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ON THE GENERATORS OF THE FIRST HOMOLOGY WITH COMPACT SUPPORTS OF THE WEIERSTRASS FAMILY IN CHARACTERISTIC ZERO

BY

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ABSTRACT. Let $\mathbf{W}_{\mathbf{Q}} = \operatorname{Proj}(\mathbf{Q}[g_2, g_3, X, Y, Z]/(\text{homogeneous ideal generated by } -Y^2Z + 4X^3 - g_2XZ^2 - g_3Z^3))$. This is said to be the Weierstrass Family over the field $\mathbf{Q}$. Then the first homology with compact supports of the Weierstrass Family is computed explicitly, i.e., it is generated by $\{C^{-k}dX \wedge dY\}_{k \geq 1}$ and $\{XC^{-k}dX \wedge dY\}_{k \geq 1}$ over the ring $\mathbf{Q}[g_2, g_3]$, where C is a polynomial $Y^2 - 4X^3 + g_2X + g_3$. When one tensors the homology of the Weierstrass Family with $\Delta^{-1}\mathbf{Q}[g_2, g_3]$, being localized at the discriminant $\Delta = g_2^3 - 27g_3^2$, over $\mathbf{Q}[g_2, g_3]$, the first homology is generated by $C^{-1}dX \wedge dY$ and $XC^{-1}dX \wedge dY$. One also obtains the first homologies with compact supports of singular fibres over $\wp = (g_2 = g_3 = 0)$ and $\wp = (g_2 = 3, g_3 = 1)$ as corollaries.

Introduction. We wish to compute the $\mathbf{Q}[g_2, g_3]$-adic homology with compact supports of the Weierstrass Family $\mathbf{W}_{\mathbf{Q}}$, where

$$\mathbf{W}_{\mathbf{Q}} = \operatorname{Proj}\left(\frac{\mathbf{Q}[g_2, g_3, X, Y, Z]}{\text{homogeneous ideal generated by } -Y^2Z + 4X^3 - g_2XZ^2 - g_3Z^3}\right).$$

We regard the graded ring $\mathbf{Q}[g_2, g_3, X, Y, Z]$ as the graded $\mathbf{Q}[g_2, g_3]$-algebra such that X, Y and Z each has degree $+1$ and all the elements of $\mathbf{Q}[g_2, g_3]$ have degree zero. Let U be the open subset of $\mathbf{W}_{\mathbf{Q}}$, "the finite points": $U = \mathbf{W}_{\mathbf{Q}} \cap \mathbf{A}^2(\operatorname{Spec}(\mathbf{Q}[g_2, g_3]))$. This is the closed subscheme of $\mathbf{A}^2(\operatorname{Spec}(\mathbf{Q}[g_2, g_3]))$ given by $Y^2 = 4X^3 - g_2X - g_3$. Then we have the long exact sequence of the homology with compact supports, $\cdots \to H^c_{h-2}(\{\text{points at } \infty\}, \mathbf{Q}[g_2, g_3]) \to H^c_h(\mathbf{W}_{\mathbf{Q}}, \mathbf{Q}[g_2, g_3]) \to H^c_h(U, \mathbf{Q}[g_2, g_3]) \to \cdots$. Since $H^c_h(\{\text{points at } \infty\}, \mathbf{Q}[g_2, g_3])$ vanishes except at $h = 0$, we have

$$H^c_h(U, \mathbf{Q}[g_2, g_3]) = \begin{cases} H^c_h(\mathbf{W}_{\mathbf{Q}}, \mathbf{Q}[g_2, g_3]), & h \neq 2, \\ \mathbf{Q}[g_2, g_3], & h = 2. \end{cases}$$

Therefore the knowledge of $H^c_h(U, \mathbf{Q}[g_2, g_3])$, $h \geq 0$, determines the homology groups of all the fibres in the family over the various points $\wp \in \operatorname{Spec}(\mathbf{Q}[g_2, g_3])$, i.e.,

$$E^2_{p,q} = \operatorname{Tor}_p^{\mathbf{Q}[g_2, g_3]}\left(H^c_q(U, \mathbf{Q}[g_2, g_3], \mathbf{K}(\wp))\right)$$

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with the abutment $H_n^c(U_\wp, \mathbf{K}(\wp))$, where $\mathbf{K}(\wp)$ is the characteristic zero residue field at $\wp \in \operatorname{Spec}(\mathbf{Q}[g_2, g_3])$.

Let us consider the unequal characteristic case. Suppose that $\mathcal{O}$ is a complete discrete valuation ring with the quotient field K and residue class field k and suppose that $\underline{A}$ is an $\mathcal{O}$-algebra. Let X be a scheme over $A = (\underline{A} \otimes_{\mathcal{O}} k)_{\text{red}}$. Suppose that $\mathbf{K}(\wp)$ is a finite field at $\wp \in \operatorname{Spec}(A)$ and let $W(\mathbf{K}(\wp))$ be the complete discrete valuation ring and denote the quotient field of $W(\mathbf{K}(\wp))$ by $K_\wp = W(\mathbf{K}(\wp)) \otimes_{\mathbf{Z}} \mathbf{Q}$. Then the zeta function of the fibre $X_\wp$ at $\wp$ is given by

$$Z_{X_\wp}(T) = \frac{\prod_{p+q=\text{odd}} P_{p,q}(T)}{\prod_{p+q=\text{even}} P_{p,q}(T)} \tag{0.1}$$

where $P_{p,q}(T)$ is the reverse characteristic polynomial of the endomorphism of

$$E_{p,q}^2 = \operatorname{Tor}_p^{\underline{A}\dagger \otimes_{\mathbf{Z}} \mathbf{Q}}\left(H_q^c(X, \underline{A}\dagger \otimes_{\mathbf{Z}} \mathbf{Q}), K_\wp\right) \tag{0.2}$$

induced by the p^rth power map, $p^r = \operatorname{card}(\mathbf{K}(\wp))$ (see pp. 448–450, [**6**]). This homological spectral sequence abuts upon $H_n^c(X_\wp, K_\wp)$. Therefore if one knows the lifted p-adic homology with compact supports of X over A, $H_h^c(X, \underline{A}\dagger \otimes_{\mathbf{Z}} \mathbf{Q})$, $h \geqslant 0$, and the zeta endomorphisms of these groups, (1) determines the zeta function of every fibre over a finite field in the algebraic family X over the ring A. These are the subjects in the forthcoming paper [**2**].

The main result of the paper is the explicitness of the generation of the first homology with compact supports of the entire Weierstrass Family $\mathbf{W_Q}$ in the characteristic zero (Theorem 1) and its consequences.

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1. In this section (notations being the same as in the Introduction) we describe explicitly the basis elements over the ring $\mathbf{Q}[g_2, g_3]$ which generate the first homology with compact supports of the Weierstrass Family over the field of rational numbers $\mathbf{Q}$, $H_1^c(U, \mathbf{Q}[g_2, g_3])$. By the definition of the lifted p-adic homology with compact supports [**6**, p. 415], applied to the characteristic zero case, we have

$$\begin{aligned} &H_1^c(U, \mathbf{Q}[g_2, g_3]) \\ &= H^3\Big(\mathbf{A}^2(\operatorname{Spec}(\mathbf{Q}[g_2, g_3])), \mathbf{A}^2(\operatorname{Spec}(\mathbf{Q}[g_2, g_3])) - U, \Gamma^*_{\mathbf{Q}[g_2, g_3]}(\operatorname{Spec}(\mathbf{Q}[g_2, g_3]))\Big). \end{aligned}$$

If one tensors $H_1^c(U, \mathbf{Q}[g_2, g_3])$ with $\Delta^{-1}\mathbf{Q}[g_2, g_3]$ over $\mathbf{Q}[g_2, g_3]$, one has the free $\Delta^{-1}\mathbf{Q}[g_2, g_3]$-module of rank two, where $\Delta = g_2^3 - 27g_3^2$. This is so because we have the universal coefficients spectral sequence

$$E_{0,1}^2 = H_1^c(U, \mathbf{Q}[g_2, g_3]) \otimes_{\mathbf{Q}[g_2, g_3]} \Delta^{-1}\mathbf{Q}[g_2, g_3] \xrightarrow{\approx} H_1^c(U, \Delta^{-1}\mathbf{Q}[g_2, g_3]),$$

and $\Delta^{-1}\mathbf{Q}[g_2, g_3]$ means that the ring $\mathbf{Q}[g_2, g_3]$ is localized at the discriminant Δ. The computation has been made even in the p-adic case in [**1**] for this open subfamily of the Weierstrass Family.

THEOREM 1. *Consider* $U = \mathbf{W}_{\mathbf{Q}} \cap \mathbf{A}^2(\mathrm{Spec}(\mathbf{Q}[g_2, g_3]))$, *which is the closed affine subscheme of* $\mathbf{A}^2(\mathrm{Spec}(\mathbf{Q}[g_2, g_3]))$. *Then the first homology with compact supports* $H_1^c(U, \mathbf{Q}[g_2, g_3])$ *is generated by* $\{C^{-l} dX \wedge dy\}_{l \geqslant 1}$ *and* $\{XC^{-l} dX \wedge dY\}_{l \geqslant 1}$ *as a* $\mathbf{Q}[g_2, g_3]$*-module.*

REMARK 1. For the pair of affine schemes

$$\mathbf{A}^2(\mathrm{Spec}(\mathbf{Q}[g_2, g_3])) \quad \text{and} \quad \mathbf{A}^2(\mathrm{Spec}(\mathbf{Q}[g_2, g_3])) - U,$$

where U is the closed subscheme corresponding to the polynomial $C = Y^2 - 4X^3 + g_2 X + g_3$ in $\mathbf{Q}[g_2, g_3, X, Y, Z]$, there is induced a long exact sequence of hypercohomology groups,

$$\cdots \xrightarrow{\partial^{n-1}} H^n(\mathbf{A}^2(A), \mathbf{A}^2(A) - U, \Gamma_A^*(\mathbf{A}^2(A))) \to H^n(\mathbf{A}^2(A), \Gamma_A^*(\mathbf{A}^2(A)))$$
$$\to H^n(\mathbf{A}^2(A) - U, \Gamma_A^*(\mathbf{A}^2(A))) \xrightarrow{\partial^n} \cdots$$

where $A = \mathrm{Spec}(\mathbf{Q}[g_2, g_3])$.

There are three first-quadrant spectral sequences induced which have the above three hypercohomology groups as their abutments:

$$\begin{cases} 'E_1^{p,q} = H^q(\mathbf{A}^2(A) - U, \Gamma_A^p(\mathbf{A}^2(A))), \\ E_1^{p,q} = H^q(\mathbf{A}^2(A), \Gamma_A^p(\mathbf{A}^2(A))), \\ ''E_1^{p,q} = H^q(\mathbf{A}^2(A), \mathbf{A}^2(A) - U, \Gamma_A^p(\mathbf{A}^2(A))). \end{cases}$$

LEMMA 1. *We have the following isomorphisms: the abutment*

$$''E^3 = H^3(\mathbf{A}^2(A), \mathbf{A}^2(A) - U, \Gamma_A^*(\mathbf{A}^2(A))) \overset{\approx}{\leftarrow} ''E_2^{2,1},$$

and

$$''E^3 \overset{\approx}{\leftarrow} 'E^2 = H^2(\mathbf{A}^2(A) - U, \Gamma_A^*(\mathbf{A}^2(A))) \overset{\approx}{\leftarrow} \mathrm{coker}('E_1^{2,0} \leftarrow 'E_1^{1,0}).$$

PROOF OF LEMMA 1. Consider the following diagram (Diagram A) with exact rows. We denote the structure sheaf of the affine scheme $\mathbf{A}^2(A) = \mathbf{A}^2(\mathrm{Spec}(\mathbf{Q}[g_2, g_3]))$ by $\mathcal{O}_{\mathbf{A}^2(A)}$. Therefore, we have $''E_1^{p,q} = 0$ unless $q = 1$, which is abutting $''E^3 = H^3(\mathbf{A}^2(A), \mathbf{A}^2(A) - U, \Gamma_A^*(\mathbf{A}^2(A)))$. Then the isomorphism $''E_2^{2,1} \to ''E^3$ in Lemma 1 follows. Furthermore, this diagram can be rewritten as Diagram B. The remaining two isomorphisms in Lemma 1 are obtained from the well-known lemma in homological algebra, i.e., from Diagram B with the exact rows we have the induced exact sequence

$$\begin{tikzcd}[column sep=small] 0 \arrow[r] & \ker d_1^{1,0} \arrow[r] & \ker 'd_1^{1,0} \arrow[r] & \ker ''d_1^{1,0} \arrow[r] & \mathrm{coker}\, d_1^{1,0} \arrow[r] \arrow[d, "\wr"] & \mathrm{coker}\, 'd_1^{1,0} \arrow[r] \arrow[d, "\wr"] & \mathrm{coker}\, ''d_1^{1,1} \arrow[r] \arrow[d, "\wr"] & 0 \\ & & & & E^2 \arrow[r] & 'E^2 \arrow[r] & ''E^3 \arrow[r] & 0 \end{tikzcd}.$$

$$\begin{tikzcd}[column sep=small]
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & H^0(\mathbf{A}^2(A), \mathbf{A}^2(A) - U, \mathcal{O}_{\mathbf{A}^2(A)}) \arrow[r] \arrow[d, "{''d_1^{0,0}}"] & H^0(\mathbf{A}^2(A), \mathcal{O}_{\mathbf{A}^2(A)}) \arrow[r] \arrow[d, "{d_1^{0,0}}"] & H^0(\mathbf{A}^2(A) - U, \mathcal{O}_{\mathbf{A}^2(A)}) \arrow[r] \arrow[d, "{'d_1^{0,0}}"] & H^1(\mathbf{A}^2(A), \mathbf{A}^2(A) - U, \mathcal{O}_{\mathbf{A}^2(A)}) \arrow[r] \arrow[d, "{''d_1^{0,1}}"] & 0 \\
0 \arrow[r] & H^0(\mathbf{A}^2(A), \mathbf{A}^2(A) - U, \Gamma_A^1(\mathbf{A}^2(A))) \arrow[r] \arrow[d, "{''d_1^{1,0}}"] & H^0(\mathbf{A}^2(A), \Gamma_A^1(\mathbf{A}^2(A))) \arrow[r] \arrow[d, "{d_1^{1,0}}"] & H^0(\mathbf{A}^2(A) - U, \Gamma_A^1(\mathbf{A}^2(A))) \arrow[r] \arrow[d, "{'d_1^{1,0}}"] & H^1(\mathbf{A}^2(A), \mathbf{A}^2(A) - U, \Gamma_A^1(\mathbf{A}^2(A))) \arrow[r] \arrow[d, "{''d_1^{1,1}}"] & 0 \\
0 \arrow[r] & H^0(\mathbf{A}^2(A), \mathbf{A}^2(A) - U, \Gamma_A^2(\mathbf{A}^2(A))) \arrow[r] \arrow[d] & H^0(\mathbf{A}^2(A), \Gamma_A^2(\mathbf{A}^2(A))) \arrow[r] \arrow[d] & H^0(\mathbf{A}^2(A) - U, \Gamma_A^2(\mathbf{A}^2(A))) \arrow[r] \arrow[d] & H^1(\mathbf{A}^2(A), \mathbf{A}^2(A) - U, \Gamma_A^2(\mathbf{A}^2(A))) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 & 0 &
\end{tikzcd}$$

DIAGRAM A

$$\begin{tikzcd}[column sep=small]
 & 0 \arrow[r] & \Gamma^1_{\mathbf{Q}[g_2,g_3]}(\mathbf{Q}[g_2,g_3,X,Y]) \arrow[r] \arrow[d, "{d_1^{1,0}}"] & \Gamma^1_{\mathbf{Q}[g_2,g_3]}(\mathbf{Q}[g_2,g_3,X,Y,C^{-1}]) \arrow[r] \arrow[d, "{'d_1^{1,0}}"] & H^1(\mathbf{A}^2(A), \mathbf{A}^2(A) - U, \Gamma_A^1(\mathbf{A}^2(A))) \arrow[r] \arrow[d, "{''d_1^{1,1}}"] & 0 \\
 & 0 \arrow[r] & \Gamma^2_{\mathbf{Q}[g_2,g_3]}(\mathbf{Q}[g_2,g_3,X,Y]) \arrow[r] \arrow[d, "\mathrm{epi}"] & \Gamma^2_{\mathbf{Q}[g_2,g_3]}(\mathbf{Q}[g_2,g_3,X,Y,C^{-1}]) \arrow[r] \arrow[d, "\mathrm{epi}"] & H^1(\mathbf{A}^2(A), \mathbf{A}^2(A) - U, \Gamma_A^2(\mathbf{A}^2(A))) \arrow[r] \arrow[d, "\mathrm{epi}"] & 0 \\
 & {} \arrow[r] & E^2 \arrow[r] & 'E^2 \arrow[r] & ''E^3 \arrow[r] & 0
\end{tikzcd}$$

DIAGRAM B

and since the $\mathbf{Q}[g_2, g_3]$-homomorphism

$$d_1^{1,0}\colon E_1^{1,0} = \Gamma^1_{\mathbf{Q}[g_2,g_3]}(\mathbf{Q}[g_2, g_3, X, Y]) \to E_1^{2,0} = \Gamma^2_{\mathbf{Q}[g_2,g_3]}(\mathbf{Q}[g_2, g_3, X, Y])$$

is an epimorphism, we have $E^2 \approx E_2^{2,0} \approx 0$. Therefore

$$'E^2 \overset{\approx}{\leftarrow} \operatorname{coker} 'd_1^{1,0} \overset{\approx}{\rightarrow} \operatorname{coker} ''d_1^{1,1} \approx ''E^3$$

as stated in Lemma 1. Q.E.D.

Hence our computation of the abutment $''E^3 = H^3(\mathbf{A}^2(A), \mathbf{A}^2(A) - U, \Gamma_A^*(\mathbf{A}^2(A)))$ is reduced to compute

$$\operatorname{coker}\left(\Gamma^1_{\mathbf{Q}[g_2,g_3]}\left(\mathbf{Q}[g_2, g_3, X, Y, C^{-1}]\right) \xrightarrow{'d_1^{1,0}} \Gamma^2_{\mathbf{Q}[g_2,g_3]}\left(\mathbf{Q}[g_2, g_3, X, Y, C^{-1}]\right)\right).$$

PROOF OF THEOREM 1. From now on we denote, "d", instead of the exterior differential, "$'d_1^{1,0}$" in the spectral sequence. We have that

(1) $$d(C^{-k}X^iY^j\,dX) = \left(-2kC^{-k-1}X^iY^{j+1} + jC^{-k}X^iY^{j-1}\right) dY \wedge dX,$$

(2) $$d(C^{-k}X^iY^j\,dY) = \left(12kC^{-k-1}X^{i+2}Y^j - g_2kC^{-k-1}X^iY^j + iC^{-k}X^{i-1}Y^j\right) dX \wedge dY,$$

in the $\mathbf{Q}[g_2, g_3]$-module $\Gamma^2_{\mathbf{Q}[g_2,g_3]}(\mathbf{Q}[g_2, g_3, X, Y, C^{-1}])$, where $C = Y^2 - 4X^3 + g_2X + g_3$, i, j and k are nonnegative integers. The equations (1) and (2) give the cohomologous relations, which are denoted by "$\sim$", as

(3) $$2kC^{-k-1}X^iY^{j+1}dX \wedge dY \sim jC^{-k}X^iY^j\,dX \wedge dY$$

and

(4) $$\left(12kC^{-k-1}X^{i+2}Y^j - g_2kC^{-k-1}X^iY^j + iC^{-k}X^{i-1}Y^j\right) dX \wedge dY \sim 0.$$

Notice that, by Lemma 1:

$$''E_2^{2,1} \overset{\approx}{\leftarrow} ''E_1^{2,1}/\operatorname{Im}\left(''E_1^{1,1} \to ''E_1^{2,1}\right)$$

and

$$''E_1^{1,1} \overset{\approx}{\leftarrow} 'E_1^{1,0}/\operatorname{Im}\left('E_1^{1,0} \leftarrow E_1^{1,0}\right),$$

where $E_1^{1,0} \approx \Gamma^1_{\mathbf{Q}[g_2,g_3]}(\mathbf{Q}[g_2, g_3, X, Y])$. Therefore it suffices to consider the integer $k \geqslant 1$ in the equations (1), (2), (3) and (4) above.

If $j = 0$ in (3), then $C^{-k-1}X^iYdX \wedge dY \sim 0$ for all $i \geqslant 0$ and $k \geqslant 1$. But (4) implies that $C^{-1}X^iYdX \wedge dY \sim 0$ for $i \geqslant 0$ since $iC^{-1}X^{i-1}YdX \wedge dY \sim g_2C^{-2}X^iYdX \wedge dY - 12C^{-2}X^{i+2}YdX \wedge dY$. Therefore,

(5) $$C^{-k}X^iYdX \wedge dY \sim 0 \quad \text{for all integers } i, k \geqslant 0.$$

For any odd integer $j > 1$ we have $C^{-k}X^iY^jdX \wedge dY \sim 0$ by combining (3) and (5) and the repeated use of (4). For example, for $j = 3$, we have $12kC^{-k-1}X^iY^3dX \wedge dY \sim 2C^{-k}X^iYdX \wedge dY$, which is cohomologous to zero by (5). Then apply (4) for $j = 3$ to get

$$iC^{-k}X^{i-1}Y^3dX \wedge dY \sim g_3kC^{-k-1}X^iY^3dX \wedge dY - 12kC^{-k-1}X^{i+3}Y^3dX \wedge dY.$$

But the right-hand side is cohomologous to zero from the above result. If $i = 0$ in (4), we then have

(6) $$12kC^{-k-1}X^2Y^jdX \wedge dY \sim g_2kC^{-k-1}Y^jdX \wedge dY$$

for all integers $k \geqslant 1$ and $j \geqslant 0$. Especially we have, for $j = 0$, $12kC^{-k-1}X^2dX \wedge dY \sim g_2kC^{-k-1}dX \wedge dY$. Then it can be plainly seen that

$$\left(C^{-k}dX \wedge dY\right)_{k\geqslant 1}, \qquad \left(XC^{-k}dX \wedge dY\right)_{k\geqslant 1} \quad \text{and} \quad \left(X^iC^{-1}dX \wedge dY\right)_{i\geqslant 2}$$

generate all the elements of the type $X^iC^{-k}dX \wedge dY$ for integers $i \geqslant 0$ and $k \geqslant 0$ over the ring $\mathbf{Q}[g_2, g_3]$ from equations (3) and (4). In particular, $X^2C^{-1}dX \wedge dY \sim X^2Y^2C^{-2}dX \wedge dY$ by (3) for letting $i = 2, j = 1$ and $k = 1$, but $X^2Y^2C^{-2}dX \wedge dY \sim Y^2C^{-2}dX \wedge dY$ by (4) for $i = 0, j = 2$ and $k = 1$; furthermore, $Y^2C^{-2}dX \wedge dY$ is cohomologous to $C^{-1}dX \wedge dY$ from (3) for $i = 0, j = 1$ and $k = 1$. Hence we have established that $X^2C^{-1}dX \wedge dY \sim C^{-1}dX \wedge dY$. Next we claim that all the elements of the type $(X^iC^{-1}dX \wedge dY)_{i \geqslant 3}$ are generated by the two elements $C^{-1}dX \wedge dY$ and $XC^{-1}dX \wedge dY$ over the ring $\mathbf{Q}[g_2, g_3]$. We have the following recursive formula for integers $i \geqslant 3$ from (3) and (4):

$$4X^iC^{-1}dX \wedge dY \sim g_2\left(\frac{1}{12(i-2)} + 1\right)X^{i-2}C^{-1}dX \wedge dY + \left(g_3 - \frac{1}{i-2}\right)X^{i-3}C^{-1}dX \wedge dY.$$

Therefore it follows from this recursive formula that $(X^iC^{-1}dX \wedge dY)_{i \geqslant 3}$ are generated by $C^{-1}dX \wedge dY$ and $XC^{-1}dX \wedge dY$ over $\mathbf{Q}[g_2, g_3]$. We have established the statement of Theorem 1 for the elements $X^iY^jC^{-k}dX \wedge dY$ with $i \geqslant 1, j = 0$ and $k \geqslant 1$. Now we need consider the elements $X^iY^jC^{-k}dX \wedge dY$ for $j = 1, 2, 3, \ldots$. As noted before, we know that if j is an odd integer, $X^iY^jC^{-k}dX \wedge dY \sim 0$. If j is an even integer, the repeated use of (3) and (4) for the elements $X^iY^jC^{-1}dX \wedge dY$, $i \geqslant 1$ and $j \geqslant 1$, provides the generation of the first homology with compact supports $H_1^c(U, \mathbf{Q}[g_2, g_3])$ of the Weierstrass Family by the elements $(C^{-k}dX \wedge dY)_{k \geqslant 1}$ and $(XC^{-k}dX \wedge dY)_{k \geqslant 1}$. Q.E.D.

PROPOSITION 1. *Assumptions and notations being the same as in Theorem 1, $H_1^c(U, \mathbf{Q}[g_2, g_3]) \otimes_{\mathbf{Q}[g_2, g_3]} (\Delta^{-1}\mathbf{Q}[g_2, g_3])$ is a free $(\Delta^{-1}\mathbf{Q}[g_2, g_3])$-module of rank two, i.e., it is generated by $XC^{-1}dX \wedge dY$ and $C^{-1}dX \wedge dY$, where Δ is the discriminant, $\Delta = g_2^3 - 27g_3^2$, and $\Delta^{-1}\mathbf{Q}[g_2, g_3]$ is localized at the discriminant Δ.*

PROOF OF PROPOSITION 1. For any integer $i \geqslant 2$ we have

$$C^{-(i-1)} = C^{-i}\left(Y^2 - 4X^3 + g_2X + g_3\right),$$

where $dX \wedge dY$ is omitted for simplicity, and from equations (3), (4) and (6) we have the following cohomologous relation for $i \geqslant 2$:

$$\frac{6i-11}{6(i-1)}C^{-(i-1)} \sim \frac{2g_2}{3}XC^{-i} + g_3C^{-i}. \tag{1.1}$$

Similarly, one has the corresponding formula for $XC^{-(i-1)}$ by the equations (3), (4) and (6):

$$\frac{6i-13}{6(i-1)}XC^{-(i-1)} \sim \frac{g_2^2}{18}C^{-i} + g_3XC^{-i}. \tag{1.2}$$

We finally have for $i \geqslant 2$,
(1.3)

$$C^{-i}dX \wedge dY \sim \frac{18}{\Delta}\left\{\frac{g_2(6i-13)}{6(i-1)}XC^{-(i-1)}dX \wedge dY - \frac{g_3(6i-11)}{4(i-1)}C^{-(i-1)}dX \wedge dY\right\}$$

from equations (1.1) and (1.2).

Equations (1.3) and (1.1) prove that $H_1^c(U, \mathbf{Q}[g_2, g_3]) \otimes_{\mathbf{Q}[g_2,g_3]}(\Delta^{-1}\mathbf{Q}[g_2, g_3])$ is generated by $XC^{-1}dX \wedge dY$ and $C^{-1}dX \wedge dY$ as a $(\Delta^{-1}\mathbf{Q}[g_2, g_3])$-module. Q.E.D.

COROLLARY 1. *Let $\mathbf{V}_{\mathbf{Q}}^0$ be the closed subfamily defined by "$g_2 = 0$" of the whole Weierstrasss Family $W_{\mathbf{Q}}$. Then the first homology with compact supports,*

$$H_1^c(\mathbf{V}_{\mathbf{Q}}^0 \cap \mathbf{A}^2(\operatorname{Spec}\mathbf{Q}[g_3]), \mathbf{Q}[g_3]),$$

is generated by $\{C^{-k}dX \wedge dY\}_{k\geqslant 1}$ and $\{XC^{-k}dX \wedge dY\}_{k \geqslant 1}$ as a $\mathbf{Q}[g_3]$-module.

PROOF. In (1.1) and (1.2) in the proof of Proposition 1, we have the following corresponding equations for the closed subfamily $\mathbf{V}_{\mathbf{Q}}^0$ defined by "$g_2 = 0$":

$$(1.1)^0 \qquad \frac{12i-22}{12(i-1)}C^{-(i-1)} \sim g_3C^{-i},$$

$$(1.2)^0 \qquad \frac{6i-13}{6(i-1)}XC^{-(i-1)} \sim g_3XC^{-i}.$$

Then the statement of Corollary 1 follows plainly from $(1.1)^0$ and $(1.2)^0$. Q.E.D.

Note 1. The equations $(1.1)^0$ and $(1.2)^0$ also show that Corollaries 2 and 3 are true.

COROLLARY 2. *The first homology with compact supports of the singular fibre $U_\wp$ over a point $\wp = (g_2 = 0, g_3 = 0) \in \operatorname{Spec}(\mathbf{Q}[g_2, g_3])$, a projective line with a cusp (or $\wp = (g_3 = 0) \in \operatorname{Spec}(\mathbf{Q}[g_3])$), $H_1^c(U_\wp, \mathbf{Q})$, is trivial.*

COROLLARY 3. *Notations being the same as in Proposition* 1,

$$H_1^c(\mathbf{V}_{\mathbf{Q}}^0 \cap \mathbf{A}^2(\operatorname{Spec}(\mathbf{Q}[g_3])), \mathbf{Q}[g_3]) \otimes_{\mathbf{Q}[g_3]}(g_3^{-1}\mathbf{Q}[g_3])$$

is generated by the two elements $C^{-1}dX \wedge dY$ and $XC^{-1}dX \wedge dY$, where $g_3^{-1}\mathbf{Q}[g_3]$ means the localization of the ring $\mathbf{Q}[g_3]$ at g_3.

REMARK 2. For a point $\wp \neq (g_3 = 0)$, $H_1^c(U_\wp, \mathbf{K}(\wp))$ is generated by $C^{-1}dX \wedge dY$ and $XC^{-1}dX \wedge dY$ as a $\mathbf{K}(\wp)$-vector space and where $\mathbf{K}(\wp)$ is the characteristic zero residue field, i.e., $U_\wp$ is an elliptic curve. Note that the open subfamily of the Weierstrass Family over $\mathbf{Z}/P\mathbf{Z}$ defined by "$\Delta \neq 0$" has been computed explicitly using the hypercohomology of a flat lifting with coefficients in the † of sheaves of differential forms, $H^1(U, (\Delta^{-1}\hat{\mathbf{Z}}_p[g_2, g_3])^\dagger \otimes_{\mathbf{Z}} \mathbf{Q})$, where $(\Delta^{-1}\hat{\mathbf{Z}}_p[g_2, g_3])^\dagger$ is the † of the localization of the ring $\hat{\mathbf{Z}}_p[g_2, g_3]$ at the discriminant $\Delta = g_2^3 - 27g_3^2$, see [**1**]. The following universal coefficient spectral sequence explains the relationship between Corollary 2 and Theorem 1.

$E_{p,q}^2 = \operatorname{Tor}_p^{\mathbf{Q}[g_2,g_3]}(H_q^c(U, \mathbf{Q}[g_2, g_3]), \mathbf{K}(\wp))$ with the abutment $H_n^c(U_\wp, \mathbf{K}(\wp))$, where $\wp = (g_2 = g_3 = 0) \in \operatorname{Spec}(\mathbf{Q}[g_2, g_3])$ and $\mathbf{Q} = \mathbf{K}(\wp)$.

COROLLARY 4. *Let* $\mathbf{V}_{\mathbf{Q}}^3$ *be the closed subfamily of the Weierstrass Family* $\mathbf{W}_{\mathbf{Q}}$, *defined by* "$g_2 = 3$". *Then* $H_1^c(\mathbf{V}_{\mathbf{Q}}^3 \cap \mathbf{A}^2(\operatorname{Spec}\mathbf{Q}[g_3]), \mathbf{Q}[g_3])$ *is generated by* $\{C^{-k}dX \wedge dY\}_{k\geqslant 1}$ *and* $\{XC^{-k}dX \wedge dY\}_{k\geqslant 1}$ *as a* $\mathbf{Q}[g_3]$-*module. Morover the first homology with compact supports of the singular fibre over the point* $\wp = (g_3 = 1)$ *in the base* $\operatorname{Spec}(\mathbf{Q}[g_3])$, *a projective line with an ordinary double point over* $\mathbf{K}(\wp)$, *is generated by one element as a* $\mathbf{K}(\wp)$-*vector space. One can then take either* $C^{-1}dX \wedge dY$ *or* $XC^{-1}dX \wedge dY$ *as the base element for the vector space.*

PROOF. We only need prove the latter statement. From equations (1.1) and (1.2), we have $(1.1)_1^3$ and $(1.2)_1^3$ as follows:

$$(1.1)_1^3 \qquad \frac{6i-11}{6(i-1)}C^{-(i-1)} \sim 2XC^{-i} + C^{-i},$$

$$(1.2)_1^3 \qquad \frac{6i-13}{6(i-1)}XC^{-(i-1)} \sim \frac{1}{2}C^{-i} + XC^{-i}.$$

Then we have $2(6i-13)XC^{-(i-1)} \sim (6i-11)C^{-(i-1)}$ for $i \geqslant 2$. Hence this vector space is one dimensional and the statement of Corollary 4 follows. Q.E.D.

Note 2. For the closed subfamily $\mathbf{V}_{\mathbf{Q}}^3$ of the Weierstrass Family we have the following equations $(1.1)^3$, $(1.2)^3$ and $(1.3)^3$:

$$(1.1)^3 \qquad \frac{6i-11}{6(i-1)}C^{-(i-1)} \sim 2XC^{-i} + g_3C^{-i},$$

$$(1.2)^3 \qquad \frac{6(i-13)}{6(i-1)}XC^{-(i-1)} \sim \frac{1}{2}C^{-i} + g_3XC^{-i},$$

$$(1.3)^3 \qquad (g_3^2-1)C^{-i} \sim \frac{1}{6(i-1)}\{g_3(6i-11)C^{-(i-1)} - 2(6i-13)XC^{-(i-1)}\},$$

for integers $i \geqslant 2$.

Note 3. This paper has been entirely in characteristic zero. The case of nonzero characteristic $p \neq 2, 3$ will appear in a forthcoming paper **[2]**, which is a generalization of the paper **[1]**, where an open subfamily "$\Delta \neq 0$" of the Weierstrass Family was studied.

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