

**ON EXTENDING FREE GROUP ACTIONS ON SPHERES
 AND A CONJECTURE OF IWASAWA**

BY

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ABSTRACT. A transfer map for Reidemeister torsion is defined and used to determine whether free actions of \mathbf{Z}/k on S^{2n+1} , $n > 1$, extend to free actions of \mathbf{Z}/hk . It is shown that for k odd, every free \mathbf{Z}/k action on S^{2n+1} , $n > 1$, extends to a free $\mathbf{Z}/2k$ action. For prime p , extension of an arbitrary free \mathbf{Z}/p action to a free \mathbf{Z}/p^2 action is reduced to a long-standing conjecture of Iwasawa.

0. Introduction and statement of results. In this paper we address the following basic question: Let π be a finite group, ρ a subgroup. When does a free action of ρ on S^{2n+1} extend to a free action of π ?

Some of our results apply in general, but we concentrate on the case where π is a cyclic group. In this case we answer the above question completely, reducing the issue to algebraic number theory. If $\rho = 1$, our results can be viewed as an addendum to [1] which solved this question for $\rho = 1$ and π an odd order cyclic group. We will now briefly summarize our results.

We write an action of π as a map $\mu: \pi \times S^{2n+1} \rightarrow S^{2n+1}$ and we say μ_1, μ_2 are equivalent if there is a homeomorphism $f: S^{2n+1} \rightarrow S^{2n+1}$ such that $f\mu_1(T, x) = \mu_2(T, f(x))$ for all $(T, x) \in \pi \times S^{2n+1}$. For π cyclic of order h , the prototype is of course the linear action of \mathbf{Z}/h on $S^{2n+1} \subset \mathbf{C}^{n+1}$ given by $\mu_0(T, (z_0, z_1, \dots, z_n)) = \exp(2\pi i/h)(z_0, z_1, \dots, z_n)$. We write $\Delta(\mu)$ for the Reidemeister torsion of the action $\mu: \pi \times S^{2n+1} \rightarrow S^{2n+1}$, following Milnor [4]. $\Delta(\mu)$ is a unit in the ring $\mathbf{Q}R_\pi$, defined by $\mathbf{Q}\pi/(\Sigma)$, where Σ denotes the sum of the elements of π . It is a basic invariant of the action. For the standard linear action mentioned above, $\Delta(\mu_0) = (T - 1)^{n+1}$ (see [4, p. 406]). Actually it is more convenient to keep tabs on the quotient

$$\Delta(\mu)/\Delta(\mu_0) = \Delta(\mu) \cdot (T - 1)^{-n-1} \in \mathbf{Q}R_\pi^\times / (\pm\pi).$$

In fact, let $R_\pi = \mathbf{Z}\pi/(\Sigma)$ with $j: (R_\pi)^\times \rightarrow \mathbf{Q}R_\pi^\times$ the natural inclusion map. Then Wall has shown [9, Theorem 14.E.3] that

0.1. LEMMA. $\Delta(\mu)/\Delta(\mu_0) = j(u)$ for some unique unit $u \in (R_\pi)^\times / (\pm\pi)$. Moreover, given a unit $u \in (R_\pi)^\times$ there is a cellular free action μ on a finite complex homotopically equivalent to S^{2n+1} such that $\Delta(\mu)/\Delta(\mu_0) = j(u) \pmod{(\pm\pi)}$.

For this reason, this unit $u \in (R_\pi)^\times / (\pm\pi)$ is called the *associated unit* of μ .

Received by the editors April 14, 1980 and, in revised form, June 24, 1981.

1980 *Mathematics Subject Classification.* Primary 57S17, 57S25, 57Q10; Secondary 18F25, 12A35.

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0002-9947/82/0000-0821/\$03.75

After a surgery calculation—carried out in Part I—we define, for each finite group π and subgroup ρ , a transfer map,

$$\begin{aligned} \text{tr}: K_1(R_\pi)/(\pm\pi) &\rightarrow K_1(R_\rho)/(\pm\rho) \quad \text{and} \\ \text{tr}: K_1(\mathbf{Q}R_\pi)/(\pm\pi) &\rightarrow K_1(\mathbf{Q}R_\rho)/(\pm\rho) \end{aligned}$$

and we prove that $\text{tr} \Delta(\mu) = \Delta(\bar{\mu})$ whenever $\mu: \pi \times X \rightarrow X$ is a free action for which the torsion is defined, and $\bar{\mu}$ is its restriction to ρ . This is Theorem 1.7, and we feel it should be of independent interest.

Now it is known that

LEMMA. $K_1(R_\pi) = R_\pi^\times$ if π is a finite cyclic group.

(For a proof, see the Remark following 1.6.B.)

Hence, if π is a cyclic group our definition of the transfer provides a homomorphism

$$\text{tr}: (R_\pi)^\times / (\pm\pi) \rightarrow (R_\rho)^\times / (\pm\rho).$$

We then prove

THEOREM 1. Let $\bar{\mu}$ be a free action of $\rho = \mathbf{Z}/k$ on S^{2n+1} , $n \geq 2$. Let $\bar{u} \in (R_\rho)^\times / (\pm\rho)$ be its associated unit. Then $\bar{\mu}$ extends to a free action of $\mathbf{Z}/hk = \pi$ if and only if \bar{u} is in the image of $\text{tr}: (R_\pi)^\times / (\pm\pi) \rightarrow (R_\rho)^\times / (\pm\rho)$.

If p is a prime number it then turns out that the problem of extending a \mathbf{Z}/p^r action to a \mathbf{Z}/p^{r+1} action is closely related to an old conjecture of Iwasawa in [2]. This conjecture states that the norm map $N: \mathbf{Z}(\zeta_{r+1})^\times \rightarrow \mathbf{Z}(\zeta_r)^\times$ is always onto, where ζ_r denotes a primitive p^r th root of unity. Iwasawa’s conjecture would follow at once from Vandiver’s conjecture (which says that p does not divide the second factor of the ideal class group of $\mathbf{Z}(\zeta_1)$). Vandiver’s conjecture is known to be true for all primes $< 125,000$. (Cf. [2, p. 556 and 12].)

The relationship of this to group actions is as follows: If $\pi = \mathbf{Z}/p^{r+1}$, $\rho = \mathbf{Z}/p^r$, define a map $\varepsilon: R_\pi \rightarrow \mathbf{Z}(\zeta_r)$ by sending T to ζ_r . ε induces $\varepsilon_*: (R_\rho)^\times \rightarrow \mathbf{Z}(\zeta_r)^\times$ sending $(\pm\rho)$ to C , the group of roots of unity.

THEOREM 2. Let $\bar{\mu}$ be a free action of \mathbf{Z}/p^r on S^{2n+1} , $n \geq 2$. Let \bar{u} be its associated unit.

(A) If $\varepsilon_*(\bar{u})$ is not in the image of $N: \mathbf{Z}(\zeta_{r+1})^\times \rightarrow \mathbf{Z}(\zeta_r)^\times / C$, then $\bar{\mu}$ does not extend to a \mathbf{Z}/p^{r+1} action.

(B) Suppose $r = 1$, and suppose Iwasawa’s conjecture holds for p . Then $\bar{\mu}$ extends to a free \mathbf{Z}/p^2 action on S^{2n+1} .

REMARK. Despite the explicit nature of Theorem 1, we can provide no example of a free \mathbf{Z}/k action which fails to extend to a free \mathbf{Z}/hk action. John Ewing has shown [11], using our results, that for each p , there is a \mathbf{Z}/p^r action which does not extend to a \mathbf{Z}/p^{r+1} action.

THEOREM 3. Let k be an odd integer. Every free \mathbf{Z}/k action on S^{2n+1} , $n > 1$, extends to a free $\mathbf{Z}/2k$ action.

Here is an outline of the rest of the paper.

Part I of §1 is a surgery theoretic calculation showing that the natural transfer map of the two surgery exact sequences involved is surjective. Part II constructs the transfer map needed to study the Reidemeister torsion; we then use this to prove Theorem 1. §2 gives the proof of Theorems 2 and 3.

We would like to give a word of thanks here to John Ewing and to John Cruthirds for a number of very useful conversations.

1. The basic condition for extending actions.

PART I. Our goal in this part of §1 is to establish

PROPOSITION 1.1. *Let L^{2n+1} be a finite Poincaré complex with $\pi_1(L) = \mathbf{Z}/hk$ and universal cover $\simeq S^{2n+1}$. Let \bar{L} denote its h -fold cover. Then the transfer map,*

$$\pi^*: S_{\text{Top}}^s(L) \rightarrow S_{\text{Top}}^s(\bar{L}),$$

is onto.

(As in Wall [9], $S_{\text{Top}}^s(L)$ denotes the Top-structures on L .)

This will be done analyzing the following map of surgery exact sequences:

$$(1.2) \quad \begin{array}{ccccccc} \tilde{L}_{2n+2}^s(\mathbf{Z}/k) & \rightarrow & S_{\text{Top}}^s(\bar{L}) & \rightarrow & NM(\bar{L}) & \xrightarrow{\bar{\sigma}} & L_{2n+1}^s(\mathbf{Z}/k) \\ \uparrow \text{tr} & & \uparrow \pi^* & & \uparrow \pi^* & & \uparrow \text{tr} \\ \tilde{L}_{2n+2}^s(\mathbf{Z}/hk) & \rightarrow & S_{\text{Top}}^s(L) & \rightarrow & NM(L) & \xrightarrow{\sigma} & L_{2n+1}^s(\mathbf{Z}/hk) \end{array}$$

Here $\tilde{L}_n^s(G) =$ the cokernel of $L_n^s(e) \rightarrow L_n^s(G)$, and NM denotes the set of bordism classes of Top-normal maps.

It is well known that such an L is homotopically equivalent to some standard lens space $L(hk, \theta_0, \theta, \dots, \theta_n)$ where θ_i are integers prime to hk . Choose such a homotopy equivalence: it yields an element of $NM(L)$ and its cover is an element of $NM(\bar{L})$. This choice then allows us to identify $NM(L)$ with $[L, G/\text{Top}]$ (see [6 or 9]) and π^* with the group homomorphism, $\pi^*: [L, G/\text{Top}] \rightarrow [\bar{L}, G/\text{Top}]$. Our first step is

LEMMA 1.3. $\pi^*: [L, G/\text{Top}] \rightarrow [\bar{L}, G/\text{Top}]$ is an epimorphism. Its kernel is a finite group of order $h^{[n/2]}$ unless k is odd and h is even. In this case its order is $2^{[(n+1)/2]}h^{[n/2]}$.

PROOF. $[L, G/\text{Top}] = [L, G/\text{Top}_{(2)}] \oplus [L, G/\text{Top}_{(\text{odd})}]$ (since both localizations are finite) and this is isomorphic to $\tilde{H}^{4*}(L; \mathbf{Z}_{(2)}) \oplus \tilde{H}^{4*+2}(L; \mathbf{Z}/2) \oplus \tilde{K}O^{\circ}(L)_{(\text{odd})}$. Similarly for \bar{L} . So the lemma is an immediate consequence of the following three statements. We first write h as $h = h_2 \cdot h_{\text{odd}}$ where h_2 is a power of 2 and h_{odd} is odd.

- (i) $\pi^*: \tilde{H}^{4*}(L; \mathbf{Z}_{(2)}) \rightarrow \tilde{H}^{4*}(\bar{L}; \mathbf{Z}_{(2)})$ is surjective; its kernel has order $h_2^{[n/2]}$.
- (ii) $\pi^*: \tilde{H}^{4*+2}(L; \mathbf{Z}/2) \rightarrow \tilde{H}^{4*+2}(\bar{L}; \mathbf{Z}/2)$ is surjective; its kernel has order $2^{[n+1/2]}$ if k is odd and h is even, and otherwise it is injective.
- (iii) $\pi^*: \tilde{K}O(L) \rightarrow \tilde{K}O(\bar{L})$ is onto; its kernel has order $(h_{\text{odd}})^{[n/2]}$.

PROOF OF (i) AND (ii). It is an easy exercise (using the fibration $S^{2n+1} \rightarrow \bar{L} \rightarrow K(\mathbf{Z}/k, 1)$) to prove, for $A = \mathbf{Z}_{(2)}$ or $\mathbf{Z}/2$, that $H^{2*}(\bar{L}; A) = (A/kA)[\bar{x}]/(\bar{x})^{n+1}$ where $\deg \bar{x} = 2$, and similarly for L . The surjectivity of $\pi^*: H^2(L; A) \rightarrow H^2(\bar{L}; A)$ is trivial from the universal coefficient theorem and the behavior of π_* on the fundamental groups. From this it is clear that π^* is onto with kernel as specified in (i) and (ii).

To prove (iii) let C be the mapping cone of π . Proceeding as above, one easily computes: $H^i(C; \mathbf{Z}_{(odd)}) = \mathbf{Z}/h_{odd}$ if i is even and $0 < i \leq 2n + 2$, and $H^i(C; \mathbf{Z}_{(odd)}) = 0$ otherwise. Hence the Atiyah-Hirzebruch spectral sequence for $KO^*(C)_{(odd)}$ collapses ($E_2^{p,q} = 0$ if p or q is odd) and we get $\tilde{K}O^1(C)_{(odd)} = 0$, $\tilde{K}O^0(C)_{(odd)} =$ a group of order $(h_{odd})^{[(n+1)/2]}$ ($E_2^{4i,-4i} = \mathbf{Z}/(h_{odd})$, $0 < i \leq [(n + 1)/2]$). so we get an exact sequence

$$0 \rightarrow \tilde{K}O^{-1}(L)_{(odd)} \xrightarrow{\pi^*} \tilde{K}O^{-1}(\bar{L})_{(odd)} \rightarrow \tilde{K}O^0(C)_{(odd)} \\ \rightarrow \tilde{K}O^0(L)_{(odd)} \xrightarrow{\pi^*} \tilde{K}O^0(\bar{L})_{(odd)} \rightarrow 0.$$

However, $\tilde{K}O^{-1}(\bar{L})_{(odd)} = 0$ if n is even (use the Atiyah-Hirzebruch spectral sequence), and for n odd $\tilde{K}O^{-1}(\bar{L}) = \mathbf{Z}_{(odd)}$ with $\text{Im}(\pi^*)$ of index (h_{odd}) . So for any n , π^* on the right is an epimorphism with kernel of order $(h_{odd})^{[n/2]}$. This proves Lemma 1.3.

Now we return to diagram (1.2) and glue together some ideas of Petrie and Wall to observe:

LEMMA 1.4. *The map*

$$\text{tr}: \tilde{L}_{2n+2}^s(\mathbf{Z}/hk) \rightarrow \tilde{L}_{2n+2}^s(\mathbf{Z}/k)$$

is surjective. Also $\tilde{L}_{2n+2}^s(\mathbf{Z}/k)$ acts freely on $S_{\text{Top}}^s(\bar{L})$ and similarly for \mathbf{Z}/hk and L .

PROOF. Following Wall [8] we write $\chi_k: L_{2n+2}^s(\mathbf{Z}/k) \rightarrow \mathbf{C}(\mathbf{Z}/k)$ for the signature map, where $\mathbf{C}(\pi)$ denotes the representation ring of π (over \mathbf{C}). According to [8], $\text{Im } \chi_k = \{4(x + (-1)^{n+1}\bar{x}) \mid x \in \mathbf{C}(\mathbf{Z}/k)\}$. Similarly for χ_{hk} . The diagram

$$\begin{array}{ccc} L_{2n+2}^s(\mathbf{Z}/k) & \xrightarrow{\chi_k} & \mathbf{C}(\mathbf{Z}/k) \\ \uparrow \text{tr} & & \uparrow p \\ L_{2n+2}^s(\mathbf{Z}/hk) & \xrightarrow{\chi_{hk}} & \mathbf{C}(\mathbf{Z}/hk) \end{array}$$

commutes where p denotes restriction. Since each representation of \mathbf{Z}/k is the restriction of a \mathbf{Z}/hk representation it is clear that $p \text{ Im } \chi_{hk} = \text{Im } \chi_k$.

But according to Wall [8], χ is injective for n odd and $\text{Ker } \chi = L_{2n+2}(e)$ for n even for these groups. It follows at once that tr is a surjection on $\tilde{L}_{2n+2}^s(\mathbf{Z}/k)$.

Now $\tilde{L}_{2n+2}^s(\mathbf{Z}/k)$ is a free abelian, and Petrie [5, 2.3 and 2.10] shows that a subgroup of $\tilde{L}_{2n+2}^s(\mathbf{Z}/k)$ acts freely on $S_{\text{Top}}^s(\bar{L})$. But the rank of this subgroup equals the rank of $\tilde{L}_{2n+2}^s(\mathbf{Z}/k)$ (as computed by Wall [7]). It follows then that $\tilde{L}_{2n+2}^s(\mathbf{Z}/k)$ itself is acting freely. This completes the proof.

The final step in this section concerns the right side of (1.2), where we now look at the kernel of $\sigma: NM(L) \rightarrow L_{2n+1}^s(\pi_1(L))$. Recall $NM(L)$ is a group via its identification with $[L, G/Top]$.

LEMMA 1.5. *The map σ is a homomorphism and the restriction of π^* to $\pi^*: Ker \sigma \rightarrow Ker \bar{\sigma}$ is an epimorphism.*

PROOF. $L_{2n+1}^s(\mathbf{Z}/hk) = 0$ unless n is odd and hk even, and in this case $L_{2n+1}^s(\mathbf{Z}/hk) = \mathbf{Z}/2$, with the surgery obstruction given by

$$\sigma(f) = \langle a_1(1 + a_2^2)^r f^*k, [L] \rangle$$

(see [9, p. 210]). Here $f \in [L, G/Top]$, $2r = n + 1$, $a_i \in H^i(L; \mathbf{Z}/2)$ is the nonzero element, and $k \in H^{4^*+2}(G/Top; \mathbf{Z}/2)$ is a certain primitive class. The primitivity of k yields immediately that σ is a homomorphism. To prove $\pi^*Ker \sigma = Ker \bar{\sigma}$ we can assume, using 1.3, that n is odd and hk is even.

Case 1. k even. Then if $\bar{f} \in Ker \bar{\sigma}$ and f is an element of $[L, G/Top]$ such that $\pi^*f = \bar{f}$ we see

$$\begin{aligned} \sigma(f) &= \langle a_1(1 + a_2^2)^r f^*k, [L] \rangle \\ &= \langle (\text{tr } \bar{a}_1)(1 + a_2^2)^r f^*k, [L] \rangle \quad \text{since } \text{tr } \bar{a}_1 = a_1 \\ &= \langle \text{tr} \{ \bar{a}_1 \cdot \pi^*((1 + a_2^2)^r f^*k) \}, [L] \rangle \quad \text{since } (\text{tr } x)y = \text{tr}(x \cdot \pi^*y) \\ &= \langle \text{tr} \{ \bar{a}_1 \cdot (1 + \bar{a}_2^2)^r \bar{f}^*k \}, [L] \rangle \quad \text{since } \pi^*a_2 = \bar{a}_2 \\ &= \langle \bar{a}_1(1 + \bar{a}_2^2)^r \bar{f}^*k, [\bar{L}] \rangle \quad \text{since } \text{tr}[L] = [\bar{L}] \\ &= \sigma(\bar{f}) = 0. \end{aligned}$$

Hence $f \in Ker \sigma$ and $\pi^*: Ker \sigma \rightarrow Ker \bar{\sigma}$ is onto in this case.

Case 2. k odd. Then $[\bar{L}, G/Top] = Ker \bar{\sigma}$ is an odd order group (as seen, for example, in the proof of 1.3) and the odd primary summand of $[L, G/Top]$ must map onto it by Lemma 1.3. But this summand is in $Ker \sigma$ since σ is a homomorphism. This completes the proof.

PROOF OF PROPOSITION 1.1. Using 1.3, 1.4 and 1.5 we see that diagram 1.2 reduces to:

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \uparrow & & \uparrow & & \\ \tilde{L}_{2n+2}^s(\mathbf{Z}/k) & \rightarrow & S_{top}^s(\bar{L}) & \rightarrow & Ker \bar{\sigma} & \rightarrow & 0 \\ & \uparrow \text{tr} & \uparrow \pi^* & & \uparrow \pi^* & & \\ \tilde{L}_{2n+2}^s(\mathbf{Z}/hk) & \rightarrow & S_{top}^s(L) & \rightarrow & Ker \sigma & \rightarrow & 0 \end{array}$$

Hence 1.1 becomes a simple diagram chase, and we are done.

PART II. We wish to make a statement concerning the behavior of the Reidmeister torsion under a transfer map so we will now define this transfer map.

1.6. *Construction.* Let A be a commutative ring (in practice $A = \mathbf{Q}$ or \mathbf{Z}). Set $AR_\pi = A(\pi)/(\Sigma)$, $AR_\rho = A(\rho)/(\bar{\Sigma})$, where ρ is a subgroup of the finite group π ,

and $\bar{\Sigma}$ is the sum of the elements of ρ . The multiplicative group $\pm\pi$ in $(A\pi)^\times$ determines a subgroup of $K_1(AR_\pi)$ denoted $(\pm\pi)$, the homomorphic image of $\pm\pi$. We construct here a homomorphism

$$\text{tr}: K_1(AR_\pi)/(\pm\pi) \rightarrow K_1(AR_\rho)/(\pm\rho).$$

$A(\pi)$ is a free left $A(\rho)$ module with base g_1, \dots, g_h , a complete set of right coset representatives of ρ in π . Let $T = \bar{\Sigma} \cdot A(\pi)$, a right ideal. $A(\pi)/T$ is a left $A(\rho)$ module with $\bar{\Sigma} \cdot (A(\pi)/t) = 0$ so that $A(\pi)/T$ becomes a left AR_ρ module. In fact, $A(\pi)/T$ is a free left AR_ρ module with base $g_1 \bmod T \cdots g_h \bmod T$. This base defines an isomorphism $A(\pi)/T \approx (AR_\rho)^h$ of left AR_ρ modules, and an induced isomorphism from $\text{End}(A(\pi)/T)$, the ring of AR_ρ endomorphisms of $A(\pi)/T$, to the matrix ring $M(h, AR_\rho)$. Denote this

$$\lambda: \text{End}(A(\pi)/T) \approx M(h, AR_\rho).$$

Now right multiplication by elements of $A(\pi)$ defines a map $A(\pi) \rightarrow \text{End}(A(\pi)/T)$ sending Σ to 0. This induces a ring homomorphism $r: AR_\pi \rightarrow \text{End}(A(\pi)/T)$.

So the ring homomorphism $\lambda r: AR_\pi \rightarrow M(h, AR_\rho)$ yields a ring homomorphism $(\lambda r)_*: M(m, AR_\pi) \rightarrow M(mh, AR_\rho)$ given by $(\lambda r)_*(a_{ij}) = (\lambda r(a_{ij}))$ (a matrix of blocks). The induced homomorphism on the K_1 level is denoted

$$\text{tr}: K_1(AR_\pi) \rightarrow K_1(AR_\rho).$$

We claim that $\text{tr}(\pm\pi) \subset (\pm\rho)$. For, if $x \in \pi$, then for each coset representative g_i we have $g_i x = \bar{x}(i) \cdot g_{j(x,i)}$ for some element $\bar{x}(i) \in \rho$ and some integer $j(x, i)$ between 1 and h . It follows that $\lambda r(\pm x) = \pm P \cdot D(\bar{x}(1), \dots, \bar{x}(h))$ where P is a permutation matrix and D denotes a diagonal matrix. Hence, $\text{tr}(\pm x) = \pm \bar{x}(1) \cdot \bar{x}(2) \cdots \bar{x}(h) \in (\pm\rho)$.

This leaves us with an induced map

$$\text{tr}: K_1(AR_\pi)/(\pm\pi) \rightarrow K_1(AR_\rho)/(\pm\rho).$$

The elementary properties below are easily checked.

1.6.A. If $A \xrightarrow{f} A'$ is a ring homomorphism, the following diagram commutes:

$$\begin{array}{ccc} K_1(AR_\pi)/(\pm\pi) & \xrightarrow{\text{tr}} & K_1(AR_\rho)/(\pm\rho) \\ \downarrow f^* & & \downarrow f^* \\ K_1(A'R_\pi)/(\pm\pi) & \xrightarrow{\text{tr}} & K_1(A'R_\rho)/(\pm\rho) \end{array}$$

1.6.B. Suppose given a ring epimorphism $\varphi: A(\pi) \rightarrow \Lambda$ with $\bar{\Lambda} = \varphi(A(\rho))$ and with $\varphi(\bar{\Sigma}) = 0$. Suppose Λ is free over $\bar{\Lambda}$ with base $\varphi(g_1), \dots, \varphi(g_h)$. Then there is a commutative square:

$$\begin{array}{ccc} K_1(AR_\pi) & \xrightarrow{\varphi_*} & K_1(\Lambda) \\ \downarrow \text{tr} & & \downarrow N \\ K_1(AR_\rho) & \xrightarrow{\bar{\varphi}_*} & K_1(\bar{\Lambda}) \end{array}$$

Here $\bar{\varphi}: AR_\rho \rightarrow \bar{\Lambda}$ is the epimorphism induced by φ , and N is the “norm” map induced by the isomorphism of rings, $M(n, \Lambda) \rightarrow M(nh, \bar{\Lambda})$ which this basis of Λ over $\bar{\Lambda}$ defines.

REMARK. If π is a finite cyclic group, $SK_1(\mathbf{Z}\pi) = \{1\}$ (see [10, p. 623]). Hence, $K_1(\mathbf{Z}\pi) = \mathbf{Z}\pi^\times$ and one sees at once that $K_1(R_\pi) = R_\pi^\times$, by examining the exact sequence of the Cartesian square:

$$\begin{array}{ccc} \mathbf{Z}\pi & \xrightarrow{\varepsilon} & \mathbf{Z} \\ \downarrow & & \downarrow \\ R_\pi & \rightarrow & \mathbf{Z}/|\pi|\mathbf{Z} \end{array}$$

Also $K_1(\mathbf{Q}R_\pi) = (\mathbf{Q}R_\pi)^\times$ because $\mathbf{Q}R_\pi$ is a product of fields.

Now we show how the torsion of a complex behaves under this transfer map.

So let L denote a finite CW complex with fundamental group π , a finite group, universal cover \tilde{L} and intermediate cover \bar{L} corresponding to a subgroup ρ of π . We assume π acts trivially on $H_*(\tilde{L}; \mathbf{Q})$. In this case, Milnor [4, p. 405], defines its Reidemeister torsion $\Delta(L)$ in $K_1(\mathbf{Q}R_\pi)/(\pm\pi)$. We shall prove

THEOREM 1.7. $\Delta(\bar{L}) = \text{tr } \Delta(L)$.

PROOF. Recall from [4] how Δ is defined. If $C(\tilde{L})$ denotes the cellular chain complex then $C(\tilde{L}) \otimes_{\mathbf{Z}\pi} \mathbf{Q}R_\pi$ (hereafter denoted C_π) is acyclic and based, over $\mathbf{Q}R_\pi$. Similarly for $C(\bar{L}) \otimes_{\mathbf{Z}\rho} \mathbf{Q}R_\rho$ (written C_ρ), over $\mathbf{Q}R_\rho$. $\Delta(L) = \prod_i [c_i/c'_i]$ where c_i denotes the basis of cells for $(C_\pi)_i$ and c'_i denotes the basis determined by the acyclicity of C_π , and $[c_i/c'_i]$ denotes the class of the change of basis matrix (c_i/c'_i) in $GL(m_i, \mathbf{Q}R_\pi)$ ($m_i = \text{rank } C_i$, over $\mathbf{Z}\pi$).

Now if g_1, \dots, g_h are a complete set of coset representatives of ρ in π , each basis c_i for $(C_\pi)_i$ determines a basis for $(C_\rho)_i$ over $\mathbf{Q}R_\rho$ —namely: $c_i g_1 \cup c_i g_2 \cup \dots \cup c_i g_h$ which we write \bar{c}_i . Observe that $\bar{c}'_i, i = 0, 1, 2, \dots$, are the bases for $(C_\rho)_i$ determined by its acyclicity. Going back to 1.6, we check that $(\lambda r)_*(c_i/c'_i) = (\bar{c}_i/\bar{c}'_i)$ in $GL(m_i h, \mathbf{Q}R_\rho)$. Hence $\text{tr}(L) = \prod_i \text{tr}[c_i/c'_i] = \prod_i \lambda r_*(c_i/c'_i) = \prod_i [\bar{c}_i/\bar{c}'_i] = \Delta(\bar{L})$ because, by its definition, it is clear that \bar{c}_i is the basis of cells for $(C_\rho)_i$ over $\mathbf{Q}R_\rho$. This completes the proof.

1.8. PROOF OF THEOREM 1. First we show that if \bar{u} is the associated unit of an action $\bar{\mu}: \mathbf{Z}/k \times S^{2n+1} \rightarrow S^{2n+1}$ which does extend to a free \mathbf{Z}/hk action μ , we have $\bar{u} = \text{tr } u$ for some $u \in (R_\pi)^\times / (\pm\pi)$. In fact u can be taken to be the associated unit of μ . To see this we calculate

$$\Delta(\mu) = (T - 1)^{n+1} j(u) \quad \text{and} \quad \Delta(\bar{\mu}) = (T^h - 1)^{n+1} j(\bar{u})$$

by 0.1. By Theorem 1.7, $\text{tr } \Delta(\mu) = \Delta(\bar{\mu})$ and $\text{tr}(T - 1)^{n+1} = (T^h - 1)^{n+1}$ (since $\Delta(\mu_0) = (T - 1)^{n+1}$). Hence we see $j\bar{u} = \text{tr } j(u) = j \text{tr } u$ by 1.6.A. But

$$j: R_\rho^\times / (\pm\rho) \rightarrow \mathbf{Q}R_\rho^\times / (\pm\rho)$$

is evidently injective. So $\text{tr } u = \bar{u}$ as required.

Conversely, now we suppose $\bar{\mu}$ is an action with associated unit \bar{u} and that $\bar{u} = \text{tr } u$ for some unit in $(R_\pi)^\times / (\pm\pi)$. By 0.1, there is a finite complex L with fundamental

group π , universal cover $\simeq S^{2n+1}$, such that $\Delta(L) = (T - 1)^{n+1}j(u)$. Let \bar{L} be the h -fold cover of L . By Theorem 1.7, $\Delta(\bar{L}) = (T^h - 1)^{n+1}j(\bar{u}) = \Delta(\bar{M})$, where \bar{M} is the orbit manifold of the action $\bar{\mu}: \rho \times S^{2n+1} \rightarrow S^{2n+1}$. But this implies that \bar{M} is simple homotopy equivalent to \bar{L} via a map $\bar{f}: \bar{M} \rightarrow \bar{L}$ say. Hence $(\bar{M}, \bar{f}) \in S_{\text{Top}}^s(\bar{L})$. But by Proposition 1.1, there is an (M, f) in $S_{\text{Top}}^s(L)$ whose cover is (\bar{M}, \bar{f}) . This means that M is the orbit space of a free action of π on \tilde{M} . Its restriction to an action of ρ is just $\bar{\mu}$. Thus $\bar{\mu}$ extends as required. This completes the proof of Theorem 1.

2. Proof of Theorem 2. We now turn to free actions of cyclic groups of order p^r with the aim of proving Theorem 2.

PROOF OF THEOREM 2(A). By Theorem 1, $\bar{\mu}$ extends to a free action μ only if $\bar{u} = \text{tr } u$ for some unit u in $(R_\pi)^\times$. But by 1.6.B the diagram below commutes. So $\bar{\mu}$ does not extend unless $\varepsilon_*(\bar{u})$ is in the image of N .

$$\begin{array}{ccc} (R_\pi)^\times & \xrightarrow{\varepsilon} & \mathbf{Z}(\zeta_{r+1})^\times \\ \downarrow \text{tr} & & \downarrow N \\ (R_\rho)^\times & \xrightarrow{\varepsilon} & \mathbf{Z}(\zeta_r)^\times \end{array}$$

PROOF OF THEOREM 2(B). If $\pi = \mathbf{Z}/p^2$, and ζ_i = the primitive p^i th root of unity, we have the exact sequence,

$$1 \rightarrow R_\pi^\times \xrightarrow{\lambda_1 \oplus \lambda_2} \mathbf{Z}(\zeta_1)^\times \oplus \mathbf{Z}(\zeta_2)^\times \xrightarrow{k} (\mathbf{Z}(\zeta_1)/(p))^\times,$$

$\lambda_i f(T) = f(\zeta_i)$, $k(f_1(\zeta_1), f_2(\zeta_2)) = f_1(\zeta_1)^{-1}f_2(\zeta_2) \pmod p$ (obtained by noting $\mathbf{Z}[T]/\Phi_1 \cdot \Phi_2 = R_\pi$, $\mathbf{Z}[T]/(\Phi_1, \Phi_2) = \mathbf{Z}(\zeta_1)/(p)$).

Now $R_\rho = \mathbf{Z}(\zeta_1)$ and it is easy to see from the definition of tr that $\text{tr} = N \circ \lambda_2: R_\pi^\times \rightarrow \mathbf{Z}(\zeta_2)^\times \rightarrow \mathbf{Z}(\zeta_1)^\times$ where N is a norm map as in 1.6.B. Iwasawa's conjecture asserts that $N: \mathbf{Z}(\zeta_2)^\times \rightarrow \mathbf{Z}(\zeta_1)^\times$ is surjective. So if Iwasawa's conjecture holds, we need only prove that tr is onto (by Theorem 1). So let $\bar{u} \in \mathbf{Z}(\zeta_1)^\times$. We are given that $\bar{u} = N(f(\zeta_2))$ for some unit $f(\zeta_2)$ in $\mathbf{Z}(\zeta_2)^\times$. Consider $z = (\bar{u}, f(\zeta_2)) \in \mathbf{Z}(\zeta_1)^\times \oplus \mathbf{Z}(\zeta_2)^\times$. It is well known that $N(a) \equiv a^p \pmod p$ if $a \in \mathbf{Z}(\zeta_2)$, so we see that

$$\begin{aligned} k(z) &= (Nf(\zeta_2)^{-1}f(\zeta_1)) \pmod p = (f(\zeta_2)^p)^{-1}f(\zeta_1^p) \pmod p \\ &\equiv (f(\zeta_2)^p)^{-1}f(\zeta_2)^p \pmod p = 1. \end{aligned}$$

So $z \in \text{Ker } k = \text{Im } \lambda_1 \oplus \lambda_2$. This implies there is an element $u \in R_\pi^\times$ such that $\lambda_1(u) = \bar{u}$, $\lambda_2(u) = f(\zeta_2)$ and so $\text{tr } u = N\lambda_2(u) = Nf(\zeta_2) = \bar{u}$. Thus tr is onto and we are done.

Now we restate and prove

THEOREM 3. *If k is an odd integer, $n > 1$, any free \mathbf{Z}/k action on S^{2n+1} extends to a free $\mathbf{Z}/2k$ action.*

PROOF. According to Theorem 1 we have to prove that $\text{tr}: (R_\pi)^\times \rightarrow (R_\rho)^\times$ is onto, where $\pi = \mathbf{Z}/2k$, $\rho = \mathbf{Z}/k$. But there is an exact sequence of units (used earlier),

$$1 \rightarrow (\mathbf{Z}\rho)^\times \xrightarrow{i_*} (R_\rho)^\times \xrightarrow{\varepsilon_*} (\mathbf{Z}/k)^\times / (\pm 1),$$

where ϵ_* is induced by the augmentation $\epsilon: R\rho \rightarrow \mathbf{Z}/k$. Hence it is enough to prove:
 (a) $\text{Im } i_* \subset \text{Im tr}$ where $i_*: (\mathbf{Z}\rho)^\times \rightarrow (R\rho)^\times$ is the map induced by projection and
 (b) $\epsilon_* \circ \text{tr}$ is onto.

The proof of (b) is elementary: Each unit of $(\mathbf{Z}/k)^\times$ is of the form $2r + 1 \pmod k$ for some integer such that $0 < 2r + 1 < 2k$. We claim that, in $(R\pi)^\times$ the unit $u = (T^{2r+1} - 1)/(T - 1)$ satisfies $\epsilon_* \text{tr } u = 2r + 1$. For $u = 1 + T + \dots + T^{2r} = \alpha + T^k\beta$ where $\alpha = 1 + T^2 + \dots + T^{2r}$, $\beta = (T^{k+1} + T^{k+3} + \dots + T^{k+2r-1})$. Note α and β lie in the image of $\mathbf{Z}\rho$. Since $\pi = \rho \times \mathbf{Z}/2$, it is an elementary exercise to see $\text{tr } u = \alpha^2 - \beta^2 \pmod{\bar{\Sigma}}$. Set $\gamma = \alpha - T^{2r}$, and note $\text{tr } u = (\gamma + T^{2r})^2 - T^{2r}(\gamma^2) = (1 - T^2)\gamma^2 + 2\gamma T^{2r} + T^{4r}$. Hence $\epsilon_* \text{tr } u = 0 \cdot \epsilon(\gamma^2) + 2\epsilon(\gamma) + 1 = 2r + 1$ since $\epsilon(\gamma) = r$. This proves (b).

To prove (a) we observe that the diagram

$$\begin{array}{ccc} \mathbf{Z}(\pi)^\times & \rightarrow & (R\pi)^\times \\ \downarrow N & & \downarrow \text{tr} \\ \mathbf{Z}(\rho)^\times & \xrightarrow{i} & (R\rho)^\times \end{array}$$

commutes, where N is the norm map, so we can concentrate on proving that N is onto. Now $\mathbf{Z}\pi \approx A(\mathbf{Z}/2)$ where $A = \mathbf{Z}(\rho)$ so we show $N: A(\mathbf{Z}/2)^\times \rightarrow A^\times$ is onto. Each element of $A(\mathbf{Z}/2)$ can be written $a + bS$, where $a, b \in A$, $S \in \mathbf{Z}/2$. Also N is given by $N(a + bS) = a^2 - b^2$. An element $v = a + bS$ is a unit of $A(\mathbf{Z}/2)$ if and only if $a + b = u$ and $a - b = u'$ are units of A ; conversely, given units u, u' in A^\times with $u \equiv u' \pmod{2A}$ they determine a unit v uniquely. Note $N(v) = uu'$.

Now, the diagram

$$\begin{array}{ccc} A^\times & \xrightarrow{r} & A/2A^\times \\ \downarrow S & & \downarrow F \\ A^\times & \xrightarrow{r} & A/2A^\times \end{array}$$

commutes where r is reduction, $S(x) = x^2$ and F is given by the Frobenius map $F(x) = x^2$. $F: A/2A \rightarrow A/2A$ is an isomorphism of rings because $A/2A$ is a product of finite fields of characteristic 2 ($A = \mathbf{Z}_{(\rho)}$). It follows that $\text{Im } r = \text{Im } F \circ r$.

Now we prove $N: A(\mathbf{Z}/2)^\times \rightarrow A^\times$ is onto. Let $w \in A^\times$. By the last paragraph, $r(w) = r(u^2)$ for some $u \in A^\times$. Write w as $w = uu'$ for some $u' \in A^\times$. Then $r(u'/u) = r(w/u^2) = 1$ so $u'/u \equiv 1 \pmod{2A}$. It follows that $u \equiv u' \pmod{2A}$ and so, as seen above, these determine a unit $v = (u + u')/2 + S(u - u')/2$, in $A(\mathbf{Z}/2)^\times$ with $Nv = uu' = w$ as required. This completes the proof.

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