TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 273, Number 2, October 1982

$C(\alpha)$ PRESERVING OPERATORS ON $C(K)$ SPACES

BY

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ABSTRACT. Let A: $C(K) \to X$ be a bounded linear operator where K is a compact Hausdorff space and X is a separable Banach space. Sufficient conditions are given for A to be an isomorphism (into) when restricted to a subspace Y of $C(K)$, such that Y is isometrically isomorphic to a space $C(\alpha)$ of continuous functions on the space of ordinal numbers less than or equal to the countable ordinal α.

1. For a compact Hausdorff space K, $C(K)$ will denote the Banach space of continuous real-valued functions on K with the supremum norm. The Greek letters α, β and γ will be reserved for countable ordinal numbers. We consider the set $[0, \alpha] = \{\gamma : 0 \leqslant \gamma \leqslant \alpha\}$ to have the order topology and denote $C([0, \alpha])$ by $C(\alpha)$. Also $C_0(\alpha)$ will denote the subspace of $C(\alpha)$ of all functions vanishing at α. The letters X and Y will denote real Banach spaces.

In this paper we give conditions on a bounded linear operator A: $C(K) \to X$, provided that X is separable, which guarantee that there is a subspace Y of $C(K)$ with Y isometric to $C_0(\alpha)$ such that A is an isomorphism (i.e. a homeomorphism into) when restricted to Y.

To state these conditions on A we define a "Szlenk type" index for operators on a $C(K)$ space. Throughout this paper B will denote a weak* metrizable bounded set of measures in $C(K)^*$. (We identify the dual space $C(K)^*$ of $C(K)$ with the regular Borel measures on K.) The notation $\mu_n \to \mu$ will always mean that the sequence of measures $\{\mu_n\}$ converges to μ in the weak* topology.

Suppose $1 \geqslant \lambda > 0$. For each countable ordinal α we will define a family $P_\alpha(\lambda, B)$ of open subsets G and associated measures $\mu_G \in B$. Let $P_0(\lambda, B) = \{(G, \mu_G)$: G is an open set in K, $\mu_G \in B$ and $|\mu_G|(G) \geqslant \lambda\}$. If $\alpha = \beta + 1$, then $P_\alpha(\lambda, B) = \{(G, \mu_G)$: $(G, \mu_G) \in P_0(\lambda, B)$ and there is a disjoint sequence $\{G_n\}$ of open subsets of G and associated measures μ_{G_n} such that $(G_n, \mu_{G_n}) \in P_\beta(\lambda, B)$, $\mu_{G_n} \to \mu_G$ and $\overline{\bigcup_n G} \subset G\}$. If α is a limit ordinal, then $P_\alpha(\lambda, B) = \{(G, \mu_G) \in P_0(\lambda, B)$: there is a disjoint sequence of open subsets $\{G_n\}$ of G with associated measures μ_{G_n} such that $\mu_{G_n} \to \mu_G$, $\overline{\bigcup_n G_n} \subset G$ and $(G_n, \mu_{G_n}) \in P_{\alpha_n}(\lambda, B)$ where $\alpha_n \nearrow \alpha\}$. The notation $\alpha_n \nearrow \alpha$ means that α_n is an increasing sequence of ordinals with $\alpha = \sup_n \alpha_n$. A simple example illustrating the sets $P_\alpha(\lambda, B)$ is contained in §4.

Received by the editors March 20, 1979 and, in revised form, July 30, 1981.

1980 *Mathematics Subject Classification*. Primary 46E15.

[1]Research on this paper was partially supported by the National Science Foundation grant number MCS 77-01690.

Let ω denote the first nonfinite ordinal. The main result of the paper is the following:

THEOREM A. *Suppose K is a compact Hausdorff space, $1 \geqslant \lambda > 0$, and $\gamma = \omega^\alpha$ where α is a countable ordinal. Let A: $C(K) \to X$ be a bounded linear operator and assume $B = \{A^*(x^*)\colon \|x^*\| \leqslant 1\}$ is weak* metrizable. (This condition on B holds if X is separable.) Assume* (1) *there is a sequence γ_n of ordinals with $\gamma_n \nearrow \gamma$ and a disjoint sequence $\{G_n\}$ of open sets with $(G_n, \mu_{G_n}) \in P_{\gamma_n}(\lambda, B)$ for some associated measures μ_{G_n}. Then there is a subspace Y of $C(K)$ with Y isometric to $C_0(\omega^\gamma)$ such that A restricted to Y is an isomorphism.*

We note that assumption (1) in Theorem A is implied by the condition $P_\gamma(\lambda, B) \neq \varnothing$. Also, if (1) is replaced by the assumption that $P_\gamma(\lambda, B) \neq \varnothing$, then the subspace Y in Theorem A can be chosen to be isometric to $C(\omega^\gamma)$.

The author is indebted to Dale Alspach (whose paper [**1**] gave rise to this paper) for many helpful conversations. His paper [**2**] contains a converse to Theorem A and also shows that the requirement that the sets $\{G_n\}$ be disjoint in assumption (1) is unnecessary.

Theorem A and Alspach's converse [**2**, Corollary 0.5] combine to give

THEOREM B. *Let A: $C(K) \to X$ be a bounded linear operator where K is a compact Hausdorff space and X is a separable Banach space. Let $\gamma = \omega^\alpha$. Then there is a subspace Y of $C(K)$ with Y isometric to $C_0(\omega^\gamma)$ and such that T restricted to Y is an isomorphism if and only if, for some $\lambda > 0$, $P_\beta(\lambda, B) \neq \varnothing$ for all $\beta < \gamma$ where $B = \{A^*(x^*)\colon \|x^*\| \leqslant 1\}$.*

The assumption that γ be of the form ω^α is essential in Theorem A; however, it is isomorphically unimportant since for any countable ordinal β there is an ordinal α such that $C(\beta)$ is isomorphic to $C(\omega^\gamma)$ where $\gamma = \omega^\alpha$ (see [**6**]).

It is hoped that Theorem A and the techniques of this paper will contribute to a solution of the following folklore question: if Y is a complemented subspace of $C(S)$ where S is a compact metric space, is Y isomorphic to $C(T)$ for some compact metric space T? Via the work of Rosenthal [**14**] and Pelczynski [**11**] (see also Benyamini [**5**]) this question has been reduced to the following:

Question. Let K be a compact metric space and let P: $C(K) \to Y \subset C(K)$ be a bounded linear projection such that Y^* is separable. Then is there a countable ordinal α such that (a) *Y contains a subspace isomorphic to $C(\alpha)$ and* (b) *Y is isomorphic to a complemented subspace of $C(\alpha)$?*

Theorem A provides a method for computing the largest ordinal α satisfying condition (a) of this question.

If Y is a subspace of X and B is a bounded subset of X^*, we say that B *norms* Y if there is a constant $\lambda > 0$ such that, for each $y \in Y$, $\sup\{|x^*(y)| : x^* \in B\} \geqslant \lambda \|y\|$. It is a simple observation that if A: $X \to Z$ is a bounded linear operator and Y is a subspace of X, then A restricted to Y is an isomorphism iff $\{A^*z^*\colon \|z^*\| \leqslant 1\}$ norms Y. Using this observation, Theorem A will be proved in §3 using the following:

THEOREM 1. *Let K be a compact Hausdorff space and let B be a symmetric weak* metrizable set of measures in the unit ball of $C(K)^*$. Let $1 \geqslant \lambda > 0$ and suppose N is an integer with $N > \frac{2}{\lambda}$. If $P_\gamma(\lambda, B) \neq \varnothing$ and if $\beta \cdot N \leqslant \gamma$, then there is a subspace Y of $C(K)$ which is normed by B such that Y is isometric to $C(\omega^\beta)$.*

In the next section the notion of a γ family of sets with measures is introduced and used to prove Theorem 1. In §3, Theorem A is derived from Theorem 1 using some of the techniques of §2. §4 has some remarks on the relationship between the sets $P_\alpha(\lambda, B)$ and the sets used in defining the Szlenk index of an operator in [**1**]. The final section shows that the theorem of Alspach and Benyamini [**17**] and Billard [**18**] on the primariness of the spaces $C(\alpha)$ for α a countable ordinal can be obtained as a corollary to the techniques of this paper.

2. γ families of sets and the proof of Theorem 1. The proof of Theorem 1 is conceptually quite simple but seems to be unavoidably technical. The proof is broken down into six steps which are outlined here. It is hoped that by referring back to this outline, the reader will be able to maintain a sense of direction as he goes through the proof.

Throughout, K will be a compact Hausdorff space, $\lambda > 0$ and B a subset of $C(K)^*$.

Step I. We begin by introducing the central concept of a γ family of sets in K with λ measures taken from B and show that if $P_\alpha(\lambda, B) \neq \varnothing$, then such a family of sets and measures must exist (Proposition 2.2). The key proposition, 2.6, is then proved, which gives three conditions (denoted $(*)$, $(**)$ and $(***)$) on a γ family with λ measures taken from B which guarantee that $C(K)$ has a subspace isometric to $C(\omega^\alpha)$ which is normed by B.

Steps II–V describe four operations on γ families with λ measures which yield new families having additional properties.

Step II. In this step it is shown that an arbitrary γ family with λ measures can be easily trimmed down (by throwing away some sets) so that what is left is still a γ family with λ measures and, in addition, condition $(**)$ is satisfied.

Step III. In this ugly and technical step it is shown (Proposition 2.8) that, given $\varepsilon > 0$, an arbitrary γ family with λ measures can be reorganized so that it has the additional property of being "towered" (defined below) and the λ measures become $\lambda - \varepsilon$ measures.

Step IV. Proposition 2.13 shows how, from a given towered γ family with λ measures, one can pick out a towered β family (β not much smaller than γ) with λ measures which, in addition, satisfies condition $(*)$. The argument used in this step only works for towered families and this is the reason the lengthy Step III had to be included.

Step V. Lemma 2.15 shows how, given $\varepsilon > 0$, a γ family with λ measures satisfying condition $(*)$, can be adjusted to give a γ family with $(\frac{\lambda}{2} - 2\varepsilon)$ measures also satisfying $(*)$ and, in addition, satisfying condition $(***)$.

Step VI. The proof of Theorem 1 is now a matter of putting together the previous steps.

STEP I. The central concept of this section is the following:

DEFINITION. Let γ be a countable ordinal. Then a family $\mathfrak{F}$ of nonempty open subsets of K is a γ *family* if for each α with $0 \leqslant \alpha \leqslant \gamma$ there is a subfamily $\mathfrak{F}_\alpha$ of $\mathfrak{F}$ such that $\mathfrak{F}$ has the following six properties:

(1) $\mathfrak{F} = \bigcup_{\alpha \leqslant \gamma} \mathfrak{F}_\alpha$ and if G_1 and G_2 are in $\mathfrak{F}$, then $G_1 \cap G_2 = \varnothing$ or $G_1 \subset G_2$ or $G_2 \subset G_1$.

(2) $\mathfrak{F}_\gamma$ is a single set which we denote by G_γ and, if $G \in \mathfrak{F}$, then $G \subset G_\gamma$, and $\overline{\bigcup \{H \in \mathfrak{F}: H \subset G, H \neq G\}} \subset G$.

(3) If $\alpha < \gamma$, $\mathfrak{F}_\alpha$ is an infinite family of disjoint open sets.

(4) If $G \in \mathfrak{F}_\beta$ and $\alpha < \beta \leqslant \gamma$, the set $\{H: H \subset G \text{ and } H \in \mathfrak{F}_\alpha\}$ is infinite.

(5) If $G \in \mathfrak{F}_{\beta+1}$, then there is a sequence G_n of disjoint subsets in $\mathfrak{F}_\beta$ such that $\overline{\bigcup G_n} \subset G$.

(6) If $G \in \mathfrak{F}_\beta$ and β is a limit ordinal, then there is a sequence $\{G_n\}$ of disjoint subsets of G such that $G_n \in \mathfrak{F}_{\beta_n}$ and β_n is an increasing sequence of ordinals with $\beta = \sup_n \beta_n$ and $\overline{\bigcup G_n} \subset G$.

This horrid definition is motivated by the following example, which we will call the *standard* γ *family* in $K = [0, \omega^\gamma]$. Let K^α denote the αth derived set of K. For $\beta_1 \in K^{\alpha+1} \setminus K^\alpha$, define $\beta_0 = \sup\{\beta \in K^{\alpha+1} \setminus K: \beta < \beta_1\}$. (If β_1 is the smallest member of $K^{\alpha+1} \setminus K^\alpha$ let $\beta_0 = 0$.) Then $\mathfrak{F}_\alpha$ consists of all sets of the form $(\beta_0, \beta_1]$ for $\beta_1 \in K^{\alpha+1} \setminus K^\alpha$. It is routine to check that $\mathfrak{F} = \bigcup_{\alpha \leqslant \gamma} \mathfrak{F}_\alpha$ is a γ family of sets in K.

This notion of a γ family of sets will be used to construct subspaces of $C(K)$ which are isometric to $C(\omega^\alpha)$. In order to get normed subspaces we need the following additional structure.

If $1 \geqslant \lambda > 0$, a γ family $\mathfrak{F}$ is called a γ *family with* λ *measures* if for each $G \in \mathfrak{F}$ there is a measure μ_G such that:

(5′) If $G \in \mathfrak{F}_{\beta+1}$, then there is a sequence $\{G_n\}$ of disjoint subsets of G with each $G_n \in \mathfrak{F}_\beta$, $\overline{\bigcup G_n} \subset G$ and $\mu_{G_n} \to \mu_G$.

(6′) If $G \in \mathfrak{F}_\beta$, where β is a limit ordinal, then there is a sequence $\{G_n\}$ of disjoint subsets of G such that $G_n \in \mathfrak{F}_{\beta_n}$, $\beta_n \nearrow \beta$, $\overline{\bigcup G_n} \subset G$ and $\mu_{G_n} \to \mu_G$.

(7) For each $G \in \mathfrak{F}$, $|\mu_G|(G) \geqslant \lambda$.

The standard γ family $\mathfrak{F}$ of sets on $[0, \omega^\gamma]$ is made into a γ family with 1 measures if for each set $(\beta_0, \beta_1] \in \mathfrak{F}$, we associate the point mass μ_{β_1}.

Our objectives in STEP I are first to show that if $P_\gamma(\lambda, B) \neq \varnothing$, then there exists a γ family of sets with λ measures taken from B (Proposition 2.2) and, secondly, to establish three conditions (Proposition 2.6) on a γ family with λ measures which are sufficient to imply the existence of a subspace of $C(K)$ isometric to $C(\omega^\gamma)$ which is normed by the measures associated with the γ family of sets.

We establish some notation and operations on γ families with λ measures which will be used repeatedly in the following. Let $\mathfrak{F}$ be a γ family with λ measures. A subset $\mathfrak{F}'$ of $\mathfrak{F}$ is a γ *subfamily with* λ *measures* if it is also a γ family with the same measures and $\mathfrak{F}'_\alpha = \mathfrak{F}_\alpha \cap \mathfrak{F}'$. If $G \in \mathfrak{F}_\alpha$, then $\mathfrak{F} | G$ is an α family with λ measures in an obvious way.

In many of the transfinite induction arguments which follow, the successor and limit ordinal cases will use parts (a) and (b), respectively, of the next lemma. The proof, which is an exercise with definitions, is omitted.

LEMMA 2.1. *Let $\{G_n\}$ be a sequence of disjoint open subsets of the open set G and $\bigcup G_n \subset G$. Suppose μ_G is a measure with $|\mu_G|(G) \geqslant \lambda$.*

Let $\mathfrak{F}_n$ be a γ_n family with λ measures such that $(\mathfrak{F}_n)_{\gamma_n} = G_n$ and $\mu_{G_n} \to \mu_G$. Suppose that either

(a) $\gamma = \beta + 1$ *and* $\gamma_n = \beta$ *for each* n, *or*

(b) $\gamma_n \nearrow \gamma$.

Let $\mathfrak{F} = \{G\} \cup (\bigcup_{n=1}^{\infty} \mathfrak{F}_n)$ and let $\mathfrak{F}_\alpha = \bigcup_n (\mathfrak{F}_n)_\alpha$ for $\alpha < \gamma$ and $\mathfrak{F}_\gamma = \{G\}$. Then $\mathfrak{F}$ is a γ family with λ measures.

PROPOSITION 2.2. *Suppose $1 \geqslant \lambda > 0$ and $(G, \mu_G) \in P_\gamma(\lambda, B)$. Then there is a γ famiiy $\mathfrak{F}$ with λ measures such that if $H \in \mathfrak{F}$ then $\mu_H \in B$ and $G = G_\gamma$ and $\mu_G = \mu_{G_\gamma}$.*

We omit the proof, which is a simple transfinite induction argument using Lemma 2.1.

We now show how a γ family of sets in K gives a subspace isometrically isomorphic to $C(\omega^\gamma)$.

For a compact metric space S, the derived sets S^α for each countable ordinal α are defined by transfinite induction as follows: Let $S^0 = S$ and if $\alpha = \beta + 1$, $s \in S^\alpha$ if there is a sequence $s_n \in S^\beta$ with $s = \lim s_n$. If α is a limit ordinal, $S^\alpha = \bigcup_{\beta<\alpha} S^\beta$ or, equivalently, $s \in S^\alpha$ if there is a sequence $\{s_n\}$ with $s_n \to s$, $s_n \in S^{\beta_n}$ and $\beta_n \nearrow \beta$.

For a set G, χ_G will denote the characteristic function of G. If $\mathfrak{F}$ is a γ family of sets, $A(\mathfrak{F})$ will denote the normed closed Banach algebra of functions (in the space of bounded functions on K) generated by the family $\{\chi_G\colon G \in \mathfrak{F}\}$. Then $A(\mathfrak{F})$ is isometrically and algebraically isomorphic to $C(T)$ for some compact metric space T. (T is metrizable since $A(\mathfrak{F})$ is separable.)

PROPOSITION 2.3. *Suppose $\mathfrak{F}$ is a γ family of sets. Then $A(\mathfrak{F})$ has a subspace isometric to $C(\omega^\gamma)$.*

This proposition follows immediately from the next two lemmas.

LEMMA 2.4. *Suppose $\mathfrak{F}$ is a γ family of sets. Then $T^\gamma \neq \varnothing$ where $C(T) = A(\mathfrak{F})$.*

PROOF. For each $G \in \mathfrak{F}$, the image of χ_G in $C(T)$ is a characteristic function of some closed and open set which we denote by $\hat{G}$. If $G_1 \subset G_2$ or $G_1 \cap G_2 = \varnothing$ then, respectively, $\hat{G}_1 \subset \hat{G}_2$ or $\hat{G}_1 \cap \hat{G}_2 = \varnothing$. The lemma follows from the next claim.

Claim. If $\alpha \leqslant \gamma$ and $G \in \mathfrak{F}_\alpha$, then $\hat{G} \cap T^\alpha \neq \varnothing$.

If $\alpha = 0$ the claim is clear since $\hat{G} \subset T^0$. If $\alpha = \beta + 1$ and $G \in \mathfrak{F}_\alpha$, then by (5) there is a sequence $\{G_n\}$ of disjoint open sets with $G_n \subset G$ and $G_n \in \mathfrak{F}_\beta$. By induction we can choose $t_n \in \hat{G}_n \cap T^\beta$. Then any limit point of the distinct sequence $\{t_n\}$ is in $\hat{G} \cap T^\alpha$. The proof, if α is a limit ordinal, is similar and the lemma is proved.

LEMMA 2.5. *Let S be a compact metric space and suppose that $S^\alpha \neq \varnothing$. Then $C(S)$ contains a subspace isometric to $C(\omega^\alpha)$.*

PROOF. We give a sketchy proof of this well-known fact. First, if S is uncountable, then S contains a subset homeomorphic to the Cantor set Δ [**8**, p. 158]. Thus $C(\Delta)$ is isometric to a subspace of $C(S)$ via an extension operator [**12**, Theorem 6.6]. A

simple argument shows that any totally disconnected compact metric space is homeomorphic to a subset of Δ. Thus $[0, \omega^\alpha] \subset \Delta$ and an extension operator gives $C(\omega^\alpha)$ isometric to a subspace of $C(\Delta)$ and thus $C(S)$.

On the other hand, suppose S is countable. Let β be the last ordinal such that $S^\beta \neq \varnothing$ and suppose S^β has n points. Then S is homeomorphic to $[0, \omega^\beta n]$ [**10**, p. 21; **9**, p. 103 or **4**, Theorem 2]. Since $S^\alpha \neq \varnothing$, $\omega^\alpha \leqslant \omega^\beta n$ and thus $[0, \omega^\alpha]$ is a subset of $[0, \omega^\beta n] = S$. An extension operator gives $C(\omega^\alpha)$ isometric to a subspace of $C(S)$ and the lemma is proved.

The next proposition, which completes STEP I, gives three conditions on a γ family of sets with λ measures taken from B, guaranteeing that a subspace isometric to $C(\omega^\gamma)$ is normed by B.

PROPOSITION 2.6. *Let B be a set of measures from the unit ball of $C(K)^*$. Let $\mathfrak{F}$ be a γ family with λ measures taken from B and suppose $1 \geqslant \lambda > 5\varepsilon > 0$. Suppose that, for each $G \in \mathfrak{F}$,*

(*) $|\mu_G|(G_\gamma \setminus G) \leqslant \varepsilon$ *and*

(**) $|\mu_G|(\bigcup\{H \in \mathfrak{F}: H \subset G \text{ and } H \neq G\}) \leqslant \varepsilon$ *and*

(***) $\mu_G(G) \geqslant \lambda$.

Then there is a subspace of $C(K)$ which is isometric to $C(\omega^\gamma)$ which is $(\lambda - 5\varepsilon)$-normed by B.

PROOF. For each $G \in \mathfrak{F}$ choose a closed set F with $\bigcup\{H \in \mathfrak{F}: H \subset G, H \neq G\} \subset F \subset G$ and $|\mu_G|(G \setminus F) \leqslant \varepsilon$. Choose $f_G \in C(K)$ such that $f_G = 1$ on F, $f_G = 0$ outside G and $0 \leqslant f_G \leqslant 1$. The desired subspace is the closed linear span of $\{f_G: G \in \mathfrak{F}\}$ which we denote by Y. We will establish that the correspondence $\chi_G \to f_G$ for $G \in \mathfrak{F}$ uniquely determines an isometry of $A(\mathfrak{F})$ onto Y and that Y is normed by B.

Step 1. $\|\Sigma_{i=1}^n a_i \chi_{G_i}\| = \|\Sigma_{i=1}^n a_i f_{G_i}\|$ for all finite collections $\{G_i\}$ from $\mathfrak{F}$ and real numbers a_i.

Let $h = \Sigma_{i=1}^n a_i \chi_{G_i}$ and $Th = \Sigma_{i=1}^n a_i f_{G_i}$. Let $x \in K$. Since for $i \neq j$, $G_i \cap G_j = \varnothing$, $G_i \subset G_j$ or $G_j \subset G_i$, we may reorder the sets $\{G_i\}$ such that $x \in G_1 \subset G_2 \subset \cdots \subset G_k$ and $G_i \cap G_k \neq 0$ for $k < i \leqslant n$. Then $h(x) = a_1 + \cdots + a_k$. Also $Th(x) = a_1 f_{G_1}(x) + a_2 + \cdots + a_k$ since $f_{G_i}(x) = 1$ for $2 \leqslant i \leqslant k$. By choosing some $y \in G_1$ such that $f_{G_1}(y) = 1$, we have $Th(y) = h(x)$ and thus $\|Th\| \geqslant \|h\|$. For the opposite inequality assume that $|Th(x)| = a_1 f_{G_1}(x) + a_2 + \cdots + a_n$. (A similar argument works for the other case.) If $a_1 \geqslant 0$ then $h(x) \geqslant |Th(x)|$ since $0 \leqslant f_{G_1}(x) \leqslant 1$. On the other hand, if $a_1 < 0$, we can choose a $y \in G_2 \setminus \bigcup_{i \neq 2} G_i$ and then $h(y) = a_2 + \cdots + a_k \geqslant |Th(x)|$. Thus $\|h\| \geqslant \|Th\|$.

Step 2. We next show that $A(\mathfrak{F})$ is the closed linear span of $\{\chi_G: G \in \mathfrak{F}\}$. Let $\mathcal{Q}$ be the smallest family of sets containing $\mathfrak{F}$ which is closed under finite unions, intersections and differences. Then $A(\mathfrak{F})$ is the closed linear span of $\{\chi_H: H \in \mathcal{Q}\}$. A technical but elementary argument shows that if $H \in \mathcal{Q}$ then $H = \bigcup_{i=1}^n X_i$, where each X_i is of the form $G_0 \setminus \bigcup_{j=1}^m G_j$ where $G_j \in \mathfrak{F}$ for $0 \leqslant j \leqslant m$ and $G_j \subset G_0$ for $1 \leqslant j \leqslant m$. (The only property of $\mathfrak{F}$ used here is that if A and B are in $\mathfrak{F}$ then $A \cap B = \varnothing$, $A \subset B$ or $B \subset A$.) Step 2 now follows and we have an isometry of $A(\mathfrak{F})$ onto Y.

Step 3. It remains to show that B norms Y.

Let $h = \sum_{i=1}^{n} a_i f_{G_i}$. Arguments as in Step 1 show that a reordering of the sets $\{G_i\}$ gives a sequence $G_1 \subset G_2 \subset \cdots \subset G_k$ such that $\|h\|$ either equals $|a_1 + \cdots + a_k|$ or $|a_2 + \cdots + a_k|$. Leaving the other cases to the reader, we finish the proof assuming that $\|h\| = a_2 + \cdots + a_k$. Let $A = \bigcup\{H \in \mathfrak{F}: H \subset G_2, H \neq G_2\}$ and let F be the closed set used in defining f_{G_2}. Then $A \subset F \subset G$ and since $f_{G_2} = 1$ on F, $h = a_2 + \cdots + a_k = \|h\|$ on $F \setminus A$. So

$$\begin{aligned}\int h\, d\mu_{G_2} &\geqslant \int_{F\setminus A} h\, d\mu_{G_2} - \|h\|\left[\,|\mu_{G_2}|\left\{(G_\gamma \setminus G_2) \cup (G_2 \setminus F) \cup A\right\}\right] \\ &\geqslant \|h\|\left(\mu_{G_2}(F\setminus A) - 3\varepsilon\right) \geqslant \|h\|\left(\mu_{G_2}(G_2) - |\mu_{G_2}|\left\{(G_2\setminus F) \cup A\right\} - 3\varepsilon\right) \\ &\geqslant \|h\|(\lambda - 5\varepsilon)\end{aligned}$$

and Y is $(\lambda - 5\varepsilon)$-normed by B. This finishes the proof.

In the following four steps we show how the crude γ family of sets with λ measures, whose existence is implied by the assumption that $P_\gamma(\lambda, B) \neq \varnothing$, can be improved by some trimming and modification to satisfy conditions $(*)$, $(**)$ and $(***)$ of the last proposition.

STEP II. The next proposition shows that condition $(**)$ is easy to obtain.

PROPOSITION 2.7. *Let $\mathfrak{F}$ be a γ family with λ measures. Then for every $\varepsilon > 0$ there is a γ subfamily $\mathfrak{F}'$ of $\mathfrak{F}$ such that*

$(**)$ *if $G \in \mathfrak{F}'$, then $|\mu_G|(\bigcup\{H: H \in \mathfrak{F}', H \subset G, H \neq G\}) \leqslant \varepsilon$.*

PROOF. We use induction on γ. Since the proposition is clear for $\gamma = 0$ and since the successor and limit ordinal cases are similar, we prove only the limit case. By (6′) let $\{G_n\}$ be a disjoint sequence of subsets of G_γ such that $G_n \in \mathfrak{F}_{\gamma_n}$, $\gamma_n \nearrow \gamma$ and $\mu_{G_n} \to \mu_{G_\gamma}$. Choose a subsequence $\{G_{n_i}\}$ such that $|\mu_{G_\gamma}|(\bigcup_i G_{n_i}) \leqslant \varepsilon$. For each i, let $\mathfrak{F}_{n_i}$ be a γ_{n_i} subfamily of $\mathfrak{F}|G_{n_i}$ such that $(**)$ is satisfied. Then $\mathfrak{F}'$ is obtained as in Lemma 2.1 using the sequences $\{G_{n_i}\}$ and $\mathfrak{F}_{n_i}$. Condition $(**)$ is immediate and the proposition is proved.

STEP III. A little more work seems necessary to get condition $(*)$. We first show in Proposition 2.8 that a γ family with λ measures can be modified to yield a "towered" γ family (defined below). Then Proposition 2.13 shows how we obtain condition $(*)$ from a towered γ family.

DEFINITION. A γ family $\mathfrak{F}$ with λ measures will be called a *towered γ family with λ measures* if it satisfies:

(8) If $G \in \mathfrak{F}_\alpha$ and $\alpha < \beta \leqslant \gamma$, then there is an $H \in \mathfrak{F}_\beta$ with $G \subset H$. (Note that (3) implies the set H is unique.)

(9) If $G \in \mathfrak{F}_\beta$ where β is a limit ordinal, then there is an $H \in \mathfrak{F}_0$ such that $\lim_{\alpha<\beta} \mu_{H_\alpha} = \mu_G$ where $H_\alpha \subset G$ is the unique set in $\mathfrak{F}_\alpha$ containing H.

The standard γ family with 1 measures on $[0, \omega^\gamma]$ is towered. However, the crude γ family with λ measures constructed under the assumption that $P_\gamma(\lambda, B) \neq \varnothing$ is not towered, since the procedure in Lemma 2.1(b) does not give a towered family.

PROPOSITION 2.8. *Let B be a metrizable set of measures and let $1 \geqslant \lambda > 0$. Suppose $\mathfrak{F}$ is a γ family with λ measures taken from B. Then for every ε, with $\lambda > \varepsilon > 0$, there exists a towered γ family $\mathfrak{F}'$ with $(\lambda - \varepsilon)$ measures taken from B such that $\mathfrak{F}'_\gamma = \mathfrak{F}_\gamma$.*

For the proof of this proposition we will need three lemmas.

LEMMA 2.9. *Let $\mathfrak{F}$ be a γ family with λ measures. Suppose the set of measures associated with $\mathfrak{F}$ is metrizable. Suppose $\alpha < \beta \leqslant \gamma$ and $G \in \mathfrak{F}_\beta$. Then there is a disjoint sequence $\{G_n\}$ of subsets of G with $G_n \in \mathfrak{F}_\alpha$ for each n such that $\mu_{G_n} \to \mu_G$ and $\overline{\bigcup_n G_n} \subset G$.*

PROOF. We use induction on β considering α to be fixed. If $\beta = \alpha + 1$ the lemma follows from (5′). Suppose the lemma holds for all ordinals smaller than β. Then by either (5′) of (6′) there is a sequence $\{G_n\}$ of disjoint subsets of G such that $\mu_{G_n} \to \mu_G$, $\overline{\bigcup G_n} \subset G$ and $G_n \in \mathfrak{F}_{\beta_n}$ where $\beta_n < \beta$. Since each μ_{G_n} is a limit of measures associated with sets in $\mathfrak{F}_\alpha$ contained in G_n, the lemma follows since the set of measures associated with $\mathfrak{F}$ is metrizable.

The proof of Proposition 2.8 is by induction; however, Lemma 2.1 does not work in the limit ordinal case since the γ family constructed there is not towered. The next lemma will be used in its place.

LEMMA 2.10. *Suppose $1 \geqslant \lambda > \varepsilon > 0$, B is a metrizable set of measures, F is an open subset of K and μ_F is a measure with $|\mu_F|(F) \geqslant \lambda$. For each $n \geqslant 1$ let $\mathfrak{F}_n$ be a towered γ_n family of λ measures taken from B such that $\gamma_n \nearrow \gamma$. Denote $(\mathfrak{F}_n)_{\gamma_n}$ by F_n and assume $\{F_n\}$ is a disjoint sequence, $\overline{\bigcup F_n} \subset F$ and $\mu_{F_n} \to \mu_F$. Suppose that for each $n \geqslant 2$ there is an $H_n \in (\mathfrak{F}_n)_{\gamma_{n-1}+1}$ such that, if $\gamma_{n-1} < \alpha \leqslant \gamma_n$ and H is the unique set in $(\mathfrak{F}_n)_\alpha$ containing H_n, then $|\mu_H|(\bigcup_{i=1}^{n-1} F_i) \leqslant \varepsilon$. Then there is a towered γ family $\mathfrak{F}$ of $(\lambda - \varepsilon)$ measures taken from B such that $\mathfrak{F}_\gamma = F$.*

PROOF. We begin by defining the sets $\mathfrak{F}_\alpha$ and associated measures. For $0 \leqslant \alpha \leqslant \gamma_1$, let $\mathfrak{F}_\alpha = \bigcup_n (\mathfrak{F}_n)_\alpha$ where the associated measures are the same and let $\mathfrak{F}_\gamma = \{F\}$ where μ_F is the measure given in the lemma. Suppose $\gamma_1 < \alpha < \gamma$. There is a unique integer k with $\gamma_{k-1} < \alpha \leqslant \gamma_k$. Let D_α be the unique set in $(\mathfrak{F}_k)_\alpha$ containing H_k and let $E_\alpha = D_\alpha \cup (\bigcup_{n=1}^{k-1} F_n)$. Define $\mathfrak{F}_\alpha = \{E_\alpha\} \cup (\bigcup_{n \geqslant k} (\mathfrak{F}_n)_\alpha) \setminus \{D_\alpha\}$. Thus $\mathfrak{F}_\alpha$ is exactly the same as the union of the $(\mathfrak{F}_n)_\alpha$ except that the set D_α is replaced by the larger set E_α. Let $\mu_{E_\alpha} = \mu_{D_\alpha}$ and let the other associated measures be the same. Finally let $\mathfrak{F} = \bigcup_{\alpha \leqslant \gamma} \mathfrak{F}_\alpha$.

$\mathfrak{F}$ satisfies conditions (1) and (2) by definition and (3) follows from the disjointness of the sets $\{F_n\}$ (thus E_α is disjoint from the other sets in $\mathfrak{F}_\alpha$).

Note that if $G \in \mathfrak{F}_\beta$ then either $G \in (\mathfrak{F}_n)_\beta$ for some n or else $G = E_\beta$. Thus the verification of the remaining conditions which $\mathfrak{F}$ must satisfy each fall into two cases.

We note two simple facts regarding $\mathfrak{F}$:

(a) If $\gamma_1 < \alpha < \beta < \gamma$, then $E_\alpha \subset E_\beta$ and E_β is the only set in $\mathfrak{F}_\beta$ containg E_α.

(b) If $\alpha < \beta$ and $G \in (\mathfrak{F}_n)_\beta$ for some n, then $E_\alpha \cap G = \varnothing$. Thus if $H \in (\mathfrak{F}_n)_\alpha$ and $H \subset G$, then $H \in \mathfrak{F}_\alpha$.

For (4) suppose $G \in \mathfrak{F}_\beta$ and $\alpha < \beta \leqslant \gamma$. If $G \in (\mathfrak{F}_n)_\beta$ for some n, then $\{H \in \mathfrak{F}_\alpha\colon H \subset G\} = \{H \in (\mathfrak{F}_n)_\alpha\colon H \subset G\}$ is infinite since $E_\alpha \cap G = \varnothing$. On the other hand, if $G = E_\beta$, then $D_\beta \subset E_\beta$ and $\{H \in \mathfrak{F}_\alpha\colon H \subset G\}$ is infinite since it contains all of the family $\{H \in \mathfrak{F}_\alpha\colon H \subset D_\beta\}$ except possibly the one set D_α.

For (6′) let $G \in \mathfrak{F}_\beta$ where β is a limit ordinal. For $G \in (\mathfrak{F}_n)_\beta$ for some n, the argument is like the one for (5′) and is omitted. So suppose $G = E_\beta$. Again since

$D_\beta \subset E_\beta$ and $D_\beta \in (\mathfrak{F}_n)_\beta$ for some n, we get a disjoint sequence $\{G_k\}$ of subsets of D_β such that $G_k \in (\mathfrak{F}_n)_{\beta_k}$, $\beta_k \nearrow \beta$ and $\mu_{G_k} \to \mu_{D_\beta} = \mu_G$. If an infinite number of the G_k's are in $\mathfrak{F}$, we are done by passing to a subsequence. So suppose G_k is not in $\mathfrak{F}$ for all but a finite number of k's. Then $G_k = D_{\beta_k}$. Passing to a subsequence we may assume $G_k = D_{\beta_k}$ for each k. By Lemma 2.9 choose $Q_k \in (\mathfrak{F}_n)_{\beta_{k-1}}$ such that $Q_k \subset G_k$ and $d(\mu_{Q_k}, \mu_{G_k}) < \frac{1}{k}$ where d is the metric on B. Then $Q_k \neq D_{\beta_{k-1}}$ since the sets $G_k = D_{\beta_k}$ are disjoint. Then $\{Q_k\}$ is a disjoint sequence in $\mathfrak{F}$ with $\mu_{Q_k} \to \mu_G$ and $Q_k \in \mathfrak{F}_{\beta_{k-1}}$ and $\beta_{k-1} \nearrow \beta$ and (6′) is established.

The verification of conditions (5), (7), (8) and (9) is straightforward and tedious, and therefore is omitted. This finishes the proof of the lemma.

The proof of the next "Rosenthal type" proposition is given in the next section.

PROPOSITION 2.11. *Let $\{G_n\}$ be a disjoint sequence of open sets in K and for each n, let $\mathfrak{F}_n$ be a γ_n family with λ measures taken from the bounded set B such that $(\mathfrak{F}_n)_{\gamma_n} = \{G_n\}$. Then for any $\varepsilon > 0$ there is a subsequence n_i such that for each i there is a γ_{n_i} subfamily $\mathfrak{F}'_{n_i}$ of $\mathfrak{F}_{n_i}$ such that if $G \in \mathfrak{F}'_{n_k}$, then $|\mu_G|(\bigcup_{i \neq k} G_{n_i}) < \varepsilon$.*

PROOF OF PROPOSITION 2.8. Using induction on γ, the proposition is clear for $\gamma = 0$, letting $\mathfrak{F}' = \mathfrak{F}$. Suppose $\gamma = \beta + 1$, $\lambda > \varepsilon > 0$ and the proposition holds for β. By (5′) choose a disjoint sequence $\{G_n\}$ of subsets of G_γ such that $\mu_{G_n} \to \mu_G$. For each n, let $\mathfrak{F}_n$ be a towered β family with $(\lambda - \varepsilon)$ measures taken from B such that $(\mathfrak{F}_n)_\beta = (\mathfrak{F} | G_n)_\beta = G_n$. The process in Lemma 2.1 gives a γ family $\mathfrak{F}'$ with $(\lambda - \varepsilon)$ measures. Conditions (8) and (9) are easily verified so $\mathfrak{F}'$ is a towered γ family with $(\lambda - \varepsilon)$ measures.

Next suppose γ is a limit ordinal and $\varepsilon > 0$ and assume the proposition holds for all $\alpha < \gamma$. We will use Lemma 2.10. First use (6′) to get a sequence $\{G_n\}$ of disjoint open sets in G_γ such that $G_n \in (\mathfrak{F})_{\gamma_n}$, $\gamma_n \nearrow \gamma$, $\mu_{G_n} \to \mu_{G_\gamma}$ and $\overline{\bigcup_n G_n} \subset G_\gamma$. Apply Proposition 2.11 to the sequence $\mathfrak{F} | G_n$ of γ_n families, and we may assume that for each n we have a γ_n family with λ measures such that $(\mathfrak{F}_n)_{\gamma_n} = G_n$, and if $G \in \mathfrak{F}_n$, then $|\mu_G|(\bigcup_{i \neq n} G_i) \leqslant \frac{\varepsilon}{2}$. Applying the inductive hypothesis to each $\mathfrak{F}_n$ yields a towered γ_n family $\mathfrak{F}'_n$ with $(\lambda - \frac{\varepsilon}{2})$ measures such that $(\mathfrak{F}'_n)_{\gamma_n} = G_n$. Lemma 2.10 is now satisfied by letting H_n, for $n \geqslant 2$, be any set in $(\mathfrak{F}_n)_{\gamma_{n-1}+1}$ and Proposition 2.8 is proved.

STEP IV. Once we have a towered family, Proposition 2.13 below shows we can pass to a restricted family (with a reduction in γ) which satisfies condition $(*)$.

The elementary proof of the next lemma is omitted.

LEMMA 2.12. *Let G be an open set and suppose $\mu_n \to \mu$. Then*

$$|\mu|(G) \leqslant \sup_n |\mu_n|(G).$$

PROPOSITION 2.13. *Let $\mathfrak{F}$ be a towered γ family with λ measures taken from a metrizable set of measures. Let $\varepsilon > 0$ and suppose N is an integer with $\frac{1}{N} < \varepsilon$. Suppose that $\|\mu_G\| \leqslant 1$ for each $G \in \mathfrak{F}$. If $\gamma \geqslant \beta N$, then there is a subset $\mathfrak{F}'$ of $\mathfrak{F}$ such that $\mathfrak{F}'$ is a towered β family with λ measures and condition $(*)$ is satisfied, i.e. if $G_\beta = \mathfrak{F}'_\beta$, then for each $G \in \mathfrak{F}'$, $|\mu_G|(G_\beta \setminus G) \leqslant \varepsilon$.*

PROOF. We begin by defining a function $f\colon \mathfrak{F}_0 \to [1, N]$. For each $G \in \mathfrak{F}_0$, consider the sequence

$$G = G_0 \subset G_1 \subset \cdots \subset G_N$$

where G_n is the unique element of $\mathfrak{F}_{\beta n}$ containing G. There must be at least one integer $n \in [1, N]$ such that

$$|\mu_G|(G_n \setminus G_{n-1}) \leqslant \tfrac{1}{N} < \varepsilon$$

since $\{G_1 \setminus G_0, G_2 \setminus G_1, \ldots, G_N \setminus G_{N-1}\}$ is a sequence of N disjoint sets and $\|\mu_G\| \leqslant 1$. Define $f(G)$ to be such an integer n.

To finish the proposition we need the following lemma.

LEMMA 2.14. *Let $\mathfrak{F}$ be a γ family with λ measures and let N be a positive integer. Let $f\colon \mathfrak{F}_0 \to [1, N]$ be an arbitrary function. Then there is a γ subfamily $\mathfrak{F}'$ with λ measures such that f is constant on $\mathfrak{F}'_0$.*

PROOF. We use induction on γ. The lemma is trivial for $\gamma = 0$. Since the limit and successor ordinal cases are similar we prove only the limit case. Suppose γ is a limit ordinal and the lemma holds for $\alpha < \gamma$. By (6′) there is a disjoint sequence $\{G_n\}$ of subsets of G_γ such that $G_n \in \mathfrak{F}_{\gamma_n}$, $\gamma_n \nearrow \gamma$ and $\mu_{G_n} \to \mu_{G_\gamma}$. Let $\mathfrak{F}_n$ be a γ_n subfamily of $\mathfrak{F} \mid G_n$ with λ measures such that f is constant on $(\mathfrak{F}_n)_0$. Then there is a subsequence n_i such that f takes the same value on each $(\mathfrak{F}_{n_i})_0$. Now apply Lemma 2.1 using the sequence $\{G_{n_i}\}$ and the lemma is proved.

We can now finish the proof of Proposition 2.13. Applying the last lemma yields a γ subfamily $\mathfrak{F}''$ of $\mathfrak{F}$ and an integer n such that if $G \in \mathfrak{F}''_0$, then $f(G) = n$, i.e.,

$$\text{(P)} \qquad |\mu_G|(G_n \setminus G_{n-1}) \leqslant \varepsilon.$$

To define $\mathfrak{F}'$, let G_β be any set in $\mathfrak{F}''_{\beta n}$. Roughly speaking, $\mathfrak{F}'$ will just be $\mathfrak{F}'' \mid G_\beta$ with all the sets in $(\mathfrak{F}'' \mid G_\beta)_\alpha$ thrown away for $\alpha < (n-1)\beta$. For each ordinal α with $0 \leqslant \alpha < \beta$, let $\mathfrak{F}'_\alpha = \{H\colon H \subset G_\beta \text{ and } H \in F''_{\beta(n-1)+\alpha}\}$ and let $\mathfrak{F}'_\beta = \{G_\beta\}$. Finally let $\mathfrak{F}' = \bigcup_{\alpha \leqslant \beta} \mathfrak{F}'_\alpha$. For $H \in \mathfrak{F}'$, μ_H is the same as μ_H when H is considered as an element of $\mathfrak{F}''$ or $\mathfrak{F}$. The fact that $\mathfrak{F}'$ is a β family with λ measures is straightforward. It remains to be shown that if $G \in \mathfrak{F}'$, then $|\mu_G|(G_\beta \setminus G) \leqslant \varepsilon$. Suppose $G \in \mathfrak{F}'_\alpha$. Then $G \in F''_{\beta(n-1)+\alpha}$. Applying Lemma 2.9, let $\{G_k\}$ be a sequence of sets in $\mathfrak{F}''_0$ with $\overline{\bigcup_k G_k} \subset G$ such that $\mu_{G_k} \to \mu_G$. Condition (P) says that $|\mu_{G_k}|(G_\beta \setminus G_n) \leqslant \frac{1}{N}$ since G_β must be the unique set in $\mathfrak{F}''_{\beta n}$ containing G_k and similarly for G. Since $\mu_{G_k} \to \mu_G$,

$$\begin{aligned} |\mu_G|(G_\beta \setminus G) &\leqslant |\mu_G|\left(G_\beta \setminus \overline{\bigcup_n G_n}\right) \leqslant \sup_n |\mu_{G_n}|\left(G_\beta \setminus \overline{\bigcup_k G_k}\right) \\ &\leqslant \sup_n |\mu_{G_n}|(G_\beta \setminus G_n) \leqslant \frac{1}{N} < \varepsilon \end{aligned}$$

by Lemma 2.12 and the proposition is proved.

STEP V. One last detail must be taken care of before we can prove Theorem 1. The difficulty is that Proposition 2.2 gives a γ family whose measures satisfy $|\mu_G|(G) \geqslant \lambda$; however, in Proposition 2.6 we need the stronger condition

$$(***) \qquad \mu_G(G) \geqslant \lambda.$$

LEMMA 2.15. *Let $1 \geqslant \lambda > 2\varepsilon > 0$ and suppose $\mathfrak{F}$ is a γ family with λ measures taken from B which satisfies $(*)$ for each $G \in \mathfrak{F}$, $|\mu_G|(G_\gamma \setminus G) \leqslant \varepsilon$. Then there is a γ subfamily $\mathfrak{F}^*$ with $(\frac{\lambda}{2} - 2\varepsilon)$ measures taken from $B \cup -B$ satisfying $(*)$ such that $\mu_G(G) \geqslant (\frac{\lambda}{2} - 2\varepsilon)$ for each $G \in \mathfrak{F}^*$.*

PROOF. For each $G \in \mathfrak{F}_0$, using the Hahn decomposition theorem choose disjoint closed sets P_G and N_G such that $P_G \cup N_G \subset G$, $|\mu_G|(G \setminus (P_G \cup N_G)) < \frac{\varepsilon}{2}$, $\mu_G(P_G) = |\mu_G|(P_G)$ and $|\mu_G|(N_G) = -\mu_G(N_G)$. Then either (A) $\mu_G(P_G) \geqslant \frac{\lambda}{2} - \frac{\varepsilon}{2}$ or (B) $\mu_G(N_G) \leqslant -\frac{\lambda}{2} + \frac{\varepsilon}{2}$. Define f: $\mathfrak{F}_0 \to \{1, 2\}$ by $f(G) = 1$ if (A) holds or $f(G) = 2$ if (A) does not hold. By Lemma 2.14 there is a γ subfamily $\mathfrak{F}'$ such that f is constant on $\mathfrak{F}_0'$. We will assume $f(G) = 1$ for all $G \in \mathfrak{F}_0'$. The proof is the same if $f(\mathfrak{F}_0') = 2$ except we change the measures to $-\mu_G$ for each $G \in \mathfrak{F}'$, and then our measures are taken from $-B$. So we assume that for each $G \in \mathfrak{F}_0'$, $\mu_G(P_G) \geqslant \frac{\lambda}{2} - \frac{\varepsilon}{2}$. Also $(*)$ still holds since $\mathfrak{F}'$ is a subfamily. Define $\mathfrak{F}^*$ as follows. For each $G \in \mathfrak{F}'$ let

$$G^* = G \setminus \overline{\bigcup \{H_N : H \subset G, H \in \mathfrak{F}_0'\}}\,.$$

Let $\mathfrak{F}^* = \{G^* : G \in \mathfrak{F}'\}$ and let $\mu_{G^*} = \mu_G$. The verification that $\mathfrak{F}^*$ is a γ family satisfying $(*)$ is routine since $G_1 \cap G_2 = \varnothing$ implies $G_1^* \cap G_2^* = \varnothing$ and $G_1 \subset G_2$ implies $G_1^* \subset G_2^*$. Now for $G^* \in \mathfrak{F}_0^*$,

$$\mu_{G^*}(G^*) = \mu_G(G \setminus N_G) \geqslant \mu_G(P_G) - |\mu_G|(G \setminus (N_G \cup P_N)) \geqslant \tfrac{\lambda}{2} - \varepsilon.$$

If $G \in \mathfrak{F}_\alpha^*$, then by Lemma 2.9 there is a sequence $\{G_n\} \subset \mathfrak{F}_0^*$ such that $\overline{\bigcup_n G_n} \subset G$ and $\mu_{G_n} \to \mu_G$. Choose $f_G \in C(K)$ with $0 \leqslant f_G \leqslant 1$, $f_G = 1$ on $\bigcup_n G_n$, $f_G = 0$ outside G and $\mu_G(G) \geqslant \int f_G \, d\mu_G - \frac{\varepsilon}{2}$. Then

$$\mu_G(G) \geqslant \lim_n \int f_G \, d\mu_{G_n} \geqslant \lim_n \left[\mu_{G_n}(G_n) - |\mu_{G_n}|(G \setminus G_n)\right] \geqslant \tfrac{\lambda}{2} - 2\varepsilon$$

by condition $(*)$ and the lemma is proved.

STEP VI.

PROOF OF THEOREM 1. If $P_\gamma(\lambda, B) \neq 0$, by Proposition 2.2 there is a γ family $\mathfrak{F}$ with λ measures taken from B. Choose a small $\varepsilon > 0$. By Proposition 2.8 we may assume $\mathfrak{F}$ is towered with $(\lambda - \varepsilon)$ measures. Then Proposition 2.13 gives a β family with $(\lambda - \varepsilon)$ measures satisfying condition $(*)$ of Proposition 2.6. Then the last lemma gives a β family with $\frac{\lambda}{2} - 3\varepsilon$ measures taken from $B \cup (-B)$ satisfying $(*)$ and such that $\mu_G(G) \geqslant \frac{\lambda}{2} - 3\varepsilon$ for each G. We may assume $(**)$ is also satisfied by Proposition 2.7. So Proposition 2.6 gives a subspace Y of $C(K)$ which is normed by B and isometric to $A(\mathfrak{F})$. By Proposition 2.3, Y contains a subspace isometric to $C(\omega^\beta)$.

3. Proof of Theorem A. In addition to the proof of Theorem A, this section closes with the proof of Proposition 2.11.

LEMMA 3.1. *Let $\gamma = \omega^\alpha$. Then for any integer $N > 0$ and any ordinal $\beta < \gamma$, $\beta \cdot N < \gamma$.*

PROOF. The proof is a simple induction on α using the following facts (see [**15**]): $\omega^{\beta+1} = \sup_n \omega^\beta \cdot n$ and, if α is a limit, then $\omega^\alpha = \sup_{\beta<\alpha} \omega^\beta$.

PROOF OF THEOREM A. Suppose $\gamma_n \nearrow \gamma$ and $\{G_n\}$ is a disjoint sequence of sets with $(G_n, \mu_{G_n}) \in P_n(\lambda, \beta)$. Choose ε with $0 < \varepsilon < \frac{\lambda}{10}$ and choose N with $N > \frac{2}{\lambda}$. For each n, by the last lemma, $\gamma_n \cdot N < \gamma$ so there is a $k > n$ with $\gamma_n \cdot N < \gamma_k$. Thus passing to a subsequence, we may assume that for each n, $\gamma_n \cdot N \leqslant \gamma_{n+1}$. Since $G_{n+1} \in P_{\gamma_{n+1}}(\lambda, B)$, by Proposition 2.2, there is a γ_{n+1} family $\mathfrak{F}_{n+1}$ with λ measures such that $(\mathfrak{F}_{n+1})_{\gamma_{n+1}} = G_{n+1}$. Applying Propositions 2.8, 2.13 and 2.7 exactly as in the proof of Theorem 1, we obtain a γ_n family $\mathfrak{F}_n'$ with $(\lambda - 4\varepsilon)$ measures such that, letting $H_n = (\mathfrak{F}_n')_{\gamma_n}$, $H_n \subset G_{n+1}$ and conditions $(*)$ and $(**)$ of Proposition 2.6 are satisfied. Next, using Proposition 2.11, and passing to a subsequence of $\{H_n\}$ and to γ_n subfamilies $\mathfrak{F}_n''$ of $\mathfrak{F}_n'$, we may assume that, in addition to $(*)$ and $(**)$ of Proposition 2.6 being satisfied for each $\mathfrak{F}_n''$, we also have for each n, if $G \in \mathfrak{F}_n'$, then $|\mu_G|(\bigcup_{k \neq n} H_k) < \varepsilon$. It is now easily checked as in Proposition 2.6, that if Y is the closed linear span of $\{\chi_G$: for some n, $G \in \mathfrak{F}_n''\}$ then Y is normed by B and Y contains a subspace isometric to $C_0(\omega^\gamma)$. This finishes the proof of Theorem A.

In order to prove Proposition 2.11 we need the next lemma.

LEMMA 3.2. *Let $\mathfrak{F}$ be a γ family with λ measures, F a measurable set in K and $\varepsilon > 0$. Suppose that for every γ subfamily $\mathfrak{F}'$ of $\mathfrak{F}$, there is a $G \in \mathfrak{F}_0'$ such that $|\mu_G|(F) \geqslant \varepsilon$. Then there is a γ subfamily $\mathfrak{F}''$ of $\mathfrak{F}$ such that if $G \in \mathfrak{F}_0''$, then $|\mu_G|(F) \geqslant \varepsilon$.*

PROOF. Using induction on γ, the lemma is trivial if $\gamma = 0$. We omit the successor ordinal case since it is similar to the limit case.

Suppose γ is a limit ordinal. Let $\{G_n\}$ be a disjoint sequence of subsets of G_γ such that $G_n \in \mathfrak{F}_{\gamma_n}$, $\gamma_n \nearrow \gamma$ and $\mu_{G_n} \to \mu_{G\gamma}$. Let $M = \{n$: for every γ_n subfamily $\mathfrak{F}_n'$ of $\mathfrak{F}|G_n$ there is a $G \in (\mathfrak{F}_n')_0$ with $|\mu_G|(F) \geqslant \varepsilon\}$. If M is infinite, then by inductive hypothesis there is for each $n \in M$ a γ_n subfamily $\mathfrak{F}_n'$ of $\mathfrak{F}|G_n$ such that if $G \in (\mathfrak{F}_n')_0$ then $|\mu_G|(F) \geqslant \varepsilon$. Applying Lemma 2.1 to the sequence $\mathfrak{F}_n'$ for $n \in M$ gives the γ subfamily $\mathfrak{F}''$ desired in the lemma. On the other hand, suppose M is finite. If $n \notin M$ then there is a γ_n subfamily $\mathfrak{F}_n'$ of $\mathfrak{F}|G_n$ such that if $G \in (\mathfrak{F}_n')_0$ then $|\mu_G|(F) < \varepsilon$. In this case apply Lemma 2.1 to the sequence $\mathfrak{F}_n$ for $n \notin M$ and we get a γ subfamily $\mathfrak{F}''$ of $\mathfrak{F}$ such that if $G \in (\mathfrak{F}'')_0$ then $|\mu_G|(F) < \varepsilon$. This contradicts the assumption of $\mathfrak{F}$ so M cannot be finite.

PROOF OF PROPOSITION 3.1. (See proof of Rosenthal's Lemma in [7, p. 18].) Partition the integers into an infinite number of disjoint infinite subsets $\{M_p\}_{p=1}^\infty$. The proposition holds if, for some p, for every $k \in M_p$ there is a γ_k subfamily $\mathfrak{F}_k'$ of $\mathfrak{F}_k$ such that if $G \in (\mathfrak{F}_k')_0$ for some $k \in M_p$, then $|\mu_G|(\bigcup\{G_i: i \in M_p, i \neq k\}) \leqslant \varepsilon$. (Note that if this inequality holds for every $G \in (\mathfrak{F}_k')_0$, then it holds for every $G \in \mathfrak{F}_k'$ by Lemmas 2.9 and 2.12.) We will obtain a contradiction from the negation of the above statement. Thus assume that for every p there is a $k(p) \in M_p$ such that for every $\gamma_{k(p)}$ subfamily $\mathfrak{F}_{k(p)}'$ of $\mathfrak{F}_{k(p)}$, there is a $G \in (\mathfrak{F}_{k(p)}')_0$ such that

$$|\mu_G|\Big(\bigcup\{G_i; i \in M_p, i \neq k(p)\}\Big) \geqslant \varepsilon.$$

Thus by Lemma 3.3 there is a $\gamma_{k(p)}$ subfamily $\mathfrak{F}_{k(p)}''$ of $\mathfrak{F}_{k(p)}$ such that for every $G \in (\mathfrak{F}_{k(p)}'')_0$,

$$|\mu_G|\Big(\bigcup\{G_i: i \in M_p, i \neq k(p)\}\Big) \geqslant \varepsilon.$$

Thus for any p, if $G \in (\mathcal{F}''_{k(p)})_0$ then $|\mu_G|(\bigcup_{i \neq p} G_{k(i)}) \leqslant Q - \varepsilon$ since $\bigcup_{i \neq p} G_{k(i)}$ is disjoint from $\bigcup \{G_i\colon i \in M_p,\ i \neq k(p)\}$ where $Q = \sup\{\|\mu\|\colon \mu \in B\}$. Apply this same procedure to the sequence $\{G_{k(p)}\}_{p=1}^{\infty}$ and $\{\mathcal{F}''_{k(p)}\}_{p=1}^{\infty}$ and get an inequality as above for a subsequence; however we get $Q - 2\varepsilon$ rather than $Q - \varepsilon$. Repeating this procedure n times, where $Q - n\varepsilon < 0$, gives a contradiction and proves the proposition.

4. The relationship between the sets $P_\alpha(\lambda, B)$ and the Szlenk index. The sets $P_\alpha(\lambda, B)$ were first considered in an attempt to understand the Szlenk index of an operator as considered in an attempt to understand the Szlenk index of an operator as considered in Alspach [**1**] (see also [**16**]). Also a forerunner of Theorem A is Theorem 0.2 of [**1**] which gives a condition in terms of the Szlenk index for an operator on a $C(K)$ space to preserve a copy of $C_0(\omega^\omega)$. Hence in this section we briefly discuss the relationship between the sets $P_\alpha(\lambda, B)$ and the sets $P^*_\alpha(\lambda, B)$ (defined below) used in the definition of the Szlenk index.

For a bounded set B in the dual X^* of a Banach space X and for $\lambda > 0$, we inductively define (following [**1**]) for each ordinal α, a Szlenk set $P^*_\alpha(\lambda, B) \subset X^*$ as follows: $P^*_0(\lambda, B) = B$ and if $\alpha = \beta + 1$ then $P^*_\alpha(\lambda, B) = \{b$: there is a sequence $(b_n)_{n=1}^{\infty} \subset P^*_\beta(\lambda, B)$ and a sequence $(a_n)_{n=1}^{\infty} \subset X$ with $\|a_n\| \leqslant 1$ such that $b_n \to b$ (weak*) and $a_n \to 0$ (weakly) and $\limsup \langle b_n, a_n\rangle \geqslant \lambda\}$. If α is a limit ordinal then $P^*_\alpha(\lambda, B) = \bigcup_{\beta<\alpha} P^*_\beta(\lambda, B)$.

The following lemma is trivial.

LEMMA 4.1. *Suppose B is a bounded set in $C(K)^*$ and $\lambda > 0$. Then if $(\mu_G, G) \in P_\alpha(\lambda, B)$ then $\mu_G \in P^*_\alpha(\lambda, B)$.*

Using arguments from [**1**] (see Lemma 1.3 and Remark 2), one can prove

LEMMA 4.2. *Let B be a bounded subset of $C(K)^*$. Then there is a $\lambda > 0$ such that $P_1(\lambda, B) \neq \varnothing$ if and only if there is an $\varepsilon > 0$ such that $P^*_1(\varepsilon, B) \neq \varnothing$.*

Finally we give a simple example to show that $P^*_2(\varepsilon, B)$ can be nonempty but $P_2(\lambda, B) = \varnothing$ for all $\lambda > 0$.

EXAMPLE. Let K be the space consisting of two sequences $\{x_n\}$ and $\{y_n\}$ with $x_n \to x_0$ and $y_n \to y_0$ where $x_0 \neq y_0$. Let $B = \{\mu \in C(K)^*\colon \|\mu\| \leqslant 1\}$ and let $\mu_{ij} = \frac{1}{2}\delta_{x_i} + \frac{1}{2}\delta_{y_j}$ where $\delta_x \in C(K)^*$ is defined by $\delta_x(F) = F(x)$ for each $F \in C(K)$. Then $\mu_{0j} \in P^*_1(\frac{1}{2}, B)$ since $\mu_{ij} \to \mu_{0j}$ (weak*) and if $F_i = \chi_{\{x_i\}}$ then $F_i \to 0$ weakly and $\langle \mu_{ij}, F_i\rangle = \frac{1}{2}$. Also $\mu_{00} \in P^*_2(\frac{1}{2}, B)$ since $\mu_{0j} \to \mu_{00}$ weak* and $\chi_{\{y_j\}} \to 0$ weakly and $\langle \mu_{0j}, \chi_{\{y_j\}}\rangle = \frac{1}{2}$ for every j. Thus $P^*_2(\frac{1}{2}, B) \neq \varnothing$. However, $P_2(\lambda, B) = \varnothing$ for every $\lambda > 0$ since $P_2(\lambda, B) \neq \varnothing$ implies there is a 2 family of sets in K and then Lemma 2.4 implies that the second derived set K^2 is nonempty. However, $K^2 = \varnothing$ (since K^1 has 2 points) and thus $P_2(\lambda, B) = \varnothing$ for every $\lambda > 0$. This completes the example.

5. Primariness of the spaces $C(\alpha)$. In this section we show how the following result of Alspach and Benyamini [**17**] and Ballard [**18**] can be obtained from the techniques of this paper.

THEOREM. *For any countable ordinal β, $C(\beta)$ is primary, i.e., if $C(\beta) = A_1 \oplus A_2$ where A_1 and A_2 are closed subspaces of $C(\beta)$, then either A_1 or A_2 is isomorphic to $C(\beta)$.*

PROOF OF THEOREM. Let $\varepsilon > 0$ be given. By [**6**], $C(\beta)$ is isomorphic to $C(K)$ with $K = [1, \alpha]$ where $\alpha = \omega^\lambda$ and $\gamma = \omega^\lambda$ for some ordinal λ. Let $P_1\colon C(K) \to C(K)$ be the projection onto A_1 with kernel A_2 and let $P_2 = I - P_1$. Let $\mathfrak{F}$ be the standard γ family for K. We consider two choices of measures for this γ family: $\mu_G^i = P_i^*\mu_G$ for $i = 1$ or 2, where μ_G is the standard measure for $G \in \mathfrak{F}$. Choose ordinals γ_n with $\gamma_n \nearrow \gamma$ and choose N with $\frac{1}{N} < \varepsilon$. Then $\gamma_n N^2 \nearrow \gamma$. Choose a disjoint sequence $\{G_n\}$ of sets from $\mathfrak{F}$ such that $G_n \in \mathfrak{F}_{\gamma_n} N^2$ and let $\mathfrak{F}_n = \mathfrak{F} | G_n$. Then $\mathfrak{F}_n$ is a towered $\gamma_n N^2$ family. By Proposition 2.13 there is a towered $\gamma_n N$ family $\mathfrak{F}_n'$ taken from $\mathfrak{F}_n$ such that $|\mu_G^1|(G_{\gamma_n N} \setminus G) \leqslant \varepsilon$ for each $G \in \mathfrak{F}_n'$. Applying Proposition 2.13 a second time gives a towered γ_n family $\mathfrak{F}_n''$ such that $|\mu_G^2|(G_{\gamma_n} \setminus G) \leqslant \varepsilon$ for each $G \in \mathfrak{F}_n''$. Since $G_{\gamma_n} \subset G_{\gamma_n N}$ we have (a) $|\mu_G^i|(G_{\gamma_n} \setminus G) \leqslant \varepsilon$ for each $G \in \mathfrak{F}_n''$ and for $i = 1, 2$. For each $G \in (F_n)_0$,

$$\mu_G^1(G) + \mu_G^2(G) = (P_1^*\mu_G + P_2^*\mu_G)(G) = \mu_G(G) = 1.$$

Thus either (A) $\mu_G^1(G) \geqslant \frac{1}{2}$ or (B) $\mu^2(G) \geqslant \frac{1}{2}$. Using Lemma 2.14 we can choose a γ_n subfamily $\mathfrak{F}_n'''$ from $\mathfrak{F}_n''$ such that (A) holds for every $G \in (\mathfrak{F}_n''')_0$ or (B) holds for every $G \in (\mathfrak{F}_n''')_0$ (also condition (a) still holds). Passing to a subsequence of the n's we may assume (A) holds for each n or (B) holds for each n. Without loss of generality assume (A) holds. Thus for each n we have a γ_n family $\mathfrak{F}_n'''$ such that $\mu_G^1(G) \geqslant \frac{1}{2}$ for each $G \in (\mathfrak{F}_n''')_0$ and

$$(*) \qquad |\mu_G^1|\left(G_{\gamma_n} \setminus G\right) \leqslant \varepsilon \quad \text{for each } G \in \mathfrak{F}_n'''.$$

In fact we have $\mu_G^1(G) \geqslant \frac{1}{2} - \varepsilon$ for all $G \in \mathfrak{F}_n'''$ since for any such G there is, by Lemma 2.9, a sequence $\{G_n\}$ in $(\mathfrak{F}_n''')_0$ with $G_n \subset G$ such that $\mu_{G_n}^1 \to \mu_G^1$ and thus

$$\mu_G^1(G) = \lim_n \mu_{G_n}^1(G) \geqslant \lim_n \left[\mu_{G_n}^1(G_n) + \mu_{G_n}^1(G - G_n)\right] \geqslant \tfrac{1}{2} - \varepsilon$$

by $(*)$. Thus each $\mathfrak{F}_n'''$ is a γ_n family with $\frac{1}{2} - \varepsilon$ measures satisfying condition $(*)$. Passing to a subfamily by Proposition 2.7 we may assume we also have for each $G \in \mathfrak{F}_n'''$,

$$(**) \qquad |\mu_G|\left(\bigcup \{H\colon H \in \mathfrak{F}_n''', H \subset G, H \neq G\}\right) \leqslant \varepsilon.$$

Finally, passing to a subsequence by Proposition 2.11 we may assume that, for each $G \in \mathfrak{F}_n'''$, $|\mu_G|(\bigcup_{k \neq n} G_{\gamma_k}) < \varepsilon$. It is not hard to verify that for ε small enough $\{\chi_G\colon G \in \mathfrak{F}_n'''$ for some $n\}$ spans a subspace isomorphic to $C_0(\omega^\gamma)$ which is normed by $P_1^*(\{\mu\colon \|\mu\| \leqslant 1\})$. Thus $P_1(C(\alpha)) = A_1$ contains a subspace isomorphic to $C(\omega^\gamma)$. Standard arguments (the "decomposition method" of Pelczynski and the main result of [**11**]) give that A_1 is isomorphic to $C(\beta)$.

BIBLIOGRAPHY

1. D. E. Alspach, *Quotients of $C[0, 1]$ with separable dual*, Israel J. Math. **29** (1978), 361–384.
2. _____, *$C(S)$ norming subsets of $C[0, 1]^*$*, Studia Math. (to appear).
3. D. E. Alspach and Y. Benyamini, *$C(K)$ quotients of separable L_∞ spaces*, Israel J. Math. (to appear).
4. J. W. Baker. *Compact spaces homeomorphic to a ray of ordinals*, Fund. Math. **76** (1972), 19–27.
5. Y. Benyamini, *An extension theorem for separable Banach spaces*, Israel J. Math. **29** (1978), 24–30.

6. C. Bessaga and A. Pelczynski, *Spaces of continuous functions*. IV, Studia Math. **19** (1960), 53–62.

7. J. Diestel and J. J. Uhl, *Vector measures*, Math. Surveys, no. 15, Amer. Math. Soc., Providence, R.I., 1977.

8. F. Hausdorff, *Set theory*, Chelsea, New York, 1962.

9. K. Kuratowski, *Topology*, Vol. II (translated from the French), Academic Press, New York and London; PWN, Warsaw, 1968.

10. S. Mazurbiewicz and W. Sierpinski, *Contributions à la topologie des ensembles dinombrables*, Fund. Math. **1** (1920), 17–27.

11. A. Pelczynski, *On $C(S)$ subspaces of separable Banach spaces*, Studia Math. **31** (1968), 513–522.

12. _____, *Linear extensions, linear averaging and their application to linear topological classification of spaces of continuous functions*, Rozprawy Mat. **58** (1968).

13. H. P. Rosenthal, *On relatively disjoint families of measures, with some applications to Banach space theory*, Studia Math. **37** (1970), 13–36.

14. _____, *On factors of $C([0, 1])$ with non-separable dual*, Israel J. Math. **13** (1972), 361–378.

15. W. Sierpinski, *Cardinal and ordinal numbers*, PWN, Warsaw, 1965.

16. W. Szlenk, *The non-existence of a separable reflexive Banach space, universal for all separable reflexive Banach spaces*, Studia Math. **30** (1968), 53–61.

17. D. Alspach and Y. Benyamini, *Primariness of spaces of continuous functions on ordinals* (to appear).

18. P. Billard, *Sur la primarité des espaces $C(\alpha)$*, Studia Math. **62** (1978), 143–162.

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