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# DIMENSION OF STRATIFIABLE SPACES

BY

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ABSTRACT. We define a subclass, denoted by $EM_3$, of the class of stratifiable spaces, and obtain several dimension theoretical results for $EM_3$ including the coincidence theorem for dim and Ind. The class $EM_3$ is countably productive, hereditary, preserved under closed maps and, furthermore, the largest subclass of stratifiable spaces in which a harmonious dimension theory can be established.

**1. Introduction.** Beyond metric spaces, the following line of generalized metric spaces has been established by many authors [**S, C, B, H, Ok**]:

$$\text{metric} \to \text{Lašnev}^1 \to M_1 \to \text{stratifiable} \to \text{paracompact } \sigma.$$

After Katětov and Morita's work for metric spaces, the first attack to this line in dimension theory was done by Leĭbo [$\mathbf{L_1}$] who proved the equality $\dim X = \operatorname{Ind} X$ for any Lašnev space $X$. Nagami extended this result by defining $L$-spaces [$\mathbf{N_3}$] and free $L$-spaces [$\mathbf{N_4}$]. Free $L$-spaces form a countably productive and hereditary class containing every Lašnev space and included in the class of $M_1$-spaces. It is now desired to develop a satisfactory dimension theory of a still larger class of generalized metric spaces, say, $M_1$-spaces or stratifiable spaces.

In this direction we define a subclass of stratifiable spaces in terms of a special kind of $\sigma$-closure-preserving collection.

DEFINITION 1.1. Let $X$ be a space. A collection $\mathfrak{E}$ of subsets of $X$ is called an *encircling net* (or, for short, *E-net*) if for any point $x$ and any open neighborhood $U$ of $x$, there exists a subcollection $\mathfrak{F}$ of $\mathfrak{E}$ such that $x \in X - \mathfrak{F}^* \subset U$ and $\mathfrak{F}^*$ is a closed set of $X$ (where $\mathfrak{F}^*$ denotes the union of the members of $\mathfrak{F}$).

By $EM_3$ we denote *the class of stratifiable spaces with $\sigma$-closure-preserving E-nets*, and by $M_3$ *the class of stratifiable spaces*.

The class $EM_3$ is countably productive, hereditary and preserved under closed maps as well as perfect maps (Corollary 3.9).

Our first main result is a characterization of members of $EM_3$ as those spaces which are the perfect (closed) images of zero-dimensional stratifiable spaces (Theorem 3.8). This means that $EM_3$ is just the maximal perfect subclass of $M_3$ in the sense of Nagami [$\mathbf{N_1}$].

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[1] A space is called a Lašnev space if it is the closed image of a metric space.

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The second main results appear in Theorems 4.2 and 4.3 and consist of the following theorems for $EM_3$:

(a) the equidimensional $G_\delta$-envelope theorem,

(b) the dimension raising theorem,[2]

(c) the decomposition theorem,

(d) the coincidence theorem for dim and Ind.

These theorems for $EM_3$ extend the corresponding theorems for free $L$-spaces [$\mathbf{N_4}$] as well as those for Lašnev spaces [$\mathbf{L_1, L_2, O_1}$].

It is an open problem whether the inclusion $EM_3 \subset M_3$ is proper. But the characterization above implies that $EM_3$ is the largest[3] subclass of $M_3$ in which the dimension raising theorem holds. We also see in Corollary 4.5 that $EM_3$ is the largest[3] subclass of $M_3$ in which the decomposition theorem and the equidimensional $G_\delta$-envelope theorem *simultaneously* hold.

Our arguments are based on Gruenhage and Junnila's result that a stratifiable space is an $M_2$-space [**G, J**]. Indeed, though we use the word "stratifiable" in view of its significance, what we need is only the existence of a $\sigma$-closure-preserving quasi-base.

*Conventions*. Throughout this paper a space is a *Hausdorff topological space*, and a map means an *onto continuous* one. Let $X$, $Y$ be spaces and let $f\colon X \to Y$ be a map. For a collection $\mathfrak{F}$ of subsets of $X$, the symbol $\mathfrak{F}^*$ denotes the union of all members of $\mathfrak{F}$, and $f(\mathfrak{F})$ means the collection of subsets of $Y$ of the form $\{f(F)\colon F \in \mathfrak{F}\}$. For a subset $Z$ of $X$ we denote by $\overline{Z}$ (or $\operatorname{Cl} Z$) the closure of $Z$, by $\operatorname{Int} Z$ the interior of $Z$, and by $\operatorname{Bd} Z$ the boundary of $Z$.

**2. Encircling nets and large encircling nets.** Encircling nets are naturally strengthened as follows:

Definition 2.1. Let $X$ be a space. A collection $\mathfrak{E}$ of subsets of $X$ is called a *large encircling net* (or, simply, an *LE-net*) if for any disjoint closed sets $C$ and $K$ of $X$, there exists a subcollection $\mathfrak{F}$ of $\mathfrak{E}$ such that $C \subset \mathfrak{F}^* \subset X - K$ and $\mathfrak{F}^*$ is a closed set of $X$.

Remarks. Since an $LE$-net is a net in the usual sense, it follows from Siwiec-Nagata [**SN**] that a space with a $\sigma$-closure-preserving $LE$-net is a $\sigma$-space. But a space with a $\sigma$-closure-preserving $E$-net is not necessarily a $\sigma$-space as will be seen in Example 2.8. On the other hand it is trivial that a regular $\sigma$-space $X$ with $\operatorname{ind} X \leqslant 0$ admits a $\sigma$-closure-preserving $E$-net, and that a normal $\sigma$-space $X$ with $\dim X \leqslant 0$ admits a $\sigma$-closure-preserving $LE$-net.

Proposition 2.2. *A metric space admits a $\sigma$-locally finite LE-net.*

Proof. Let $M$ be a metric spaces and $\{\mathfrak{E}_i\colon i = 1, 2, \dots\}$ a sequence of locally finite closed covers of $M$ such that, for each $i$, the diameter of each member of $\mathfrak{E}_i$ is smaller than $1/i$. Let $C$, $K$ be disjoint closed sets of $M$ and put

$$\mathfrak{F}_i = \{E \in \mathfrak{E}_i\colon E \cap C \neq \varnothing \text{ and } E \cap K = \varnothing\}.$$

[2] The dimension raising theorem for a topological class $\mathcal{C}$ is: If $X \in \mathcal{C}$ and $\dim X \leqslant n$, then $X$ is the image of a space $X_0 \in \mathcal{C}$ with $\dim X_0 \leqslant 0$ under a perfect map of order not greater than $n + 1$.

[3] When using this word we take no account of infinite-dimensional spaces in the sense of dim.

It is then clear that $\bigcup_{i=1}^{\infty}\mathfrak{F}_i^*$ is a closed set of $X$ including $C$ but not meeting $K$. Hence $\bigcup_{i=1}^{\infty}\mathfrak{E}_i$ is a $\sigma$-locally finite $LE$-net on $M$, which completes the proof.

PROPOSITION 2.3. *The property of having a $\sigma$-closure-preserving LE-net is preserved under closed maps.*

We thus have

PROPOSITION 2.4. *A Lašnev space admits a $\sigma$-closure-preserving LE-net, and hence it is a member of $EM_3$.*

LEMMA 2.5. *If $\mathfrak{E}$ is an E-net (resp. LE-net) on a space, then $\{\bar{E}: E \in \mathfrak{E}\}$ is an E-net (resp. LE-net) on the space.*

PROPOSITION 2.6. *The property of having a $\sigma$-closure-preserving E-net is countably productive, hereditary and preserved under perfect maps.*

PROOF. Let $X_i$, $i = 1, 2, \ldots$, be spaces with $\sigma$-closure-preserving $E$-nets $\mathfrak{E}_i$. It is then clear that

$$\left\{E_j \times \prod_{i=1, i\neq j}^{\infty} X_i : E_j \in \mathfrak{E}_j, j = 1, 2, \ldots\right\}$$

is a $\sigma$-closure-preserving $E$-net on $\prod_{i=1}^{\infty} X_i$.

By the preceding lemma it is obvious that the property is hereditary.

Let $X$ be a space with a $\sigma$-closure-preserving $E$-net $\mathfrak{E}$ and let $f\colon X \to Y$ be a perfect map onto a space $Y$. By Lemma 2.5 we may assume that every finite intersection of members of $\mathfrak{E}$ is again a member of $\mathfrak{E}$. To show that $f(\mathfrak{E})$ is an $E$-net on $Y$ let $y \in Y$ and let $U$ be an open neighborhood of $y$. There exist subcollections $\mathfrak{E}_i$, $1 \leq i \leq k$, of $\mathfrak{E}$ such that $f^{-1}(y) \subset X - \bigcap_{i=1}^{k}\mathfrak{E}_i^* \subset f^{-1}(U)$ and $\mathfrak{E}_i^*$ is a closed set of $X$. It then follows from assumption that $f(\bigcap_{i=1}^{k}\mathfrak{E}_i^*)$ is a closed set of $Y$ written as a union of members of $f(\mathfrak{E})$ such that $y \in Y - f(\bigcap_{i=1}^{k}\mathfrak{E}_i^*) \subset U$. This completes the proof.

PROPOSITION 2.7. *Let $X$ be a space (resp. a semistratifiable space). Then the following statements are equivalent:*

(1) *$X$ admits a $\sigma$-closure-preserving LE-net (resp. E-net).*

(2) *$X$ admits a $\sigma$-locally finite LE-net (resp. E-net).*

(3) *$X$ admits a $\sigma$-discrete LE-net (resp. E-net).*

PROOF. It follows from Lemma 2.5 and a remark above that a space with a $\sigma$-closure-preserving $LE$-net admits a $\sigma$-closure-preserving net of closed sets, and therefore it is semistratifiable. Hence the proposition is immediate from Lemma 2.5 and the following fact, which is essentially due to Siwiec and Nagata [**SN**]: *Let $X$ be a semistratifiable space and $\mathfrak{E}$ a $\sigma$-closure-preserving collection of closed sets of $X$. Then there exists a $\sigma$-discrete collection $\mathfrak{F}$ of closed sets of $X$ such that each member of $\mathfrak{E}$ is a union of members of $\mathfrak{F}$.*

As for famous pathological spaces, we have the following results which imply particularly that the existence of $\sigma$-closure-preserving $E$-nets does not mean, in general, that of $\sigma$-closure-preserving $LE$-nets (but, for stratifiable spaces, the former means the latter as will be seen in Theorem 3.8).

EXAMPLES 2.8. (1) *The Michael line $I(M)$ has a $\sigma$-discrete E-net, but does not have a $\sigma$-closure-preserving LE-net.*

(2) *The same is true for the Sorgenfrey line $R(S)$.*

(3) $[0, \omega_1]$ *does not admit a $\sigma$-closure-preserving E-net.*

(4) *The quotient space $I(M)/Q$ obtained by identifying the rational points in $I(M)$ does not admit a $\sigma$-closure-preserving E-net. In particular the property of having a $\sigma$-closure-preserving E-net is not preserved under closed maps.*

PROOF. (1) and (2) (simultaneously). Let $\mathfrak{F}$ be a $\sigma$-discrete net of closed sets in the unit interval $I$ (resp. the real line $R$) with the usual topology. It is easy to see that $\mathfrak{F}$ is a $\sigma$-discrete $E$-net on $I(M)$ (resp. $R(S)$). But $I(M)$ (resp. $R(S)$) does not admit a $\sigma$-closure-preserving $LE$-net because it is not a $\sigma$-space.

(3) For any $\sigma$-closure-preserving collection $\mathfrak{F}$ of $[0, \omega_1]$, $\mathfrak{F}$ fails to be an $E$-net at $\omega_1$; indeed, $\mathrm{Cl}(\{\bar{F}: F \in \mathfrak{F}, \omega_1 \notin \bar{F}\}^*) \cap \{\omega_1\} = \varnothing$.

(4) If $I(M)/Q$ had a $\sigma$-closure-preserving $E$-net, then every point in $I(M)/Q$, in particular the quotient image of $Q$, would be a $G_\delta$-set of $I(M)/Q$; but this is impossible because $Q$ is not a $G_\delta$-set of $I(M)$.

## 3. Characterizations of $EM_3$.

LEMMA 3.1 [$\mathbf{O_2}$, LEMMA 3.1]. *Let $X$ be a submetrizable space (that is, $X$ admits a weaker metric topology), and let $\mathfrak{U}$ be a $\sigma$-discrete collection of cozero sets of $X$. Then there exist a metric space $M$ and a one-to-one map $f\colon X \to M$ such that $f(U)$ is an open set of $M$ for every $U \in \mathfrak{U}$.*

The following lemma plays a fundamental role in this paper.

LEMMA 3.2. *Let $X$ be a paracompact $\sigma$-space and let $\mathfrak{F} = \bigcup_{i=1}^\infty \mathfrak{F}_i$ be a collection of closed sets of $X$ such that $\mathfrak{F}_i$ is closure-preserving for each $i$. Then there exist a metric space $M$ and a one-to-one map $f\colon X \to M$ such that $f(F)$ is a closed set of $M$ for every $F \in \mathfrak{F}$ and such that $f(\mathfrak{F}_i)$ is closure-preserving in $M$ for every $i$.*

PROOF. Let $\mathfrak{B} = \bigcup_{i=1}^\infty \mathfrak{B}_i$ be a net of $X$ consisting of closed sets such that $\mathfrak{B}_i$ is discrete for each $i$. For each $i$ let $\mathfrak{V}_i = \{V_i(B): B \in \mathfrak{B}_i\}$ be a discrete collection of open sets of $X$ such that $B \subset V_i(B)$ for each $B \in \mathfrak{B}_i$. For $i, j = 1, 2, \ldots$, $B \in \mathfrak{B}_i$, put

$$W_i^j(B) = V_i(B) \cap \left(X - \left\{F: F \in \mathfrak{F}_j, F \cap B = \varnothing\right\}^*\right).$$

Then $W_i^j(B)$ is an open set of $X$, and $\{W_i^j(B): B \in \mathfrak{B}_i\}$ is discrete in $X$. Hence Lemma 3.1 applies to give a metric space $M$ and a one-to-one map $f\colon X \to M$ such that $f(W_i^j(B))$ is an open set of $M$ for every $B \in \mathfrak{B}_i$, $i, j = 1, 2, \ldots$. It is then obvious that for each $i$, $f(\mathfrak{F}_i)$ is a closure-preserving collection of closed sets of $M$. This completes the proof.

DEFINITION 3.3. Let $X \in EM_3$ and let $\{\mathfrak{F}, \mathfrak{V}, \mathfrak{E}, \mathfrak{S}\}$ be a quartet of collections of subsets of $X$. The quartet is called an *E-quartet* if we can write $\mathfrak{F} = \bigcup_{i=1}^\infty \mathfrak{F}_i$, $\mathfrak{V} = \bigcup_{i=1}^\infty \mathfrak{V}_i$, $\mathfrak{E} = \bigcup_{i=1}^\infty \mathfrak{E}_i$, $\mathfrak{S} = \bigcup_{i=1}^\infty \mathfrak{S}_i$ and if the following four conditions are satisfied:

($1_q$) *$\mathfrak{F}$ is a net on $X$ consisting of closed sets.*

($2_q$) *For each* $i$, $\mathcal{V}_i$ *is a discrete collection of open sets of* $X$ *written as* $\mathcal{V}_i = \{V_i(F)$: $F \in \mathcal{F}_i\}$ *in such a manner that* $F \subset V_i(F)$ *for each* $F \in \mathcal{F}_i$.

($3_q$) $\mathcal{E}$ *is an E-net on* $X$ *consisting of closed sets and* $\mathcal{E}_i$ *is closure-preserving for each* $i$.

($4_q$) $\mathcal{S}$ *is a quasi-base*[4] *for* $X$ *consisting of closed sets and* $\mathcal{S}_i$ *is closure-preserving for each* $i$.

By Heath [**H**], Gruenhage [**G**] and Junnila [**J**], each member of $EM_3$ admits an $E$-quartet.

DEFINITION 3.4. Let $X$ be a member of $EM_3$ with an $E$-quartet $\{\mathcal{F}, \mathcal{V}, \mathcal{E}, \mathcal{S}\}$. A map $f$: $X \to Y$ onto a normal space $Y$ is called an *E-map* with respect to the $E$-quartet if the following five conditions are satisfied:

($0_f$) $f$ *is one-to-one.*

($1_f$) $f(F)$ *is a closed set for every* $F \in \mathcal{F}$.

($2_f$) $f(V)$ *is an open set for every* $V \in \mathcal{V}$.

($3_f$) $f(E)$ *is a closed set for every* $E \in \mathcal{E}$, *and* $f(\mathcal{E}_i)$ *is closure-preserving in* $Y$ *for every* $i$.

($4_f$) $f(S)$ *is a closed set for every* $S \in \mathcal{S}$, *and* $f(\mathcal{S}_i)$ *is closure-preserving in* $Y$ *for every* $i$.

Noting that $\{X - V: V \in \mathcal{V}\}$ is a $\sigma$-closure-preserving collection of closed sets of $X$, we have the following result by virtue of Lemma 3.2.

PROPOSITION 3.5. *Let* $X$ *be a member of* $EM_3$. *Then for any E-quartet of* $X$ *there exist a metric space* $M$ *and an E-map* $f$: $X \to M$ *with respect to the E-quartet.*

The following lemma is well known (see, for example, [**E**, 2.3.16]).

LEMMA 3.6. *Let* $X$ *be a space and let* $C$, $K$ *be disjoint closed sets of* $X$. *Let* $\mathcal{U}$ *be a countable open cover of* $X$ *such that for each* $U \in \mathcal{U}$, *either* $\overline{U} \cap C = \varnothing$ *or* $\overline{U} \cap K = \varnothing$. *Then* $C$ *and* $K$ *are separated by a closed set* $S$ *such that* $S \subset \{\operatorname{Bd} U: U \in \mathcal{U}\}^*$.

Now we have the following result frequently used later.

PROPOSITION 3.7. *Let* $X$ *be a member of* $EM_3$ *with an E-quartet* $\{\mathcal{F}, \mathcal{V}, \mathcal{E}, \mathcal{S}\}$. *Let* $f$: $X \to Y$ *be an E-map with respect to the E-quartet onto a normal space* $Y$. *Then* $\operatorname{Ind} X \leq \operatorname{Ind} Y$.

PROOF. The proof is by induction on $\operatorname{Ind} Y$. If $Y = \varnothing$ then the proposition is trivial. Suppose that the proposition is valid when $\operatorname{Ind} Y \leq n - 1$ and consider the case of $\operatorname{Ind} Y = n$. To show $\operatorname{Ind} X \leq n$, let $C$, $K$ be disjoint closed sets of $X$. For the time being, fix a point $x$ in $X - C$ arbitrarily. We show that there exists an open neighborhood $W$ of $x$ such that $\overline{W} \cap C = \varnothing$ and $\operatorname{Ind} \operatorname{Bd} W \leq n - 1$. Let $\mathcal{E}(x)$ be a subcollection of $\mathcal{E}$ such that $x \in X - \mathcal{E}(x)^* \subset X - C$ and $\mathcal{E}(x)^*$ is a closed set. Write $\mathcal{E}(x) = \bigcup_{i=1}^{\infty} \mathcal{E}_i(x)$ where $\mathcal{E}_i(x) \subset \mathcal{E}_i$. Put $\mathcal{S}_i(x) = \{S \in \mathcal{S}_i: S \cap \mathcal{E}(x)^* = \varnothing\}$

[4] A collection $\mathcal{S}$ of subsets of a space $X$ is called a quasi-base for $X$ if for any point $x$ and any open neighborhood $U$ of $x$ there exists a member $S$ of $\mathcal{S}$ such that $x \in \operatorname{Int} S \subset S \subset U$.

and $\mathbb{S}(x) = \bigcup_{i=1}^{\infty}\mathbb{S}_i(x)$. Fix $i_0$ so that $x \in \operatorname{Int} \mathbb{S}_{i_0}(x)^*$. By ($3_{\mathrm{f}}$) and ($4_{\mathrm{f}}$) there exist open sets $O_j, j = 1, 2, \ldots$, of $Y$ such that

$$f\left(\bigcup_{i=1}^{j} \mathbb{S}_i(x)^*\right) \cup f\left(\mathbb{S}_{i_0}(x)^*\right) \subset O_j \subset \overline{O}_j \subset Y - f\left(\mathfrak{S}_j(x)^*\right)$$

and

$$\operatorname{Ind} \operatorname{Bd} O_j \leqslant n - 1.$$

Define $W = \bigcap_{j=1}^{\infty} f^{-1}(O_j)$. Then

$$x \in W \subset \overline{W} \subset \bigcap_{j=1}^{\infty} f^{-1}\left(\overline{O}_j\right) \subset X - \mathfrak{S}(x)^* \subset X - C.$$

To show that $W$ is open, let $x' \in W$. Since $x' \in X - \mathfrak{S}(x)^*$ and $\mathfrak{S}(x)^*$ is a closed set, it follows from ($4_{\mathrm{q}}$) that $x' \in \operatorname{Int} \mathbb{S}_m(x)^*$ for some $m$. Then

$$x' \in \bigcap_{j=1}^{m-1} f^{-1}\left(O_j\right) \cap \operatorname{Int} \mathbb{S}_m(x)^* \subset W,$$

which implies that $W$ is open. To show $\operatorname{Ind} \operatorname{Bd} W \leqslant n - 1$, note that, for any subset $Z$ of $X$, $f|Z\colon Z \to f(Z)$ is again an $E$-map with respect to the $E$-quartet $\{\mathfrak{F}|Z, \mathfrak{V}|Z, \mathfrak{S}|Z, \mathbb{S}|Z\}$ on $Z$. Hence we may apply induction hypothesis to obtain $\operatorname{Ind} f^{-1}(\operatorname{Bd} O_j) \leqslant n - 1$, $j = 1, 2, \ldots$, which yields

$$\begin{aligned}
\operatorname{Ind} \operatorname{Bd} W &\leqslant \operatorname{Ind}\left(\bigcup_{j=1}^{\infty} \operatorname{Bd} f^{-1}\left(O_j\right)\right) \\
&= \max\left\{\operatorname{Ind} \operatorname{Bd} f^{-1}\left(O_j\right) : j = 1, 2, \ldots\right\} \\
&\leqslant \max\left\{\operatorname{Ind} f^{-1}\left(\operatorname{Bd} O_j\right) : j = 1, 2, \ldots\right\} \leqslant n - 1.
\end{aligned}$$

Hence $W$ is a required open neighborhood of $x$; we have thus finished "local" separation.

Now put

$$\mathfrak{F}_i(C) = \left\{F \in \mathfrak{F}_i : F \subset W \text{ for some open set } W \text{ with } \overline{W} \cap C = \varnothing \text{ and } \operatorname{Ind} \operatorname{Bd} W \leqslant n - 1\right\}.$$

Then by ($1_{\mathrm{q}}$) and by the "local" separation above, we have $\bigcup_{i=1}^{\infty}\mathfrak{F}_i(C)^* = X - C$. For each $F \in \mathfrak{F}_i(C)$, fix such a $W$ and denote it by $W_i(C, F)$. On the other hand, by ($1_{\mathrm{f}}$) and ($2_{\mathrm{f}}$), there exist open sets $H_i(F)$, $F \in \mathfrak{F}_i$, of $Y$ such that $f(F) \subset H_i(F) \subset \operatorname{Cl} H_i(F) \subset f(V_i(F))$ and $\operatorname{Ind} \operatorname{Bd} H_i(F) \leqslant n - 1$ (where the set $V_i(F)$ is as in Definition 3.3($2_{\mathrm{q}}$)). By induction hypothesis again,

$$\operatorname{Ind} \operatorname{Bd} f^{-1}\left(H_i(F)\right) \leqslant \operatorname{Ind} f^{-1}\left(\operatorname{Bd} H_i(F)\right) \leqslant n - 1.$$

Put for each $F \in \mathfrak{F}_i(C)$,

$$D_i(C, F) = W_i(C, F) \cap f^{-1}\left(H_i(F)\right).$$

Then

$$\operatorname{Ind} \operatorname{Bd} D_i(C, F) \leqslant \max\left\{\operatorname{Ind} \operatorname{Bd} W_i(C, F), \operatorname{Ind} \operatorname{Bd} f^{-1}\left(H_i(F)\right)\right\} \leqslant n - 1.$$

Put $D_i(C) = \{D_i(C, F): F \in \mathcal{F}_i(C)\}^*$. Since $D_i(C, F) \subset V_i(F)$, $(2_q)$ implies that $\{D_i(C, F): F \in \mathcal{F}_i(C)\}$ is discrete. Thus $\operatorname{Ind} \operatorname{Bd} D_i(C) \leq n - 1$, $i = 1, 2, \ldots$. By the same discreteness and by the fact $D_i(C, F) \subset W_i(C, F) \subset \operatorname{Cl} W_i(C, F) \subset X - C$, we have $C \cap \operatorname{Cl} D_i(C) = \varnothing$ for every $i = 1, 2, \ldots$. We also obtain $\bigcup_{i=1}^\infty D_i(C) = X - C$ because $\bigcup_{i=1}^\infty \mathcal{F}_i(C)^* = X - C$.

Quite similarly we can obtain open subsets $D_i(K)$, $i = 1, 2, \ldots$, such that $\operatorname{Ind} \operatorname{Bd} D_i(K) \leq n - 1$, $K \cap \operatorname{Cl} D_i(K) = \varnothing$ and $\bigcup_{i=1}^\infty D_i(K) = X - K$. Hence, applying Lemma 3.6, we have a closed set $B$ separating $C$ and $K$ such that

$$B \subset \left( \bigcup_{i=1}^{\infty} \operatorname{Bd} D_i(C) \right) \cup \left( \bigcup_{i=1}^{\infty} \operatorname{Bd} D_i(K) \right).$$

By the countable sum theorem for Ind, we have $\operatorname{Ind} B \leq n - 1$. Thus $\operatorname{Ind} X \leq n$, which completes the proof of Proposition 3.7.

We can now prove a characterization theorem for $EM_3$.

THEOREM 3.8. *The following statements about a space $X$ are equivalent*:

(1) *$X$ is a stratifiable space with a $\sigma$-closure-preserving $E$-net.*

(2) *$X$ is the perfect image of a stratifiable space $X_0$ with* $\dim X_0 \leq 0$.

(3) *$X$ is the closed image of a stratifiable space $X_0$ with* $\operatorname{ind} X_0 \leq 0$.

(4) *$X$ is a stratifiable space with a $\sigma$-closure-preserving $LE$-net.*

PROOF. The implications (2) → (3) and (4) → (1) are obvious. To show (1) → (2) let $X$ be a member of $EM_3$ with an $E$-quartet $\{\mathcal{F}, \mathcal{V}, \mathcal{E}, \mathcal{S}\}$. By Proposition 3.5 there exists an $E$-map $f: X \to M$ onto a metric space $M$ with respect to $\{\mathcal{F}, \mathcal{V}, \mathcal{E}, \mathcal{S}\}$. By Morita [**M**], $M$ is the image of a metric space $P$ with $\dim P \leq 0$ under a perfect map $g$. Now let $T$ be the fiber product of $P$ and $X$ with respect to $g$ and $f$, that is,

$$T = \{(p, x) \in P \times X: g(p) = f(x)\}$$

with the topology induced from $P \times X$. Let $t_P$, $t_X$ be the restrictions to $T$ of the projections from $P \times X$ onto $P$ and $X$, respectively. We thus have the following commutative diagram:

$$\begin{array}{ccc} X & \xleftarrow{t_X} & T \\ f\downarrow & & \downarrow t_P \\ M & \xleftarrow[g]{} & P \end{array}$$

It is a well-known property of fiber products that the perfectness of $g$ implies the perfectness of $t_X$ (see [**Pe**, Lemma 7.5.13]). $T$ is stratifiable by [**C**, Theorems 2.3, 2.4]. Hence what should be proved is the zero-dimensionality of $T$. By Proposition 2.2, $P$ admits an $E$-quartet $\{\mathcal{F}_P, \mathcal{V}_P, \mathcal{E}_P, \mathcal{S}_P\}$. Now define

$$\mathcal{F}_T = \left\{t_P^{-1}(F_P) \cap t_X^{-1}(F): F_P \in \mathcal{F}_P, F \in \mathcal{F}\right\},$$

$$\mathcal{V}_T = \left\{t_P^{-1}(V_P) \cap t_X^{-1}(V): V_P \in \mathcal{V}_P, V \in \mathcal{V}\right\},$$

$$\mathcal{S}_T = \left\{t_P^{-1}(S_P) \cap t_X^{-1}(S): S_P \in \mathcal{S}_P, S \in \mathcal{S}\right\}, \quad \text{and}$$

$$\mathcal{E}_T = \left\{t_P^{-1}(E_P): E_P \in \mathcal{E}_P\right\} \cup \left\{t_X^{-1}(E): E \in \mathcal{E}\right\}.$$

Then it is easy to see that the quartet $\{\mathcal{F}_T, \mathcal{V}_T, \mathcal{E}_T, \mathcal{S}_T\}$ is an $E$-quartet of $T$. Furthermore, the map $t_P$ is an $E$-map with respect to $\{\mathcal{F}_T, \mathcal{V}_T, \mathcal{E}_T, \mathcal{S}_T\}$ because, in general, $t_P(t_P^{-1}(P') \cap t_X^{-1}(X')) = P' \cap g^{-1} \circ f(X')$ for any $P' \subset P$ and $X' \subset X$, and because $f$ is an $E$-map with respect to $\{\mathcal{F}, \mathcal{V}, \mathcal{E}, \mathcal{S}\}$. Hence, applying Proposition 3.7, we have $\operatorname{Ind} T \leqslant 0$. Thus the implication (1) $\to$ (2) has been proved.

To show (3) $\to$ (4) let $X_0$ be a stratifiable space with $\operatorname{ind} X_0 \leqslant 0$ and let $f\colon X_0 \to X$ be a closed map. Note that every net on $X_0$ is an $E$-net; hence $X_0$ is a member of $EM_3$ by Heath **[H]**. It now follows from the implication (1) $\to$ (2) that $X_0$ is the image of a stratifiable space $X_1$ with $\dim X_1 \leqslant 0$ under a perfect map $h$. Since every net on $X_1$ is an $LE$-net, it follows from Heath **[H]** again that $X_1$ admits a $\sigma$-closure-preserving $LE$-net. Hence, applying Proposition 2.3 to the closed map $f \circ h$, we see that $X$ admits a $\sigma$-closure-preserving $LE$-net. On the other hand $X$ is stratifiable by Borges **[B**, Theorem 3.1]. This completes the proof of Theorem 3.8.

COROLLARY 3.9. *The class $EM_3$ is countably productive, hereditary and preserved under closed maps.*

PROOF. This is immediate from Theorem 3.8, Proposition 2.6 and the analogous result for $M_3$ due to Ceder **[C]** and Borges **[B]**.

A topological class $\mathcal{C}$ is called *perfect* (Nagami **[N$_1$]**, also see **[N$_2$]**) if it is countably productive, hereditary, preserved under perfect maps, included in the class of normal spaces, and every member of $\mathcal{C}$ is the perfect image of a zero-dimensional (in the sense of dim) member of $\mathcal{C}$. Theorem 3.8 and Corollary 3.9 say

COROLLARY 3.10. *The class $EM_3$ is the maximal perfect subclass of $M_3$.*

Recently Itō **[I]** has presented a free $L$-space, a certain closed image of which is not a free $L$-space. But we have

COROLLARY 3.11. *Every closed image of a free L-space is a member of $EM_3$.*

PROOF. By Nagami **[N$_4$**, Theorem 2.10] and Theorem 3.8, every free $L$-space is a member of $EM_3$ (it is also easy to directly prove that every free $L$-space admits a $\sigma$-closure-preserving $E$-net). Hence this corollary is immediate from Corollary 3.9.

**4. Dimension for** $EM_3$. We begin with the equidimensional $G_\delta$-envelope theorem. To show this, the following lemma is useful.

LEMMA 4.1 (OKA **[O$_4$**, LEMMA 3.3]). *Let $X$ be a hereditarily normal space and let $f$: $X \to L$ be a map onto a metric space $L$. Then for any subset $Y \subset X$, there exist a $G_\delta$-set $Z$ of $X$, a metric space $M$ and maps $g\colon Z \to M$, $h\colon M \to f(Z)$ such that*

(i) $Y \subset Z$,

(ii) $\dim g(Y) \leqslant \dim Y$ *and*

(iii) $f | Z = h \circ g$.

THEOREM 4.2. *Let $X \in EM_3$ and let $Y$ be a subset of $X$ with* $\dim Y \leqslant n$. *Then there exists a $G_\delta$-set $G$ of $X$ such that $Y \subset G$ and* $\dim G \leqslant n$.

PROOF. Let $f\colon X \to L$ be an $E$-map onto a metric space $L$ with respect to an $E$-quartet, say $\{\mathcal{F}, \mathcal{V}, \mathcal{E}, \mathcal{S}\}$, on $X$. By the above lemma there exist a $G_\delta$-set $Z$ of $X$, a metric space $M$ and maps $g$: $Z \to M$, $h$: $M \to f(Z)$ satisfying (i), (ii), (iii) above. Since $\dim g(Y) \leqslant n$ and $M$ is metrizable, we can find a $G_\delta$-set $H$ of $M$ such that $g(Y) \subset H$ and $\dim H \leqslant n$ (see, for example, [**E**, 4.1.19]). Define $G = g^{-1}(H)$. Then $G$ is a $G_\delta$-set of $Z$, and hence of $X$. To show $\dim G \leqslant n$, note that $g \mid G$ is an $E$-map with respect to $\{\mathfrak{F} \mid G, \mathfrak{V} \mid G, \mathfrak{E} \mid G, \mathfrak{S} \mid G\}$ because $f \mid G$ is so and because $f \mid G = h \circ g \mid G$ by (iii). Hence by Proposition 3.7 we have $\operatorname{Ind} G \leqslant \operatorname{Ind} H$. Consequently

$$\dim G \leqslant \operatorname{Ind} G \leqslant \operatorname{Ind} H = \dim H \leqslant n,$$

as required. This completes the proof.

The following theorem occupies the central position in dimension theory of $EM_3$. The key argument of the proof has already appeared in the proof of Theorem 3.8.

THEOREM 4.3. *The following statements about a space $X$ are equivalent*:

(1) $X \in EM_3$ *and* $\dim X \leqslant n$.

(2) *$X$ is the image of a stratifiable space $X_0$ with* $\dim X_0 \leqslant 0$ *under a perfect map of order not greater than* $n + 1$.

(3) *$X$ is a stratifiable space which is the union of $G_\delta$-sets $X_i$*, $1 \leqslant i \leqslant n + 1$, *with* $\dim X_i \leqslant 0$.

(4) $X \in EM_3$ *and* $\operatorname{Ind} X \leqslant n$.

PROOF. (1) $\to$ (2). Let $X$ be a member of $EM_3$ such that $\dim X \leqslant n$. Let $\{\mathfrak{F}, \mathfrak{V}, \mathfrak{E}, \mathfrak{S}\}$ be an $E$-quartet of $X$. By Proposition 3.5 there exist a metric space $L$ and an $E$-map $f$: $X \to L$ with respect to the $E$-quartet. By Pasynkov's factorization theorem [**P**, Theorem 29], there exist a metric space $M$ and maps $g$: $X \to M$, $h$: $M \to L$ such that $\dim M \leqslant n$ and $f = h \circ g$. It then follows from Morita [**M**] that $M$ is the image of a metric space $P$ with $\dim P \leqslant 0$ under a perfect map $r$ such that $\operatorname{ord} r \leqslant n + 1$. Let $T$ be the fiber product of $P$ and $X$ with respect to $r$ and $g$, and let $t_p$, $t_X$ be the restrictions to $T$ of the projections from $P \times X$ onto $P$ and $X$, respectively. We thus obtain the following commutative diagram:

$$\begin{array}{ccccc} X & = & X & \xleftarrow{t_X} & T \\ f\downarrow & & g\downarrow & & \downarrow t_P \\ L & \xleftarrow[h]{} & M & \xleftarrow[r]{} & P \end{array}$$

It is obvious that $t_X$ is a perfect map of order not greater than $n + 1$ and that $T$ is a stratifiable space. Note that $g$ is an $E$-map with respect to $\{\mathfrak{F}, \mathfrak{V}, \mathfrak{E}, \mathfrak{S}\}$ because $f$ is so and $f = h \circ g$. Now, as in the proof of Theorem 3.8, $t_P$ is also an $E$-map with respect to a certain $E$-quartet of $T$, and hence $\dim T \leqslant 0$ by Proposition 3.7.

(2) $\to$ (3). Let $t$: $X_0 \to X$ be a perfect map from a stratifiable space $X_0$ with $\dim X_0 \leqslant 0$ onto a space $X$ such that $\operatorname{ord} t \leqslant n + 1$. Put $Y_i = \{x \in X: |t^{-1}(x)| = i\}$, $1 \leqslant i \leqslant n + 1$. It then follows from Nagami [$\mathbf{N_2}$, Lemma 4] that $\dim Y_i \leqslant 0$ for each $i = 1, 2, \ldots, n + 1$. Since $X$ is a member of $EM_3$ by Theorem 3.8, we may apply Theorem 4.2 to obtain $G_\delta$-sets $X_i$, $1 \leqslant i \leqslant n + 1$, such that $\dim X_i \leqslant 0$ and $Y_i \subset X_i$.

The implication (4) $\to$ (1) is trivial.

Finally the implication (3) $\to$ (4) is assured by the following theorem (but the fact $\operatorname{Ind} X \leq n$ only is direct from (3) as a consequence general for hereditarily normal spaces).

THEOREM 4.4. *Let $X$ be a normal $\sigma$-space expressed as the finite union of $G_\delta$-sets $X_i$, $1 \leq i \leq k$, such that* $\dim X_i \leq 0$. *Then $X$ admits a $\sigma$-closure-preserving LE-net.*

PROOF. The proof is by induction on $k$. When $k = 1$, the theorem is trivial. Now suppose that the theorem is valid when $k = m - 1$, and consider the case $k = m$. Put $Y_m = X - X_m$. Then by induction hypothesis and Lemma 2.5, the normal $\sigma$-space $Y_m$ admits a $\sigma$-closure-preserving $LE$-net, say $\mathfrak{G}$, consisting of closed sets of $Y_m$. Write $Y_m = \bigcup_{i=1}^\infty C_i$ with closed sets $C_i$ such that $C_i \subset C_{i+1}$, and put $\mathfrak{G}_i = \mathfrak{G} \mid C_i$. Let $\mathfrak{F}$ be a $\sigma$-locally finite net of $X$. Now consider the $\sigma$-closure-preserving collection $\bigcup_{i=1}^\infty \mathfrak{G}_i \cup \mathfrak{F}$ of $X$. To show that the collection is an $LE$-net on $X$, let $C$, $K$ be disjoint closed sets of $X$. Since $X$ is hereditarily normal and $\operatorname{Ind} X_m \leq 0$, there exists a closed set $S$ separating $C$ and $K$ such that $S \cap X_m = \varnothing$. Represent $X$ as the disjoint union $V \cup S \cup W$, where $V$ and $W$ are open sets of $X$ including $C$ and $K$ respectively. Write $V = \bigcup_{i=1}^\infty V_i$ with open sets $V_i$ such that $\overline{V_i} \subset V_{i+1}$ for every $i$. For each $i$ take a subcollection $\mathfrak{F}_i$ of $\mathfrak{G}_i$ such that

$$(W \cup S) \cap C_i \subset \mathfrak{F}_i^* \subset C_i - (\overline{V_i} \cup C)$$

and $\mathfrak{F}_i^*$ is a closed set of $C_i$. Now put

$$B = W \cup \left( \bigcup_{i=1}^{\infty} \mathfrak{F}_i^* \right).$$

It is easy to see that $B$ is a closed set of $X$ including $K$ and not meeting $C$. Since $W$ is the union of some members of $\mathfrak{F}$, $B$ is the union of some members of $\bigcup_{i=1}^\infty \mathfrak{G}_i \cup \mathfrak{F}$. Thus $\bigcup_{i=1}^\infty \mathfrak{G}_i \cup \mathfrak{F}$ is a $\sigma$-closure-preserving $LE$-net on $X$. This completes the proof of Theorem 4.4 and, therefore, of Theorem 4.3.

REMARK. Slightly modifying the above proof, we can weaken the condition "$X_i$ is $G_\delta$" in Theorem 4.4 to "$X_i$ is either $G_\delta$ or $F_\sigma$".

As a trivial version of Theorem 4.4, we have the following result which tells us that the dimension theory does not work well in the remainder $M_3 - EM_3$.

COROLLARY 4.5. *Let $X$ be a normal $\sigma$-space not admitting a $\sigma$-closure-preserving LE-net. Then either*

(1) *$X$ cannot be decomposed into finitely many zero-dimensional (in the sense of* dim) *subsets, or*

(2) *there exists a zero-dimensional (in the sense of* dim) *subset of $X$ not admitting an equidimensional $G_\delta$-envelope.*

As an immediate consequence of Theorem 4.3, we have

COROLLARY 4.6. *Let $X$ be a stratifiable space with* $\operatorname{ind} X \leq 0$. *Then* $\dim X = \operatorname{Ind} X$.

REMARK. This result, however, is generalized to paracompact $\sigma$-spaces in my recent paper [**O$_5$**].

We conclude this section with the following result, an immediate consequence of Corollary 3.11 and Theorem 4.3.

COROLLARY 4.7. *Let $X$ be the closed image of a free $L$-space. Then* $\dim X = \operatorname{Ind} X$.

**5. Other spaces admitting $\sigma$-closure-preserving $E$-nets.** Let $\mathcal{C}$ be a topological property. A space is called *peripherally* $\mathcal{C}$ if every point in the space admits an open neighborhood base, the boundary of each member of which is $\mathcal{C}$.

THEOREM 5.1. (1) *A peripherally $\sigma$-discrete, paracompact $\sigma$-space admits a $\sigma$-closure-preserving $E$-net.*

(2) *A peripherally $\sigma$-compact, stratifiable space admits a $\sigma$-closure-preserving $E$-net.*

PROOF. We shall prove (1) and (2) simultaneously. Let $\mathfrak{F}$ be a $\sigma$-locally finite net (resp. a $\sigma$-closure-preserving quasi-base) of $X$ consisting of closed sets. To show that $\mathfrak{F}$ itself is an $E$-net on $X$ let $x$ be a point of $X$ and $V$ an open neighborhood of $x$. Take an open set $U$ such that $x \in U \subset \overline{U} \subset V$ and $\operatorname{Bd} U$ is $\sigma$-discrete (resp. $\sigma$-compact). Write $U = \bigcup_{i=1}^{\infty} U_i$ with open sets $U_i$ such that $\overline{U}_i \subset U_{i+1}$ for every $i$. Write $\operatorname{Bd} U = \bigcup_{i=1}^{\infty} C_i$ with discrete (resp. compact) closed sets $C_i$, $i = 1, 2, \ldots$. There exists, for each $i$, a discrete (resp. finite) subcollection $\mathfrak{F}_i$ of $\mathfrak{F}$ such that $C_i \subset \mathfrak{F}_i^* \subset X - (\overline{U}_i \cup \{x\})$. Then $\bigcup_{i=1}^{\infty} \mathfrak{F}_i^* \cup (X - \overline{U})$ is a closed set of $X$ including $X - V$, not meeting $\{x\}$ and expressed as a union of members of $\mathfrak{F}$. Thus $\mathfrak{F}$ is an $E$-net of $X$, which completes the proof.

Now we have the following generalization of Corollary 4.6.

COROLLARY 5.2. *Let $X$ be a peripherally $\sigma$-compact* (*or peripherally $\sigma$-discrete*) *stratifiable space. Then* $\dim X = \operatorname{Ind} X$.

We next verify a countable sum theorem for $\sigma$-closure-preserving $LE$-nets.

THEOREM 5.3. *Let $X$ be a normal space expressed as the countable union of closed sets $X_i$, $i = 1, 2, \ldots$, each of which admits a $\sigma$-closure-preserving $LE$-net. Then $X$ has a $\sigma$-closure-preserving $LE$-net.*

PROOF. Note that $X$ is perfectly normal because each $X_i$ is. Let $\mathcal{E}_i$ be a $\sigma$-closure-preserving $LE$-net of $X_i$. It is clear that $\bigcup_{i=1}^{\infty} \mathcal{E}_i$ is $\sigma$-closure-preserving in $X$. To show that $\bigcup_{i=1}^{\infty} \mathcal{E}_i$ is an $LE$-net, let $C$ and $K$ be disjoint closed sets of $X$. Write $X - C = \bigcup_{i=1}^{\infty} V_i$ with open sets $V_i$ such that $\overline{V}_i \subset V_{i+1}$. For each $i$ let $\mathfrak{F}_i$ be a subcollection of $\mathcal{E}_i$ such that $\mathfrak{F}_i^*$ is a closed set of $X_i$ and $C \cap X_i \subset \mathfrak{F}_i^* \subset X_i - (K \cup \overline{V}_i)$. It is then obvious that $\bigcup_{i=1}^{\infty} \mathfrak{F}_i^*$ is a closed set of $X$ and $C \subset \bigcup_{i=1}^{\infty} \mathfrak{F}_i^* \subset X - K$. This completes the proof.

The following result is immediate from Theorem 5.3, Proposition 2.2 and Ceder [**C**, Theorem 8.3].

COROLLARY 5.4. *A chunk complex* (*and hence a $CW$-complex*) *is a member of $EM_3$.*

We list several unsolved problems below.

*Problem* 5.5. (1) *Does every stratifiable space admit a σ-closure-preserving E-net*? By virtue of Theorem 3.8, this is equivalent to:

(2) (Nagami [$\mathbf{N_1}$, Problem 4]) *Is every stratifiable space a perfect image of a zero-dimensional* (*in the sense of* dim) *stratifiable space*?

The author also does not know whether the inclusion $EM_3 \subset M_1$ (or $M_1 \subset EM_3$) holds or not.

*Problem* 5.6. *Let X be a paracompact σ-space admitting a σ-closure-preserving E-net. Then*:

(1) *Does the equality* $\dim X = \operatorname{Ind} X$ *hold*?

(2) *Is X a perfect image of a zero-dimensional* (*in the sense of* dim) *paracompact σ-space*? More weakly:

(3) *Does X admit a σ-closure-preserving LE-net*?

In the specific case of $\operatorname{ind} X \leqslant 0$, (1) admits an affirmative answer by the inequality $\operatorname{Ind} X \leqslant \dim X + \operatorname{ind} X$ for every nonempty paracompact σ-space $X$ [$\mathbf{O_5}$]; (2) is also affirmative, that is, *a paracompact σ-space of* ind $\leqslant 0$ *is the perfect image of a paracompact σ-space of* dim $\leqslant 0$.

To outline the proof, let $X$ be a nonempty paracompact σ-space with $\operatorname{ind} X = 0$. Let $\mathcal{F} = \bigcup_{i=1}^{\infty} \mathcal{F}_i$ and $\mathcal{V} = \bigcup_{i=1}^{\infty} \mathcal{V}_i$ be as in Definition 3.3. Let $f\colon X \to M$ be a one-to-one map onto a metric space $M$ such that $f(\mathcal{V}_i^*)$ is open and $f(\mathcal{F}_i^*)$ is closed for every $i$. In [$\mathbf{O_5}$, Lemma 5] it is proved that, in general, $\operatorname{Ind} X \leqslant \operatorname{Ind} M + \operatorname{ind} X$ for any such map $f\colon X \to M$. The metric space $M$ is the image of a metric space $L$ with $\dim L = 0$ under a perfect map $g$. Let $T$ be the fiber product of $L$ and $X$ with respect to $g$ and $f$. Let $t_L$, $t_X$ be the restrictions to $T$ of the projections from $L \times X$ onto $L$ and $X$, respectively. Then, since the map $t_L$ is of the "same type" as $f$, we have $\operatorname{Ind} T \leqslant \operatorname{Ind} L + \operatorname{ind} T = \operatorname{ind} T$. But, in the present case, $\operatorname{ind} T \leqslant \operatorname{ind}(L \times X) = 0$; hence $\operatorname{Ind} T = 0$. It is clear that $T$ is a paracompact σ-space and $t_X$ is a perfect map. This completes the proof.

*Problem* 5.7. *Let X be a stratifiable space expressed as the union of countably many metrizable* ($G_\delta$-) *subsets. Does the equality* $\dim X = \operatorname{Ind} X$ *hold*? *More strongly, does X admit a σ-closure-preserving E-net*? (A space of this type is a natural generalization of a Lašnev space in view of Lašnev's well-known decomposition theorem [**La**].)

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