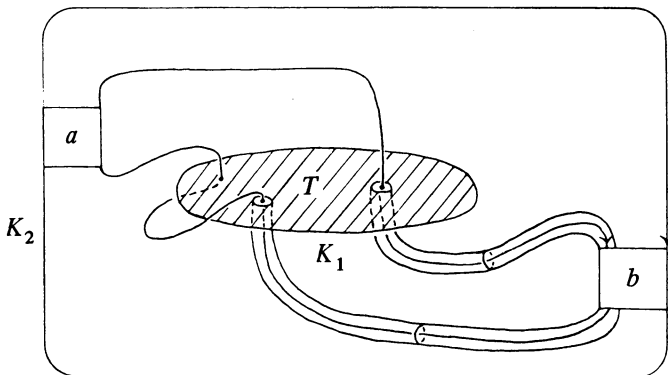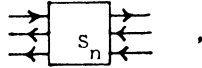

FIGURE 25

Since the linking number of the two components is 1, $\hat{K}_1$ has one component, and so the lift $\hat{T}$ of T to the p-fold cover of K_2 is a connected p-fold cover of T branched on one point. Thus

$$\chi(\hat{T}) = p\chi(T) - (p - 1) = 1 - 2p$$

and so the genus of $\hat{T}$ is p. But $\hat{T}$ is a Seifert surface for $\hat{K}_1$, and so the degree of $\Delta_{K_1}(t)$ as a polynomial in t with constant coefficient is $\leqslant 2p$, i.e., by (6),

$$(10) \qquad\qquad\qquad \text{degree } \nabla_{\hat{K}_1} \leqslant 2p. \quad \square$$

THEOREM 21. *degree* $\nabla_{\hat{K}_2} = 2(p-1)(b-1)$.

COROLLARY 22. *degree* $\nabla_{\hat{K}_2} >$ *degree* $\nabla_{\hat{K}_1}$ *for* $b \geqslant 4,\ p \geqslant 2$.

PROOF. The corollary follows immediately from the theorem and from (10). To prove the theorem, we first draw $\hat{K}_2$ and then use Conway's calculus to calculate its ∇. Cutting along the obvious disk bounded by K_1, we obtain

FIGURE 26

To draw $\hat{K}_2$ we place p of these blocks end-to-end and connect:

We can now use room theory to calculate $\nabla_{\hat{K}_2}$ inductively. First let

$$S_n = \begin{array}{c}\boxed{b}\ \boxed{a}\ \boxed{b}\ \boxed{a}\ \boxed{b}\ \boxed{a}\ \cdots\ \boxed{b}\ \boxed{a}\end{array}.$$

When convenient, we will write this room as

so that

(11)

Further, define skein elements $S_{i,n}$ by

(12)

We shall follow two conventions. First, since the orientations always lead to arrows pointing into the top of the box labeled b and out of that labeled a, we will not always draw them in the pictures. Second, we will use an asterisk $(*)$ as before to indicate a skein move.

From the picture of $\hat{K}_2$ on the previous page, we see that we desire $S_{0,p}^N$. But by (11),

To use Proposition 15 for $(A + B)^N$, we need to know A^N, B^N, A^D, and B^D. Thus we must compute $S_{1,n}^N$ and $S_{1,n}^D$.

$$S_{1,n}^{N} = \left[\boxed{S_n}\right]^{N} = \left[\boxed{S_{n-1}}\,\boxed{b}\,\boxed{a}\right]^{N} = \boxed{S_{n-1}}\,\boxed{b}\,\boxed{a}\;*$$

$$= \boxed{S_{n-1}}\,\boxed{b}\,\boxed{a} \;\oplus\; \boxed{S_{n-1}}\,\boxed{b}\,\boxed{a}$$

$$= \boxed{b}\,\boxed{a}\,\boxed{S_{n-2}}\,\boxed{b}\,\boxed{a} \;\oplus\; \boxed{S_{n-1}}\,\boxed{b}\,\boxed{a}$$

$$= \left[\boxed{b}\,\boxed{a} \;+\; \boxed{S_{n-2}}\,\boxed{b}\,\boxed{a}\right]^{N} \;\oplus\; \boxed{S_{n-1}}\,\boxed{b}\,\boxed{a}$$

$$= \left(\boxed{b}\,\boxed{a} \;+\; S_{0,n-1}\right)^{N} \;\oplus\; S_{0,n}^{N}.$$

$$S_{1,n}^{D} = \left[\boxed{S_n}\right]^{D} = \left[\boxed{S_{n-1}}\,\boxed{b}\,\boxed{a}\right]^{D} = \boxed{S_{n-1}}\,\boxed{b}\,\boxed{a}$$

$$\overset{\text{isotopy}}{=} \boxed{S_{n-1}}\,\boxed{b}\,\boxed{a} = \left[\boxed{S_{n-1}} \;+\; \boxed{b}\,\boxed{a}\right]^{N}$$

$$= \left(S_{2,n-1} \;+\; \boxed{b}\,\boxed{a}\right)^{N}.$$

$$S_{2,n}^{N} = \left[\boxed{S_n}\right]^{N} = \left[\boxed{S_{n-1}}\,\boxed{b}\,\boxed{a}\right]^{N} = \boxed{S_{n-1}}\,\boxed{b}\,\boxed{a}\;*$$

$$= \boxed{S_{n-1}}\,\boxed{b}\,\boxed{a} \;\oplus\; \boxed{S_{n-1}}\,\boxed{b}\,\boxed{a}$$

$$= S_{1,n-1}^{D} \,\#\, \boxed{b}\,\boxed{a} \;\oplus\; S_{1,n}^{D}.$$

$$S_{2,n}^{D} = \left[\boxed{S_n}\right]^{D} = \boxed{S_n} = \left[\boxed{S_n}\right]^{D} = S_{1,n}^{D}.$$

And finally,

$$S^D_{0,n} = \left[\boxed{S_n}\right]^D = \underbrace{\boxed{S_{n-1}}\boxed{b\;a}} = S^D_{1,n-1} \# \boxed{b\;a} .$$

Summarizing,

(13)
$$
\begin{aligned}
S^N_{0,n} &= (S_{1,n-1} + \boxed{b\;a})^N, \\
S^D_{0,n} &= S_{1,n-1} \# \boxed{b\;a}, \\
S^N_{1,n} &= (\boxed{b\;a} + S_{0,n-1})^N \oplus S^N_{0,n}, \\
S^D_{1,n} &= (S_{2,n-1} + \boxed{a\;a})^N, \\
S^N_{2,n} &= (S^D_{1,n-1} \# \boxed{b\;a}) \oplus S^D_{1,n}, \\
S^D_{2,n} &= S^D_{1,n}.
\end{aligned}
$$

Before evaluating ∇ of these links, note that $a + b = 1$ and so, by Lemma 9, $\nabla_a = \nabla_{1-b} = \nabla_{b-1}$. Furthermore, letting $Q = \nabla_b \nabla_{b-1}$, we see that Example 6(a) can be written as

$$(\boxed{b\;a})^D = \nabla_b \nabla_a = \nabla_{b-1} \nabla_b = Q,$$

$$(\boxed{b\;a})^N = 1.$$

Note that, by Lemma 9, degree $Q = 2b - 3$.

So, using our formulas for $\oplus$, $\#$, $(A + B)^N$ and $(A + B)^D$, we can eliminate the S_2's and rewrite (16) as

(14)
$$
\begin{aligned}
S^N_{0,p} &= QS^N_{1,p-1} + S^D_{1,p-1}, \qquad S^D_{0,p} = QS^D_{1,p-1}, \\
S^N_{1,p-1} &= S^D_{0,p-2} + QS^N_{0,p-2} + zS^N_{0,p-1}, \\
S_{1,p-1} &= Q^2 S^D_{1,p-3} + zQS^D_{1,p-2} + S^D_{1,p-2}.
\end{aligned}
$$

The examples in §6 can also be written in terms of Q:

(6c) $\qquad\qquad S^N_{0,2} = 2zQ + 1,$

(6d) $\qquad\qquad S^D_{0,2} = zQ^2 + Q,$

(6e) $\qquad\qquad S^N_{1,2} = 2(z^2 + 1)Q + z,$

(6f) $\qquad\qquad S^D_{1,2} = Q^2(1 + z^2) + 2zQ + 1,$

(6b) $\qquad\qquad S^D_{1,1} = zQ + 1.$

We now use these recursion formulas to gain a formula for degree $\nabla_{\hat{K}_2}$.

PROPOSITION 23. *Consider the polynomials in* (14) *as polynomials in Q with coefficients in $\mathbf{Z}[z]$. Then for each of these polynomials,*

(1) *The term containing the highest power of Q has coefficient containing the highest power of z; hence this term is of highest degree if considered as a polynomial in z alone.*

(2) *These distinguished terms are as follows*:

polynomial	term
$S_{0,p}^N$	$Q^{p-1}[p-1]$
$S_{0,p}^D$	$Q^p[p-1]$
$S_{1,p}^N$	$Q^{p-1}[p]$
$S_{1,p}^D$	$Q^p[p]$

where we are writing $[n]$ *for a polynomial in z of degree n with nonnegative coefficients.*

PROOF. This is a straightforward induction using (14) and the examples in §6. The proposition is true for $p = 2$ by the examples. We give details only for the case of $S_{0,p}^N$. Assume true for $k \leqslant p$; then by (14),

$$S_{0,p+1}^N = QS_{1,p}^N + S_{1,p}^D$$
$$= Q(Q^{p-1}[p]) + Q^p[p] + \text{lower order terms}$$
$$= Q^p[p] + \text{lower order terms}$$

since the coefficients in $[p]$ are nonnegative.

Clearly, (1) and (2) are still satisfied by $S_{0,p+1}^N$. The other cases follow similarly. □

So we finally have our

THEOREM 24. $\deg \nabla_{\hat{K}_2} = \deg S_{0,p}^N = 2(p-1)(b-1)$.

PROOF. By Proposition 23,

$$\deg S_{0,p}^N = p - 1 + \deg Q^{p-1}$$
$$= p - 1 + (p-1)(2b-3) = 2(p-1)(b-1). \quad \square$$

REMARKS. (1) As mentioned before, a similar link provides an example of such a family with linking number zero (the proof is entirely similar to that given above).

(2) Murasugi [M2] gives formulas for the polynomials and signatures of $\hat{K}_1$ and $\hat{K}_2$ in terms of those of the link; in our case, the signatures are identical.

(3) Drawing $\hat{K}_1$ is possible but messy.

We now show that, as promised, we can produce for each $p \geqslant 2$ an infinite family of pairs of distinct knots with homeomorphic p-fold covers.

LEMMA 25. *The p-fold cyclic covers of S^3 branched along $\hat{K}_1$ and $\hat{K}_2$ are homeomorphic.*

PROOF. Let M be the p-fold cover of $S^3 - L$ corresponding to the map φ:

$$\pi_1(S^3 - L) \xrightarrow{\pi} H_1(S^3 - L) = \mathbf{Z}_1 \oplus \mathbf{Z}_2 \to \mathbf{Z}_1 \to \mathbf{Z}/p$$
$$\underbrace{\hspace{7cm}}_{\varphi}$$

where $\mathbf{Z}_i \cong \mathbf{Z}$. Then $\pi_1(M) = G \subseteq \pi_1(S^3 - L)$ is the kernel of φ and M is the complement of $\hat{K}_2$ in the p-fold cyclic cover of $S^3 - K_1$.

Now, $\pi(G) \supseteq (0, \mathbf{Z}_2)$, so there is a homeomorphism $G \to \mathbf{Z}_2$. Let $\overline{M}$ be the p-fold cover of M corresponding to the map $G \to \mathbf{Z}_2 \overset{P_2}{\to} \mathbf{Z}/p$. Then $\overline{M}$ is a p^2-fold cover of $S^3 - L$. For let $\pi_1(\overline{M}) = \overline{G} \subseteq G$. We have a commutative diagram:

$$
\begin{array}{ccccccc}
G & \hookrightarrow & \pi_1(S^3 - L) & \to & \mathbf{Z}_1 \oplus \mathbf{Z}_2 & \to & \mathbf{Z}_1 \overset{P_1}{\to} \mathbf{Z}/p \\
\cup| & & \cup| & & & & \\
\overline{G} & \to & G & \to & \mathbf{Z}_2 & \overset{P_2}{\to} & \mathbf{Z}/p
\end{array}
$$

Note that $\pi_1(S^3 - L)/\overline{G} \cong \pi_1(S^3 - L)/G \oplus G/\overline{G} = \mathbf{Z}/p \oplus \mathbf{Z}/p$, and that $\overline{M}$ is a cover of $S^3 - L$ corresponding to the map

$$
\pi_1(S^3 - L) \to H_1(S^3 - L) \overset{P_1 \oplus P_2}{\to} \mathbf{Z}/p \oplus \mathbf{Z}/p.
$$

But the same map would be obtained had we started with the cover of $S^3 - K_2$ and formed the corresponding p^2-fold cover of $S^3 - L$. $\square$

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