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STABLE ORBITS OF DIFFERENTIABLE GROUP ACTIONS

BY

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ABSTRACT. We prove that a compact orbit of a smooth Lie group action is stable provided the first cohomology space vanishes for the normal representation at some (equivalently, every) point of the orbit. When the orbit is a single point, the acting group need only be compactly generated and locally compact for this conclusion to hold. Applied to foliations, this provides a sufficient condition for the stability of a compact leaf and includes the stability theorems of Reeb and Thurston and of Hirsch as cases.

1. Introduction. The dynamics of an iterated diffeomorphism or a smooth flow have been the subject of heavy research, particularly since 1960, but the dynamics of a more general group action remain poorly understood. This paper approaches some basic questions in the general case. Namely, we provide a condition under which a stationary point of an action will persist when the action is perturbed. This condition is generalized to one insuring the stability of a compact orbit.

The notion of hyperbolicity has proved the key to questions of stability in the cases of actions by $\mathbf{Z}$ or $\mathbf{R}$. For actions by other groups, we would hope to find conditions analogous to hyperbolicity in the sense that they facilitate analysis to a comparable extent. However, we should not expect these conditions to resemble hyperbolicity too closely, for they should reflect the algebra of the particular group being studied. We might begin by looking at a compact orbit of some action. What first-order infinitesimal conditions on the action along that orbit allow us to determine the presence or absence of nearby orbits of the same type? Also, what conditions insure that this compact orbit persists when the action is perturbed? These questions are closely related. Having a satisfactory answer, we may then search for conditions under which the local dynamics near such an orbit may be determined and under which the local picture is stable. It seems reasonable to postpone any attempts at global analysis until after these local questions are answered.

The present paper attempts the first step of this program. Our infinitesimal criterion, condition VC, for the stability of a stationary point p is a condition on the cohomology of the tangent space at p as a G-module, where G is the acting group. If G is a compactly generated Lie group, say, the verification of condition VC for a particular G-module is a matter of linear algebra. When $G = \mathbf{Z}$, for example, this

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condition means that one is not an eigenvalue of the generating linear transformation. Our basic result, proved in §2, is the following:

THEOREM 2.1. *If a stationary point is VC, then the set of all stationary points nearby is a submanifold which is stable under perturbations of the action.*

Stability of compact orbits is taken up in §4. The result obtained there, Theorem 4.1, although a fair generalization of Theorem 2.1, is incomplete in one respect. That is, there may be infinitesimal information along a compact orbit which causes the orbit to be stable and yet does not fall under the hypotheses of Theorem 4.1.

§3 concerns the stability of a compact leaf of a foliation. The original result in this area, proved by G. Reeb **[9]** and generalized by W. Thurston **[11]**, says that under certain conditions a neighborhood of the compact leaf will be foliated as a stable product foliation. At the other extreme, the stability theorem of M. Hirsch **[3]** states that if certain conditions hold, then every perturbation of the given foliation has exactly one leaf which projects diffeomorphically onto the given leaf in some tubular neighborhood. These results are cases of the following:

THEOREM 3.1. *If the linear holonomy along a compact leaf L is VC, then the leaves which project diffeomorphically onto L in a tubular neighborhood form a submanifold with a product foliation. This structure is stable under perturbations of the foliation.*

Under stronger hypotheses, the set of leaves which cover L in some tubular neighborhood, and whose fundamental groups project to contain a given subgroup of finite index in $\pi_1(L)$, form a foliated bundle over L. The fiber is a union of intersecting planes, and as before the structure is stable.

§5 comprises a few applications of the previous results to stable properties of linear representations.

A word about the topology on $\mathrm{Act}^\nu(G, M)$, $1 \leqslant \nu \leqslant \infty$, where G is a topological group and M is a smooth manifold. We give $\mathrm{Diff}^\nu(M)$ the weak C^ν topology **[4**, p. 35]. This induces a compact open topology on $C^0(G, \mathrm{Diff}^\nu(M))$, and we assign to $\mathrm{Act}^\nu(G, M)$ the topology it inherits as a subset of the latter.

The basic Theorem 2.1 was conjectured by Morris Hirsch in a course he gave at Berkeley in 1978. I am grateful to him and to my advisor, Charles Pugh, for their interest and guidance.

2. Stability of a stationary point. In this section we investigate the stability of a stationary point of a group action. We prove that a stationary point is stable if the first cohomology space for the isotropy representation at that point vanishes.

A point x of M is said to be stationary for the action α if $\alpha(g)x = x$ for all $g \in G$. It is said to be a stable stationary point if, for every neighborhood of x in M, all actions near α have a stationary point in that neighborhood. As a model, we might take our manifold to be a finite-dimensional vector space E and our action to be a continuous representation $\rho\colon G \to GL(E)$. The origin of E is then stationary. In fact, this model is of general interest, because the dynamics of a differentiable action α near a stationary point x resemble those of the isotropy representation $g \mapsto D\alpha(g)\big|_x$ of G on T_xM.

The simplest way to perturb a linear action $\rho\colon G \to GL(E)$ is to compose each mapping $\rho(g)$ with a translation by $u(g) \in E$. This results in an action (which will be an affine representation of G) precisely when the cocycle condition $u(gh) = \rho(g)u(h) + u(g)$ is satisfied for all $g, h \in G$. The affine representation obtained will have a stationary point $-e \in E$ if and only if $u(g) = \rho(g)e - e$ for all $g \in G$. A mapping u of the latter kind is called a coboundary, and the quotient of the vector space of (continuous) cocycles for ρ by the subspace of coboundaries is the (continuous) cohomology space $H^1(G, E)$. Thus the condition that $H^1(G, E) = 0$ means that every affine representation of G with linear part ρ has a stationary point. When this condition holds, we say that the representation ρ is VC (for "vanishing cohomology"), and we say that a stationary point x of a differentiable group action is VC if the isotropy representation at x is so.

THEOREM 2.1. *Let x be a VC stationary point of $\alpha \in \mathrm{Act}^\nu(G, M)$, where G is compactly generated and locally compact, $1 \leqslant \nu \leqslant \infty$, and x is interior to M. Then x is stable. In fact, there exist C^ν embeddings $\varphi_\beta\colon T_xM \to M$ for β near α such that*:

(1) $\varphi_\alpha(0) = x$, $D\varphi_\alpha|_0 = \mathrm{id}_{T_xM}$;

(2) $\varphi_\beta(T_xM)^G = \{y \in \text{image } \varphi_\beta : y \text{ is stationary for } \beta\}$; *and*

(3) φ_β *varies continuously with β (in the C^ν topologies).*

Here $(T_xM)^G$ is the set of vectors stationary for the isotropy representation $D\alpha|_x$.

COROLLARY 2.2 *Let ρ be a continuous representation of a compactly generated, locally compact group G on a finite-dimensional vector space E. The origin of E is stable with respect to perturbations in* $\mathrm{Act}^\nu(G, E)$, $1 \leqslant \nu \leqslant \infty$, *if and only if ρ is VC.*

Our main result, Theorem 2.1, is a slight generalization of Theorem C of [**10**], in which G was assumed to be a compactly generated Lie group and ν was taken to be one. We postpone the proof until the end of this section, where we also prove a similar result for the case that x lies in the boundary of M.

Theorem 2.1 says nothing new or deep about actions by the group $\mathbf{Z}$ of integers. Still, this group serves as a good test case. Let $\rho\colon \mathbf{Z} \to GL(E)$ be a representation. Each choice of $u(1) \in E$ determines a unique cocycle u, which is given by

$$u(n) = \begin{cases} \left(\rho(1)^0 + \rho(1)^1 + \cdots + \rho(1)^{n-1}\right)u(1) & \text{if } n > 0, \\ 0 & \text{if } n = 0, \\ -\left(\rho(1)^n + \rho(1)^{n+1} + \cdots + \rho(1)^{-1}\right)u(1) & \text{if } n < 0. \end{cases}$$

This cocycle will be a coboundary if and only if the equation $u(1) = \rho(1)e - e$ has a solution $e \in E$. Thus ρ is VC exactly when one fails to be an eigenvalue of the generating transformation $\rho(1)$. (In fact, Morris Hirsch [**5, 6**] shows that for any nilpotent group which admits a nontrivial homomorphism into the real numbers, condition VC is equivalent to the condition that zero be the only stationary vector of the representation.)

Clearly a stationary point of an action by $\mathbf{Z}$ may be stable without being VC; for instance, it may yet be an isolated fixed point of nonzero index. Thus stability cannot always be determined solely on the basis of first-order infinitesimal information. Surprisingly, even local information may not suffice. Morris Hirsch has given the following example in which a stationary point x has an $\alpha(G)$-invariant neighborhood U such that x is stable but is unstable with respect to perturbations of $\alpha(G)\,|_U$. Let G be the group of orientation-preserving affine transformations of the real line. We write (a, b) for the transformation $x \to e^a x + b$. Let X and Y be the left-invariant vector fields on G whose values at $(0,0) = 1_G$ are $\partial/\partial a$ and $\partial/\partial b$, respectively. Then $[X, Y] = Y$, and $\mathrm{Act}^\infty(G, M)$ may be identified with $\{(\overline{X}, \overline{Y}) \in \mathrm{X}^\infty(M) \times \mathrm{X}^\infty(M) : [\overline{X}, \overline{Y}] = -\overline{Y}\}$ when M is compact. Let M be the unit circle in $\mathbf{C}$, and let $\overline{X}$ be a C^∞ vector field on M, vanishing only at ± 1, with derivative 1 at $+1$ and derivative -2 at -1. Let $\overline{Y}$ be the zero vector field and $\overline{\overline{X}}, \overline{\overline{Y}}$ smooth perturbations of $\overline{X}, \overline{Y}$, respectively. Then $\overline{\overline{X}}$ vanishes at two points, which we may take to be ± 1, and the derivative of $\overline{\overline{X}}$ at -1 is approximately -2. If $[\overline{\overline{X}}, \overline{\overline{Y}}] = -\overline{\overline{Y}}$, a computation reveals that $\overline{\overline{Y}}$ must vanish near -1, hence on $M \setminus \{+1\}$, and hence on all of M. Thus $+1$ is stable for the original action. On the other hand, the restriction of the original action to $U = M \setminus \{-1\}$ is conjugate by a C^∞ diffeomorphism to the action $\alpha(a, b)x = e^a x$ of G on the real line. The origin is not stable for α, since the action $\alpha_\varepsilon(a, b)x = e^a x + \varepsilon b$ has no stationary point when $\varepsilon \neq 0$.

For actions by $\mathbf{Z}$, any VC stationary point is isolated and has nonzero index, so it is stable with respect to C^0 perturbations of the action. For other groups, however, a VC stationary point may be unstable in the C^0 sense. Let G and α_ε be as above. For $\varepsilon > 0$, let f_ε be the diffeomorphism $x \mapsto \varepsilon x + x^3$ of the real line. The actions

$$\beta_\varepsilon(a, b)x = \left(f_\varepsilon^{-1} \circ \alpha_\varepsilon(a, b) \circ f_\varepsilon\right)x$$

are C^∞ and converge in $\mathrm{Act}^0(G, \mathbf{R})$ to the C^∞ action $\beta(a, b)x = e^{a/3}x$ as ε approaches zero. The origin is a VC stationary point for β, but β_ε has no stationary point when $\varepsilon > 0$.

The following will be of use in §3.

THEOREM 2.3. *Let G be a finitely generated discrete group, H a subgroup of finite index in G, and x a stationary point of $\alpha \in \mathrm{Act}^\nu(G, M)$ which is interior to M, $1 \leqslant \nu \leqslant \infty$. Suppose that x is VC for the restriction $\alpha\,|_H$. Let P denote the subset $\bigcup_{g \in G} D\alpha(g)\,|_x (T_x M)^H$ of $T_x M$. For β near α, there exist C^ν embeddings $\psi_\beta \colon T_x M \to M$ such that*:

(1) $\psi_\alpha(0) = x$, $D\psi_\alpha\,|_0 = \mathrm{id}_{T_x M}$;
(2) $\psi_\beta(P)$ is *$\beta(G)$-invariant*;
(3) $\beta(g)\psi_\beta(p) = \psi_\beta(D\alpha(g)\,|_x p)$ *for all $p \in P$ and $g \in G$; and*
(4) ψ_β *varies continuously with β.*

Since H has finite index and $(T_x M)^H$ is a vector subspace of $T_x M$, P is a finite union of vector subspaces. When H is normal, P is simply $(T_x M)^H$ itself. In any case, P is certainly invariant under the isotropy representation $D\alpha\,|_x$, and Theorem 2.3 says that the action of G on P occurs as an embedded constituent of all actions near α.

PROOF OF THEOREM 2.3. Let J be the largest normal subgroup of G which is contained in H. Then J has finite index in G. We choose a complete set of representatives $\{s_1,\ldots,s_m\}$ for its cosets. Notice that for every $g \in G$, J acts trivially on $D\alpha(g)\,|_x(T_xM)^H$. Thus P is contained in the G-invariant subspace $(T_xM)^J$, and we may find an inner product on T_xM whose restriction to $(T_xM)^J$ is G-invariant.

Being of finite index, H is finitely generated. By Theorem 2.1, there is a continuous family of C^ν embeddings $\varphi_\beta\colon T_xM \to M$ for β near α such that

$$(*) \qquad \varphi_\beta(T_xM)^H = \{y \in \text{image}\, \varphi_\beta : y \text{ is stationary for } \beta\,|_H\}.$$

Also, by the simple Lemma 2.4 to be proved, there is a similar family of embeddings $\bar{\varphi}_\beta\colon T_xM \to M$ with $\bar{\varphi}_\beta(T_xM)^J \supseteq \{y \in \text{image}\, \bar{\varphi}_\beta : y$ is stationary for $\beta\,|_J\}$. We may then easily arrange that the maps φ_β enjoy both these properties. That is, we may also assume that

$$(**) \qquad \varphi_\beta(T_xM)^J \supseteq \{y \in \text{image}\, \varphi_\beta : y \text{ is stationary for } \beta\,|_J\}.$$

Let V be a small neighborhood of x, so that the mapping

$$f_\beta(y) = \frac{1}{m}\sum_{i=1}^{m} D\alpha(s_i)\,|_x \cdot \varphi_\beta^{-1}\beta(s_i)^{-1}y$$

is defined for all $y \in V$ and all β near α. Since the derivative of f_α at x is id_{T_xM}, we may shrink V and obtain that f_β is an embedding for β near enough to α. Let T be the union of a finite set of generators for H with the set $\{s_1,\ldots,s_m\}$. We may find a neighborhood W of x such that $\beta(t)W \subseteq V$ for all $t \in T$ and all β near α.

Now suppose an action β near α, a point $y \in W$ which is stationary for $\beta\,|_H$, and $t \in T$ are given. The condition $s_{\sigma(i)}s_i^{-1}t \in J$ defines a permutation σ of $\{1,\ldots,m\}$. Since $J \subseteq H$, we have

$$\begin{aligned} f_\beta(\beta(t)y) &= \frac{1}{m}\sum_{i=1}^{m} D\alpha(s_i)\,|_x \cdot \varphi_\beta^{-1}\beta(s_i^{-1}t)y \\ &= D\alpha(t)\,|_x\left(\frac{1}{m}\sum_{i=1}^{m} D\alpha\big(t^{-1}s_is_{\sigma(i)}^{-1}\big)\,|_x \cdot D\alpha(s_{\sigma(i)})\,|_x \cdot \varphi_\beta^{-1}\beta(s_{\sigma(i)})^{-1}y\right). \end{aligned}$$

No matter the action, the set of points left stationary by J is always G-invariant. In particular, $\beta(s_{\sigma(i)})^{-1}y$ is stationary for J, and by $(**)$, φ_β^{-1} takes this point into $(T_xM)^J$. We then have that

$$\begin{aligned} f_\beta(\beta(t)y) &= D\alpha(t)\,|_x\left(\frac{1}{m}\sum_{i=1}^{m} D\alpha(s_{\sigma(i)})\,|_x \cdot \varphi_\beta^{-1}\beta(s_{\sigma(i)})^{-1}y\right) \\ &= D\alpha(t)\,|_x \cdot f_\beta(y). \end{aligned}$$

In particular, $f_\beta(y) \in (T_xM)^H$.

Let B be a metric ball about the origin of T_xM such that $f_\beta(W) \supseteq B$ for all β near α. Since $W \subseteq \text{image}\, \varphi_\beta$, it follows that $B \cap f_\beta\{y \in W\colon y$ is stationary for $\beta\,|_H\}$ is a closed submanifold of $B \cap (T_xM)^H$, without boundary and of full dimension. It must then be all of $B \cap (T_xM)^H$. The union over $t \in T$ of the sets

$D\alpha(t)\,|_x(B \cap (T_xM)^H)$ is $B \cap P$. It follows that $f_\beta^{-1}(B \cap P)$ is $\beta(G)$-invariant and that f_β is a conjugacy on this set.

We define the mappings ψ_β to be $f_\beta^{-1} \circ r$, where r is a fixed diffeomorphism from T_xM onto B of the form $r(z) = \rho(\|z\|)z$ with $\rho(\|z\|) = 1$ for small $\|z\|$. Q.E.D.

When P spans T_xM, Theorem 2.3 suggests that α might actually appear to be linear in a proper choice of coordinates at x. This is not always the case. Also, examples exist which show that the hypothesis of Theorem 2.3 cannot be replaced by the hypothesis that the restriction mapping $H^1(G, T_xM) \to H^1(H, T_xM)$ is zero.

The proof of Theorem 2.1 begins with the following lemma.

LEMMA 2.4. *Let x be a stationary point of $\alpha \in \mathrm{Act}^\nu(G, M)$, $1 \leqslant \nu \leqslant \infty$, where G is an arbitrary topological group and x is interior to M. There are C^ν embeddings $\bar{\varphi}_\beta$: $T_xM \to M$ for β near α such that*:

(1) $\bar{\varphi}_\alpha(0) = x$, $D\bar{\varphi}\,|_0 = \mathrm{id}_{T_xM}$;

(2) $\bar{\varphi}_\beta(T_xM)^G \supseteq \{y \in$ *image* $\bar{\varphi}_\beta : y$ *is stationary for* $\beta\}$; *and*

(3) $\bar{\varphi}_\beta$ *varies continuously with* β.

PROOF. We can find $g_1, \ldots, g_k \in G$, with k at most the codimension of $(T_xM)^G$, such that $(T_xM)^G$ is precisely the set of common fixed vectors of $D\alpha(g_1)\,|_x$ $\ldots, D\alpha(g_k)\,|_x$. Let φ: $T_xM \to M$ be an embedding with $\varphi(0) = x$ and $D\varphi\,|_0 = \mathrm{id}_{T_xM}$, and consider the mapping

$$f_\beta(z) = \left(\varphi^{-1}\beta(g_1)\varphi z - z, \ldots, \varphi^{-1}\beta(g_k)\varphi z - z\right).$$

When β is near α, f_β maps a neighborhood of the origin in T_xM into $T_xM \times \cdots \times T_xM$. If φz is stationary for β, then clearly $f_\beta(z) = 0$. Let π be a projection from $T_xM \times \cdots \times T_xM$ onto the range E of $Df_\alpha\,|_0$. Since the derivative of πf_α (as a mapping into E) is surjective at the origin with kernel $(T_xM)^G$, the lemma follows from the implicit function theorem. Q.E.D.

Now let G, α, and x be as in Theorem 2.1, and let $\{\bar{\varphi}_\beta\}$ be as in Lemma 2.4. Our goal is to show that, if β is near α and z is near the origin in $(T_xM)^G$, then $\bar{\varphi}_\beta(z)$ is stationary for β. Provided this holds, we obtain the mappings φ_β sought in Theorem 2.1 by composing $\bar{\varphi}_\beta$ with a fixed radial diffeomorphism from T_xM onto a neighborhood of the origin in T_xM.

Let $\bar{\beta}$ denote the local action on T_xM defined by $\bar{\beta}(g) = \bar{\varphi}_\beta^{-1}\beta(g)\bar{\varphi}_\beta$. For each compact set $K \subseteq G$, there is a neighborhood U of the origin in T_xM such that $\bar{\beta}(g)$ is defined on U for all $g \in K$ and all β near α. Furthermore, $\bar{\beta}\,|_K \in C^0(K, C^\nu(U, T_xM))$ depends continuously upon β. We choose a compact neighborhood L of the identity which generates G and an inner product on T_xM and define E to be the orthogonal complement to $(T_xM)^G$. For β near α and $z \in T_xM$, let $\mu(\beta, z) = \max_{g \in L}\|\bar{\beta}(g)z - z\|$, provided this makes sense. Clearly $\mu(\beta, z) = 0$ if and only if $\bar{\varphi}_\beta z$ is stationary for β.

Claim. Let $\beta_1, \beta_2, \ldots,$ be actions converging to α, and let $z_1, z_2, \ldots,$ be points in T_xM converging to the origin. Suppose that for each i, $\mu(\beta_i, z_i) \neq 0$. Then for sufficiently large i, z_i fails to be a local minimum of the restriction of $\mu(\beta_i, \ )$ to the coset of E containing z_i.

This claim implies Theorem 2.1. To see this, notice that the origin is a strict local minimum of the restriction of $\mu(\alpha, \ )$ to E. It follows that, for each action β near α and each coset X of E near E itself, the restriction of $\mu(\beta, \ )$ to X has a local minimum z near $X \cap (T_xM)^G$. According to the claim, $\mu(\beta, z)$ must be zero when β is very close to α and z is very close to the origin. Then $\bar{\varphi}_\beta z$ is stationary for β, and in fact $\{z\} = X \cap (T_xM)^G$.

To establish the claim, we consider the functions $u_i(g) = \bar{\beta}_i(g)z_i - z_i$, where $g \in G$. Given a compact set $K \subseteq G$, u_i will be defined on K when i is large enough. Suppose that $g, h \in G$ and an index i are such that $\bar{\beta}_i(h)z_i$ is defined and that $\bar{\beta}_i(g)$ is defined on the segment $[z_i, \bar{\beta}_i(h)z_i]$ from z_i to $\bar{\beta}_i(h)z_i$. We have

$$\|u_i(gh) - D\alpha(g)\,|_x u_i(h) - u_i(g)\| = \left\| \int_{[z_i, \bar{\beta}_i(h)z_i]} \left(D\bar{\beta}_i(g)\,|_y - D\alpha(g)\,|_x \right) \overrightarrow{dy} \right\|$$
$$\leqslant \|u_i(h)\| \cdot \max_{y \in [z_i, \bar{\beta}_i(h)z_i]} \|D\bar{\beta}_i(g)\,|_y - D\alpha(g)\,|_x\|.$$

We conclude that for each compact set $K \subseteq G$ and each $\varepsilon > 0$, the inequality

$$(*) \qquad \|u_i(gh) - D\alpha(g)\,|_x u_i(h) - u_i(g)\| \leqslant \varepsilon \|u_i(h)\|$$

holds (and makes sense) for all $g, h \in K$ and all sufficiently large i.

Set $\mu_i = \mu(\beta_i, z_i) = \max_{g \in L} \|u_i(g)\|$. Let $L_1 \subseteq L$ be a neighborhood of the identity such that $\|D\alpha(g)\,|_x - \mathrm{id}_{T_xM}\| < \frac{1}{4}$ for all $g \in L_1$, and for $n = 2, 3, \ldots,$ let L_n be a neighborhood of the identity with $(L_n)^n \subset L_1$. Finally, let i_0 be such that

$$\|D\bar{\beta}_i(g)\,|_y - \mathrm{id}_{T_xM}\| \leqslant 1/2$$

for all $i \geqslant i_0$, $y \in [z_i, \bar{\beta}_i(h)z_i]$, and $g, h \in L_1$. Suppose that $g \in L_n$, $i \geqslant i_0$, and $0 \leqslant k \leqslant n-1$. Integrating $D\bar{\beta}_i(g^k) - \mathrm{id}_{T_xM}$ along the segment from z_i to $\bar{\beta}_i(g)z_i$, we see that

$$\|\bar{\beta}_i(g^{k+1})z_i - \bar{\beta}_i(g^k)z_i - u_i(g)\| \leqslant \tfrac{1}{2}\|u_i(g)\|.$$

Summing over k, we then obtain $\|u_i(g^n) - nu_i(g)\| \leqslant \frac{1}{2}n\|u_i(g)\|$. Thus

$$(**) \qquad \|u_i(g)\| \leqslant 2\|u_i(g^n)\|/n \leqslant 2\mu_i/n$$

for all $g \in L_n$ and $i \geqslant i_0$.

From $(*)$ and $(**)$, it follows that the sequence $w_i = \mu_i^{-1}u_i$, eventually defined on each compact subset of G, is uniformly bounded and equicontinuous on compact subsets of G. By the Arzela theorem, some subsequence, which we may legitimately assume to be the sequence w_i itself, converges uniformly on compact sets to a continuous limit function w. Clearly $\max_{g \in L} \|w(g)\| = 1$, and by $(*)$ w is a cocycle for the isotropy representation at x. Let $e \in T_xM$ be such that $w(g) = D\alpha(g)\,|_x e - e$ for all $g \in G$; clearly e may be chosen from E, so we do so. Writing

$$\left.\frac{d}{dt}\right|_{t=0} \mu_i^{-1}\|\bar{\beta}_i(g)(z_i + te) - (z_i + te)\|^2 = 2\left\langle w_i(g), D\bar{\beta}_i(g)\,|_{z_i} e - e \right\rangle$$
$$= 2\left\langle w_i(g), w(g) \right\rangle + 2\left\langle w_i(g), \left(D\bar{\beta}_i(g)\,|_{z_i} - D\alpha(g)\,|_x \right) e \right\rangle$$

for $g \in L$, we see that this derivative converges uniformly on L to the function $2\|w(g)\|^2$. When i is sufficiently large, it follows that

$$\frac{d}{dt}\bigg|_{t=0} \mu_i^{-1}\|\bar{\beta}_i(g)(z_i + te) - (z_i + te)\|^2 > \frac{1}{8}$$

for all $g \in L$ such that $\|w_i(g)\| \geqslant \frac{1}{2}$. For such a value of i, continuity of the derivatives involved assures that we can find $t_0 < 0$ such that if $t_0 < t < 0$, then

$$\begin{aligned} (1) \qquad \|\bar{\beta}_i(g)(z_i + te) - (z_i + te)\|^2 &< \|\bar{\beta}_i(g)z_i - z_i\|^2 + t\mu_i/8 \\ &\leqslant \mu_i^2 + t\mu_i/8 \quad \text{if } \|w_i(g)\| \geqslant 1/2, \end{aligned}$$

and

$$\begin{aligned} (2) \qquad \|\bar{\beta}_i(g)(z_i + te) - (z_i + te)\|^2 &< \|\bar{\beta}_i(g)z_i - z_i\|^2 + \mu_i^2/4 \\ &< \mu_i^2/2 \quad \text{if } \|w_i(g)\| < 1/2. \end{aligned}$$

In particular, z_i fails to be a local minimum in its coset of E of the function $\mu(\beta_i, z) = \max_{g \in L}\|\bar{\beta}_i(g)z - z\|$. This establishes our claim and hence establishes Theorem 2.1

Finally, we consider the stability of a stationary point x which lies in the boundary of M. Note that the isotropy representation at such a point x leaves invariant the interior closed half-space T_xM^+ and its boundary $T_x\partial M$.

THEOREM 2.5. *Let $x \in \partial M$ be a stationary point of $\alpha \in \mathrm{Act}^\nu(G, M)$, $1 \leqslant \nu \leqslant \infty$, where G is compactly generated and locally compact. If x is VC for the induced action $\partial\alpha \in \mathrm{Act}^\nu(G, \partial M)$, then x is stable. Moreover, if either*

(a) $(T_xM)^G \subseteq T_x\partial M$ and x is VC for $\partial\alpha$, or

(b) $(T_xM)^G \not\subseteq T_x\partial M$ and x is VC for α,

then there exist C^ν embeddings φ_β: $(T_xM^+, T_x\partial M) \to (M, \partial M)$ satisfying conditions (1) *and* (3) *of Theorem* 2.1 *and the condition*

(2′) $\varphi_\beta(T_xM^+ \cap (T_xM)^G) = \{y \in \mathrm{image}\,\varphi_\beta : y$ is stationary for $\beta\}$.

We should mention that condition (b) forces x to be VC for $\partial\alpha$ as well. The first assertion, a trivial consequence of Theorem 2.1, is included for completeness. The hypotheses on the isotropy representation here, although not in general necessary, are nevertheless sharp, as the following corollary demonstrates.

COROLLARY 2.6. *Let E^+ be a closed linear half-space of a finite-dimensional vector space E, G a compactly generated and locally compact group, and ρ: $G \to GL(E)$ a continuous representation leaving E^+ invariant. The origin is stable with respect to perturbations in $\mathrm{Act}^\nu(G, E^+)$, $1 \leqslant \nu \leqslant \infty$, if and only if $\partial\rho$: $G \to GL(\partial E^+)$ is VC. Moreover, the stronger conclusions of Theorem* 2.5 *hold if and only if either*

(a) $E^G \subseteq \partial E^+$ and $\partial\rho$ is VC, or

(b) $E^G \not\subseteq \partial E^+$ and ρ is VC.

PROOF OF COROLLARY 2.6. Suppose that $\partial\rho$ is not VC. Let u: $G \to \partial E^+$ be a cocycle for $\partial\rho$ which is not a coboundary. The affine representation σ of G on E defined by $\sigma(g)e = \rho(g)e + u(g)$ leaves E^+ invariant, and its stationary points constitute an affine subspace of E which does not interect ∂E^+. Furthermore, the affine representation $\bar{\sigma}(g)e = \rho(g)e - u(g)$ has the same properties, and the stationary sets of σ and $\bar{\sigma}$ are negatives of each other. Thus at least one of σ and $\bar{\sigma}$ has no stationary point in E^+. Replacing u by εu in this argument, where ε is a small positive number, we see that the origin is unstable.

It remains to show that the stronger conclusions of Theorem 2.5 fail when $E^G \not\subseteq \partial E^+$ and ρ is not VC. Since they certainly fail when $\partial\rho$ is not VC, we may assume that $H^1(G, \partial E^+) = 0$. From the exact sequence

$$\cdots \to H^1(G, \partial E^+) \to H^1(G, E) \to H^1(G, E/\partial E^+) \to \cdots$$

and the assumption that $H^1(G, E) \neq 0$, it follows that $H^1(G, E/\partial E^+) \neq 0$. Since G acts trivially on $E/\partial E^+$, this is equivalent to the statement that G admits a nontrivial continuous homomorphism into the additive group of real numbers. Let u be such a homomorphism and v a vector in $E^G/\partial E^+$. Each vector of E can be uniquely expressed as $e + tv$ with $e \in \partial E^+$ and $t \in \mathbf{R}$. We define

$$\rho_\varepsilon(g)(e + tv) = \rho(g)e + t\exp(\varepsilon u(g))v.$$

For each $\varepsilon \neq 0$, ρ_ε leaves E^+ invariant and has no stationary point in the interior of E^+. Q.E.D.

PROOF OF THEOREM 2.5. (a) Suppose that $(T_xM)^G \subseteq T_x\partial M$ and x is VC for $\partial\alpha$. Theorem 2.1 provides a family of embeddings ψ_β: $T_x\partial M \to \partial M$ such that $\psi_\beta(T_xM)^G = \{y \in \text{image}\,\psi_\beta : y \text{ is stationary for } \partial\beta\}$. Using a collar on M, we extend these maps to embeddings $\bar{\psi}_\beta$: $T_xM^+ \to M$. Now let $g_1, \ldots, g_k \in G$ be such that $(T_xM)^G$ is the set of common fixed vectors of $D\alpha(g_1)\,|_x, \ldots, D\alpha(g_k)\,|_x$. Elementary estimates show that no point of the form $\bar{\psi}_\beta(z)$, with β near α and $z \in T_xM^+ \setminus T_x\partial M$ near the origin, can be fixed by all of $\beta(g_1), \ldots, \beta(g_k)$. Composing $\bar{\psi}_\beta$ with a fixed radial diffeomorphism from T_xM onto a small neighborhood of the origin, we then obtain the desired mappings φ_β.

(b) Suppose that $(T_xM)^G \not\subseteq T_x\partial M$. By transversality, Lemma 2.4 then has a working analogue, namely that there exist embeddings $\bar{\varphi}_\beta$: $T_xM^+ \to M$ with $\bar{\varphi}_\beta(T_x\partial M) \subseteq \partial M$ and $\bar{\varphi}_\beta(T_xM^+ \cap (T_xM)^G) \supseteq \{y \in \text{image}\,\bar{\varphi}_\beta : y \text{ is stationary for } \beta\}$. The proof then proceeds as before. Q.E.D.

3. Stability of compact leaves of a foliation. Our goal is to establish a stability theorem for compact leaves of a foliation. This amounts to a fairly direct application of the results of §2, except that the action involved, the holonomy along a compact leaf, is only a local action.

Let $\mathrm{Fol}_k^\nu(M)$ denote the space of C^ν foliations of M of codimension k, where M is a smooth manifold without boundary and $1 \leqslant \nu \leqslant \infty$. This space carries a natural weak C^ν topology, which is described in [3]. A compact leaf L of $\mathcal{F} \in \mathrm{Fol}_k^\nu(M)$ is said to be stable if, for every tubular neighborhood U of L, all foliations near $\mathcal{F}$ have a leaf contained in U which is mapped diffeomorphically onto L by the tubular neighborhood projection. If we wish to determine whether L is stable, or if we wish to analyze the structure of $\mathcal{F}$ near L, the natural object to study is the holonomy along L. We postpone a formal discussion of the holonomy; for the moment, it suffices to state that the linear holonomy at $x \in L$ is a representation of $\pi_1(L, x)$ on the normal space $E_x = T_x M / T_x L$.

THEOREM 3.1. *Let L be a compact leaf of $\mathfrak{F} \in \mathrm{Fol}_k^\nu(M)$, $1 \leqslant \nu \leqslant \infty$. For $x \in L$, let $E_x^\pi \subseteq E_x$ denote the set of vectors stationary for the linear holonomy at x. Suppose that the linear holonomy at some (equivalently, every) point $x \in L$ is VC. Then L is stable. In fact, there exist C^ν embeddings $\Phi_{\mathfrak{G}}: E_x^\pi \times L \to M$ for $\mathfrak{G}$ near $\mathfrak{F}$ such that:*

(1) *$\Phi_{\mathfrak{F}}(0, y) = y$ for all $y \in L$;*

(2) *$\Phi_{\mathfrak{G}}(e \times L)$ is a leaf of $\mathfrak{G}$ for all $e \in E_x^\pi$;*

(3) *for some tubular neighborhood U of L, every leaf of $\mathfrak{G}$ which is contained in U and projects diffeomorphically onto L is of the form $\Phi_{\mathfrak{G}}(e \times L)$; and*

(4) *$\Phi_{\mathfrak{G}}$ varies continuously with $\mathfrak{G}$.*

This unifies the stability theorems of Reeb and Hirsch, wherein the dimensions of E_x^π are k and zero, respectively.

COROLLARY 3.2 (REEB **[9]**, THURSTON **[11]**; ALSO LANGEVIN AND ROSENBERG **[8]**). *Let L be a compact leaf of $\mathfrak{F} \in \mathrm{Fol}_k^\nu(M)$. Suppose that $H^1(L, \mathbf{R}) = 0$ and that the linear holonomy along L is trivial. Then $\mathfrak{F}$ saturates a neighborhood of L as the product foliation $L \times E_x$, and the same is true of all foliations near $\mathfrak{F}$.*

COROLLARY 3.3 (HIRSCH **[3]**). *Let L be a compact of $\mathfrak{F} \in \mathrm{Fol}_k^\nu(M)$ and $x \in L$. Suppose that the linear holonomy of some central element $g \in \pi_1(L, x)$ fixes only the origin. Then for some tubular neighborhood U of L, all foliations near $\mathfrak{F}$ have exactly one leaf which is contained in U and projects diffeomorphically onto L.*

The hypothesis of Corollary 3.2 is equivalent to the hypothesis that $\pi_1(L)$ admits no nontrivial homomorphism into $\mathbf{R}$. Since a cocycle for the trivial representation on E_x is no more than a homomorphism into E_x, this implies that the linear holonomy is VC. Thus Corollary 3.2 follows from Theorem 3.1. Similarly, the hypotheses of the Hirsch stability theorem imply that the linear holonomy is VC. In this case, E_x^π is zero.

To define the holonomy, let x and y be points on a leaf L of $\mathfrak{F} \in \mathrm{Fol}_k^\nu(M)$. Construct disks D_x and D_y of dimension k which are transverse to $\mathfrak{F}$. Let γ be a curve from x to y in L. For a sufficiently small neighborhood U of y in D_y, there is a continuous mapping $\Gamma: U \times [0,1] \to M$ with the properties that $\Gamma(u, 1) = u$ for all u, $\Gamma(y, t) = \gamma(t)$ for all t, $\Gamma(u, t)$ lies in the same leaf as u for all u and t, and $\Gamma(u, 0) \in D_x$ for all u. For any such Γ, the mapping $u \mapsto \Gamma(u, 0)$ is C^ν, and its germ at y depends only on the homotopy class rel$\{0, 1\}$ of γ. In particular, when $x = y$ and $D_x = D_y$, $\pi_1(L, x)$ is represented as a group of germs from (D_x, x) to (D_x, x). We will say that this homomorphism $\pi_1(L, x) \to \mathrm{Germ}((D_x, x), (D_x, x))$ is the holonomy of L at x. Strictly speaking, however, the holonomy at x should be described as the equivalence class of such homomorphisms, where D_x is allowed to vary. Let D_x' be another transverse disk at x, and let φ denote the canonical germ from (D_x', x) to (D_x, x) corresponding to the constant curve at x. Then for each

$g \in \pi_1(L, x)$, the following diagram of germs commutes:

$$\begin{tikzcd} (D'_x, x) \arrow[r, "\varphi"] \arrow[d, "\text{holonomy}"', "\text{along } g"] & (D_x, x) \arrow[d, "\text{holonomy}", "\text{along } g"'] \\ (D'_x, x) \arrow[r, "\varphi"] & (D_x, x). \end{tikzcd}$$

Similarly, for arbitrary choices (D_x, x) and (D_y, y), each homotopy class of curves from x to y in L determines a natural isomorphism $\pi_1(L, y) \to \pi_1(L, x)$ and a natural germ $(D_y, y) \to (D_x, x)$ which intertwine the respective holonomies.

The derivative of the holonomy at x provides a linear representation of $\pi_1(L, x)$ on $T_x D_x$. Conjugating with the isomorphism $T_x D_x \xrightarrow{\cong} E_x = T_x M / T_x L$, we obtain the linear holonomy representation at x. This linear holonomy is independent of the choice of D_x, for if φ and D'_x are as above, it is easy to see that the composition

$$E_x \to T_x D'_x \xrightarrow{D\varphi|_x} T_x D_x \to E_x$$

is the identity. Similarly, there are nearly canonical intertwining isomorphisms between the linear holonomies at various points of L. In particular, the linear holonomy is either VC at every point of L or at no point of L.

What comes up in the proof of Theorem 3.1 is a local action of $\pi_1(L, x)$ on a transverse disk D_x. Let G be a discrete group and N a smooth manifold. A local action of G on N consists of a collection of embeddings $\alpha(g)$, $g \in G$, from (possibly empty) open sets $U(g) \subseteq N$ into N together with a collection of open sets $U(g, h) \subseteq N$, $g, h \in G$. We require that $\alpha(1_G) = \mathrm{id}_N$ and that $\alpha(g)\alpha(h)y = \alpha(gh)y$ whenever $y \in U(g, h)$; in particular, both sides of the latter equation are defined. A point x of N is stationary for the local action α if $x \in U(g)$, $x \in U(g, h)$, and $\alpha(g)x = x$ for all $g, h \in G$. The topology on $\mathrm{Act}^{\nu}_{\mathrm{loc}}(G, N)$ is generated by open sets of two kinds. One kind is obtained by choosing a compact set $K \subseteq N$ and an element g of G. There is a natural topology on the space of all C^{ν} maps from (varying) neighborhoods of K into N; it controls derivatives on K and is not a T_0 topology. For each open set W in this space, we declare the set of all local actions α with $\alpha(g) \in W$ to be open. The other kind is obtained by choosing a compact set $K \subseteq N$ and a pair $g, h \in G$. The set of all local actions α with $K \subseteq U_{\alpha}(g, h)$ is then declared to be open.

Theorem 2.1 holds with slight modification for local actions. Specifically, let $x \in N$ be a VC stationary point for $\alpha \in \mathrm{Act}^{\nu}_{\mathrm{loc}}(G, N)$, $1 \leqslant \nu \leqslant \infty$. Let S be a finite set of generators for G. Embeddings φ_{β}: $T_x N \to N$ exist for $\beta \in \mathrm{Act}^{\nu}_{\mathrm{loc}}(G, N)$ near α such that image $\varphi_{\beta} \subseteq U_{\beta}(g)$ for all $g \in S$ and such that $\varphi_{\beta}(T_x N)^G = \{y \in$ image $\varphi_{\beta}: \beta(g)y = y$ for all $g \in S\}$. The Proofs of 2.1 and 2.4 apply almost verbatim.

To begin the proof of Theorem 3.1, let L be a compact leaf of $\mathcal{F} \in \mathrm{Fol}^{\nu}_k(M)$ such that the linear holonomy along L is VC. We choose a point $x \in L$, which is to be fixed for the rest of the discussion. Let U be a C^{ν} tubular neighborhood of L, fibered by the disks D_y, $y \in L$, and let p: $U \to L$ be the projection. We may choose U so that the fibers D_y are transverse to all foliations near $\mathcal{F}$, and we only consider foliations near enough to $\mathcal{F}$ that this holds.

For each $g \in \pi_1(L, x)$, choose a loop γ_g representing g; γ_1 is required to be the constant loop. Let $\mathcal{G}$ be a foliation near $\mathcal{F}$. For each point $z \in D_x$, γ_g may be partially lifted to a curve $\tilde{\gamma}$ in U, with $\tilde{\gamma}(1) = z$, such that $\tilde{\gamma}$ stays in the leaf of $\mathcal{G}$ through z. This lift may be definable on all of $[0, 1]$, or it may be definable only on some maximal interval $(t_0, 1]$ with $0 \leqslant t_0 < 1$. We define $U_{\mathcal{G}}(g) \subseteq D_x$ to be the set of all points z for which the former occurs and define $\alpha_{\mathcal{G}}(g)$: $U_{\mathcal{G}}(g) \to D_x$ as mapping z to $\tilde{\gamma}(0)$. Similarly, for each pair $g, h \in G$, let $H_{g,h}$ be a homotopy from $\gamma_g\gamma_h$ to γ_{gh} with fixed endpoints. Let $U_{\mathcal{G}}(g, h)$ denote the set of all points $z \in D_x$ such that $H_{g,h}$ lifts to a map $\tilde{H}$ into the leaf of $\mathcal{G}$ through z with $\tilde{H}(1, 1) = z$. The essential properties of this construction are that $\alpha_{\mathcal{G}}$ is a local action of $\pi_1(L, x)$ on D_x, x is stationary for $\alpha_{\mathcal{F}}$, and $\alpha_{\mathcal{G}}$ varies continuously with $\mathcal{G}$. See [**2** and **3**] for details.

A leaf of $\mathcal{G}$ which is contained in U and projects diffeomorphically onto L will intersect D_x in a point which is stationary for $\alpha_{\mathcal{G}}$. To deduce 3.1 from 2.1, we need some kind of converse. Place a Riemannian metric on L and set $r = \max_{y \in L} d(x, y)$. Let $S \subseteq \pi_1(L, x)$ denote those elements representable by a loop of length less than $4r$; S is finite and generates $\pi_1(L, x)$. By "loop" we now mean piecewise smooth loop, and we assume that the loops γ_g were chosen to be piecewise smooth.

LEMMA 3.4. *There exists $\rho > 0$ such that every loop at x of length less than $4r$ can be homotoped (with fixed end points) to the corresponding loop γ_g, $g \in S$, through loops of length no greater than ρ.*

PROOF. Let $\varepsilon > 0$ be such that any two points of L within ε of each other can be connected by a unique geodesic of length less than ε. For $s \in [0, 1]$ and $y, z \in L$ with $d(y, z) < \varepsilon$, let $f(y, z; s)$ be the point at a proportion s of the way along the short geodesic from y to z. Then f is smooth. Let A be the maximum of $\|\partial f/\partial y\|$ over $\{(y, z; s) : d(y, z) \leqslant \varepsilon/2 \text{ and } s \in [0, 1]\}$.

Any loop at x may be parameterized by $[0, 1]$ in proportion to arc length. Among loops so parameterized and of length less than $4r$, there is a finite set $c_1, \ldots, c_n$ which is $(\varepsilon/2)$-dense; that is, for any such loop c, there exists i with $d(c(t), c_i(t)) \leqslant \varepsilon/2$ for all $t \in [0, 1]$. We can then homotope c to c_i by the homotopy $c_s(t) = f(c(t), c_i(t); s)$. For each $s \in [0, 1]$, the length of c_s is less than $8rA$.

For $i = 1, \ldots, n$, choose an arbitrary piecewise smooth homotopy from c_i to the corresponding loop γ_g, $g \in S$. Composing, we have homotoped c to γ_g through loops whose lengths are bounded independently of c. Q.E.D.

Let ρ be as provided by the lemma. There is a neighborhood W of x in D_x such that for all $z \in W$ and all $\mathcal{G}$ near $\mathcal{F}$, every loop at x of length less than or equal to ρ may be lifted to a path in a leaf of $\mathcal{G}$ beginning at z. In particular, $\alpha_{\mathcal{G}}(g)$ is defined on W for all $g \in S$. Now let $z \in W$ be such that $\alpha_{\mathcal{G}}(g)z = z$ for all $g \in S$. We map L into the leaf of $\mathcal{G}$ through z as follows. Given $y \in L$, connect x to y by a path in L of length less than $2r$. Lift this path to a path in the leaf through z beginning at z, and denote the end point of the left by $\psi(y)$. Since $\rho \geqslant 4r$, this construction at least makes sense. Furthermore, ψ is well defined. To see this, consider two paths γ, γ' from x to y of length less than $2r$. We can homotope the loop $\gamma'\gamma^{-1}$ at x to the representative γ_g, $g \in S$, of its homotopy class through loops of length less than or equal to ρ. Let H be such a homotopy; $H(0, t) = \gamma'\gamma^{-1}(t)$, and $H(1, t) = \gamma_g(t)$. Then H can be lifted to a map $\tilde{H}$ into the leaf through z with $\tilde{H}(s, 1) = z$ for all $s \in [0, 1]$. Since $\tilde{H}(0, 1) = z$, it follows that $\tilde{H}(0, 0) = z$. Thus the definition of $\psi(y)$ using either the path γ or the path γ' produces $\psi(y) = \tilde{H}(0, \frac{1}{2})$. Once we know ψ is well defined, it is obvious that ψ is a local diffeomorphism. Since ψ is one-to-one, it follows that ψ is a diffeomorphism from L onto the leaf of $\mathcal{G}$ through z. So the leaf of $\mathcal{G}$ through z is contained in U and projects diffeomorphically onto L.

Conversely, if $z \in W$ and $\alpha_{\mathcal{G}}(g)z \neq z$ for some $g \in S$, then the leaf of $\mathcal{G}$ through z, even if it happens to be contained in U, does not project diffeomorphically onto L. Now we apply Theorem 2.1 for local actions. If $\mathcal{G}$ is near $\mathcal{F}$, there is an embedding $\varphi_{\mathcal{G}}$: $E_x \to D_x$ such that image $\varphi_{\mathcal{G}} \subseteq U_{\mathcal{G}}(g)$ for all $g \in S$ and such that $\varphi_{\mathcal{G}} E_x^\pi = \{z \in \text{image } \varphi_{\mathcal{G}} : \alpha_{\mathcal{G}}(g)z = z \text{ for all } g \in S\}$. We may assume that image $\varphi_{\mathcal{G}} \subseteq W$ for $\mathcal{G}$ near $\mathcal{F}$. Define $\Phi_{\mathcal{G}}$: $E_x^\pi \times L \to U$ by

$$\Phi_{\mathcal{G}}(e, y) = \psi_{\mathcal{G}}(\varphi_{\mathcal{G}} e, y).$$

Here $\psi_{\mathcal{G}}$ refers to the diffeomorphism from L onto a leaf of $\mathcal{G}$ which was constructed above. Conditions (1), (2), and (4) of Theorem 3.1 are now satisfied. To obtain condition (3), let U' be a subtubular neighborhood of U—that is, a family of disks $D'_y \subseteq D_y$ about each y in L—such that $D'_x \subseteq$ image $\varphi_{\mathcal{G}}$ for all $\mathcal{G}$ near $\mathcal{F}$. Any leaf of $\mathcal{G}$ which is contained in U' and project diffeomorphically onto L must intersect D'_x at a point which is fixed by $\alpha_{\mathcal{G}}(g)$ for all $g \in S$. But then this leaf is $\Phi_{\mathcal{G}}(e \times L)$ for some $e \in E_x^\pi$. This completes the proof of Theorem 3.1.

By the same kind of argument, Theorem 2.3 leads to the following result concerning foliations. Let L be a compact leaf of $\mathcal{F} \in \mathrm{Fol}_k^r(M)$ and $x \in L$. Let H be a subgroup of finite index in $\pi = \pi_1(L, x)$ such that the linear holonomy restricted to H is VC. Set $P = \bigcup_{g \in \pi} \rho(g) E_x^H$, where ρ is the linear holonomy. Now form the flat bundle X over L, with fiber P, whose holonomy is given by ρ. That is, X is the quotient of $\tilde{L} \times P$ by the right action defined by $(y, p)g = (yg, \rho(g)^{-1}p)$ for $y \in \tilde{L}$, $p \in P$, and $g \in \pi$. Here $\tilde{L}$ denotes a universal covering space of L, and the right action of π on $\tilde{L}$ is that by covering transformations. (We should point out that unless π is abelian, there is no natural isomorphism between π and the group of covering transformations, but only a natural family of isomorphisms. Any one will suffice here.) Since the action on $\tilde{L} \times P$ preserves the foliation $\{\tilde{L} \times p\}_{p \in P}$, X is transversely foliated. Now the same arguments as before show that X occurs as an embedded constituent of all foliations near $\mathcal{F}$. Furthermore, the embeddings may be chosen so that for some tubular neighborhood U of L, every leaf of a foliation near $\mathcal{F}$ which is contained in U and covers L with a group containing H must in fact be contained in the embedded copy of X.

4. Stability of compact orbits. In §2, we presented a criterion for the stability of a stationary point. Here, we ask what is logically the next question: how do those results generalize to the problem of stability of a compact orbit?

Throughout this section, G denotes a compactly generated Lie group, $\mathfrak{g}$ its Lie algebra, and M a smooth manifold. For simplicity, we assume that M has no boundary. Let α be an action of G on M and L a compact orbit of α. We say that L is stable if, for every tubular neighborhood U of L, every action near α has an orbit which is contained in U and which projects diffeomorphically onto U by the tubular neighborhood projection.

The isotropy representation at a point $x \in L$ splits as

$$0 \to T_xL \to T_xM \to E_x = T_xM/T_xL \to 0.$$

Let H be the isotropy group of x. Note that the linear surjection $\alpha_* |_x\colon \mathfrak{g} \to T_xL$ intertwines the adjoint representation of H on $\mathfrak{g}$ with the isotropy representation. (α_* is the infinitesimal generator of α.) Then T_xL is equivalent to the H-module $\mathfrak{g}/\mathfrak{h}$, where $\mathfrak{h}$ is the Lie algebra of H. We refer to the representation of H on E_x as the normal representation.

THEOREM 4.1. *Let $\alpha \in \mathrm{Act}^\nu(G, M)$, $1 \leqslant \nu \leqslant \infty$, a compact orbit L of α, and $x \in L$ be given. Let H be the isotropy group of x, and suppose that the normal representation of H is VC. Then L is stable. Moreover, there exist C^ν embeddings $\Phi_\beta\colon E_x^H \times L \to M$ for β near α such that*:

(1) *$\Phi_\alpha(0, y) = y$ for all $y \in L$;*

(2) *$\Phi_\beta(e \times L)$ is an orbit of β for all $e \in E_x^H$;*

(3) *for some tubular neighborhood U of L, every orbit of β which is contained in U and projects diffeomorphically onto L is of the form $\Phi_\beta(e \times L)$; and*

(4) *Φ_β varies continuously with β.*

This sounds just like Theorem 3.1. When H is discrete, the orbits of α near x do constitute a foliation, and if, in addition, G is simply connected, Theorem 4.1 follows from Theorem 3.1. The essential difference in the general case is that orbits near L may have dimension greater than the dimension of L, and thus H does not naturally act, even in a local way, on a disk transverse to L at x. Notice that the hypothesis that the normal representation be VC is independent of the choice of $x \in L$.

Clearly there is a stronger sense in which L might be said to be stable—namely, if all actions near α have nearby orbits which are of the same algebraic type, rather than the same topological type, as L. If H is the isotropy group of $x \in L$, this is to say that for every action near α, there exists a point near x whose isotropy group is H. The appropriate condition for this stronger kind of stability to occur seems to be simply that $H^1(H, T_xM) = 0$. Theorem 4.1 then has a glaring failure; it does not include this stronger stability as a special case. That is, the normal representation may fail to be VC even when the entire isotropy representation is VC. Perhaps a more appropriate hypothesis, including both cases, would be that the induced linear transformation $H^1(H, T_xM) \to H^1(H, E_x)$ be zero. At the moment, we do not know if this works. To test such a hypothesis, it would seem necessary to investigate the space of deformations of H in G.

The hypotheses of Theorem 2.1 are quite satisfactory in one sense. Given a representation of G which is not VC, one can realize this as the isotropy representation at an unstable stationary point of some action. Such an action is in fact the representation itself. This suggests a similar test of satisfaction for the current problem. Consider an exact sequence of H-modules (linear representations of H on a vector space) $0 \to \mathfrak{g}/\mathfrak{h} \to F \to E \to 0$, where H is a closed subgroup of the Lie group G. Can this sequence always be realized as that of the isotropy representation at some point of a G-action having H as its isotropy group? Perhaps not. Still, among all sequences which can be so realized, it would seem desirable to have a canonical realization. The construction of such a model would then probably be simple enough to reveal necessary and sufficient conditions for stability of the compact orbit under perturbations of the model. That is, we would have an analogue of Corollary 2.2.

We now begin the proof of Theorem 4.1.

Place an inner product on $\mathfrak{g}$ and let B be a small open ball about the origin in $\mathfrak{h}^{\perp}$. Specifically, we need to assume that the mapping $b \mapsto \alpha(\exp b)x$ embeds $2B$ into L. Choose an embedding φ: $E = E_x \to M$ such that $\varphi(0) = x$, the composition

$$E \xrightarrow{D\varphi|_x} T_x M \to E$$

is the identity, and the image $\varphi(E)$ is a small, well-behaved disk. For actions β near α, we define coordinates on the neighborhood $\beta(\exp B)\varphi(E)$ of x by

$$(b, e) \mapsto \beta(\exp b)\varphi(e), \qquad b \in B, e \in E.$$

We denote the corresponding local action of G on $B \times E$ as $\bar{\beta}$; $\bar{\beta}$ varies continuously with β. For simplicity, let us assume that $E^H = 0$. That is, the normal representation indicates that α has no other orbits near L which project diffeomorphically onto L in a tubular neighborhood. The transition from this case to the general case can be made by mimicking the arguments of Theorem 2.1; rather than looking for a local minimum of a certain function on E, we would look for a local minimum of its restriction to a coset of some complement of E^H in E.

Let π: $B \times E \to E$ be the projection. If we assign to $h \in H$ the transformation $\pi\bar{\beta}(h)$, we do not obtain a local action on E, but we obtain a kind of approximation to a local action. Our goal is then to find, for all β near α, a point e near the origin of E such that

$$(*) \qquad \pi\bar{\beta}\,(h)(0, e) = e \quad \text{for all } h \in H_0.$$

Here H_0 is some compact neighborhood of the identity in H which generates H. Because G is compactly generated and G/H is compact, H_0 exists.

Condition $(*)$ makes sense, at least, since there is some neighborhood of $(0,0)$ in $B \times E$ on which $\bar{\beta}(h)$ is defined for all β near α and all $h \in H_0$. Let us see why $(*)$ suffices to prove Theorem 4.1. First, if $(*)$ holds, the orbit of β through $\varphi(e)$ has dimension equal to that of L. Now choose $g_1, \ldots, g_n \in G$ such that $G = \bigcup_{i=1}^{n} g_i(\exp B)H$. Let p be the projection in some tubular neighborhood of L. When β is near α and e is near 0, the mappings from B into L defined by $b \mapsto p\beta(g_i \exp b)\varphi(e)$, $i = 1, \ldots, n$, are C^ν embeddings, and their images cover L. In particular, the sets $\beta(g_i \exp B)\varphi(e)$, $i = 1, \ldots, n$, are relatively open subsets in the intrinsic topology of the orbit through $\varphi(e)$, and each projects diffeomorphically into L. We wish to show that if β is near enough to α and e is near enough to 0, then the restriction of p to the set $P = \bigcup_{i=1}^{n} \beta(b_i \exp B)\varphi(e)$ is one-to-one. It will follow that P is a relatively open and closed subset of the orbit through $\varphi(e)$ which projects diffeomorphically onto L. Suppose we stipulate that $g_1 = 1_G$. Then $\varphi(e) \in P$. If P_0 is the component of P containing $\varphi(e)$, then P_0 is invariant under H and under the identity component G_0 of G. Since $\{g_1, \ldots, g_n\}HG_0 = G$, it follows that P is the entire orbit.

To show that $p \mid P$ is one-to-one, we need to make another restriction on B, one which could have been made at the outset. We need to assume that for some $\varepsilon > 0$, the mapping from εB into L defined by $c \mapsto \alpha(\exp b \exp c)x$ is an embedding for all b in the closure $\bar{B}$ of B. It follows that if β is near α, e is near 0, $1 \leqslant i \leqslant n$, and $b \in B$, then the mapping $c \mapsto p\beta(\exp b \exp c)\varphi(e)$ embeds $\frac{1}{2}\varepsilon B$ into L. In particular, this mapping is one-to-one. Now let K be a compact neighborhood in H of $H \cap (\bigcup_{i,j=1}^{n} (g_i \exp \bar{B})^{-1} g_j \exp \bar{B})$. Note that $K \subseteq H_0^m$ for some m. It is easy to show that there exists $\delta > 0$ such that if e satisfies $(*)$ with e near 0 and β near α, then for all $d \in \delta B$ and all $h \in K$, there exists $c = c(h, d) \in \frac{1}{2}\varepsilon B$ with

$$\beta(h \exp d)\varphi(e) = \beta(\exp c)\varphi(e).$$

Now we put these pieces together. The equation

$$\alpha(g_i \exp b)x = \alpha(g_j \exp b')x, \quad \text{for some } 1 \leqslant i,\ j \leqslant n, b, b' \in \bar{B},$$

implies that $(g_i \exp b)^{-1}(g_j \exp b') \in H$. The set $K \exp \delta B \subseteq G$ is a neighborhood of all such values. Therefore if β is near enough to α, and e is near enough to 0, the equation

$$p\beta(g_i \exp b)\varphi(e) = p\beta(g_j \exp b')\varphi(e)$$

implies that $(g_i \exp b)^{-1} g_j \exp b' = h \exp d$ for some $h \in K$ and some $d \in \delta B$. Then

$$\beta(g_j \exp b')\varphi(e) = \beta(g_i \exp b)\beta(h \exp d)\varphi(e) = \beta(g_i \exp b \exp c(h, d))\varphi(e),$$

where $c(h, d) \in \frac{1}{2}\varepsilon B$. Since this point projects to $p\beta(g_i \exp b)\varphi(e)$, it follows from our choice of ε that $c(h, d) = 0$. That is, $\beta(g_j \exp b')\varphi(e) = \beta(g_i \exp b)\varphi(e)$.

Thus to prove Theorem 4.1, we need only to show that for all β near α there exists a point e near the origin of E such that $(*)$ holds. The strategy here is the same as in the proof of Theorem 2.1. Place an inner product on E and set $\mu(\beta, e) = \max_{h \in H_0} \| \pi\bar{\beta}(h)(0, e) - e \|$, provided this makes sense. Since the origin is a strict local minimum of $\mu(\alpha, \ )$, there will be a local minimum of $\mu(\beta, \ )$ near the origin when β is near α. As before, we will argue that if $\beta_1, \beta_2, \ldots$ converge to α and $e_1, e_2, \ldots$ converge to the origin in E with $\mu(\beta_i, e_i) \neq 0$, then e_i fails to be a local minimum of $\mu(\beta_i, \ )$ when i is large.

For $h \in H$, define $b_i(h) \in B$ and $u_i(h) \in E$ by the equation $\bar{\beta}_i(h)(0, e_i) = (b_i(h), e_i + u_i(h))$. Let $\mu_i = \mu(\beta_i, e_i) = \max_{h \in H_0} \| u_i(h) \|$. We wish to show that some subsequence of $\{\mu_i^{-1} u_i\}$ converges, uniformly on compact sets, to a cocycle for the normal representation ρ. The apparent problem is that $b_i(h)$ may be very large relative to μ_i, making estimates difficult.

Let K be a compact subset of H and $g, h \in K$. Integrating $\pi D\bar{\beta}_i(g)$ along the segment from $(b_i(h), e_i)$ to $(b_i(h), e_i + u_i(h))$, we obtain

$$\begin{aligned}\|u_i(gh) - \rho(g)u_i(h) - u_i(g)\| = \Big\| &\pi\bar{\beta}_i(g)(b_i(h), e_i) - \pi\bar{\beta}_i(g)(0, e_i) \\ &+ \int_{(b_i(h), e_i)}^{(b_i(h), e_i + u_i(h))} \left(\pi D\bar{\beta}_i(g)\big|_y - \rho(g)\right) \overrightarrow{dy}\Big\| \\ &\leqslant \|\pi\bar{\beta}_i(g)(b_i(h), e_i) - \pi\bar{\beta}_i(g)(0, e_i)\| + c\|u_i(h)\|.\end{aligned}$$

Here c is the maximum of $\|\pi D\bar{\beta}_i(g) - \rho(g)\|$ along that segment. It may be bounded by a constant $\varepsilon_{i,K}$ depending only on i and K, with $\varepsilon_{i,K}$ tending to zero as i tends to infinity. (Note that the expressions here make sense when i is large, independent to the choice of $g, h \in K$.) To estimate the first term above, we introduce $m_i = \|\pi\bar{\beta}_{i*}|_{(0, e_i)}\|$, where $\bar{\beta}_{i*}$ is the infinitesimal generator of $\bar{\beta}_i$. When H is discrete, m_i equals to zero. For $0 \leqslant \tau \leqslant 1$, consider

$$\begin{aligned}\frac{d}{dt}\Big|_{t=\tau} \pi\bar{\beta}_i(g)(tb_i(h), e_i) &= \frac{d}{dt}\Big|_{t=\tau} \pi\bar{\beta}_i(g\exp[tb_i(h)])(0, e_i) \\ &= \pi \cdot D\bar{\beta}_i(g\exp[\tau b_i(h)])\big|_{(0,e_i)} \cdot \bar{\beta}_{i*}(b_i(h))\big|_{(0,e_i)}.\end{aligned}$$

For brevity, let $\bar{g}$ denote $g\exp[\tau b_i(h)]$. Now decompose $\mathfrak{g}$ as a direct sum $V \oplus W$ as follows. First, V is the inverse image of E under $\bar{\beta}_{i*}|_{(0,e_i)}$. The dimension of V equals that of $\mathfrak{h}$, and V tends to $\mathfrak{h}$ as i tends to infinity. Secondly, W is chosen so as to be contained in the kernel of $\pi \cdot D\bar{\beta}_i(\bar{g}) \cdot \bar{\beta}_{i*}|_{(0,e_i)}$. Specifically, W is the image of $\mathfrak{h}^\perp$ under the linear isomorphism of $\mathfrak{g}$ given by

$$z \mapsto \frac{d}{dt}\Big|_{t=0} (\bar{g})^{-1}\exp(b_i(\bar{g}) + tz)(\exp b_i(\bar{g}))^{-1}\bar{g}.$$

We should point out that, whereas V depends only on i, W depends on τ, g, $b_i(\bar{g})$, and $b_i(h)$. Observe that when $b_i(g) = b_i(h) = 0$, $W \cap \mathfrak{h} = 0$. Thus we may find some cone about $\mathfrak{h}$ in $\mathfrak{g}$ such that W intersects this cone only at the origin, no matter $\tau \in [0, 1]$ and $g, h \in K$, provided $b_i(g)$ and $b_i(h)$ are sufficiently small. The upshot is that when i is large enough, the subspaces V and W of $\mathfrak{g}$ are not too close together, no matter τ, g, and h. That is, there exist constants $C_K > 0$ and i_K depending only on K such that for $i \geqslant i_K$ we have

$$b_i(h) = v + w, \qquad v \in V, w \in W,$$

where $\|v\| \leqslant C_K\|b_i(h)\|$. Then for all $\tau \in [0, 1]$,

$$\begin{aligned}\left\|\frac{d}{dt}\Big|_{t=\tau} \pi\bar{\beta}_i(g)(tb_i(h), e_i)\right\| &= \|\pi \cdot D\bar{\beta}_i(g\exp[\tau b_i(h)])\big|_{(0,e_i)}\bar{\beta}_{i*}(v + w)\big|_{(0,e_i)}\| \\ &\leqslant \|D\bar{\beta}_i(g\exp[\tau b_i(h)])\big|_{(0,e_i)}\| \cdot m_i C_K\|b_i(h)\| \\ &\leqslant m_i\bar{C}_K\|b_i(h)\|,\end{aligned}$$

where $\overline{C}_K$ is the product of C_K with some uniform upper bound for the derivative factor. In summary, we have proved the following:

LEMMA 4.2. *For each compact set $K \subseteq H$, there exist $\overline{C}_K > 0$, i_K, and $\varepsilon_{i,K} \to 0$ such that for all $i \geqslant i_K$ and all $g, h \in K$,*

$$\|u_i(gh) - \rho(g)u_i(h) - u_i(g)\| \leqslant m_i\overline{C}_K \|b_i(h)\| + \varepsilon_{i,K}\|u_i(h)\|.$$

We now use Lemma 4.2 to show that m_i is not much greater than μ_i when i is large. Let K be the exponential image of the closed ball of radius r about the origin in $\mathfrak{h}$, with r small enough that

(1) $K \subseteq H_0$, and

(2) $\|\rho(g) - \mathrm{id}_E\| < \frac{1}{4}$ for all $g \in K$.

Let $\xi \in \mathfrak{h}$ with $\|\xi\| = r$. For $s, t \in [0, 1]$, we have

$$\|u_i(\exp(s+t)\xi) - \rho(\exp s\xi)u_i(\exp t\xi) - u_i(\exp s\xi)\|$$
$$\leqslant m_i\overline{C}_K\|b_i(\exp t\xi)\| + \varepsilon_{i,K}\|u_i(\exp t\xi)\|.$$

Since both sides vanish when $t = 0$, we may differentiate at $t = 0$ to obtain

$$\left\| \frac{d}{dt}\bigg|_{t=s} u_i(\exp t\xi) - \rho(\exp s\xi)\pi\overline{\beta}_{i*}(\xi)\,\big|_{(0,e_i)} \right\|$$
$$\leqslant m_i\overline{C}_K\|\pi_B\overline{\beta}_{i*}(\xi)\,\big|_{(0,e_i)}\| + \varepsilon_{i,K}\|\pi\overline{\beta}_{i*}(\xi)\,\big|_{(0,e_i)}\|.$$

Let i be large enough that $\overline{C}_K\|\pi_B\overline{\beta}_{i*}(\xi)\,|_{(0,e_i)}\| < \frac{1}{4}r$, independent of $\xi \in \mathfrak{h}$ with $\|\xi\| = r$, and that $\varepsilon_{i,K} < \frac{1}{4}$. Now choose ξ with $\|\pi\overline{\beta}_{i*}(\xi)\,|_{(0,e_i)}\| = m_ir$. We have

$$\left\| \frac{d}{dt}\bigg|_{t=s} u_i(\exp t\xi) - \pi\overline{\beta}_{i*}(\xi)\,\big|_{(0,e_i)} \right\|$$
$$\leqslant \|(\rho(\exp s\xi) - \mathrm{id}_E)\pi\overline{\beta}_{i*}(\xi)\,\big|_{(0,e_i)}\| + m_ir/4 + m_ir/4 \leqslant 3m_ir/4.$$

Integrating from $s = 0$ to $s = 1$, we obtain

$$\|u_i(\exp\xi) - \pi\overline{\beta}_{i*}(\xi)\,\big|_{(0,e_i)}\| \leqslant 3m_ir/4,$$

so

$$\|u_i(\exp\xi)\| \geqslant m_ir/4.$$

Since $\exp\xi \in H_0$, we have shown the following:

LEMMA 4.3. *There exists $r > 0$ and i_0 such that $m_i \leqslant 4r^{-1}\mu_i$ when $i \geqslant i_0$.*

Set $w_i = \mu_i^{-1}u_i$. Lemmas 4.2 and 4.3 together show that for each compact set $K \subseteq H$,

$$(*) \qquad \|w_i(gh) - \rho(g)w_i(h) - w_i(g)\| \leqslant 4r^{-1}\overline{C}_K\|b_i(h)\| + \varepsilon_{i,K}\|w_i(h)\|$$

for all $g, h \in K$ and all $i \geqslant \max(i_K, i_0)$. Also, a review of the estimates leading to Lemma 4.3 establishes that

$$(**) \qquad \|w_i(\exp t\xi)\| \leqslant 7t$$

for all $i \geqslant i_0$, $t \in [0, 1]$, and $\xi \in \mathfrak{h}$ with $\|\xi\| = r$. The sequence $\{w_i\}$ is therefore eventually uniformly bounded and equicontinuous on each compact subset of H. It is clear from $(*)$ that the limit of a convergent subsequence will be a cocycle for ρ. By the same arguments as in the proof of Theorem 2.1, we may conclude that when i is large, e_i fails to be a local minimum of $\mu(\beta_i, \ )$. This completes the proof of Theorem 4.1.

5. Stable properties of linear representations. We began this investigation by considering a special case—perturbing a linear action into an affine action by adding a translational part. We end by applying our general results to the family of linear representations.

To avoid constant repetition of the hypotheses, let us fix a compactly generated, locally compact group G and a finite-dimensional vector space E. $\mathrm{Hom}(G, GL(E))$ denotes the space of continuous linear representations of G on E with the topology it inherits as a subset of $\mathrm{Act}^\infty(G, E)$. Since $GL(E)$ is an embedded subgroup of $\mathrm{Diff}^\infty(E)$, this matches any other natural definition one might give for the topology on $\mathrm{Hom}(G, GL(E))$.

THEOREM 5.1. *The VC representations constitute an open subset of* $\mathrm{Hom}(G, GL(E))$.

Since a representation which is not VC may be perturbed into an affine action having no stationary point at all, Theorem 5.1 follows directly from Corollary 2.2. The stronger conclusions of Theorem 2.1 also provide the following:

THEOREM 5.2. *Let* $\rho \in \mathrm{Hom}(G, GL(E))$ *be VC. If* σ *is near* ρ, *then* E_σ^G *has the same dimension as* E_ρ^G *and is near* E_ρ^G *in the appropriate Grassmannian manifold.*

If a representation ρ has an invariant subspace F, it may be of some interest to know if all representations near ρ have an invariant subspace near F. We view F as a stationary point of the action of G on the Grassmannian manifold which is induced by ρ. The tangent space at F to the Grassmannian manifold is canonically isomorphic to $L(F, E/F) = F^* \otimes (E/F)$. If $\rho_F\colon G \to GL(F)$ denotes the restriction and $\bar\rho\colon G \to GL(E/F)$ the quotient representation, then it is easily verified that the isotropy representation of G at F corresponds to $\rho_F^* \otimes \bar\rho$ via this canonical isomorphism. (The tensor product $\rho_F^* \otimes \bar\rho$ may also be viewed as the representation on $L(F, E/F)$ defined by $gA = \bar\rho(g)A\rho_F(g)^{-1}$.) Thus F is stable provided $\rho_F^* \otimes \bar\rho$ is VC. Since F has an invariant complement if and only if $\rho_F \otimes \bar\rho^*$ is VC, we have the following result.

THEOREM 5.3. *Let F be an invariant subspace of* $\rho \in \mathrm{Hom}(G, GL(E))$. *If the induced representation of G on $F^* \otimes (E/F)$ is VC, then every representation near ρ has an invariant subspace near F. If the induced representations on $F^* \otimes (E/F)$ and $F \otimes (E/F)^*$ are both VC, then there is a continuous choice of splittings of representations near ρ, with F being one of the summands for the splitting of ρ.*

Finally, we ask: under what conditions is a representation ρ rigid? That is, when is it true that, for every representation σ near ρ, there exists $A \in GL(E)$ with $\sigma(g) = A\rho(g)A^{-1}$ for all $g \in G$? This is equivalent to saying that A is stationary for the action on $GL(E)$ defined by $gB = \sigma(g)B\rho(g)^{-1}$. When $\sigma = \rho$, the identity is clearly stationary, and the isotropy representation at the identity is $\mathrm{Ad} \circ \rho\colon G \to GL(\mathfrak{gl}(E))$. Thus ρ is rigid if $\mathrm{Ad} \circ \rho$ is VC. The same argument holds for rigidity of homomorphisms from G into any Lie group. This well known result is proved by A. Weil **[12]**. Also, D. Coppersmith proves a stronger version in **[1]**.

THEOREM 5.4. *Let* $\rho \in \mathrm{Hom}(G, H)$, *where* H *is a Lie group. If* $\mathrm{Ad} \circ \rho\colon G \to GL(\mathfrak{h})$ *is VC, then* ρ *is rigid, and* $\mathrm{Hom}(G, H)$ *is a manifold near* ρ.

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