

## ***AF* ALGEBRAS WITH DIRECTED SETS OF FINITE DIMENSIONAL \*-SUBALGEBRAS**

BY

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**ABSTRACT.** We characterize the unital *AF* algebras whose families of finite dimensional \*-subalgebras are directed by inclusion. A representation theorem for the algebras of this class allows us to classify them up to \*-isomorphisms.

**Introduction.** Hofmann and Thayer remarked in [6] that the family of all finite dimensional \*-subalgebras of the commutative *AF* algebras or of the *C\**-algebra obtained by adjoining a unit to the ideal of compact operators on a separable Hilbert space is directed by inclusion. They also asked if this property is shared by all *AF* algebras.

We shall say that an *AF* algebra has the directed set property (d.s.p.) if its collection of finite dimensional \*-subalgebras is directed by inclusion. It is not hard to provide an example of an *AF* algebra without the d.s.p. We propose to characterize the unital *AF* algebras with the d.s.p. The characterization which we obtain is in terms of the topology of the spectrum and the primitive quotients of the algebra. It turns out actually that the class of *AF* algebras with the d.s.p. is quite restricted. In §4 we give a complete set of invariants up to \*-isomorphisms for the algebras of this class.

Our notation follows [3]. In particular, the space of a representation  $\pi$  is denoted  $H_\pi$ . If  $H$  is an infinite dimensional Hilbert space we shall denote by  $LC(H)$  the ideal of all compact operators on  $H$  and by  $\widetilde{LC}(H)$  the *C\**-algebra generated by  $LC(H)$  and the identity operator on  $H$ . In addition to that, we shall use the following notations for a natural number  $n$ :

$$\begin{aligned}\alpha_n &= 2^{-1} \left\{ 1 + [1 - 4(n+1)^{-2}]^{-1/2} \right\}, \\ \beta_n &= (n+1)^{-1}, \\ \gamma_n &= 2^{-1} \left\{ 1 - [1 - 4(n+1)^{-2}]^{-1/2} \right\}.\end{aligned}$$

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**1. Examples.** In this section we shall present some examples of *AF* algebras without the d.s.p. which will be relevant later.

**EXAMPLE 1.1.** Let *A* be the *C\**-algebra of all sequences of  $2 \times 2$  complex matrices which converge to a matrix of the form

$$\begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix}$$

for some complex number  $\alpha$ . It is clearly an *AF* algebra and if we denote  $p = \{p_n\}_{n=1}^\infty, q = \{q_n\}_{n=1}^\infty$ , where

$$p_n = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad q_n = \begin{pmatrix} \alpha_n & \beta_n \\ \beta_n & \gamma_n \end{pmatrix}, \quad n = 1, 2, \dots,$$

then *p* and *q* are projections in *A*. We have

$$p_n q_n p_n = \begin{pmatrix} \alpha_n & 0 \\ 0 & 0 \end{pmatrix}, \quad n = 1, 2, \dots,$$

so  $pqp = \{p_n q_n p_n\}_{n=1}^\infty$  has an infinite spectrum. Thus, the *C\**-algebra of *A* generated by *p* and *q* is infinite dimensional which implies that *A* cannot have the d.s.p.

**EXAMPLE 1.2.** Let *A* be the *C\**-algebra generated by all the compact operators on an infinite dimensional separable Hilbert space and a projection whose range is infinitely dimensional and infinitely codimensional. It is easily seen that *A* is an *AF* algebra which has a *C\**-subalgebra *\**-isomorphic to the *C\**-algebra of Example 1.1. Hence *A* does not have the d.s.p. Of course, we use here the trivial fact that an *AF* subalgebra of an *AF* algebra with the d.s.p. inherits this property from the larger algebra.

**EXAMPLE 1.3.** Let now *A* be the *UHF* algebra of type  $\{2^n\}$  in the terminology of [5], that is, *A* is the closure of an increasing sequence  $\{A_n\}_{n=1}^\infty$  of factors of type  $I_{2^n}$ , all of which contain the unit of *A*. Choose matrix units  $\{e_{i,j}^n\}_{i,j=1}^{2^n}$  for  $A_n$  such that  $e_{i,j}^n = e_{i,j}^{n+1} + e_{i+2^n, j+2^n}^{n+1}$  for  $1 \leq i, j \leq 2^n, n = 1, 2, \dots$ . Consider the following elements of *A*:

$$\begin{aligned} x_n = & \sum_{k=1}^{n-1} [\alpha_k e_{2^k-1, 2^k-1}^{k+1} + \beta_k (e_{2^k, 2^k-1}^{k+1} + e_{2^k-1, 2^k}^{k+1}) + \gamma_k e_{2^k, 2^k}^{k+1}] \\ & + \alpha_n e_{2^n-1, 2^n-1}^n + \beta_n (e_{2^n, 2^n-1}^n + e_{2^n-1, 2^n}^n) + \gamma_n e_{2^n, 2^n}^n, \end{aligned}$$

defined for each natural number *n*. Using the relations

$$\begin{aligned} (e_{2^k-1, 2^k-1}^{k+1} + e_{2^k, 2^k}^{k+1})(e_{2^m-1, 2^m-1}^{m+1} + e_{2^m, 2^m}^{m+1}) &= 0 \quad \text{for } k \neq m, \\ (e_{2^k-1, 2^k-1}^{k+1} + e_{2^k, 2^k}^{k+1})(e_{2^n-1, 2^n-1}^n + e_{2^n, 2^n}^n) &= 0 \quad \text{for } k < n, \end{aligned}$$

one checks easily that  $\{x_n\}_{n=1}^\infty$  is a sequence of projections in *A*. Now, if  $m > n$ , we have

$$e_{2^n-1, 2^n-1}^n = \sum_{k=n}^{m-1} e_{2^k-1, 2^k-1}^{k+1} + e_{2^m-1, 2^m-1}^m$$

and similar equalities for  $e_{2^n-1,2^n}^n, e_{2^n,2^{n-1}}^n$  and  $e_{2^n,2^n}^n$ . Hence

$$\begin{aligned} x_m - x_n &= \sum_{k=n}^{m-1} [(\alpha_k - \alpha_n)e_{2^k-1,2^k-1}^{k+1} \\ &\quad + (\beta_k - \beta_n)(e_{2^k-1,2^k}^{k+1} + e_{2^k,2^k-1}^{k+1}) + (\gamma_k - \gamma_n)e_{2^k,2^k}^{k+1}] \\ &\quad + (\alpha_m - \alpha_n)e_{2^{m-1},2^{m-1}}^m + (\beta_m - \beta_n)(e_{2^{m-1},2^m}^m + e_{2^m,2^{m-1}}^m) \\ &\quad + (\gamma_m - \gamma_n)e_{2^m,2^m}^m. \end{aligned}$$

From this we infer that

$$\|x_m - x_n\| = \max_{n \leq k \leq m} \left\| \begin{pmatrix} \alpha_k - \alpha_n & \beta_k - \beta_n \\ \beta_k - \beta_n & \gamma_k - \gamma_n \end{pmatrix} \right\|.$$

Consequently,  $\{x_n\}_{n=1}^\infty$  is a Cauchy sequence and  $x = \lim_{n \rightarrow \infty} x_n$  is a projection in  $A$ . Now,

$$e_{1,1}^1 x_n e_{1,1}^1 = \sum_{k=1}^{n-1} \alpha_k e_{2^k-1,2^k-1}^{k+1} + \alpha_n e_{2^n-1,2^n-1}^n,$$

thus  $\alpha_n$  belongs to the spectrum of  $e_{1,1}^1 x_n e_{1,1}^1$  for  $m \geq n$ . By the upper semicontinuity of the spectrum it follows that the spectrum of  $e_{1,1}^1 x e_{1,1}^1$  contains the sequence  $\{\alpha_n\}_{n=1}^\infty$ . Hence, the  $C^*$ -subalgebra of  $A$  generated by the projections  $e_{1,1}^1$  and  $x$  is infinite dimensional.

**2. Preliminary results.** The following proposition allows us to restrict the discussion to unital  $C^*$ -algebras.

**PROPOSITION 2.1.** *Let  $A$  be an AF algebra with the d.s.p. Then  $\tilde{A}$ , the  $C^*$ -algebra obtained by adjoining a unit to  $A$ , is an AF algebra with the d.s.p.*

**PROOF.**  $A = \overline{\bigcup_{n=1}^\infty A_n}$ , where  $\{A_n\}_{n=1}^\infty$  is an increasing sequence of finite dimensional  $*$ -subalgebras of  $A$  and  $\tilde{A} = \bigcup_{n=1}^\infty \tilde{A}_n$ . Let  $B, C$  be finite dimensional  $*$ -subalgebras of  $\tilde{A}$ . By [1, Lemma 2.5], there are unitary elements  $u$  and  $v$  in  $\tilde{A}$  such that

$$B \subset u^* \tilde{A}_n u = \widetilde{u^* A_n u}, \quad C \subset v^* \tilde{A}_n v = \widetilde{v^* A_n v}$$

for some  $n$ . Let  $D$  be a finite dimensional  $*$ -subalgebra of  $A$  which contains  $u^* A_n u$  and  $v^* A_n v$ . Clearly we have  $B, C \subset \tilde{D}$ .

The next four results prepare the ground for the proof of the characterization theorem.

**LEMMA 2.2.** *Let  $A$  be an AF algebra with the d.s.p. and  $\varphi$  a  $*$ -homomorphism of  $A$  onto the  $C^*$ -algebra  $B$ . Then  $B$  is an AF algebra with the d.s.p.*

**PROOF.** Suppose  $B_1$  and  $B_2$  are finite dimensional  $*$ -subalgebras of  $B$ . By the proof of [7, Theorem 2.4], there are finite dimensional  $*$ -subalgebras  $A_1, A_2$  of  $A$  such that  $\varphi(A_j) = B_j, j = 1, 2$ . Let  $C$  be a finite dimensional  $*$ -subalgebra of  $A$  which contains  $A_1$  and  $A_2$ . Obviously  $\varphi(C)$  is a finite dimensional  $*$ -subalgebra of  $B$  which contains  $B_1$  and  $B_2$ .

**PROPOSITION 2.3.** *An AF algebra with the d.s.p. is postliminal.*

PROOF. Let us assume, by contradiction, that the  $AF$  algebra  $A$  with the d.s.p. is not postliminal. By [7, Corollary to Theorem 1.1], there is an  $AF$  subalgebra  $B$  of  $A$  and a quotient map of  $B$  onto the  $UHF$  algebra of Example 1.3. This contradicts the conclusion of Lemma 2.2.

Now we strengthen considerably the preceding result.

PROPOSITION 2.4. *Let  $A$  be a unital  $AF$  algebra with the d.s.p. and  $\pi$  an irreducible representation of  $A$ . Then either  $\pi$  is finite dimensional or  $\pi(A)$  is the  $C^*$ -algebra generated by the ideal of all compact operators and the identity operator on  $H_\pi$ .*

PROOF. Suppose  $H_\pi$  is infinite dimensional. By Proposition 2.3,  $\pi(A)$  contains  $\widetilde{LC}(H_\pi)$ . If  $\pi(A) \neq \widetilde{LC}(H_\pi)$  then  $\pi(A)$  must contain a projection  $p$  with infinite dimensional range and infinite dimensional null space. By Lemma 2.2,  $\pi(A)$  has the d.s.p. while Example 1.2 shows that its  $C^*$ -subalgebra generated by  $LC(H_\pi)$  and  $p$  does not have the d.s.p., a contradiction.

COROLLARY. 2.5. *Let  $A$  be a unital  $AF$  algebra with the d.s.p. Every primitive ideal of  $A$  is contained in exactly one maximal two-sided ideal  $M$ . If a maximal two-sided ideal  $M$  contains properly a primitive ideal of  $A$  then  $\dim A/M = 1$ .*

**3. The characterization theorem.** Throughout this section  $A$  will denote an  $AF$  algebra. If it is unital then  $e$  will always denote its unit. If  $A$  has the d.s.p. then it follows from Proposition 2.3 and [3, 4.3.7] that  $\text{Prim } A$  and  $\hat{A}$  are homeomorphic. We shall freely use for one of these spaces a result established for the other. Their topology is described in the coming sequence of lemmas.

LEMMA 3.1. *If  $A$  has the d.s.p. and Hausdorff spectrum then every  $\pi \in \hat{A}$  with  $\dim H_\pi > 1$  is isolated in  $\hat{A}$ .*

PROOF. Assume that  $\pi \in \hat{A}$ ,  $\dim H_\pi > 1$  but  $\pi$  is not isolated in  $\hat{A}$ . Since  $\hat{A}$  is second countable by [3, 3.3.4], there is a sequence  $\{\pi_n\}_{n=1}^\infty$  of distinct elements of  $\hat{A}$  which converges to  $\pi$ . The set  $\{\pi_n\}_{n=1}^\infty \cup \{\pi\}$  is closed in  $\hat{A}$  which is Hausdorff. Lemma 2.2 allows us to pass to a quotient so we shall suppose that  $\hat{A} = \{\pi_n\}_{n=1}^\infty \cup \{\pi\}$ . Remark that by [3, 4.7.15] and Proposition 2.3,  $A$  is liminal. Choose matrix units  $\{e_{i,j}\}_{i,j=1}^2$  in  $\pi(A)$ . A result of Fell [4, Theorem 3.1] provides us with elements  $\{x_{i,j}\}_{i,j=1}^2$  in  $A$  such that  $\pi(x_{i,j}) = e_{i,j}$  for  $1 \leq i, j \leq 2$  and  $\{\pi_n(x_{i,j})\}_{i,j=1}^2$  are matrix units in  $\pi_n(A)$  for  $n \geq N$ . Without loss of generality we shall suppose that  $N = 1$ . Now, for  $f \in C(\hat{A})$  and  $x \in A$  let  $f \cdot x$  denote the element  $y \in A$  given by [3, 10.5.6]:  $\rho(y) = f(\rho)\rho(x)$  for each  $\rho \in \hat{A}$ . Consider the  $*$ -subalgebra of  $A$  consisting of all the elements of the form  $\sum_{i,j=1}^2 f_{i,j} \cdot x_{i,j}$  where  $\{f_{i,j}\}_{i,j=1}^2 \subset C(\hat{A})$ ,  $f_{1,2}(\pi) = f_{2,1}(\pi) = f_{2,2}(\pi) = 0$ . It is plainly  $*$ -isomorphic to the algebra of Example 1.1, a contradiction.

LEMMA 3.2. *Let  $M$  be a maximal two-sided ideal of the unital algebra  $A$  with the d.s.p. The set of all primitive ideals of  $A$  which are contained in  $M$  is countable and closed in  $\text{Prim}(A)$ .*

PROOF. We may suppose that  $M$  contains properly at least one primitive ideal; let  $\{P_\alpha; \alpha \in \mathfrak{A}\}$  be the collection of all the primitive ideals properly included in  $M$ . By

Corollary 2.5, it is enough to show that  $M$  is the unique maximal ideal containing  $\bigcap_{\alpha \in \mathfrak{Q}} P_\alpha$ .

Lemma 2.2 allows us to work with  $A / \bigcap_{\alpha \in \mathfrak{Q}} P_\alpha$  instead of  $A$  so we shall suppose

$$(1) \quad \bigcap_{\alpha \in \mathfrak{Q}} P_\alpha = \{0\}$$

and prove that  $M$  is the only maximal two-sided ideal of  $A$ .

Choose an open dense subset  $U$  of  $\text{Prim}(M) = \text{Prim}(A) \setminus \{M\}$  which is Hausdorff in its relative topology (cf. [3, 4.4.5]). It follows from the assumptions made on  $M$  that  $U$  is open and dense in  $\text{Prim}(A)$  as well. By (1), the set  $\{P_\alpha : P_\alpha \in U\}$  is dense in  $\text{Prim}(A)$ . Let us denote by  $\pi_\alpha$ , the unique irreducible representation of  $A$  whose kernel is  $P_\alpha$ . According to Proposition 2.4,  $\pi_\alpha(M) = LC(H_{\pi_\alpha})$ . The representation  $\pi$  of  $A$  defined by

$$\pi(x) = \sum_{P_\alpha \in U} \bigoplus \pi_\alpha(x), \quad x \in A,$$

is a  $*$ -isomorphism. Now  $U$  is the primitive ideal space of some ideal of  $A$  contained in  $M$ . We conclude from Lemma 3.1 that each  $P_\alpha$  which belongs to  $U$  is isolated in  $U$  and  $\{P_\alpha : P_\alpha \in U\}$  is countable since  $A$  is separable. It is easily seen then that  $\pi(M)$  contains the restricted direct sum  $\sum_{P_\alpha \in U} \bigoplus LC(H_{\pi_\alpha})$ . We claim that  $\pi(M) = \sum_{P_\alpha \in U} \bigoplus LC(H_{\pi_\alpha})$ . Otherwise  $M$  must contain a nonzero projection  $p$  such that  $\pi_\alpha(p)$  is finite dimensional for every  $\alpha \in \mathfrak{Q}$  and  $\pi_\alpha(p) \neq 0$  for infinitely many  $\alpha$ 's with  $P_\alpha \in U$ . But then one can exhibit in  $\pi(M)$  a  $C^*$ -subalgebra  $*$ -isomorphic to that of Example 1.1. Since  $\pi(A)$  is obtained from  $\pi(M) = \sum_{P_\alpha \in U} \bigoplus LC(H_{\pi_\alpha})$  by adjoining to it the identity operator and the algebra thus obtained has only one maximal two-sided ideal and its primitive ideal space is countable, the proof is complete.

LEMMA 3.3. *Suppose the unital algebra  $A$  has the d.s.p. and let  $\pi \in \hat{A}$  with  $\dim H_\pi > 1$ . Then  $\pi$  is isolated in  $\hat{A}$ .*

PROOF. Let  $E$  be the set of all  $\rho \in \hat{A}$  such that  $\rho^{-1}(0)$  is contained in the same maximal two-sided ideal as  $\pi^{-1}(0)$ . Remark that if  $\pi^{-1}(0)$  is a maximal ideal then by Proposition 2.4,  $E = \{\pi\}$ . We infer from Lemma 3.2 that  $E$  is closed. Assume that  $\pi$  is not isolated in  $\hat{A}$ . It follows from the proof of Lemma 3.2 that  $\pi \in \overline{\hat{A} \setminus E}$ . Choose a decreasing basis of neighbourhoods  $\{U_n\}_{n=1}^\infty$  for  $\pi$ . The set  $U_1 \cap (\hat{A} \setminus E)$  is open, nonvoid and by [3, 3.9.4] it contains a separated point  $\pi_1$  of  $\hat{A}$ . From Proposition 2.4 we deduce that  $\overline{\{\pi_1\}}$  may contain at most one point different from  $\pi_1$  and in any case  $\pi \notin \overline{\{\pi_1\}}$ . Thus  $\pi$  and  $\pi_1$  may be separated. Since  $\{\pi_1\}$  is open in  $\overline{\{\pi_1\}}$  there is no loss of generality in supposing that  $\pi_1$  has an open neighbourhood  $V_1$  such that  $V_1 \cap U_2 = \emptyset$ ,  $V_1 \cap \overline{\{\pi_1\}} = \{\pi_1\}$ . Now  $U_2 \cap (\hat{A} \setminus E) \neq \emptyset$ , open and contains a separated point  $\pi_2$  of  $\hat{A}$ . We have  $\pi \notin \overline{\{\pi_2\}}$  so we may suppose that  $\pi_2$  has an open neighbourhood  $V_2$  such that  $V_2 \subset U_2$ ,  $V_2 \cap U_3 = \emptyset$  and  $V_2 \cap \overline{\{\pi_2\}} = \{\pi_2\}$ . Thus, an induction argument will yield a sequence  $\{\pi_n\}_{n=1}^\infty$  in  $\hat{A}$  such that  $\pi$  is a limit of it and each  $\overline{\{\pi_m\}}$  is open in the closure of  $\{\pi_n\}_{n=1}^\infty$ .

For convenience we shall suppose from now on that  $\hat{A} = \overline{\{\pi_n\}_{n=1}^\infty}$ . Let  $p$  be a projection in  $A$  satisfying  $\pi(p) \neq 0 \neq \pi(e - p)$ . We infer from [3, 3.3.2] that  $\pi_n(p) \neq 0 \neq \pi_n(e - p)$  for  $n > N$  and we may take  $N = 1$ . For each  $n$  choose operators  $r_n, s_n$  and  $u_n$  on  $H_{\pi_n}$  such that  $r_n$  and  $s_n$  are one dimensional projections,  $r_n \leq \pi_n(p)$ ,  $s_n \leq \pi_n(e - p)$  and  $u_n u_n^* = r_n, u_n^* u_n = s_n$ . The open set  $\{\pi_n\}_{n=1}^\infty$  is the spectrum of some closed two-sided ideal  $I$  of  $A$  and since  $\hat{I}$  is discrete there are  $r, s, u \in I$  such that

$$\pi_n(r) = \alpha_n r_n, \quad \pi_n(s) = -\gamma_n s_n, \quad \pi_n(u) = -\beta_n u_n, \quad n = 1, 2, \dots$$

Then  $\pi_n[p - (r + s + u + u^*)]$  is a projection for each  $n$  and since  $\{\pi_n\}_{n=1}^\infty$  is dense in  $\hat{A}$  we infer that  $p - (r + s + u + u^*)$  is a projection in  $A$ . Now it is readily seen that  $\alpha_n$  belongs to the spectrum of  $\pi_n\{p[p - (r + s + u + u^*)]p\}$ . Thus the spectrum of  $p[p - (r + s + u + u^*)]p$  is infinite, a fact which is incompatible with the d.s.p.

LEMMA 3.4. *Suppose  $A$  is unital with the d.s.p. Then each  $\pi \in \hat{A}$  with  $\dim H_\pi < \infty$  is separated from every point of  $\hat{A} \setminus \{\rho \in \hat{A} : \rho^{-1}(0) \subset \pi^{-1}(0)\}$ .*

PROOF. The conclusion follows immediately from Lemma 3.3 if  $\dim H_\pi > 1$  so we may suppose  $\dim H_\pi = 1$ . Let  $\pi' \in \hat{A} \setminus \{\rho \in \hat{A} : \rho^{-1}(0) \subset \pi^{-1}(0)\}$  and assume that  $\pi$  and  $\pi'$  are not separated. Again, this cannot happen if  $\dim H_{\pi'} > 1$  so we have  $\dim H_{\pi'} = 1$ . Since  $\{\rho \in \hat{A} : \dim H_\rho = 1\}$  is Hausdorff in its relative topology [3, 3.6.4], there are open neighbourhoods  $V, V'$  of  $\pi, \pi'$  respectively, such that

$$(1) \quad V \cap V' \cap \{\rho \in \hat{A} : \dim H_\rho = 1\} = \emptyset.$$

Choose neighbourhood bases  $\{U_n\}_{n=1}^\infty, \{U'_n\}_{n=1}^\infty$  for  $\pi, \pi'$  respectively. There is no loss of generality in supposing  $U_{n+1} \subset U_n \subset V, U'_{n+1} \subset U'_n \subset V'$  for every  $n$  and

$$(2) \quad U_1 \subset \hat{A} \setminus \{\rho \in \hat{A} : \rho^{-1}(0) \subset \pi^{-1}(0)\},$$

$$(3) \quad U'_1 \subset \hat{A} \setminus \{\rho \in \hat{A} : \rho^{-1}(0) \subset \pi^{-1}(0)\}.$$

Our pair  $\{\pi, \pi'\}$  is not separated so we may choose  $\pi_1 \in U_1 \cap U'_1$ ; by (2) and (3) we have  $\pi \notin \{\pi_1\}, \pi' \notin \{\pi_1\}$ . Suppose we have already chosen mutually distinct representations,  $\pi_i \in U_{n_i} \cap U'_{n_i}, 1 \leq i \leq k$ , with  $1 = n_1 < n_2 < \dots < n_k$ . Since  $\{\pi, \pi'\} \cap \{\pi_i\} = \emptyset$  for  $1 \leq i \leq k$ , there is  $n_{k+1} > n_k$  such that  $(U_{n_{k+1}} \cup U'_{n_{k+1}}) \cap \{\pi_1, \pi_2, \dots, \pi_k\} = \emptyset$ . There is a representation  $\pi_{k+1} \in U_{n_{k+1}} \cap U'_{n_{k+1}}$ . Thus we may construct inductively a sequence  $\{\pi_k\}_{k=1}^\infty$  which converges to  $\pi$  and  $\pi'$ . Lemma 3.3 and (1) imply that each  $\{\pi_k\}$  is open in  $\hat{A}$ .

The two-sided ideals of  $A$  are generated by projections [1, 3.3],  $\pi^{-1}(0) \neq \pi'^{-1}(0)$  and  $\pi'^{-1}(0)$  is maximal; hence there is a projection  $p \in A$  with  $\pi(p) \neq 0, \pi'(p) = 0$ . Then  $\pi'(e - p) \neq 0$ . By [3, 3.3.2] there is  $N$  such that  $\pi_k(p) \neq 0 \neq \pi_k(e - p)$  for  $k \geq N$ . From here on one may argue as in the last part of the proof of Lemma 3.3 in order to arrive at a contradiction.

LEMMA 3.5. *With the same assumptions on  $A$  as above  $\{\pi \in \hat{A} : \dim H_\pi < \infty\}$  is a nonvoid compact Hausdorff space in the relative topology.*

PROOF. The set  $\{\pi \in \hat{A} : \dim H_\pi < \infty\}$  is closed in  $\hat{A}$  by Lemma 3.3 and nonempty by Proposition 2.4. Lemma 3.4 implies that it is Hausdorff.

LEMMA 3.6. *Let  $A$  be unital and have the d.s.p. Define a map  $\varphi$  from  $\hat{A}$  to  $E = \{\rho \in \hat{A} : \dim H_\rho < \infty\}$  as follows: for each  $\pi \in \hat{A}$  we let  $\varphi(\pi)$  be the unique irreducible representation of  $A$  whose kernel is a maximal two-sided ideal containing  $\pi^{-1}(0)$ . Then  $\varphi$  is a continuous retraction.*

PROOF. Clearly, if  $\pi \in E$ , then  $\varphi(\pi) = \pi$ . Let now  $F$  be a closed subset of  $E$ . We want to show that each  $\pi \notin \varphi^{-1}(F)$  has a neighbourhood contained in  $\hat{A} \setminus \varphi^{-1}(F)$ . By Lemma 3.3 this is clear if  $\dim H_\pi > 1$ .

So suppose now that  $\pi \notin \varphi^{-1}(F)$  and  $\dim H_\pi = 1$ . For each  $\rho \in F$  Lemma 3.4 yields disjoint open neighbourhoods  $U_\rho, V_\rho$  of  $\pi, \rho$  respectively. By using the compactness of  $F$  we get open sets  $U_{\rho_1}, U_{\rho_2}, \dots, U_{\rho_n}$  and  $V_{\rho_1}, V_{\rho_2}, \dots, V_{\rho_n}$  such that  $\pi \in U = \bigcap_{k=1}^n U_{\rho_k}, F \subset \bigcup_{k=1}^n V_{\rho_k} = V$  and  $U_{\rho_k} \cap V_{\rho_k} = \emptyset, 1 \leq k \leq n$ . Clearly,  $\varphi(\rho) \in \overline{\{\rho\}}$  for each  $\rho \in \hat{A}$ . Thus the open set  $V$  contains  $\varphi^{-1}(F)$  and  $U$  is a neighbourhood of  $\pi$  disjoint from  $\varphi^{-1}(F)$ .

THEOREM 3.7. *Let  $A$  be a unital AF algebra. Then  $A$  has the d.s.p. if and only if the following conditions are fulfilled:*

- (i) every  $\pi \in \hat{A}$  is either finite dimensional or  $\pi(A)$  is  $\widetilde{LC}(H_\pi)$ ;
- (ii) each  $\pi \in \hat{A}$  with  $\dim H_\pi > 1$  is isolated in  $\hat{A}$ ;
- (iii)  $E = \{\pi \in \hat{A} : \dim H_\pi < \infty\}$  is Hausdorff in the relative topology and if for each  $\pi \in \hat{A}$  one denotes by  $\varphi(\pi)$  that element of  $\hat{A}$  whose kernel is the unique maximal two-sided ideal of  $A$  which contains  $\pi^{-1}(0)$  then  $\varphi$  is a continuous map of  $\hat{A}$  onto  $E$ .

PROOF. That a unital AF algebra with the d.s.p. satisfies (i), (ii) and (iii) has already been proved in Proposition 2.4 and Lemmas 3.3, 3.5 and 3.6. So we shall suppose that  $A$  satisfies the above conditions and prove that it has the d.s.p. The unit of  $A$  will be denoted by  $e$ .

First we prove that each  $\pi \in E$  is separated from every point  $\hat{A} \setminus \varphi^{-1}(\pi)$ . Indeed, let  $\pi' \in \hat{A} \setminus \varphi^{-1}(\pi)$ . If  $H_{\pi'}$  is infinite dimensional then  $\hat{A} \setminus \{\pi'\}$  and  $\{\pi'\}$  are disjoint open sets containing  $\pi, \pi'$  respectively. Now, if  $\pi' \in E$ , there are relatively open sets  $V, V'$  in  $E$  with  $V \cap V' = \emptyset, \pi \in V$  and  $\pi' \in V'$ . But then  $\varphi^{-1}(V), \varphi^{-1}(V')$  are open in  $\hat{A}$ , disjoint and  $\pi \in \varphi^{-1}(V), \pi' \in \varphi^{-1}(V')$ .

Let  $p \in A$  be a projection. We claim that  $\pi(p) = \pi(e)$  or  $\pi(p) = 0$  for all  $\pi \in \hat{A}$  except possibly finitely many representations  $\{\pi_i\}_{i=1}^n \cup \{\pi'_j\}_{j=1}^m \subset \hat{A} \setminus E$  for which  $\pi_i(p) \neq \pi_i(e)$  but it has finite codimension,  $\pi'_j(p) \neq 0$  but is finite dimensional and finitely many representations  $\{\rho_k\}_{k=1}^r$  with  $1 < \dim H_{\rho_k} < \infty$  for which one may have  $0 \neq \rho_k(p) \neq \rho_k(e)$ . Indeed, the function  $\pi \rightarrow \|\pi(p)\|$  is continuous on the closed set  $E' = \{\pi \in \hat{A} : \dim H_\pi = 1\}$ ; hence, there are disjoint closed sets  $E_0, E_1$  such that  $E_0 = \{\pi \in E' : \pi(p) = 0\}, E_1 = \{\pi \in E' : \pi(p) = \pi(e)\}$  and  $E_0 \cup E_1 = E'$ . By (i) and (ii), for each  $\sigma \in \hat{A}$ , the set  $[\hat{A} \setminus \varphi^{-1}(\sigma)] \cup \{\sigma\}$  is closed; it is the spectrum of some quotient of  $A$ . It follows from what has been established at the beginning of the proof and [3, 3.9.4] that each  $\sigma \in E_0$  ( $\sigma \in E_1$ ) has a relatively open neighbourhood  $V_\sigma$  in  $[\hat{A} \setminus \varphi^{-1}(\sigma)] \cup \{\sigma\}$  such that  $\pi(p) = 0$  ( $\pi(p) = \pi(e)$ ) for every  $\pi \in V_\sigma$ .  $E_0$  and  $E_1$  are compact so there are  $\{\sigma_l\}_{l=1}^s$  in  $E_0$  and  $\{\sigma'_l\}_{l=s+1}^t$  in  $E_1$  such that  $E_0 \subset \bigcup_{l=1}^s V_{\sigma_l}, E_1 \subset \bigcup_{l=s+1}^t V_{\sigma'_l}$ . It remains to see what is  $\pi(p)$  for  $\pi \in \hat{A} \setminus \bigcup_{l=1}^t V_{\sigma_l}$ . Now, the set  $U_l = V_{\sigma_l} \cup \varphi^{-1}(\sigma_l)$  is open by (ii). Put  $U = \bigcup_{l=1}^t U_l$ . Then  $\hat{A} \setminus U$  is closed and consists only of isolated points. It is therefore finite and its

elements are the exceptional representations  $\rho_k$  mentioned above. If  $\pi \in \varphi^{-1}(\sigma_l)$ ,  $1 \leq l \leq s$ , then  $\pi(p)$  is a finite dimensional projection. Applying [3, 3.3.7] to a suitable quotient of  $A$  we find that  $\pi(p) \neq 0$  only for finitely many  $\pi \in \varphi^{-1}(\sigma_l)$ . Thus the set of those  $\pi \in \bigcup_{l=1}^s \varphi^{-1}(\sigma_l)$  for which  $\pi(p) \neq 0$  is finite.

Similarly, one finds that the set of the representations  $\pi \in \bigcup_{l=s+1}^t \varphi^{-1}(\sigma_l)$  for which  $\pi(p) \neq \pi(e)$  is finite. Since  $\sigma_l(p) = \sigma_l(e)$  in this case,  $\pi(p)$  must have finite codimension. We have  $\hat{A} \setminus \bigcup_{l=1}^t V_{\sigma_l} = (\hat{A} \setminus U) \cup (\bigcup_{l=1}^t \varphi^{-1}(\sigma_l))$  so the claim has been proved.

Now let  $B$  be a finite dimensional \*-subalgebra of  $A$  containing  $e$ . We shall prove that  $B$  is contained in a finite dimensional \*-subalgebra of  $A$  which is \*-isomorphic to the direct sum of finite dimensional \*-subalgebras  $\{C_i\}_{i=1}^m, \{D_j\}_{j=1}^n$  and a commutative \*-subalgebra generated by mutually orthogonal projections  $\{p_k\}_{k=1}^r$  that satisfy:

- (1) there are distinct  $\{\pi_i\}_{i=1}^m \subset \hat{A} \setminus E$  such that  $\pi_i(C_i) \subset LC(H_{\pi_i})$  and  $\pi(C_i) = \{0\}$  for  $\pi \in \hat{A}, \pi \neq \pi_i$ ;
- (2) there are distinct  $\{\rho_j\}_{j=1}^n$  with  $1 < \dim H_{\rho_j} < \infty$  such that  $\pi(D_j) = \{0\}$  if  $\pi \in \hat{A}, \pi \neq \rho_j$ ;
- (3)  $p_k C_i = \{0\}, p_k D_j = \{0\}$  for every  $i, j, k$ ;  $\pi(p_k) = \pi(e)$  for some  $k$  whenever  $\pi \in \hat{A} \setminus (\{\pi_i\}_{i=1}^m \cup \{\rho_j\}_{j=1}^n)$ .

Indeed, let  $\{q_k\}_{k=1}^r$  be a maximal family of mutually orthogonal projections in  $B$ . Then  $\sum_{k=1}^r q_k = e$ . There are only finitely many  $\pi \in \hat{A} \setminus E$  such that for some  $k, 1 \leq k \leq r, \pi(q_k)$  is a nonzero finite dimensional projection. Denote them  $\{\pi_i\}_{i=1}^m$ , all of them distinct. There are only finitely many  $\pi \in \hat{A}$  with  $1 < \dim H_{\pi} < \infty$  such that for some  $k, 1 \leq k \leq r, \pi(q_k)$  is a nontrivial projection. Let  $\{\rho_j\}_{j=1}^n$  be these exceptional finite dimensional irreducible projections.

By (ii), the finite dimensionality of  $\rho_j$  and [3, 4.1.11, 3.2.3]  $A$  decomposes as the direct sum of two ideals  $I_j^1$  and  $I_j^2$  such that  $\rho_j(I_j^1) = \rho_j(A)$  and  $\pi(I_j^1) = \{0\}$  if  $\pi \neq \rho_j$ . Clearly,  $I_j^1$  is finite dimensional. Choose a \*-subalgebra  $D_j$  of  $I_j^1$  such that  $\rho_j(D_j) = \rho_j(B)$ . Each  $\pi_i(B)$  is the direct sum of a finite dimensional \*-algebra  $M_i \subset LC(H_{\pi_i})$  and the one-dimensional  $C^*$ -algebra generated by  $\pi_i(p_k)$  for some  $k, \pi_i(p_k)$  being a projection of finite codimension. The set  $\{\pi_i\}$  is open and closed in the spectrum of the kernel of  $\varphi(\pi_i)$  so there is a \*-subalgebra  $C_i$  of  $A$  such that  $\pi_i(C_i) = M_i$  and  $\pi(C_i) = \{0\}$  for  $\pi \in \hat{A}, \pi \neq \pi_i$ . Let now  $p_k \in A$  be the projection that satisfies

$$\pi(p_k) = \begin{cases} \pi(q_k), & \pi \in \hat{A}(\{\pi_i\}_{i=1}^m \cup \{\rho_j\}_{j=1}^n), \\ \pi(q_k), & \pi \in \{\pi_i\}_{i=1}^m, \pi(e - q_k) \in LC(H_{\pi}), \\ 0, & \text{otherwise.} \end{cases}$$

It is straightforward to check that the  $C^*$ -subalgebra generated by  $\{C_i\}_{i=1}^m, \{D_j\}_{j=1}^n$  and  $\{p_k\}_{k=1}^r$  is finite dimensional, contains  $B$  and meets all other requirements.

Suppose now that  $B$  is a finite dimensional \*-subalgebra of  $A$  and  $p \in A$  a projection. We claim that the  $C^*$ -subalgebra generated by  $B$  and  $p$  is finite dimensional. We may suppose  $e \in B$ . Let  $\{C_i, \pi_i\}_{i=1}^m, \{D_j, \rho_j\}_{j=1}^n$  and  $\{p_k\}_{k=1}^r$  be the objects described above for  $B$ . To simplify the notation we shall suppose that  $B$  is

the  $C^*$ -subalgebra generated by  $\{C_i\}_{i=1}^m, \{D_j\}_{j=1}^n$  and  $\{p_k\}_{k=1}^r$ . Fix  $i$ ; there is one and only one  $k$  such that  $\pi_i(p_k)$  has finite codimension  $H_{\pi_i}$ . Call it  $k_i$ . The  $C^*$ -subalgebra of  $\pi_i(A)$  generated by  $\pi_i(B)$  and  $\pi_i(p)$  is the direct sum of a finite dimensional  $*$ -algebra  $N_i \subset LC(H_{\pi_i})$  and the one-dimensional algebra generated by a projection  $x_i$  of finite codimension in  $H_{\pi_i}$ . This  $x_i$  is the greatest lower bound of  $\pi_i(p_{k_i})$  and  $\pi_i(p)$  when the latter has finite codimension or the greatest lower bound of  $\pi_i(p_{k_i})$  and  $\pi_i(e - p)$  when  $\pi_i(p)$  is finite dimensional. Let  $C'_i$  be that  $C^*$ -subalgebra of  $A$  which satisfies  $\pi_i(C'_i) = N_i, \pi(C'_i) = \{0\}$  for  $\pi \in \hat{A}, \pi \neq \pi_i$ . Given  $j, 1 \leq j \leq n$ , let  $D'_j$  be the  $C^*$ -subalgebra of  $A$  for which  $\pi(D'_j) = 0$  if  $\pi \in \hat{A}, \pi \neq \rho_j$  and  $\rho_j(D'_j)$  is the  $*$ -subalgebra of  $\rho_j(A)$  generated by  $\rho_j(B)$  and  $\rho_j(p)$ . We denote by  $p'_k$  the projection of  $A$  defined by

$$\pi(p'_k) = \begin{cases} \pi(p_k), & \pi \in \hat{A} \setminus \{\pi_i\}_{i=1}^n, \\ x_i, & \pi = \pi_i \text{ and } k = k_i, \\ \pi_i(p_k) = 0, & \pi = \pi_i \text{ and } k \neq k_i. \end{cases}$$

Consider the projection  $p' \in A$  given by

$$\pi(p') = \begin{cases} \pi(p), & \pi \in \hat{A} \setminus (\{\pi_i\}_{i=1}^n \cup \{\rho_j\}_{j=1}^m), \\ x_i, & \pi = \pi_i, \pi_i(p) \text{ has infinite codimension,} \\ 0, & \text{otherwise.} \end{cases}$$

Then  $\{p'_k\}_{k=1}^r$  and  $p'$  are commuting projections and each of them is orthogonal to  $\{C'_i\}_{i=1}^m$  and  $\{D'_j\}_{j=1}^n$ . Clearly the  $C^*$ -subalgebra of  $A$  generated by  $\{C'_i\}_{i=1}^m, \{D'_j\}_{j=1}^n, \{p'_k\}_{k=1}^r$  and  $p'$  is finite dimensional and contains  $B$  and  $p$ .

Now let  $B_1, B_2$  be two finite dimensional  $*$ -subalgebras of  $A$ . There are finitely many projections in  $B_2$  such that every element of  $B_2$  is a linear combination of them. An induction argument based upon the previous claim will show that the  $C^*$ -algebra generated by  $B_1$  and  $B_2$  is finite dimensional. This concludes the proof of the theorem.

The continuity of the map  $\varphi$  is crucial for the sufficiency part of the above theorem. Indeed, it may happen that a unital  $AF$  algebra satisfies (i), (ii) and  $\{\pi \in A: \dim H_\pi < \infty\}$  is Hausdorff but  $A$  does not have the d.s.p. This is shown by the following example.

EXAMPLE 3.8. Let  $H$  be a separable infinite dimensional Hilbert space whose identity operator is denoted by  $e$ . Fix a projection  $p$  with one-dimensional range in  $H$ . Consider the  $C^*$ -algebra of all sequences  $x = \{\lambda_n e + x_n\}_{n=1}^\infty$  where  $\{\lambda_n\}_{n=1}^\infty$  is a convergent sequence of complex numbers and  $\{x_n\}_{n=1}^\infty$  is a sequence from  $LC(H)$  which converges to some scalar multiple of  $p$ , say  $\alpha p$ . The norm is, of course,  $\|x\| = \sup_n \|\lambda_n e + x_n\|$ . Then  $A$  is a unital  $AF$  algebra and  $\hat{A} = \{\pi_n\}_{n=1}^\infty \cup \{\rho_n\}_{n=1}^\infty \cup \{\pi, \rho\}$  where

$$\begin{aligned} \pi_n(x) &= \lambda_n e + x_n, & \rho_n(x) &= \lambda_n, \\ \pi(x) &= \alpha + \lim_{n \rightarrow \infty} \lambda_n, & \rho(x) &= \lim_{n \rightarrow \infty} \lambda_n. \end{aligned}$$

The topology of  $\hat{A}$  is as follows:  $\{\pi_n\}$  and  $\{\overline{\pi_n} = \{\pi_n, \rho_n\}\}$  are open sets for each  $n$ . A neighbourhood basis for  $\pi$  is the family of all the sets  $\{\pi\} \cup \{\pi_n\}_{n=N}^\infty$ ,  $N = 1, 2, \dots$ , while a neighbourhood basis for  $\rho$  consists of the collection  $\{\rho\} \cup \{\pi_n\}_{n=N}^\infty \cup \{\rho_n\}_{n=N}^\infty$ ,  $N = 1, 2, \dots$ . Clearly  $A$  satisfies (i), (ii) of Theorem 3.7 and the subset of all the finite dimensional irreducible representations is Hausdorff in the relative topology induced by  $\hat{A}$ . However,  $A$  does not have the d.s.p. since it contains a  $C^*$ -subalgebra  $*$ -isomorphic to that of Example 2.1 (alternatively, the map  $\varphi$  is not continuous).

**4. Representation and classification.** We describe below a way to construct unital  $AF$  algebras with d.s.p. Then we shall show that every unital  $AF$  algebra with the d.s.p. can be represented in this manner. Part of the coming discussion is valid in a more general framework but we preferred to keep the setting as close to our needs as practicable.

Let  $T$  be a compact topological space. From now on we shall make the following blanket assumption on  $T$ .

**ASSUMPTION 4.1.** There is a nonvoid closed subset  $T' \subset T$  which is Hausdorff in the relative topology and a continuous retraction  $\varphi$  of  $T$  onto  $T'$  such that whenever  $\varphi(t) \neq t$ ,  $\{t\}$  is an open set and  $\overline{\{t\}} = \{t, \varphi(t)\}$ .

Let now  $\mathcal{Q} = \{A(t) : t \in T\}$  be a family of  $C^*$ -algebras which satisfies

**ASSUMPTION 4.2.** For each  $t \in T'$ ,  $A(t)$  is a finite dimensional factor; the point  $t \in T'$  is isolated in  $T$  if the dimension of  $A(t)$  is greater than one. For  $t \in T \setminus T'$ ,  $A(t) = \widetilde{LC}(H_t)$  where  $H_t$  is some separable infinite dimensional Hilbert space.

The unit of each  $A(t)$  is denoted by  $e_t$ . If  $A(t)$  is one-dimensional (e.g. when  $t \in T'$  and  $\varphi^{-1}(t) \neq \{t\}$ ) we shall identify it with the complex field.

Consider now all the mappings  $a : T \rightarrow \cup \{A(t) : t \in T\}$  which have the following properties:

- (1)  $a(t) \in A(t)$  for every  $t \in T$ ;
- (2) if  $t' \in T'$ ,  $t \in \varphi^{-1}(t') \setminus \{t'\}$  then  $a(t) - a(t')e_t$  is a compact operator; for every  $\varepsilon > 0$  the set  $\{t \in \varphi^{-1}(t') \setminus \{t'\} : \|a(t) - a(t')e_t\| \geq \varepsilon\}$  is finite;
- (3) for each  $t' \in T'$  with  $\dim A(t') = 1$  and each  $\varepsilon > 0$  there is a relatively open neighbourhood  $U(t', \varepsilon)$  of  $t'$  in  $T \setminus \{s : \varphi(s) = t', s \neq t'\}$  such that  $\|a(t) - a(t')e_t\| < \varepsilon$  if  $t \in U(t', \varepsilon)$ .

The set of all such mappings will be denoted  $A$ . Remark that there is  $e \in A$  such that  $e(t) = e_t$  for every  $t \in T$ .

The proofs of the following four statements are routine so will be omitted.

**LEMMA 4.3.** For each  $a \in A$  the function  $t \rightarrow a(t)$  is bounded on  $T$ .

**PROPOSITION 4.4.** With the operations defined pointwisely and the norm given by  $\|a\| = \sup\{\|a(t)\| : t \in T\}$ ,  $A$  is a  $C^*$ -algebra whose unit is  $e$ .

**LEMMA 4.5.** If  $f : T \rightarrow \mathbb{C}$  is continuous,  $a \in A$  and  $b(t) = f(t)a(t)$ ,  $t \in T$ , then  $b \in A$ .

If  $a, b$  and  $f$  are as above we shall use the notation  $f \cdot a$  for  $b$ .

LEMMA 4.6. Let  $t_0 \in T$  and  $a_0 \in A(t_0)$ . There is  $a \in A$  such that  $a(t_0) = a_0$ .

$A$  will be called the  $C^*$ -algebra associated with the pair  $(T, \mathcal{Q})$ .

LEMMA 4.7. Suppose  $I$  is a closed two-sided ideal of  $A$  and for  $t \in T$  denote  $I(t) = \{a(t) : a \in I\}$ . For a fixed  $t' \in T'$  denote  $E = \{t \in T : \varphi(t) = t', t \neq t', I(t) \neq \{0\}\}$ . Assume that the mapping  $a : T \rightarrow \cup \{A(t) : t \in T\}$  satisfies

- (1)  $a(s) = 0$  for  $s \in T \setminus E$ ;
- (2)  $a(t) \in LC(H_t)$  if  $t \in E$  and  $\{t \in E : \|a(t)\| \geq \epsilon\}$  is finite for each  $\epsilon > 0$ .

Then  $a \in I$ .

PROOF. It is easily seen that for  $t \in E$  and  $x \in LC(H_t)$  there is  $b \in I$  such that

$$b(s) = \begin{cases} x, & s = t, \\ 0, & s \neq t. \end{cases}$$

Now let  $a$  be as above. Clearly  $a \in A$ . Fix  $\epsilon > 0$  and suppose that  $\{t_1, t_2, \dots, t_n\} = \{t \in E : \|a(t)\| \geq \epsilon\}$ . By the above remark, there are  $\{a_j\}_{j=1}^n \subset I$  such that

$$a_j(s) = \begin{cases} a(t_j), & s = t_j, \\ 0, & s \neq t_j, \end{cases}$$

$j = 1, 2, \dots, n$ . Then  $\|a - \sum_{j=1}^n a_j\| < \epsilon$  and  $\sum_{j=1}^n a_j \in I$ .

PROPOSITION 4.8. Let  $I$  be a closed two-sided ideal of  $A$  and  $I(t) = \{a(t) : a \in I\}$ ,  $t \in T$ . Then  $I = \{a \in A : a(t) \in I(t), t \in T\}$ .

PROOF. The proof follows that of [3, 10.4.2]. We have to show only  $I \subset \{a \in A : a(t) \in I(t), t \in T\}$  so let  $a \in A$  satisfy  $a(t) \in I(t)$  for each  $t \in T$  and  $\epsilon > 0$ . For each  $t \in T$  there is  $a_t \in I$  with  $a(t) = a_t(t)$ . There are  $\{t_i\}_{i=1}^n \subset T'$  and relatively open neighbourhoods  $U_i$  of  $t_i$  in  $T \setminus \{t : \varphi(t) = t_i, t \neq t_i\}$  such that  $\cup_{i=1}^n (U_i \cap T') = T'$  and  $\|a(t) - a_{t_i}(t)\| < \epsilon$  if  $t \in U_i$ . Let  $\{f_i\}_{i=1}^n$  be a partition of the unity on  $T'$ , subordinated to the cover  $\{U_i \cap T'\}_{i=1}^n$  and denote  $g_i = f_i \circ \varphi$ ,  $1 \leq i \leq n$ . The mapping  $b = \sum_{i=1}^n g_i \cdot a_{t_i} = \sum_{i=1}^n (g_i \cdot e) a_{t_i}$  is clearly in  $I$ . By the previous lemma there is  $b' \in I$  such that

$$b'(t) = \begin{cases} 0, & \varphi(t) \neq t_i, \\ 0, & t = t_i, \\ (a - b)(t) - (a - b)(t_i)e_i, & \varphi(t) = t_i, t \neq t_i, 1 \leq i \leq n. \end{cases}$$

Then  $b + b' \in I$  and it is easily checked that  $\|a - (b + b')\| < \epsilon$ . Hence  $a \in I$ .

For every  $t \in T$  let  $I_t = \{a \in A : a(t) = 0\}$ . It is obvious that  $I_t$  is a closed two-sided ideal in  $A$ .

PROPOSITION 4.9. The map  $t \rightarrow I_t$  is a one-to-one correspondence from  $T$  onto  $\text{Prim}(A)$ .

PROOF. The proof is almost the same as that of [3, 10.4.3]. The modifications which should be made are obvious so we omit the details.

Remark that  $I_t(s) = LC(H_s)$  if  $\varphi(s) = t, s \neq t$ . Also, if  $t \in T \setminus T'$  then  $I_{\varphi(t)} = \{0\}$  while  $I_s = LC(H_s)$  whenever  $s \in T \setminus T'$  and  $\varphi(s) = \varphi(t)$ .

From Proposition 4.9 we deduce immediately

**PROPOSITION 4.10.** *A is postliminal. For a fixed  $t \in T$ ,  $a \rightarrow a(t)$  is an irreducible representation of A and every irreducible representation of A is of this form.*

From Assumption 4.1 one can easily infer

**LEMMA 4.11.** *Let  $S \subset T$ . Then*

$$\bar{S} = (\overline{S \cap T'}) \cup \{ \overline{\varphi(s)} : s \in S \} \cup (S \setminus T').$$

By using this lemma, one can prove, as in [3, 10.4.4],

**PROPOSITION 4.12.** *The map  $t \rightarrow I_t$  is a homeomorphism of T onto  $\text{Prim}(A)$ .*

**LEMMA 4.13.** *If T has a countable basis then A is separable.*

**PROOF.** We claim that given  $a \in A$  and  $\epsilon > 0$  there is  $f \in C(T)$  such that  $\|a(t) - f(t)e_t\| < \epsilon$  for all  $t \in T$  except, possibly, finitely many isolated points. Since  $C(T)$ , which can be identified with  $C(T')$ , is separable by the assumption of the lemma and each  $A(t)$  is separable, the conclusion is evident once we prove the claim.

For each  $t' \in T'$  with  $\dim A(t') = 1$  let  $U(t')$  be a relatively open neighbourhood of  $t'$  in  $T \setminus \{s : \varphi(s) = t', s \neq t'\}$  such that  $\|a(t) - a(t')e_t\| < \epsilon$  for  $t \in U(t')$ . Then

$$\{U(t') \cap T' : t' \in T', \dim A(t') = 1\} \cup \{t' : t' \in T', \dim A(t') > 1\}$$

is an open cover of  $T'$ . Let  $\{U(t'_i) \cap T'\}_{i=1}^n \cup \{t'_j\}_{j=n+1}^m$  be a subcover of it. Here, of course,  $\dim A(t'_i) = 1$  for  $1 \leq i \leq n$ ,  $1 < \dim A(t'_j) < \infty$  for  $n + 1 \leq j \leq m$ . Choose a partition of unity  $\{f_k\}_{k=1}^m$  on  $T'$  subordinated to this subcover and put  $g_k = f_k \circ \varphi$ . Recall that  $\|a(s) - a(t'_i)e_s\| < \epsilon$  for all  $s \in \varphi^{-1}(t'_i)$  except finitely many at most. It is then easily seen that one has

$$\left\| a(t) - \left[ \sum_{i=1}^n a(t'_i) f_i(t) \right] e_t \right\| < \epsilon$$

for all  $t \in T$  except possibly, finitely many isolated points.

**THEOREM 4.14.** *Suppose T and Q satisfy Assumptions 4.1 and 4.2. If T has a countable basis of compact open sets then the algebra A described above is a unital AF algebra with the d.s.p.*

**PROOF.** A is separable and postliminal by Lemma 4.13 and Proposition 4.10. It follows from Proposition 4.12 and [2] that A is an AF algebra. It has the d.s.p by Theorem 3.7.

**THEOREM 4.15.** *Let A be a unital AF algebra with the d.s.p. Then  $\hat{A}$  and  $\mathcal{Q} = \{\pi(A) : \pi \in \hat{A}\}$  satisfy Assumptions 4.1 and 4.2, respectively. For  $a \in A$  the mapping  $\hat{a}$  defined by  $\hat{a}(\pi) = \pi(a)$ ,  $\pi \in \hat{A}$ , belongs to the  $C^*$ -algebra B associated with the pair  $(\hat{A}, \mathcal{Q})$  and  $a \rightarrow \hat{a}$  is a \*-isomorphism of A onto B.*

**PROOF.** Clearly Theorem 3.7 insures that the needed assumptions are fulfilled. Let  $a \in A$ . Obviously  $\hat{a}$  satisfies condition (1) imposed on the elements of B. It satisfies

also the condition (2) by the proof of Lemma 3.2. Suppose now that  $\pi' \in \hat{A}$ ,  $\dim H_{\pi'} = 1$ . Let  $\varphi$  be the map given by Theorem 3.7. The set

$$\hat{A} \setminus \{\pi \in \hat{A} : \varphi(\pi) = \pi', \pi \neq \pi'\}$$

is closed in  $\hat{A}$ . Thus it is the spectrum of a quotient of  $A$ . By [3, 3.9.4] and Lemma 3.4, given  $\varepsilon > 0$ , there is a relative neighbourhood  $U$  of  $\pi'$  in this set such that  $\|\hat{a}(\pi) - \hat{a}(\pi')\pi(e)\| = \|\pi[a - \pi'(a)e]\| < \varepsilon$  for  $\pi \in U$ . Thus  $\hat{a} \in B$ .

Obviously  $a \rightarrow \hat{a}$  is a \*-isomorphism of  $A$  onto some  $C^*$ -subalgebra of  $B$ . Let  $f_1, f_2$  be two pure states of  $B$ ,  $f_1 \neq f_2$ . By Proposition 4.10 there are  $\pi_1, \pi_2 \in \hat{A}$  and unit vectors  $\xi_1 \in H_{\pi_1}$ ,  $\xi_2 \in H_{\pi_2}$  such that

$$f_j(b) = \langle b(\pi_j)\xi_j, \xi_j \rangle, \quad b \in B, j = 1, 2.$$

Moreover, if  $\pi_1 = \pi_2$  then  $\xi_1, \xi_2$  span different one-dimensional subspaces. By Kadison's transitivity theorem [3, 2.8.3] there is an  $a \in A$  such that

$$\langle \pi_1(a)\xi_1, \xi_1 \rangle \neq \langle \pi_2(a)\xi_2, \xi_2 \rangle.$$

Thus the image of  $A$  by the above \*-isomorphism separates the pure states of  $B$ . The conclusion follows now from the Stone-Weierstrass theorem for postliminal algebras [3, 11.1.8].

**COROLLARY 4.16.** *Let  $A_1$  and  $A_2$  be two unital AF algebras with the d.s.p. Then  $A_1$  and  $A_2$  are \*-isomorphic if and only if there is a homeomorphism  $\psi$  of  $\hat{A}_1$  onto  $\hat{A}_2$  such that  $\pi$  and  $\psi(\pi)$  have the same dimension for each  $\pi \in \hat{A}_1$ .*

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