SPECTRA OF INVARIANT UNIFORM
AND TRANSFORM ALGEBRAS

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ABSTRACT. For G a locally compact abelian group, any closed invariant proper subalgebra of $C_0(G)$ has analytic discs in its spectrum. Related results are given for $A(G)$ and $B(G)$.

For a compact group G it is an easy consequence of Bishop's generalized Stone-Weierstrass theorem and some well-known facts on representations that a proper (doubly) invariant uniform algebra on G cannot have G as its spectrum. (Indeed, Haar measure of a nontrivial subgroup must be multiplicative.) The corresponding result for closed subalgebras of $C_0(G)$, G locally compact abelian, seemingly required more harmonic analysis (and, originally, the argument yielding Bishop's result); but more emerged, namely, *that the spectra of (nonzero) proper closed invariant subalgebras of $C_0(G)$ contain analytic discs.* This follows easily from the following more specific result.

THEOREM 1. *Let A be a separating translation invariant closed proper subalgebra of $C_0(G)$, G l.c.a. Then*:

(1) *there is a compact subgroup H of G whose normalized Haar measure m is multiplicative on A, so $a \to m * a$ is a multiplicative projection of A onto its subalgebra $A \cap C_0(G/H)$, while*

(2) *there is a nontrivial homomorphism $\sigma: \mathbf{R} \to G/H$ for which $[A \cap C_0(G/H)] \circ \sigma$ consists of boundary value functions (on $\mathbf{R}$ or $\mathbf{R}/\mathbf{Z}$) of continuous functions analytic on the closed half-plane or disc.*

Alternatively, $A \cap C_0(G/H)$ consists of functions analytic with respect to the induced flow on G/H; in particular we have analyticity relative to a flow on G when G has no nontrivial compact subgroups. Of course one can exploit such analyticity; for example, if $f \in C_0(\mathbf{R}^n)$ has $f^{-1}(c)$ *of positive Lebesgue measure for some $c \neq 0$ then f and its translates generate $C_0(\mathbf{R}^n)$ as an algebra.* Such corollaries are given in §2, with the first section devoted to the proof of our theorem.

After a revised version of this paper was essentially complete I found a result of J. L. Taylor [T1, Lemma 2] (extending, and using, an earlier result of Gleason and Rieffel [Ri, Theorem 6.4]), which is closely related to Theorem 2 below; at least two thirds of the proof of Theorem 1 is devoted to covering essentially the same ground

(by an apparently different route). Unfortunately Taylor errs in passing from the semicharacters obtained by Rieffel to the homomorphism he used, so that Taylor's result is incorrect as stated;[2] of course the correct reading of Taylor's proof is consequently made more difficult. A much stronger result was given later by Taylor [T3, Theorem 4.5.2.], but this requires more knowledge of measure algebras than one might expect from many readers interested in uniform algebras. As a consequence I have not altered this part of the original presentation, but all my results should properly be regarded as corollaries to the Gleason-Rieffel-Taylor result. (For the well-equipped reader the later Taylor result yields the neatest approach although all use the structure of l.c.a. groups.)

There remains the possibility of replacing $C_0(G)$ throughout by other algebras, in particular by the usual algebras of Fourier transforms, $A(G)$ and $B(G)$. It was only when prompted by this question from the referee that I noticed its simplicity for $A(G)$, and proceeding to $B(G)$, turned to Taylor's works. The results have been added in a final section.

1. Essentially as an aside, we shall start with the case of G compact nonabelian because there the antisymmetric approach joins so easily with the harmonic analysis needed; the reader unfamiliar with or not interested in the latter can just as well skip to 1.1. (As the referee has pointed out, more detailed results have been given by Björk [**B**] and Gichev [**Gi**] for compact Lie groups.)

To begin, from invariance it is easy to see the maximal sets of antisymmetry for A are all cosets of a compact normal subgroup H (cf. [**Gl.2**, p. 127]) and as we shall see the Haar measure m of H is multiplicative on A. Indeed from Bishop's theorem [**G**] and the fact that A is proper we know H is nontrivial while $B = A \mid H$ is closed and proper. Of course it is an invariant algebra on H and thus the span in $C(H)$ of all the entries in certain of H's unitary matricial representations. If u_{ij} and u'_{kl} are two such entries, in U, U' say, and U is not the trivial representation (so $m(u_{ij}) = 0$) then m annihilates $u_{ij}u'_{kl}$ too since it is an entry in $U \otimes U'$ which cannot contain the trivial representation unless U' is equivalent to $\overline{U}$ [**N**, p. 438], a contradiction to antisymmetry. Thus all trigonometric polynomials in B with zero constant term form an ideal I in the dense subalgebra B_p of B formed by its trigonometric polynomials, and I is annihilated by m, as in I^- (an ideal in B of course). Since any element of B annihilated by m can be approximated by elements of I we clearly must have $B = \mathbf{C} \oplus I^- = \mathbf{C} \oplus (m^{-1}(0) \cap B)$, so m provides an element of the spectrum of A not in G, as desired.

1.1. In the analogous situation on an l.c.a. group G (with dual group $G\hat{} = \Gamma$), we need not of course have any compact subgroups, and thus no potentially multiplicative (nontrivial) normalized Haar measure. Moreover, while antisymmetry was used

in this setting originally, and gave the direction our proof still follows, it has been replaced by a quite elementary argument.

Let A^{-*} denote the w^* closure of A in $L^\infty(G)$. We shall view the dual group $\Gamma = G\hat{\ }$ as a subset of $L^\infty(G)$, and accordingly write the operation as multiplication. Then

$$(1) \qquad\qquad S_0 = \Gamma \cap A^{-*}$$

is a subsemigroup of Γ (since A is a multiplicative subsemigroup on which the w^* topology is separately continuous) which is closed since Γ has the relative w^* topology. Moreover, S_0 contains the spectrum[3] sp(a) of $a \in A$, and this fact will provide the basis for our proof; we shall show there is a homomorphism ρ of an open subgroup Γ_1 of Γ into $\mathbf{R}$ which is nonnegative (and nontrivial) on $S_0 \cap \Gamma_1$; because of the Paley-Wiener theorem this yields analytic behavior for the subalgebra of elements constant modulo the subgroup $\Gamma_1^\perp$ of G, which essentially proves our theorem. Thus the main effort is to produce ρ.

To begin, note that our closed subsemigroup S_0 of Γ is proper; if $\mu \neq 0$ is a measure orthogonal to our proper invariant algebra $A \subset C_0(G)$ then $\mu * f \neq 0$ for some f in $L^1(G)$, and $\mu * f$, still orthogonal, is in fact orthogonal to A^{-*}. Since A^{-*} cannot contain Γ without containing its w^* closed span, $L^\infty(G)$, S_0 is proper.

Consequently, $L = S_0 \cap S_0^{-1}$ is a closed proper subgroup; if the intersection is empty we let $L = \{1\}$. Now we reduce to the case in which L is trivial: if $\pi_0: \Gamma \to \Gamma_0 = \Gamma/L$ then $S = \pi_0 S_0$ is a closed proper subsemigroup of Γ_0 with

$$(2) \qquad\qquad S \cap S^{-1} \subset \{1\}.$$

Indeed, since S_0 is proper we have a $\gamma \in \Gamma \backslash S_0$ and for any open neighborhood V of such a γ missing S_0 we have $VL \cap S_0 = \varnothing$ since otherwise $V \cap S_0 \supset V \cap S_0 L \neq \varnothing$. Thus S is both closed and proper as asserted. Evidently to obtain a homomorphism ρ of an open subgroup Γ_1 of Γ into $\mathbf{R}$ which is ≥ 0 and somewhere positive on $S_0 \cap \Gamma$, it suffices to do the same for S and Γ_0.

1.2. We should note that S generates Γ_0 because S_0 generates Γ, i.e., $(S_0 S_0^{-1})^- = \Gamma$. Indeed, if the latter fails there is an x in G orthogonal to $S_0 S_0^{-1}$ and not the identity of G. So for any f in $L^1(G)$, $f_x - f$ annihilates $(S_0 S_0^{-1})^- \subset L^\infty(G)$, hence annihilates any element of the w^* closed span of $(S_0 S_0^{-1})^-$. But the latter contains A since, being a closed subgroup, $(S_0 S_0^{-1})^-$ is a set of synthesis by Reiter's theorem [**H, R**]. Thus $f_x * a = f * a = (f * a)_x$ for all f in $L^1(G)$ and $a \in A$, so a dense subalgebra of A cannot separate x and the identity, and A cannot.

1.3. We now proceed to show there is an open subgroup Γ_1 of Γ_0 and homomorphism $\rho: \Gamma_1 \to \mathbf{R}$ which is nonnegative on $\Gamma_1 \cap S$ and somewhere positive. To do so we shall use the fact that $\Gamma_0 = \Gamma_* \times \mathbf{R}^n$ where Γ_* has a compact open subgroup K and $n \geq 0$ [**Ru, We**].

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[3] sp(a) $= \bigcap\{\hat{\mu}^{-1}(0): \mu * a = 0, \mu \in L^1(G_0)\}$ (where L^1 can be replaced by M); dually sp(a) $= \Gamma \cap$ span$\{a_x: x \in G_0\}^{-*}$. Thus $S = A^{-*} \cap \Gamma \supset (\bigcup_{a \in A} \text{sp}(a))$, but in fact one has equality: for γ not in the second set one has $\mu \in L^1$ with $\hat{\mu} \equiv 0$ near the second so $\mu * a = 0$, $a \in A$, by spectral synthesis for open sets, whence γ is separated from A^{-*}.

Let

$$(3) \qquad \pi: \Gamma_0 = \Gamma_* \times \mathbf{R}^n \to \Gamma_*/K$$

denote the natural homomorphism and $S_1 = \pi S$; since $\Gamma' = \Gamma_*/K$ is discrete and SS^{-1} dense in Γ_0, evidently $S_1 S_1^{-1} = \Gamma'$. We shall consider the two possible cases:

$$(\text{i}) \qquad\qquad S_1 \neq \Gamma' \quad \left(\text{so } S_1 \cap S_1^{-1} \neq \Gamma' \right),$$

$$(\text{ii}) \qquad\qquad S_1 = \Gamma' \quad \left(\text{so } S_1^{-1} = \Gamma' \text{ as well} \right).$$

1.4. In case (i) let S_1' denote the semigroup $\{1\} \cup S_1$, so that $L' = S_1' \cap S_1'^{-1}$ is a nonempty subgroup of Γ', and proper: for at most the identity has been added to the intersection, and $S_1 \cap S_1^{-1} = \Gamma' \backslash \{1\}$ implies $1 = \gamma\gamma^{-1} \in S_1 \cap S_1^{-1}$ for any $\gamma \neq 1$ (such a γ exists unless Γ' is a singleton, and then only case (ii) can occur since $S_1 \neq \varnothing$). Now let $\pi_1: \Gamma' \to \Gamma'/L'$, $S_2 = \pi_1 S_1'$. Then $S_2 \cap S_2^{-1} = \{1\}$ since $\gamma L' = \gamma_1^{-1} L'$, $\gamma, \gamma_1 \in \Gamma'$, implies $\gamma\gamma_1 \in L'$, so $\gamma\gamma_1 = \gamma_2^{-1}$, $\gamma_2 \in S_1$, whence $\gamma = (\gamma_1\gamma_2)^{-1} \in S_1 \cap S_1^{-1} = L'$. Among the subsemigroups S' of Γ' containing S_2 and satisfying

$$(4) \qquad\qquad S' \cap (S')^{-1} = \{1\}$$

we have a maximal one, S_3 say. This has the property that every $\gamma \in S_3$, $\gamma \neq 1$, is of infinite order because of (4) since $\gamma^n = 1$ with $n > 1$ least implies $\gamma^{n-1} = \gamma^{-1} \in S_3 \cap S_3^{-1} = \{1\}$. In addition, by maximality $\gamma \in \Gamma' \backslash (S_3 \cup S_3^{-1})$ is of finite order; for if not, either no power of γ lies in $S_3 \cup S_3^{-1}$ and we can enlarge S_3 to the subsemigroup generated by γ and S_3 and preserve (4) (since $\gamma^k s = \gamma^{-l} s_1^{-1}$ for $k, l > 0$, $s, s_1 \in S_3$ implies $\gamma^{k+l} = (ss_1)^{-1}$), or some power lies in S_3 or S_3^{-1}. But $\gamma^k \in S_3$ (resp. S_3^{-1}) allows us also to adjoin γ to S_3 (resp. S_3^{-1}); for $\gamma^j s = \gamma^{-l} s_1^{-1}$ implies $(\gamma^{j+l})^k = (ss_1)^{-k} \in S_3 \cap S_3^{-1}$ so $= 1$.

Thus every element of $S_3 \cup S_3^{-1}$ (save 1) is of infinite order while every element of the complement is of finite order. So if Γ'' denotes the subgroup of elements of finite order the image S_4 of S_3 in Γ'/Γ'' has the property that $S_4 \cap S_4^{-1} = \{1\}$ and $S_4 \cup S_4^{-1} = \Gamma'/\Gamma''$. Hence S_4 provides a total ordering of Γ'/Γ'', making Γ'/Γ'' an ordered abelian group. Selecting any element $\gamma_0 \neq 1$ of the image $\tilde{S}$ of S in S_4 the subset of Γ'/Γ'' bounded by γ_0 in that order,

$$(5) \qquad\qquad \Gamma''' = \bigcup_n \left(\gamma_0^{-n} S_4 \cap \gamma_0^n S_4^{-1} \right),$$

forms a subgroup of Γ'/Γ'' whose complement meets S_4, or better, the homomorphic image $\tilde{S}$ of S itself, in an ideal of that semigroup, as is easily seen (elements which are infinitely large relative to γ_0 in the ordering multiply elements of S_4 to yield infinitely large elements). By dividing out the subgroup of infinitely small elements in Γ'' we obtain an archimedian ordered group so we have a nonnegative homomorphism of Γ''' into $\mathbf{R}$ sending γ_0 into $1 \in \mathbf{R}$ say; the inverse of Γ''' back in Γ_0 itself yields the desired open subgroup Γ_1 on which we have a homomorphism ρ into $\mathbf{R}$ sending $S \cap \Gamma_1$ into $\mathbf{R}_+$, with some positive value there. Equivalently, pulling back to Γ from $\Gamma_0 = \pi_0 \Gamma = \Gamma/L$ we have a homomorphism (still called ρ) of an open subgroup $\pi_0^{-1}\Gamma_1$ of Γ into $\mathbf{R}$ which is nonnegative on $S_0 \cap \pi_0^{-1}\Gamma_1$ and somewhere positive. Finally, if m denotes the normalized Haar measure of the

compact subgroup $H = (\pi_0^{-1}\Gamma_1)$ of G (whose dual is the discrete group $\Gamma/\pi_0^{-1}\Gamma_1$), the fact that $\tilde{S}\setminus\Gamma'''$ is an ideal in $\tilde{S}$, hence that $S\setminus\Gamma_1$ is one in S, and $S_0\setminus\pi_0^{-1}\Gamma_1$ is one in S_0, shows m is multiplicative on A as we next note.

1.5. Indeed the invariant algebra $(A\,|\,H)^-$ on the compact group H is the uniform closure of the combined spectra of the elements $a\,|\,H$ of $A\,|\,H$, while $\mathrm{sp}(a\,|\,H) \subset \pi(\mathrm{sp}(a))^-$ where $\pi\colon \Gamma \to \Gamma/(\pi_0^{-1}\Gamma_1)^\perp = H^\wedge$ is the natural map [**R**, p. 143]. Because $H^\wedge$ is discrete,

$$\mathrm{sp}(a\,|\,H) \subset \pi\,\mathrm{sp}(a) \subset \pi S_0,$$

and $A\,|\,H \subset \mathrm{span}\,\pi S_0$. Since, as functions on H,

$$\pi S_0 = S_0\,|\,H = \left(S_0 \cap \pi^{-1}\Gamma_1\right)|\,H \cup \left(S_0\setminus\pi_0^{-1}\Gamma_1\right)|\,H$$

while $m(\gamma) = m(1) = 1$ on the first set in the union and $\equiv 0$ on the ideal formed by the second set, m must be multiplicative on their uniform span, hence on $A\,|\,H$, and A.

1.6. We can now turn to case (ii), where

$$\pi\colon \Gamma_0 = \Gamma_* \times \mathbf{R}^n \to \Gamma_*/K = \Gamma'$$

maps S and S^{-1} onto Γ'. Recalling (2), that $S \cap S^{-1} \subset \{1\}$, we can now note that since a closed subsemigroup of a compact abelian group is a group [**W, Gl.1**], this implies $(K \times \{0\}) \cap S$ contains at most the identity. As a first consequence of this if $n = 0$, Γ' is nontrivial (since S cannot reduce to the identity). Indeed if $n = 0$ (so $\Gamma = \Gamma_*$) and if $\gamma \in \Gamma\setminus K$ then there is a $k \in K$ for which $\gamma k \in S$ since π maps S onto Γ'; similarly there is a $k' \in K$ with $\gamma^{-1}k' \in S$. But then $kk' = \gamma k\gamma^{-1}k' \in K \cap S \subset \{1\}$ so $k' = k^{-1}$ and thus $\gamma^{-1}k^{-1} \in S$ so $\gamma k \in S \cap S^{-1} \subset \{1\}$ despite $\gamma \notin K$. So $n \geqslant 1$.

Next, if

$$\pi_2\colon \Gamma_* \times \mathbf{R}^n \to \mathbf{R}^n \quad \text{and} \quad S_* = (K \times \mathbf{R}^n) \cap S$$

we have $(\pi_2 S_*)^- \cap (\pi_2 S_*^{-1})^- = (\pi_2 S_*)^- \cap ((\pi_2 S_*)^-)^{-1}$ containing at most the identity of $\mathbf{R}^n$; for r therein implies we have (k_j, r_j) and (k_j', r_j') in S^* with $r_j \to r$, $r_j' \to -r$. Passing to subnets we can assume $k_j \to k$, $k_j' \to k'$ in K, so that $(kk', 0) \in S_* = (K \times \mathbf{R}^n) \cap S$, hence lies in $(K \times \{0\}) \cap S \subset \{1\}$. Thus $k' = k^{-1}$, so (k, r) and $(k^{-1}, -r)$ both lie in $S_* \subset S$, so coincide with the identity by (3), and $r = 0$ as asserted.

Actually, $S_* \neq \varnothing$ since S maps onto $\Gamma' = \Gamma_*/K$ so $S_{**} = (\pi_2 S_*)^-$ is a nonempty closed subsemigroup of $\mathbf{R}^n$ with $S_{**} \cap (-S_{**}) \subset \{0\}$. Moreover, the set of positive dilates of S_{**}, $\mathbf{R}_+ S_{**}$, meets its negative at most in the identity; for $c, c' > 0$, s, s' in S_{**}, $cs = -c's'$, or $s = -(c'/c)s'$, cannot hold if c'/c is rational since then $ks = -ls'$ for positive integers k, l, whence both lie in $S_{**} \cap (-S_{**})$, hence $= 0$; and if c'/c is irrational S_{**} contains $\{ks - l(c'/c)s\colon k, l \in \mathbf{Z}_+\}$ which is dense in the line through 0 and s, so $s \neq 0$ implies S_{**}, and so $S_{**} \cap (-S_{**})$, contains a line through 0. Thus

$$(6) \qquad \left(-\mathbf{R}_+ S_{**}\right) \cap \left(\mathbf{R}_+ S_{**}\right) \subset \{0\}.$$

Now suppose S_{**} contains a maximal independent set of $k \leqslant n$ elements. Then in their vector span V over $\mathbf{R}$, $\mathbf{R}_+ S_{**}$ has interior, so (6) implies $(\mathbf{R}_+ S_{**})^-$ misses the k-dimensional interior of its negative, and thus $(\mathbf{R}_+ S_{**})^-$ is a proper closed cone in V, hence in $\mathbf{R}^n$. Consequently we have a nontrivial linear functional ψ on $\mathbf{R}^n$ which is nonnegative on $S_{**} = \pi_2((K \times \mathbf{R}^n) \cap S)^-$ and indeed positive at some point of $\pi_2 S$.

For the moment let $\Gamma^* = K \times \mathbf{R}^n$. Then for $\gamma, \gamma' \in \Gamma_0$ trivially

$$(7) \qquad (S \cap \gamma\Gamma^*)(S \cap \gamma'\Gamma^*) \subset S \cap \gamma\gamma'\Gamma^*,$$

so if we set

$$h(\gamma) = \inf\{\psi(\pi_2(\gamma')): \gamma' \in S \cap (\gamma\Gamma^*)\}$$

then $h(1) \geqslant 0$ and (7) implies (since each set is nonvoid) that

$$(8) \qquad h(\gamma) + h(\gamma') \geqslant h(\gamma\gamma').$$

In particular $h(\gamma) + h(\gamma^{-1}) \geqslant h(1) \geqslant 0$, and $h(\gamma)$ is always finite, while h is constant on cosets mod Γ^*.

Now consider

$$\Sigma = \{(\gamma, t) \in \Gamma_* \times \mathbf{R}: t > h(\gamma)\}.$$

From (8) we conclude Σ is a subsemigroup of $\Gamma_* \times \mathbf{R}$; moreover $\Sigma \cap \Sigma^{-1} = \varnothing$ since $(\gamma, t) \in \Sigma \cap \Sigma^{-1}$ says $t > h(\gamma)$ and $-t > h(\gamma^{-1})$, so $0 > h(\gamma) + h(\gamma^{-1}) \geqslant h(1) \geqslant 0$. Now among the subsemigroups Σ' of $\Gamma_* \times \mathbf{R}$ containing Σ with $\Sigma' \cap (\Sigma')^{-1}$ containing at most the identity, we have a maximal subsemigroup Σ^*, and exactly as in case (i) (cf. 1.4), we conclude that any element of Σ^*, other than the identity, has infinite order, while $(\Gamma_* \times \mathbf{R}) \backslash (\Sigma^* \cup \Sigma^{*-1})$ consists of elements of finite order (which necessarily have second coordinate zero). As before, if $\Gamma^{**} = K \times \mathbf{R}$

$$(\Sigma^* \cap \gamma\Gamma^{**}) \cdot (\Sigma^* \cap \gamma'\Gamma^{**}) \subset \Sigma^* \cap \gamma\gamma'\Gamma^{**}, \qquad \gamma, \gamma' \in \Gamma^* \times \mathbf{R},$$

which shows that with $H(\gamma) = \inf\{\pi_2\gamma': \gamma' \in \Sigma^* \cap \gamma\Gamma^{**}\}$ we have

$$(9) \qquad H(\gamma\gamma') \leqslant H(\gamma) + H(\gamma')$$

again. From this it follows that $H(1) \geqslant 0$ if it is finite; but $\Sigma \supset \Gamma_* \times (h(1), \infty)$, or $\Gamma_* \times (-\infty, -h(1)) \subset \Sigma^{-1} \subset \Sigma^{*-1}$, yields finiteness of $H(1)$, hence of $H(\gamma)$ since $H(\gamma) + H(\gamma^{-1}) \geqslant H(1)$. But now H is a homomorphism; for $\varepsilon > 0$, writing $\gamma = (\eta, t) \in \Gamma_* \times \mathbf{R}$ and $H(\eta)$ for $H(\gamma)$ (which is independent of t) we have $(\eta, H(\eta) - \varepsilon) \in (\Sigma^*)^{-1}$ provided $H(\eta) - \varepsilon \neq 0$, so $-H(\eta) + \varepsilon \geqslant H(\eta^{-1})$, whence $-H(\eta) \geqslant H(\eta^{-1})$ for all $\eta \in \Gamma_*$. But then $0 \leqslant H(1) \leqslant H(\eta) + H(\eta^{-1}) \leqslant 0$ follows so $H(\eta^{-1}) = -H(\eta)$, and thus

$$H(\eta) = H(\eta\eta_1\eta_1^{-1}) \leqslant H(\eta\eta_1) + H(\eta_1^{-1}) = H(\eta\eta_1) - H(\eta_1)$$

which combines with (9) (written in our abused notation as $H(\eta\eta_1) \leqslant H(\eta) + H(\eta_1)$) to yield our assertion.

Now we simply have to note that $h \geqslant H$ automatically, so that (recalling the definition of h) $\psi(\pi_2(\gamma)) \geqslant H(\gamma)$, $\gamma \in S$; thus $\psi \circ \pi_2 - H$ provides a homomorphism of Γ_0 into $\mathbf{R}$, nonnegative on S and somewhere positive since $\psi(\gamma_0) > 0$ for

some γ_0 in $(K \times \mathbf{R}^n) \cap S$ while $H(\gamma_0) = H(1) = 0$. (Since $H^{-1}(\mathbf{R}_+)$ has interior, it is easy to see H is continuous; for discontinuity at $\gamma_0 = 1$ would imply $\overline{\lim}_{\gamma \to \gamma_0} H(\gamma)$ $= +\infty$ and $\underline{\lim}_{\gamma \to \gamma_0} H(\gamma) = -\infty$, and by translation this would hold at any γ_0.)

1.7. We have now obtained in each case a compact subgroup H (trivial in case (ii)) whose normalized Haar measure m is multiplicative on A along with a nontrivial homomorphism $\rho: (G/H)\hat{\ } \to \mathbf{R}$ which is ≥ 0 and somewhere > 0 on S_0. Dually we have a map $\hat{\rho}: \mathbf{R} \to G/H$ which provides our flow on G/H by translation. Evidently $a \to m * a$ maps A multiplicatively onto its subalgebra $m * A$ of functions constant on cosets mod H, and it remains to see each such a has $a \circ \hat{\rho}$, the boundary value function of a function analytic in a half-plane or disc.

For this it suffices to see that[4] $\mathrm{sp}(a \circ \hat{\rho}) \subset \mathbf{R}_+$. Now

$$\mathrm{sp}(a \circ \hat{\rho}) = \cap \left\{ \hat{\mu}^{-1}(0) : \mu \in M(R), \mu * (a \circ \hat{\rho}) = 0 \right\},$$

while $\mu * (a \circ \hat{\rho}) = 0$ iff $\hat{\rho}^*\mu * a = 0$, where $\hat{\rho}^*\mu$ is the image of μ on G/H. Moreover $(\hat{\rho}^*\mu)\hat{\ }(\gamma) = \hat{\mu}(\gamma \circ \hat{\rho}) = \hat{\mu} \circ \rho(\gamma)$. Suppose $t < 0$. Then we have $\mu \in L^1(\mathbf{R}) \subset M(\mathbf{R})$ with $\hat{\mu}(t) = 1$, $\hat{\mu} \equiv 0$ near $\mathbf{R}_+$, so $(\hat{\rho}^*\mu)\hat{\ } = \hat{\mu} \circ \rho \equiv 0$ near $\rho^{-1}(\mathbf{R}_+) \supset S \supset \mathrm{sp}(a)$. Thus $\hat{\rho}^*\mu * a = 0$ by spectral synthesis for open sets, so $\mu * (a \circ \hat{\rho}) = 0$. Since $\hat{\mu}(t) = 1$, $t \notin \mathrm{sp}(a \circ \hat{\rho})$, and our proof is complete.

The two cases we were forced to consider in 1.3 correspond exactly to the two possibilities for a representation $\hat{\rho}$ of $\mathbf{R}$ in an l.c.a. group [**We**, p. 96]; either (i) $\hat{\rho}(\mathbf{R})^-$ is a compact subgroup or (ii) $\hat{\rho}$ is an isomorphism. (Case (i) leads to a $\hat{\rho}$ which factors through a discrete group, so $\hat{\rho}(\mathbf{R})^- = H$ is compact (cf. 1.4), and in case (ii) this cannot be the case since we have a line in Γ mapping onto $\mathbf{R}$ under $\hat{\rho}$ and since $\mathbf{R} \to \Gamma \to \mathbf{R}$ is topological so is the dual.) We shall make use of this dichotomy later.

Finally it is easy to see the above proof applies equally well to a strictly closed subalgebra of $C(G)$ if we simply replace closure in C_0 everywhere by strict closure in C. Thus

THEOREM 1′. *Let A be a separating translation invariant strictly closed proper subalgebra of $C(G)$, G l.c.a. Then the conclusions of Theorem 1 apply (with C in place of C_0).*

1.8. Clearly a large portion of our proof yields an assertion about certain closed subsemigroups of l.c.a. groups Γ. Call $E \subset \Gamma$ a half of Γ if there is a nontrivial homomorphism ρ of Γ into an l.c.a. group Π which is totally ordered by its closed subsemigroup Π_+ of nonnegative elements, while $E = \rho^{-1}\Pi_+$. As a direct corollary of our proof we have

THEOREM 2. *A closed proper generating subsemigroup S of an l.c.a. group Γ is contained in a half of Γ.*

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[4] This is a well-known consequence of the Paley-Wiener theorem; indeed if $\mathrm{sp}(\varphi) \subset [0, \infty)$ and $\{f_n\}$ is an approximate identity with $f_n \in L^1 \cap L^2(\mathbf{R})$ and having support in $(-1/n, 1/n)$ then $\hat{f}_n\varphi \in L^2 \cap L^\infty(\mathbf{R})$ has $\mathrm{sp}(\hat{f}\varphi) \subset [-1/n, \infty)$ so that $\varphi_n = \exp(in^{-1} \cdot)\hat{f}_n\varphi \in H^2 \cap L^\infty = H^\infty$ by Paley-Wiener, with L^∞-norm $\leq \|\varphi\|_\infty$. Since $\hat{f}_n \to 1$ uniformly on compacta it is easy to conclude φ is the boundary value function of any normal family limit of $\{\varphi_n\}$ on $\mathcal{K}_+$.

Actually Π is **R** or discrete, corresponding to our two cases. Note that if we let A be the strictly closed span of S then A is an invariant algebra, and proper in $C(G)$; indeed since $\gamma \in \Gamma \backslash S$ can be separated from S by a measure in $L^1(G)$, $S = \Gamma \cap A$. Thus *any* closed proper subsemigroup S of Γ arises as $A^- \cap \Gamma$ for a subalgebra of $C(G)$ under the strict topology. This is certainly not the case for subalgebras of $C_0(G)$ (where A^- is to be taken as the w^* closure in $L^\infty(G)$): S must be locally an M set, i.e., for any closed neighborhood N of each γ in S there must be $\varphi \in C_0$, $\varphi \neq 0$, with $\mathrm{sp}(\varphi) \subset N \cap S$. (Indeed some $a \in A$ has $\mathrm{sp}(a) \cap N^0 \neq \varnothing$ (cf. footnote 3) so $\varphi = f * a$ will do for some f in L^1.) Conversely such an S will be $A^{-*} \cap \Gamma$ for the subalgebra A of $C_0(G)$ generated by all φ in C_0 with $\mathrm{sp}(\varphi) \subset S$.

We should note that, at least in the strict setting, A need not be determined by S when G is noncompact precisely because of the failure of spectral synthesis.[5]

2. We can now easily prove

COROLLARY 1. *If A is a (strictly) closed proper invariant subalgebra of $C_0(G)$ (resp. $C(G)$) then the spectrum of A contains analytic discs.*

In fact the discs will appear as an extension to the disc D, or upper half-plane $\mathcal{H}_+$, of $\hat{\rho}$, so our analytic disc will be "bounded" by a line or circle in our group or a quotient thereof.

Suppose for the moment A separates G so our theorems apply. Then either $\hat{\rho}$ is not 1-1 and $a \circ \hat{\rho}$ is the restriction to ∂D of a function in the disc algebra $A(D)$ for $a \in A_1 = m * A$, or $a \circ \hat{\rho} \in H^\infty(\mathcal{H}_+)$. In either case if λ_z is the usual representing measure on ∂D or $\partial\mathcal{H}_+$ we know the analytic extension of $a \circ \hat{\rho}$ is given by

$$(10) \qquad z \to \lambda_z(a \circ \hat{\rho}) = \int a(\hat{\rho}(t))\lambda_z(dt);$$

since λ_z is appropriately multiplicative the map into $A(D)$ or $H^\infty(\mathcal{H}_+)$ is also, and $a \to \lambda_z(a \circ \hat{\rho})$ is a multiplicative linear functional on A_1. To obtain the corresponding element of the spectrum of A we need only apply our multiplicative projection first:

$$a \to \int m * a(\hat{\rho}(t))\lambda_z(dt) = \int m * a(x)\hat{\rho}^*\lambda_z(dx)$$

$$= \iint a(x+y)m(dy)\hat{\rho}^*\lambda_z(dx) = m * \hat{\rho}^*\lambda_z(a)$$

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[5] If K is a compact set of nonsynthesis and $x \in G$ has $nx \to \infty$ then the closed subsemigroup S generated by $Nx + K$ (for some large N) will meet a neighborhood V of $Nx + K$ in just that set. Let μ be a measure with support precisely $Nx + K$, A the strictly closed invariant subalgebra of $C(G)$ generated by $\hat{\mu}$ (so $A \cap \Gamma = S$) and let $\varphi \in C(G)$ have $\mathrm{sp}(\varphi) \subset Nx + K$ and not lie in the strictly closed span of $Nx + K$ in $C(G)$ (which exists by failure of synthesis). Then $\varphi \notin A$ since if $a_\delta \to \varphi$ strictly and $f \in L^1(G)$ has $\hat{f} \equiv 1$ near $Nx + K$ but $\equiv 0$ off V then $f * a_\delta \to f * \varphi = \varphi$ strictly, while $f * a_\delta$ lies in the strictly closed span of $Nx + K$ since f annihilates span $(S \backslash (Nx + K))$. Consequently, the strictly closed invariant algebra B generated by all ψ in $C(G)$ with spectra in $Nx + K$ contains A properly and has $B \cap \Gamma = S$.

provides our element of the spectrum with $m * \hat{\rho}^{*}\lambda_z$ our representing measure (familiar from [AS]). Now $z \to m * \hat{\rho} * \lambda_z$ provides our analytic disc once we see this map is 1-1; since $a \circ \hat{\rho}$ nonconstant implies (10) is nonconstant we of course have the map 1-1 on subdiscs of D or $\mathcal{K}_{+}$, but in fact we have global 1-1-ness. This is easier to see in the disc case: if the rotation invariant algebra $A \circ \hat{\rho}$ does not separate $z_1 \neq z_2$ in D and $|z_1| = |z_2| = r$ then $t = z_1/z_2 \in \partial D$ is a period of our functions on $|z| = r$ so $f \in A \circ \hat{\rho}$ implies $f(z) - f(tz) \equiv 0$ on $|z| = r$, so on D, and $A \circ \hat{\rho}$ cannot separate ∂D. Again if $r_1 = |z_1| < |z_2| = r_2$ then invariance again implies $f \in A \circ \hat{\rho}$ has the same range on $|z| = r_1$ as on $|z| = r_2$, so any nonconstant f attains its maximum modulus over $|z| \leqslant r_2$ at an interior point.

In the nondisc case, if $z_1 \neq z_2$ in $\mathcal{K}_{+}$ are not separated and $\operatorname{Im} z_1 = \operatorname{Im} z_2$ then $z_1 - z_2 \in \mathbf{R}$ is a period of $f \in A \circ \hat{\rho}$ on the horizontal line $\operatorname{Im} z = \operatorname{Im} z_1$ so a period of f as before, and A cannot separate $\hat{\rho}(\mathbf{R})$. If $0 \leqslant \operatorname{Im} z_1 < \operatorname{Im} z_2$ we have two subcases corresponding to the possibilities for $\hat{\rho}$, cited earlier [We, p. 96]; either $\hat{\rho}$ is an isomorphism or $\hat{\rho}(\mathbf{R})^{-}$ is a compact subgroup. In the first subcase $f \in A \circ \hat{\rho} \in C_0(\mathbf{R})$ so its analytic extension to $\mathcal{K}_{+}$ vanishes at ∞ by a well-known Phragmén-Lindelöf theorem [Ti, 5.63], and $f(\cdot + z_1)$ thus assumes its maximum modulus at some z in $\mathbf{R}$, hence at some z in $\mathbf{R} + (z_2 - z_1)$, interior to $\mathcal{K}_{+}$. In the second subcase $A \circ \hat{\rho}$ consists of almost periodic functions. If $0 = \operatorname{Im} z_1$, so we can assume $z_1 = 0$, we can choose any nonconstant $f \in A \circ \hat{\rho}$ of norm 1 and x_n in $\mathbf{R}$ with $|f(-x_n)| \to 1$; then $\{f_{x_n}\}$ has a uniform cluster point g in $A \circ \hat{\rho}$ with $\|g\| = 1 = |g(0)| = |g(z_1)|$ and since g cannot separate $z_1 = 0$ and z_2, g assumes its maximum modulus at an interior point so is constant. But that is impossible since any g in the closed orbit $\{f_x : x \in \mathbf{R}\}^{-}$ of f has precisely the same orbit as f. Finally, in case $0 < \operatorname{Im} z_1$ we have only to replace f by $f_1 = \lambda_{z_1} * f \in A \circ \hat{\rho}$; then f_1 cannot separate 0 and $z_2 - z_1$ and we argue as before.

There remains the possibility that A does not separate the points of G. But then any unseparated pair differ by a common period of the elements of A; if $A \subset C_0(G)$ the set P of periods forms a compact subgroup of G and A can be viewed as a subalgebra of $C_0(G/P)$ which separates. If instead A lies in $C(G)$ then compactness of P is not needed; A is a subalgebra of $C(G/P)$ which is evidently strictly closed, separating, invariant and proper.

2.1. One can exploit analyticity by excluding its consequences at boundaries. For example for $f \in C_0(G)$ suppose that for each nontrivial (continuous but not necessarily open) representation $\sigma: \mathbf{R} \to G$ there is a coset of the algebraic subgroup $\sigma(\mathbf{R})$ of G which meets $f^{-1}(c)$ in a set of positive finite linear measure, $c = c_\sigma$. Then the appropriate translate of f cannot have $f \circ \sigma$ in H^{∞}, and thus the subalgebra of $C_0(G)$ generated by f and all its translates is $C_0(G/P)$, P again the compact group of periods of f. More generally we can replace c_σ by a set $E \subset \mathbf{C}$ of logarithmic capacity zero by a theorem of Frostman [F, p. 97; Ts, VIII.44]. Thus

COROLLARY 2. *Suppose* $f \in C_0(G)$ *has* (*) *for each representation* $\sigma: \mathbf{R} \to G$ *a translate* f_x *for which* $\sigma^{-1}(f_x^{-1}(E_\sigma))$ *has positive finite linear measure for some set* $E_\sigma \subset \mathbf{C}$ *of zero capacity. Then* f *and its translates generate* $C_0(G/P)$ *as an algebra.*

(In all the remaining examples P will be trivial.)

A simple condition insuring (∗) when $G = \mathbf{R}^n$ is that $f^{-1}(E)$ have positive Lebesgue measure in $\mathbf{R}^n$, for E of zero capacity with $0 \notin E$. For now each σ is an isomorphism and by Fubini some coset of any line $\sigma(\mathbf{R})$ in $\mathbf{R}^n$ meets $f^{-1}(E)$ in a set of positive measure, necessarily finite since $0 \notin E$. Thus we have

COROLLARY 3. *Let $E \subset \mathbf{C}$ have capacity zero, $0 \notin E$. Suppose $f \in C_0(\mathbf{R}^n)$ has*[6] *$f^{-1}(E)$ of positive measure. Then f and its translates generate $C_0(\mathbf{R}^n)$ as an algebra.*

When σ must be an isomorphism, as when $G = \mathbf{R}^n$, the fact that $a \circ \sigma \in C_0(\mathbf{R})$ implies its analytic extension is continuous on $\mathcal{H}_+ \cup \{\infty\}$, as noted earlier; alternatively $a \circ \sigma$ can be viewed as pulling back into the disc algebra via the standard map of D onto $\mathcal{H}_+$. Consequently we can exploit the argument principle.

COROLLARY 4. *Suppose G has no nontrivial compact subgroups, $0 \neq f \in C_0(G)$, and for each nonzero homomorphism $\sigma: \mathbf{R} \to G$*

(a) *there are x_1, x_2 in G and z in $\mathbf{C}$ for which the winding numbers $\operatorname{Ind}(f_{x_i} \circ \sigma, z)$ exist and are of opposite sign.*

Then f and its translates generate C_0 as an algebra.

Alternatively in (a) one can assume there is one translate and two z's, using the fact that $\operatorname{Ind}(h, z)$ is of one sign for all z for which it exists for h or $\bar{h}$ in $A(D)$. There are of course a variety of replacements for (a). For example if $d\lambda = 2dt/\pi(1 + t^2)$, which represents i on $H^\infty(\mathcal{H}_+)$,

(a′) *there are $x \in G$ and $z \in \mathbf{C}$ for which $\int |\log f_x \circ \sigma - z| \, d\lambda = -\infty$.*

Again just the range of $f_x \circ \sigma$ can be wrong, as in

(a″) *there is an x in G for which no bounded connected open set U in $\mathbf{C}$ has $\partial U \subset f_x \circ \sigma(\mathbf{R} \cup \{\infty\}) \subset U^-$.*

Finally, assuming f is smooth enough to yield $f_x \circ \sigma(\mathbf{R})$ *nowhere dense*, one could assume

(a‴) *there is an x in G and self-homeomorphism τ of $\mathbf{R}$ for which $f_x \circ \sigma \circ \tau$ is even and 0 at 0.*

Here no point $z \in \mathbf{C} \setminus f_x \circ \sigma(\mathbf{R} \cup \{\infty\})$ can lie in the range of an analytic extension since for all such z, $\operatorname{Ind}(f_x \circ \sigma, z) = \operatorname{Ind}(f_x \circ \sigma \circ \tau, z) = 0$. Needless to say we could vary these variants of (a) with our σ.

The Phragmén-Lindelöf result used in Corollary 4 in fact extends to show no bounded analytic function on D, continuous on $D^- \setminus \{1\}$, can have distinct one sided limits on ∂D at 1 [Ti, 5.64]. Consequently one can form a strict density analogue of the above sort of condition.

COROLLARY 5. *Suppose $f \in C(\mathbf{R}^n)$ has, for each nonzero homomorphism $\sigma: \mathbf{R} \to \mathbf{R}^n$, a translate f_x for which $f_x \circ \sigma$ has distinct limits at $\pm\infty$. Then the algebra A generated by f and its translates is strictly dense in $C(\mathbf{R}^n)$.*

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[6] For f smooth we could equally well assume some partial g of f is bounded and continuous and has $g^{-1}(E)$ of positive measure. (For by induction such partials are bounded pointwise limits of difference quotients of elements of A^-.)

Evidently such an f has no periods, so A separates $\mathbf{R}^n$. Finally, assuming $f \in C(\mathbf{R}^n)$ has no periods one has exactly the strict analogues of Corollaries 2 and 3; and supposing equal limits at $\pm \infty$ for $f_x \circ \sigma$, an analogue of Corollary 4 as well.

2.2. For $G = \mathbf{R}^n$ and A separating, at least when the semigroup (formed from the strict closure A^- in C)

$$\mathbb{S} = \Gamma \cap A^- = \mathbf{R}^{n\wedge} \cap A^-$$

is nice enough the entire spectrum of A is the union of analytic discs (of the type described in Corollary 1) and $\mathbf{R}^n$; for example *if $\mathbb{S}$ is a set of synthesis with dense interior*.

Indeed, suppose $\varphi \in M_A$, so φ is represented by a probability measure λ, and thus has a strictly continuous extension to A^- given by λ. Then $\tau : \gamma \to \lambda(\gamma)$ is multiplicative on $\mathbb{S}$, continuous, and so a semicharacter of $\mathbb{S}$. Moreover τ never vanishes. For firstly, if $\tau \equiv 0$ then $\varphi = 0$ since $\mathbb{S}$ is of synthesis. So the relatively open set $\mathbb{S} \setminus \tau^{-1}(0) \neq \varnothing$ and thus $\mathbb{S}^0 \setminus \tau^{-1}(0)$ is a nonvoid open set in $\Gamma = \mathbf{R}^n$; let $\gamma_0 + B \subset \mathbb{S}^0 \setminus \tau^{-1}(0)$, where B is a small ball centered at 0. Now suppose $\gamma \in \tau^{-1}(0)$. Then because $\tau^{-1}(0)$ and $\mathbb{S}^0 \setminus \tau^{-1}(0)$ are both semigroups and $\tau^{-1}(0)$ an ideal, for $k \in \mathbf{Z}_+$

$$\gamma + k(\gamma_0 + B) \subset \tau^{-1}(0) \quad \text{and} \quad k(\gamma_0 + B) \subset \mathbb{S}^0 \setminus \tau^{-1}(0)$$

and therefore $(\gamma + k\gamma_0 + kB) \cap (k\gamma_0 + kB) = \varnothing$ so $\gamma \notin kB - kB = 2kB$ for all k. So $\tau^{-1}(0) = \varnothing$ as asserted.

As a consequence τ has an extension $\bar{\tau}$ to all of the group $\mathbb{S} - \mathbb{S}$ (which coincides with Γ since $\mathbb{S} - \mathbb{S}$ is an open dense subgroup); setting $\bar{\tau}(\gamma_1 - \gamma_2) = \tau(\gamma_1)/\tau(\gamma_2)$, we have a well-defined homomorphism of Γ into $(\mathbf{C} \setminus (0), \cdot)$ and if again $\gamma_0 + B \subset \mathbb{S}^0$ then for $\gamma = \gamma_1 - \gamma_2 \in \mathbb{S} - \mathbb{S}$, on $\gamma + B = \gamma_1 - \gamma_2 + B = \gamma_1 + \gamma_0 + B - (\gamma_0 + \gamma_2)$,

$$\bar{\tau}(\gamma + b) = \tau(\gamma_1 + \gamma_0 + b)/\tau(\gamma_0 + \gamma_2)$$

and so is continuous since $\gamma_1 + \gamma_0 + B \subset \mathbb{S}$.

Now we can write $\tau = \tau_1 \tau_2$ with τ_i continuous and multiplicative, $|\tau_1| \equiv 1$ and $\tau_2 > 0$. So $\tau_1(\gamma) = \langle x, \gamma \rangle$ for some $x \in \mathbf{R}^n = G$, and, since Γ is simply connected, $\tau_2(\gamma) = e^{-\rho(\gamma)}$, where ρ is a homomorphism into the additive reals, ≥ 0 on $\mathbb{S}$ since $|\tau(\gamma)| \leq 1$ for $\gamma \in \mathbb{S}$. If ρ is trivial, so $\lambda(\gamma) = \tau(\gamma) = \tau_1(\gamma) = \langle x, \gamma \rangle = \delta_x(\gamma)$, then δ_x and λ represent the same strictly continuous functional on the strictly closed span of $\mathbb{S}$. Since $\mathbb{S}$ is a set of synthesis that set coincides with A^-, and φ is evaluation at x. On the other hand if ρ is nontrivial, hence positive somewhere on $\mathbb{S}$, and λ_z is the Cauchy measure representing $z \in \mathcal{K}_+$ on $H^\infty(\mathcal{K}_+)$ then since $\frac{2}{\pi} \int \bar{e}^{itx}/(1 + t^2)\, dt = \lambda_i(e^{-ix\cdot}) = e^{-|x|}$,

$$\hat{\rho}^* \lambda_i(\gamma) = \lambda_i(\gamma \circ \hat{\rho}) = \lambda_i(\rho(\gamma)) = e^{-\rho(\gamma)}, \qquad \gamma \in \mathbb{S},$$

so $\delta_x * \hat{\rho}^* \lambda_i(\gamma) = \tau(\gamma) = \lambda(\gamma)$, and thus $\delta_x * \hat{\rho}^* \lambda_i$ represents φ on A, again by synthesis. Now $z \to \delta_x * \hat{\rho}^* \lambda_z$ provides our analytic disc through φ in M_A; for the measure, being multiplicative on span $\mathbb{S}$, must be multiplicative on A^-, while $f : z \to \delta_x * \hat{\rho}^* \lambda_z(a)$ is analytic for $a \in A$ since sp f lies in $\mathbf{R}_+$ exactly as in the last step in the proof of Theorem 1.

The above shows that, under our special hypotheses not only that any $\varphi \in M_A$ gives rise to a semicharacter of $\mathbb{S}$ but any continuous semicharacter τ of $\mathbb{S}$, which

trivially provides a multiplicative linear functional on span $\mathbb{S}$, is represented by a probability measure, and thus extends to a multiplicative linear functional on (span $\mathbb{S}$)$^- = A^-$; hence M_A can be identified with the set $\mathbb{S}$ of semicharacters of $\mathbb{S}$. (The same applies if A is a strictly closed subalgebra of $C(\mathbf{R}^n)$ and we interpret M_A to be the set of strictly continuous multiplicative linear functionals.) Clearly, in this case, Theorem 1 adds only the fact that $M_A \backslash \mathbf{R}^n \neq 0$.

3. For a subalgebra A of $A(G)$ or $B(G)$ translation invariance amounts to invariance of the corresponding subalgebra A_1 of $L^1(\Gamma)$ or $M(\Gamma)$ under multiplication by characters, or trigonometric polynomials. By a familiar argument the last implies (and is trivially implied by) the fact that A_1 is an L-subalgebra; $\mu \in A_1$ and $\nu \ll |\mu|$ imply $\nu \in A_1$. As a consequence we easily obtain the analogue of Theorem 1 for $A \subset A(G)$.

THEOREM 3. *If $A \subset A(G)$ is a closed proper translation invariant subalgebra either $A = (L^1(\Gamma_0))\hat{\ }$ for an open subgroup Γ_0 of Γ, or the conclusions of Theorem* 1 *apply, and M_A contains analytic discs.*

Again our analytic discs are bounded by homomorphs of $\mathbf{R}$ in G or a quotient of G; thus Corollaries 2 through 5 apply equally well with $A(G)$ in place of $C_0(G)$. (For example, here Corollary 3 asserts that if $f \in L^1(\mathbf{R}^n)$ has $\hat{f}^{-1}(1)$ of positive measure then the closed subalgebra of $L^1(\mathbf{R}^n)$ generated by $\{e^{i(t,\cdot)}f: t \in \mathbf{R}^n\}$, or by $\{\chi_U f: U$ Borel in $\mathbf{R}^n\}$, is all of $L^1(\mathbf{R}^n)$.)

In this context separation by A can be achieved by replacing G by G/P for the (again necessarily compact) group of periods P of A; of course in the theorem $\Gamma_0 = P^\perp$ and in our corollaries we obtain $\Gamma = \Gamma_0$ again because P must be trivial when $G = \mathbf{R}^n$.

For the proof let S_μ denote the closed support in Γ of μ in A_1, and S the closure of the union of all such S_μ, $\mu \geq 0$. Because $S_{\mu*\nu} = (S_\mu + S_\nu)^-$ for two nonnegative measures, S is a semigroup; since S contains a set of positive Haar measure it has interior and generates an open subgroup Γ_0 of Γ. For simplicity we can take $A_1\hat{\ } = A$ as subalgebra of $A(G/\Gamma_0^\perp)$; replacing G and Γ by $G/\Gamma_0^\perp$ and Γ_0, we can thus view A as also separating G, and S as generating Γ.

Suppose $S \neq \Gamma$. By Theorem 2 we have a (continuous) homomorphism ρ_1 of Γ onto Π, an ordered group which is either $\mathbf{R}$ or discrete, ≥ 0 on S, and nontrivial on S since S generates Γ. In case Π is discrete we can follow ρ_1 by a nonnegative homomorphism into $\mathbf{R}$ of an open subgroup, which composes with ρ_1 to yield a homomorphism ρ of an open subgroup Γ_1 of Γ into $\mathbf{R}$, with $\rho(S \cap \Gamma_1) \subset \mathbf{R}_+$, $\neq \{0\}$: we have only to select $s_0 \in S$ with $\pi_0 = \rho_1(s_0) > 0$, note that the subgroup of Π of elements bounded by π_0 (as in (5)), modulo its subgroup of elements infinitely small relative to π_0, is archimedian, and thus order isomorphic to a subgroup of $\mathbf{R}$.

Hence if S is proper we have a continuous homomorphism ρ of an open subgroup Γ_1 of Γ for which $\rho(S \cap \Gamma_1) \geq 0$, $\neq \{0\}$, while $S\backslash\Gamma_1$ is an ideal in S. Since sp $\hat{\mu} \subset S_\mu$ we see $(m * \hat{\mu}) \circ \hat{\rho}$ extends to be analytic on a half-plane or disc as before via 1.5, 1.7 and the proof of Corollary 1 where m is Haar measure of $\Gamma_1^\perp$. (Alternatively from

Theorem 2 we could produce a positive semicharacter χ of S, zero on $\Gamma_1 \backslash S$, and follow Taylor by setting

$$\varphi_z(\mu) = \int \chi^z \, d\mu, \qquad \mathrm{Re}\, z \geqslant 0;$$

since $\chi = e^{-\rho}$ on Γ_1 for ρ as above we obtain $\varphi_{iy}(\mu) = \int_{\Gamma_1} e^{-iy\rho(\gamma)} \mu(d\gamma) = \int_{\Gamma_1} \langle \hat{\rho}(y), \gamma \rangle \mu(d\gamma) = (\mu_{\Gamma_1})\hat{\,}(\hat{\rho}(y)) = (m * \hat{\mu}) \circ \hat{\rho}(y)$, showing we would obtain the same analytic disc.)

There remains the possibility that $S = \Gamma$. Then because A_1 is an L-algebra in any neighborhood N_γ of γ we can find the support of a nonzero element of A_1, which we can take to correspond to a nonnegative bounded L^1 function f; choosing f_1 supported by N_γ and f_2 supported by N_1 we have the open support of the continuous function $f_1 * f_2 \subset N_1 + N_\gamma$. Thus S contains a dense open subset U, the union of the open supports of the $\mu \geqslant 0$ in A_1 corresponding to continuous nonnegative elements of L^1 in A_1. Since U is trivially a semigroup, and dense, it must be G: $V = (-U) \cap U \neq \varnothing$, so U contains the neighborhood $W = V + V = V - V$ of 1, whence $(\gamma + W) \cap U \neq \varnothing$ implies $\gamma \in U$.

Now since evidently $\chi_U \cdot L^1 \subset A_1$ follows from the definition of U and the fact that A_1 is an L-algebra, we are done.

In the case of an invariant subalgebra of $B(G)$ and the corresponding L-subalgebra of A of $M(\Gamma)$ the situation is more complicated, and analytic discs may exist and lie far from our group G; thus only special forms of our corollaries obtain. We now rely heavily on the work of Taylor, and it is only because of that work that we can almost precisely describe the settings in which analytic discs arise (though not in a completely transparent manner). The sole gap that remains would be closed by a negative answer to a question left open by Taylor [**T2**, p. 112].

Because a generalized character of A whose modulus is nonidempotent immediately leads to an analytic disc in the spectrum, if such discs fail to exist we can locate A in the sum of two subalgebras of $M(\Gamma)$ described by Taylor in [**T2**]. Let $\mathfrak{T}$ denote the set of all locally compact group topologies τ for Γ at least as fine as the original, with Γ_τ the corresponding l.c.a. group. The radical of $L^1(\Gamma_\tau)$, $L^1(\Gamma_\tau)^{1/2}$, is the intersection of all maximal ideals of $M(\Gamma_\tau)$ containing $L^1(\Gamma_\tau)$. We view all such measures as lying in $M(\Gamma)$, and set $M_1(\Gamma) = \Sigma_\tau L^1(\Gamma_\tau)^{1/2}$, which is the first subalgebra. (The sum is in fact a direct sum of mutually singular subspaces.) The second of Taylor's subalgebras $M_2(\Gamma)$ is more recondite; it consists of those measures which, carried to the structure semigroup, are supported by the union of all maximal subgroups (as are those in $M_1(\Gamma)$) but which vanish on each such subgroup; whether $M_2(\Gamma)$ can be nontrivial is the question left open by Taylor. Excluding generalized characters of A with nonidempotent moduli implies, by [**T2**, 2.3] applied to $M = M(\Gamma)$, that

(11) $$A \subset M_1(\Gamma) + M_2(\Gamma);$$

what one can conclude when no analytic discs exist is that if $A_\tau = A \cap L^1(\Gamma)^{1/2}$, then for each τ in $\mathfrak{T}$ either:

(a) $A_\tau = 0$,

(b) $L^1(\Gamma_\tau^*) \subset A_\tau \subset L^1(\Gamma_\tau^*)^{1/2}$, *where Γ_τ^* is an open subgroup of Γ_τ, or*

(c) $A_\tau \cap L^1(\Gamma_\tau) = 0$, *and the support semigroup of A_τ is a group.*

(Here the *support semigroup* of an L-algebra is simply the closure of the union of the closed supports of its elements, necessarily a semigroup.) In all of these cases the support semigroup of A_τ is a group; conversely that condition implies (a), (b) or (c) holds and, at least when $A \cap M_2(\Gamma) = 0$, that no analytic discs exist.

THEOREM 4. *Let A be an L-subalgebra of $M(\Gamma)$. If the spectrum of A contains no analytic discs then A is the sum of the two mutually singular L-subalgebras $A \cap M_1(\Gamma)$, $A \cap M_2(\Gamma)$, where each component $A_\tau = A \cap L^1(\Gamma_\tau)^{1/2}$ for $\tau \in \mathfrak{T}$ has its support semigroup (void or) a group, or satisfies (a), (b) or (c).*

Conversely these conditions imply the elements of $A \cap M_1(\Gamma)$ are constant on any analytic disc in the spectrum, so all such discs necessarily arise in the spectrum of the ideal $A \cap M_2(\Gamma)$ of A.

Suppose we have no analytic discs in the spectrum so (11) holds. Recall that for each τ in $\mathfrak{T}$ and each μ in $M(\Gamma)$ we can uniquely write $\mu = \mu_\tau + \mu'_\tau$ where μ_τ is carried by a τ σ-compact set and μ'_τ vanishes on all τ compacta, while $\mu \to \mu_\tau$ is multiplicative; thus viewing $M(\Gamma_\tau)$ as a subset of $M(\Gamma)$ we can imbed the spectrum of $A \cap M(\Gamma_\tau)$ into that of $A \cap M(\Gamma)$. Morover, since $L^1(\Gamma_\tau)^{1/2}$ is an ideal of $M(\Gamma_\tau)$ the spectrum of $A_\tau = A \cap L(\Gamma_\tau^1)^{1/2}$ imbeds in that of $A \cap M(\Gamma_\tau)$, hence in that of $M(\Gamma)$. So for each τ the spectrum of A_τ contains no analytic disc.

Because of Theorem 3 we can assert $A_\tau \cap L^1(\Gamma_\tau) = 0$ or $A_\tau \cap L^1(\Gamma_\tau) = L^1(\Gamma_\tau^*)$ for an open subgroup Γ_τ^* of Γ_τ; for if both fail then the spectrum of $A_\tau \cap L^1(\Gamma_\tau)$ contains analytic discs by Theorem 3, and since this algebra is an ideal in A_τ, so must the spectrum of A_τ. When the second alternative holds Γ_τ^* must be the support semigroup of A_τ since otherwise we have elements of the ideal $A_\tau \cap L^1(\Gamma_\tau)$ not supported by Γ_τ^*; and because $L^1(\Gamma_\tau^*)^{1/2} = L^1(\Gamma_\tau)^{1/2} \cap M(\Gamma_\tau^*)$ holds[7] for any open subgroup Γ_τ^*, the right-hand inclusion in (b) follows, and (b) holds.

Now if (a), (b) and (c) fail then $A_\tau \cap L^1(\Gamma_\tau) = 0$ but $A_\tau = A \cap L^1(\Gamma_\tau)^{1/2} \neq 0$ and the support semigroup S of A is not a group. So exactly as in the proof of Theorem 3 we can produce a positive semicharacter χ of S with values in $(0, 1)$ on an open subset of S, and

$$\varphi_z(a) = \int \chi^z \, da, \qquad \operatorname{Re} z \geq 0,$$

provides an analytic disc.

Conversely, suppose the support semigroup of A_τ is always void or a group Γ_τ^0. If $A_\tau \cap L^1(\Gamma_\tau) \neq 0$ then Γ_τ^0 must be open, and by just the argument of Theorem 3,

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[7] Trivially, $L^1(\Gamma_\tau^*)^{1/2} \subset L^1(\Gamma_\tau)^{1/2} \cap M(\Gamma_\tau^*)$ since any multiplicative linear functional on $M(\Gamma_\tau)$ vanishing on $L^1(\Gamma_\tau)$ restricts to $M(\Gamma_\tau^*)$ to vanish on $L^1(\Gamma_\tau^*)$, hence vanishes on $L^1(\Gamma_\tau^*)^{1/2}$. For the reverse, it suffices to see each element φ of the spectrum of $M(\Gamma_\tau^*)$ vanishing on $L^1(\Gamma_\tau^*)$ is such a restriction. But $\{\mu \in M(\Gamma_\tau): \mu|_{\gamma\Gamma_\tau^*} \in \delta_\gamma * \varphi^{-1}(0), \gamma \in \Gamma_\tau\}$, the set of measures coinciding with translates of $\varphi^{-1}(0)$ on translates of Γ_τ^*, is a closed proper ideal containing $L^1(\Gamma_\tau)$, and so lies in the kernel of a ψ in the spectrum of $M(\Gamma_\tau)$. Clearly ψ annihilates $L^1(\Gamma_\tau)$, and since $\psi^{-1}(0) \supset \varphi^{-1}(0)$ and cannot contain $M(\Gamma_\tau^*)$ unless it contains all translates $\delta_\gamma * M(\Gamma_\tau^*)$, hence all $M(\Gamma_\tau)$, $\psi | M(\Gamma_\tau^*) = \varphi$.

$A_\tau \cap L(\Gamma_\tau) = L^1(\Gamma_\tau^0)$, so (b) follows for $\Gamma_\tau^0 = \Gamma_\tau^*$ as before. On the other hand, if $A_\tau \cap L^1(\Gamma_\tau) = 0$ clearly (a) or (c) holds.

Now Taylor [T3, p. 84] calls A balanced if $A \cap L^1(\Gamma_\tau)$ is always 0 or $L^1(\Gamma_\tau^*)$ for a (necessarily open) subgroup Γ_τ^*, $\tau \in \mathfrak{T}$; thus our alternatives (a)–(c) imply A is balanced. So by [T3, 9.1.2] for a in A_τ its spectrum in A_τ coincides with its spectrum in $M(\Gamma)$, or $\mathrm{sp}_{A_\tau} a = \mathrm{sp}_{M(\Gamma)} a$. Because of our homomorphism $\mu \to \mu_\tau$ the latter coincides with $\mathrm{sp}_{M(\Gamma_\tau)} a$, which is precisely $\hat{a}(\hat{\Gamma_\tau}) \cup \{0\}$, essentially by the definition of $L^1(\Gamma_\tau)^{1/2}$. Thus $\mathrm{sp}_{A_\tau} a = \hat{a}(\hat{\Gamma_\tau}) \cup \{0\}$, and the image of $\hat{\Gamma_\tau}$ in the spectrum of A_τ provides a boundary; indeed since A_τ is an L-algebra, $\hat{A_\tau}$ on $\hat{\Gamma_\tau}$ is translation invariant, so the image is that of the quotient group $\hat{\Gamma_\tau}/\Gamma_\tau^{*\perp}$, which, again by invariance, must yield the Šilov boundary.

Suppose then that some element a_0 of A_τ has its Gelfand transform $\tilde{a}_0$ nonconstant on our analytic disc $z \to \varphi_z$. In case (b) each $\tilde{a}$ in $L^1(\Gamma_\tau^*)\tilde{}$ is constant on our disc since those with $\hat{a}$ real valued span $L^1(\Gamma_\tau^*) \subset A_\tau$, and these yield analytic functions with range $\mathrm{sp}_{A_\tau} a = \hat{a}(\hat{\Gamma_\tau}) \cup \{0\} \subset \mathbf{R}$, and so are constant. Thus a_0 must lie outside $L^1(\Gamma_\tau^*)$. But if $a \in L^1(\Gamma_\tau^*)$, $(a * a_0)\tilde{}$ constant on our disc implies $\tilde{a} = 0$ on our disc, so φ_z vanishes on $L^1(\Gamma_\tau^*)$, and by definition it vanishes on all $L^1(\Gamma_\tau^*)^{1/2}$, so case (b) cannot arise, and we must have case (c).

Now since the support semigroup of A_τ is a group Γ_τ^* we can construct an element $a_1 \geqslant 0$ whose support Γ_1 is itself an open subgroup of Γ_τ^* while $a_0 \ll a_1$. Indeed replacing a_1 by a power series in a_1 we can assume $a_1 * a_1 \ll a_1$.

Consequently, from [T3, 9.2.2] each φ_z in our disc is given on $L^1(a_1)$ by $a \to \int f \, da$, $f \in L^\infty(a_1)$, with $f(\gamma\gamma') = f(\gamma)f(\gamma')$ a.e. $a_1 \times a_1$. But because $\hat{\Gamma_\tau}/\Gamma_\tau^{*\perp}$ provides the Šilov boundary, φ_z is represented by a probability measure $\nu_z \in M(\hat{\Gamma_\tau}/\Gamma_\tau^{*\perp})$:

$$\varphi_z(a) = \int \hat{a} \, d\nu_z = \int \hat{\nu}_z \, da, \qquad a \in A_\tau.$$

Thus $\hat{\nu}_z(\gamma\gamma') = \hat{\nu}_z(\gamma)\hat{\nu}_z(\gamma')$ a.e. $a_1 \times a_1$, so for $\gamma \notin N$ this holds for $\gamma' \notin N_\gamma$ where N and N_γ are a_1-null. Since N and N_γ are necessarily nowhere dense in Γ_1 and $\hat{\nu}_z$ is continuous we conclude $\hat{\nu}_z$ is multiplicative on all Γ_1. Finally since Γ_1 can be any (large) σ-compact open subgroup of Γ_τ^*, $\hat{\nu}_z$ is multiplicative on all of Γ_τ^*, hence is a character. Thus ν_z must be a point mass on the dual group $\hat{\Gamma_\tau}/\Gamma_\tau^{*\perp}$, and so our disc lies entirely in the Šilov boundary.

Now the identity of $\hat{\Gamma_\tau}/\Gamma_\tau^{*\perp}$ is a generalized peak point for $\hat{A_\tau}$ since for our a_1, $\hat{a}_1(1) = \|\hat{a}_1\|_\infty = \|a_1\|$ while $\hat{a}_1(x) = \|a_1\|$ implies $x = 1$ on the support of a_1, an arbitrary (large) σ-compact open subgroup of Γ_τ^*. By translation invariance each point of $\hat{\Gamma_\tau}/\Gamma_\tau^{*\perp}$ is a generalized peak point; since no point of an analytic disc can be one we have our contradiction, completing our proof.

Note that when no analytic discs exist we can also conclude the support semigroup of $A \cap M_2(\Gamma)$ is a group; otherwise we obtain an analytic disc in its spectrum from a positive semicharacter as before, which imbeds in the spectrum of A since $A \cap M_2(\Gamma)$ is an ideal. But we cannot obtain a converse, as with A_τ, since $\mathrm{sp}_{M(\Gamma)} a$ is no longer easily related to a Fourier-Stieltjes transform.

Finally, we should note that our corollaries, formulated in terms of just the transform algebra $A \subset B(G)$, can really only hold when $A \subset L^1(\Gamma)^{1/2}$, since a disc

nonconstant on A_τ has its boundary in a homomorph of **R** in a quotient of $\hat{\Gamma_\tau}$, which is strictly larger than G. And in case $A \subset L^1(\Gamma)^{1/2}$, our conclusion (if analyticity must fail) is less precise.

REMARK (ADDED IN PROOF DECEMBER 29, 1982). One can regard Theorem 2 as an analogue in the group setting of the existence of support functionals for cones in Banach spaces. In this sense there is an analogue of the separation theorem which follows from the proof of Theorem 1, viz.: *if S is a closed generating subsemigroup of an l.c.a. group Γ which contains the identity, and $x \in \Gamma$ has $x^n \notin S$ for all $n \geq 1$, then there is a half of Γ containing S but not x.*

One need only note that (in our proof) before the first real enlargement of $S = S_1$ or, more properly of $S_2 = \pi_1 S_1$, $y = \pi_1 x$ can be adjoined to S_2^{-1} so as to preserve (4), since $y^n s^{-1} = y^{-k} s'$ for $s, s' \in S_2$ and $n, k \geq 0$ implies $y^{n+k} = s$, $s' \in S_2$, which is excluded by hypothesis. Then the ρ and Π of Theorem 2 have $S \subset \rho^{-1} \Pi_+$ and $x \notin \rho^{-1}(\Pi_+)$.

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