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SIGN-EMBEDDINGS OF l_1^n

BY

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ABSTRACT. If $(e_i)_{i=1}^n$ are vectors in a real Banach space with $\|e_i\| \leq 1$ and $\text{Average}_{\varepsilon_i = \pm 1} \|\sum_{i=1}^n \varepsilon_i e_i\| \geq \delta n$, where $\delta > 0$, then there is a subset $A \subseteq \{1, \ldots, n\}$ of cardinality $m \geq cn$ such that $(e_i)_{i \in A}$ is K-equivalent to the standard l_1^m basis, where $c > 0$ and $K < \infty$ depend only on δ. As a corollary, if $1 < p < \infty$ and l_1^n is K-isomorphic to a subspace of $L_p(X)$, then l_1^m $(m \geq cn)$ is K'-isomorphic to a subspace of X, where $c > 0$ and $K' < \infty$ depend only on K and p.

We prove the following two theorems.

THEOREM 1. *Let $0 < \delta < 1$. There exist $\beta > 0$ and $c > 0$, depending only on δ, such that if $(e_i)_{i=1}^n$ are vectors in a real Banach space X with $\|e_i\| \leq 1$ for all i and*

$$\underset{\varepsilon_i = \pm 1}{\text{Average}} \left\| \sum_{i=1}^n \varepsilon_i e_i \right\| \geq \delta n,$$

then there is $A \subseteq \{1, \ldots, n\}$, $|A| = m \geq cn$, such that $(e_i)_{i \in A} \overset{\beta^{-1}}{\sim}$ unit vector basis of l_1^m. As $\delta \uparrow 1$, we may choose $\beta \uparrow 1$ and $c \to \frac{1}{2}$, or we may choose $c \uparrow 1$ with $\beta \downarrow 0$.

The average in the statement of the theorem is over all sequences of signs $\varepsilon = (\varepsilon_i)_{i=1}^n$ in $\{-1, 1\}^n$. The statement "$(e_i)_{i \in A} \overset{\beta^{-1}}{\sim}$ unit vector basis of l_1^m" means that

$$\left\| \sum_{i \in A} a_i e_i \right\| \geq \beta \sum_{i \in A} |a_i| \quad \text{for all scalars } (a_i)_{i \in A}.$$

We denote the cardinality of a set S by $|S|$. Our notation agrees with that in the books of Lindenstrauss and Tzafriri [**5, 6**].

This problem was suggested to us by Haskell Rosenthal, in the case that $\|\sum_{i=1}^n \varepsilon_i e_i\| \geq \delta n$ for all choices of signs ε. We may interpret this statement to say that l_1^n sign-embeds into X. Sign-embeddings of L^1 are discussed in forthcoming papers of Rosenthal [**11**] and Bourgain and Rosenthal [**1**], where it is proved that a sign-embedding of L^1 fixes a copy of l_1.

The proof of Theorem 1 shows that we may take $c = 2^{-19}\delta^2/[\log(4/\delta)]^2$ and $\beta = 2^{-13}\delta^3$. We may state the result slightly differently as follows: Let

$$M = \underset{\varepsilon_i = \pm 1}{\text{Average}} \left\| \sum_{i=1}^n \varepsilon_i e_i \right\|,$$

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so we may take $\delta = M/n$. Then

$$|A| \geqslant 2^{-19} M^2/n[\log(4n/M)]^2.$$

This shows a connection with results of G. Pisier [9] and V. Milman [8], which are for very special cases. Pisier shows that if the e_i are characters for a compact abelian group, then we may take $|A| \geqslant kM^2/n$ for some absolute constant $k > 0$ and with the equivalence constant an absolute constant. Milman shows that if the e_i are real-valued functions on an arbitrary set T such that $|e_i(t)| = 1$ for all t, then we may take $|A| \geqslant kM^2/n \log n$, and the equivalence constant is 1; that is, we have an isometry with $l_1^{|A|}$. Our second theorem shows that actually we may obtain $|A| \geqslant kM^2/n \log(2n/M)$ in Milman's case. This gives an isometry for a percentage subset if M is a percentage of n.

THEOREM 2. *Let $(e_i)_{i=1}^n$ be real-valued functions on a set T such that $|e_i(t)| = 1$ for all $t \in T$ and all i. Let $M = \text{Average}_{\varepsilon_i = \pm 1} \|\Sigma_{i=1}^n \varepsilon_i e_i\|$, where $\|f\| = \sup_{t \in T} |f(t)|$ for a bounded function $f\colon T \to R$. Then there exists $A \subseteq \{1, \ldots, n\}$ with*

$$|A| \geqslant kM^2/n \log(2n/M),$$

where $k > 0$ is an absolute constant, such that $(e_i)_{i \in A} \overset{1}{\sim}$ unit vector basis of $l_1^{|A|}$.

Related results and questions. W. B. Johnson observed that as a consequence of Theorem 1 it follows that if l_1^n is K-isomorphic to a subspace of $L_2(X)$, then l_1^m $(m \geqslant cn)$ is K'-isomorphic to a subspace of X, where $c > 0$ and $K' < \infty$ depend only on K. S. Szarek has proved (using Theorem 1) that this is true with $L_p(X)$ in place of $L_2(X)$, for any $p > 1$.

COROLLARY. *If $1 < p < \infty$ and l_1^n is K-isomorphic to a subspace of $L_p(X)$, then l_1^m $(m \geqslant cn)$ is K'-isomorphic to a subspace of X, where $c > 0$ and $K' < \infty$ depend only on K and p.*

S. Szarek has shown that a result analogous to Theorem 1 holds for sign-embeddings of l_∞^n for all signs, but not for average sign-embeddings.

If the hypothesis in Theorem 1 is strengthened to $\|\Sigma_{i=1}^n \varepsilon_i e_i\| \geqslant \delta n$ for all choices of signs, we do not know if it is possible to have both $c \uparrow 1$ and $\beta \uparrow 1$ as $\delta \uparrow 1$.

We also do not know if the theorems hold for complex Banach spaces.

Our first step in the proof of the Theorem 1 is to observe that if $0 < \delta' < \delta$, there is a set of sequences of signs $\mathfrak{S} \subseteq \{-1, 1\}^n$ such that

$$\left\| \sum_{i=1}^n \varepsilon_i e_i \right\| \geqslant \delta' n \quad \text{for all } \varepsilon \in \mathfrak{S}$$

and

$$|\mathfrak{S}| \geqslant 2^n(\delta - \delta')/(1 - \delta').$$

For if $\|\Sigma_{i=1}^n \varepsilon_i e_i\| < \delta' n$ for $2^n[1 - (\delta - \delta')/(1 - \delta)]$ sequences of signs, we would have

$$\underset{\varepsilon_i = \pm 1}{\text{Average}} \left\| \sum_{i=1}^n \varepsilon_i e_i \right\| < \left[1 - \frac{\delta - \delta'}{1 - \delta'}\right] \delta' n + n \frac{\delta - \delta'}{1 - \delta'} = \delta n,$$

a contradiction.

For each $\varepsilon \in \mathfrak{S}$, there exists f^ε in the unit ball of X^* such that $f^\varepsilon(\sum_{i=1}^n \varepsilon_i e_i) \geq \delta' n$. Let $f_i^\varepsilon = f^\varepsilon(e_i)$. Thus $(f_i^\varepsilon)_{i=1}^n$ is in the unit ball of l_∞^n.

We need a probabilistic lemma and a combinatorial lemma. First we recall the following consequence of Kolmogorov's exponential bounds theorem.

LEMMA 1 (SEE LOÈVE [7]). *Let $-1 \leq c_i \leq 1$, $i = 1,\dots,n$, and let $(\varepsilon_i)_{i=1}^n$ be independent random variables with $P[\varepsilon_i = 1] = P[\varepsilon_i = -1] = \frac{1}{2}$. If $\alpha > 0$,*

$$P\left[\sum_{i=1}^n \varepsilon_i c_i \geq \alpha n\right] \leq \exp\left(\frac{-n\alpha^2}{4}\right).$$

A generalization of Kolmogorov's result to uniformly bounded martingale differences has been used by Schechtman [13] and Johnson and Schechtman [3] in other embedding problems.

LEMMA 2. *If B is the closed ball in l_∞^n of radius $r < \delta'$ centered at $(c_i)_{i=1}^n$, where $-1 \leq c_i \leq 1$, then*

$$|\{\varepsilon \in \mathfrak{S} : (f_i^\varepsilon) \in B\}| \leq 2^n \exp\big(-n(\delta' - r)^2/4\big).$$

PROOF. If $\varepsilon \in \mathfrak{S}$ and $(f_i^\varepsilon) \in B$, then

$$\left|\sum_{i=1}^n \varepsilon_i f_i^\varepsilon - \sum_{i=1}^n \varepsilon_i c_i\right| \leq nr,$$

so

$$\sum_{i=1}^n \varepsilon_i c_i \geq \sum_{i=1}^n \varepsilon_i f_i^\varepsilon - nr \geq (\delta' - r)n.$$

By Lemma 1, this can happen for at most the stated number of ε.

LEMMA 3 (SAUER [12] AND SHELAH [14]). *Let S be a set with $|S| = n$. If $\mathcal{G} \subseteq 2^S$, the power set of S, and*

$$|\mathcal{G}| > \sum_{i=0}^{m-1} \binom{n}{i},$$

then there exists $A \subseteq S$, $|A| = m$, such that

$$\{G \cap A : G \in \mathcal{G}\} = 2^A.$$

This result was used by Milman in [8]. The largest m for which $\exists A$ with $|A| = m$ and $\{G \cap A: G \in \mathcal{G}\} = 2^A$ is called the density of $\mathcal{G}$. Karpovsky and Milman [4] have proved generalizations of this result.

Our next step in the proof of the theorem will be to discard some of the norming functionals (f_i^ε) so that the remaining ones can be grouped into very many classes which are separated by a definite amount in l_∞^n.

For convenience assume $\delta' = 2^{-p}$ where p is an integer. Write $[-1, 1] = \bigcup_k J_k$ where the J_k are disjoint ordered intervals of length $\delta'/2$, $k = 1,\dots,2^{p+2}$; and for $1 < k < 2^{p+2}$, write $J_k = \bigcup_m L_{k,m}$ where the $L_{k,m}$ are disjoint ordered intervals of length $(\delta')^3/64$. Note that

$$\text{length}(L_{k,m}) = \left[(\delta')^2/32\right]\text{length}(J_k).$$

Let $\mathfrak{S}^0 = \mathfrak{S}$. We define $\mathfrak{S}^i$ inductively, $i = 1, \ldots, n$. Assume $\mathfrak{S}^{i-1}$ has been defined. Let

$$\mathfrak{S}_k^i = \left\{\varepsilon \in \mathfrak{S}^{i-1} : f_i^\varepsilon \in J_k\right\}, \qquad 1 \leqslant k \leqslant 2^{p+2},$$
$$\mathfrak{S}_{k,m}^i = \left\{\varepsilon \in \mathfrak{S}^{i-1} : f_i^\varepsilon \in L_{k,m}\right\}, \qquad 1 < k < 2^{p+2}.$$

Thus

$$\mathfrak{S}^{i-1} = \bigcup_{k=1}^{2^{p+2}} \mathfrak{S}_k^i; \qquad \mathfrak{S}_k^i = \bigcup_m \mathfrak{S}_{k,m}^i \quad \text{for } 1 < k < 2^{p+2};$$

and these are disjoint unions. So for each $1 < k < 2^{p+2}$, there is $m(i,k)$ such that

$$|\mathfrak{S}_{k,m(i,k)}^i| \leqslant \left[(\delta')^2/32\right] |\mathfrak{S}_k^i|,$$

so

$$\left|\bigcup_{k=2}^{2^{p+2}-1} \mathfrak{S}_{k,m(i,k)}^i\right| \leqslant \left[(\delta')^2/32\right] |\mathfrak{S}^{i-1}|.$$

Let

$$\mathfrak{S}^i = \mathfrak{S}^{i-1} - \bigcup_{k=2}^{2^{p+2}-1} \mathfrak{S}_{k,m(i,k)}^i;$$

thus

$$|\mathfrak{S}^i| \geqslant \left[1 - (\delta')^2/32\right] |\mathfrak{S}^{i-1}|.$$

It follows that

$$|\mathfrak{S}^n| \geqslant \left[1 - (\delta')^2/32\right]^n |\mathfrak{S}|.$$

Let $I_{i,k}$ be the interval between $L_{k,m(i,k)}$ and $L_{k+1,m(i,k+1)}$, $k = 1, \ldots, 2^{p+2} - 1$ (for notational convenience, we have let $L_{k,m} = \{-1\}$ when $k = 1$ and $L_{k,m} = \{1\}$ when $k = 2^{p+2}$). Thus length$(I_{i,k}) < \delta'$.

Each length n sequence (k_i), where k_i is an integer between 1 and $2^{p+2} - 1$, defines a neighborhood

$$N[(k_i)] = \left\{(g_i) \in l_\infty^n : g_i \in I_{i,k_i}, i = 1, \ldots, n\right\}.$$

This is a subset of a ball in l_∞^n of radius $\delta'/2$, so by Lemma 2,

$$|\{\varepsilon \in \mathfrak{S}^n : (f_i^\varepsilon) \in N[(k_i)]\}| \leqslant 2^n \exp\left(-n(\delta')^2/16\right).$$

So by the "pigeonhole principle", the number of distinct (k_i) for which $(f_i^\varepsilon) \in N[(k_i)]$ for some $\varepsilon \in \mathfrak{S}^n$ is at least

$$\begin{aligned} &|\mathfrak{S}^n| 2^{-n} \exp\left(n(\delta')^2/16\right) \\ &\quad \geqslant \left[1 - (\delta')^2/32\right]^n [(\delta - \delta')/(1 - \delta')] 2^n \cdot 2^{-n} \exp\left(n(\delta')^2/16\right) \\ &\quad \geqslant [(\delta - \delta')/(1 - \delta')] \exp\left(n(\delta')^2/48\right) \geqslant \exp\left(n(\delta')^2/64\right) \end{aligned}$$

if $n > 192\log[(1-\delta')/(\delta-\delta')]/(\delta')^2$. (The second inequality in the above string of inequalities follows from $1-\frac{3}{4}x \geqslant \exp(-x)$ for $0 \leqslant x \leqslant \frac{1}{2}$.) Let

$$\mathfrak{N} = \{(k_i): (f_i^\varepsilon) \in N[(k_i)] \text{ for some } \varepsilon \in \mathfrak{S}^n\}.$$

The neighborhoods $\{N[(k_i)]: (k_i) \in \mathfrak{N}\}$ define the separated classes of norming functionals that we were after:

$$|\mathfrak{N}| \geqslant \exp\left(n(\delta')^2/64\right)$$

for sufficiently large n.

Next we will use the combinatorial lemma successively, $p+2$ times if necessary, to produce a "large" (i.e. a percentage independent of n) subset of the coordinates $\{1,\dots,n\}$ such that we can always find a pair of functionals, separated on each of these coordinates, whose differences may be made to have any given choice of signs on these coordinates. We present the first iteration in detail, and the argument will become clear.

For each $(k_i) \in \mathfrak{N}$, let

$$G[(k_i)] = \{i: 0 < k_i \leqslant 2^{p+2}/2\}; \qquad \mathfrak{G} = \{G[(k_i)]: (k_i) \in \mathfrak{N}\}.$$

Thus each class of norming functionals defines the set of coordinates on which the functionals of that class have values in the lower half of the possible values. Different (k_i) may give rise to the same set, so $|\mathfrak{G}|$ may be small.

If

$$|\mathfrak{G}| \geqslant |\mathfrak{N}|^{1/(p+2)} \geqslant \exp\left[n(\delta')^2/64(p+2)\right],$$

then using Lemma 3, we shall show that $\exists A \subseteq \{1,\dots,n\}$ such that

$$|A| \geqslant n(\delta')^2/\left[128(p+2)\log\left(64(p+2)/(\delta')^2\right)\right],$$

and $\forall B \subseteq A$, there is a $G \in \mathfrak{G}$ such that $B = G \cap A$. Here we use the inequality

$$\sum_{i=0}^{\alpha n-1}\binom{n}{i} \leqslant \left[\alpha^\alpha(1-\alpha)^{(1-\alpha)}\right]^{-n} \quad \text{for } \alpha \leqslant 1/2,$$

which can be proved using Stirling's formula (or see Chernoff [2]). Using Lemma 3, we can find A with $|A| \geqslant \alpha n$ where $\alpha \leqslant \frac{1}{2}$ is chosen as large as possible satisfying

$$\exp\left[n(\delta')^2/64(p+2)\right] > \left[\alpha^\alpha(1-\alpha)^{(1-\alpha)}\right]^{-n};$$

i.e.,

$$(\delta')^2/64(p+2) > -\log\left[\alpha^\alpha(1-\alpha)^{(1-\alpha)}\right].$$

We show that this holds if

$$\alpha = (\delta')^2/\left[128(p+2)\log\left(64(p+2)/(\delta')^2\right)\right],$$

which will verify our statement above. We want to solve the inequality $-\alpha\log\alpha - (1-\alpha)\log(1-\alpha) < \gamma$, where $\gamma = (\delta')^2/64(p+2)$. Note that $-(1-\alpha)\log(1-\alpha) \leqslant \alpha$ for $0 \leqslant \alpha < 1$, so it suffices to have $-\alpha(\log\alpha - 1) < \gamma$. Let $\alpha = \gamma/2\log(1/\gamma)$.

Putting in this value for α, we need only verify that

$$-[\gamma/2\log(1/\gamma)][\log\gamma - \log(2\log(1/\gamma)) - 1] < \gamma,$$

that is,

$$\log(2\log(1/\gamma)) + 1 < \log(1/\gamma),$$

which clearly holds if $\gamma < e^{-3}$, which is certainly true in our case, so our stated value of α does indeed satisfy the required inequality.

Now if $(a_i)_{i\in A}$ are scalars, let $B = \{i \in A\colon a_i < 0\}$. Let $(k_i) \in \mathfrak{N}$ such that $G[(k_i)] \cap A = B$, and let $(k_i') \in \mathfrak{N}$ such that $G[(k_i')] \cap A = A - B$. Let $f_i^\varepsilon \in N[(k_i)]$ and $f_i^{\varepsilon'} \in N[(k_i')]$ with ε and ε' in $\mathfrak{S}^n$. Then

$$f_i^\varepsilon - f_i^{\varepsilon'} \leqslant -(\delta')^3/64, \qquad i \in B,$$

$$f_i^\varepsilon - f_i^{\varepsilon'} \geqslant (\delta')^3/64, \qquad i \in A - B,$$

since the intervals $I_{i,k}$ have gaps between them of size of $(\delta')^3/64$. Thus

$$\left\| \sum_{i\in A} a_i e_i \right\| \geqslant \frac{f^\varepsilon - f^{\varepsilon'}}{\|f^\varepsilon - f^{\varepsilon'}\|} \sum_{i\in A} a_i e_i \geqslant \sum_{i\in A} |a_i| \frac{(\delta')^3}{128}.$$

So A satisfies the conclusion of the theorem, with $\beta = (\delta')^3/128$.

If instead $|\mathcal{G}| < |\mathfrak{N}|^{1/(p+2)}$, then there is $A \subseteq \{1,\dots,n\}$ and $\mathfrak{N}^1 \subseteq \mathfrak{N}$ such that $\{i\colon 0 < k_i \leqslant 2^{p+2}/2\} = A$ for each $(k_i) \in \mathfrak{N}^1$, where

$$|\mathfrak{N}^1| \geqslant |\mathfrak{N}|^{(p+1)/(p+2)} \geqslant \exp\left[n(\delta')^2(p+1)/64(p+2)\right].$$

We set $l_i^1 = 0$ if $i \in A$, $l_i^1 = 2^{p+2}/2$ if $i \notin A$; $u_i^1 = 2^{p+2}/2$ if $i \in A$, $u_i^1 = 2^{p+2}$ if $i \notin A$; so $l_i^1 < k_i \leqslant u_i^1$, $i = 1,\dots,n$, for each $(k_i) \in \mathfrak{N}^1$, and $u_i^1 - l_i^1 = 2^{p+2}/2$. We are ready for the second iteration.

For each $(k_i) \in \mathfrak{N}^1$, let

$$G^1[(k_i)] = \{i\colon l_i^1 < k_i \leqslant (l_i^1 + u_i^1)/2\}; \qquad \mathcal{G}^1 = \{G^1[(k_i)]\colon (k_i) \in \mathfrak{N}^1\}.$$

If

$$|\mathcal{G}^1| \geqslant |\mathfrak{N}^1|^{1/(p+1)} \geqslant \exp\left[n(\delta')^2/64(p+2)\right],$$

then we get, by Lemma 3, the same conclusion as we did in the first iteration, with the same constants.

If $|\mathcal{G}^1| < |\mathfrak{N}^1|^{1/(p+1)}$, then there is $A \subseteq \{1,\dots,n\}$ and $\mathfrak{N}^2 \subseteq \mathfrak{N}^1$ such that $\{i\colon l_i^1 < k_i \leqslant (l_i^1 + u_i^1)/2\} = A$ for each $(k_i) \in \mathfrak{N}^2$, and

$$|\mathfrak{N}^2| \geqslant |\mathfrak{N}^1|^{p/(p+1)} \geqslant \exp\left[n(\delta')^2 p/64(p+2)\right].$$

We set $l_i^2 = l_i^1$ if $i \in A$, $l_i^2 = (l_i^1 + u_i^1)/2$ if $i \notin A$; $u_i^2 = (l_i^1 + u_i^1)/2$ if $i \in A$, $u_i^2 = u_i^1$ if $i \notin A$; so $l_i^2 < k_i \leqslant u_i^2$, $i = 1,\dots,n$, for each $(k_i) \in \mathfrak{N}^2$, and $u_i^2 - l_i^2 = 2^{p+2}/4$.

It is clear how to continue. For the $(p+2)$th iteration, if it is reached, we would have $u_i^{p+1} - l_i^{p+1} = 2$, so $(l_i^{p+1} + u_i^{p+1})/2 - l_i^{p+1} = 1$. Since $G^{p+1}[(k_i)] = \{i\colon l_i^{p+1} < k_i \leqslant (l_i^{p+1} + u_i^{p+1})/2\}$ for $(k_i) \in \mathfrak{N}^{p+1}$, we have $(k_i) = (k_i')$ iff $G^{p+1}[(k_i)] = G^{p+1}[(k_i')]$, when (k_i) and (k_i') are in $\mathfrak{N}^{p+1}$. Thus

$$|\mathcal{G}^{p+1}| = |\mathfrak{N}^{p+1}| \geqslant |\mathfrak{N}^p|^{1/2} \geqslant |\mathfrak{N}^{p-1}|^{(2/3)(1/2)} \geqslant \cdots \geqslant |\mathfrak{N}|^{1/(p+2)},$$

so we find A satisfying the conclusion of the theorem, just as in the first iteration.

Now choose the integer p such that $\delta' = 2^{-p}$ satisfies $\delta/4 < \delta' \leqslant \delta/2$. From $\delta' > \delta/4$ we see that (recall $\beta = (\delta')^3/128$) we may take $\beta \geqslant 2^{-13}\delta^3$. A computation using $\delta' = 2^{-p}$ and $\delta' > \delta/4$ in the above formula for α shows that we may take

$$c \geqslant 2^{-19}\delta^2/[\log(4/\delta)]^2.$$

(This c works trivially if $n \leqslant 192\log[(1-\delta')/(\delta-\delta')]/(\delta')^2$, using $\delta' \leqslant \delta/2$, which settles our earlier restriction.) Of course we made no attempt to get the sharpest constants from this argument. This completes the proof of the first part of the theorem.

REMARK. In the proof, we have produced a large subset $A \subseteq \{1,\ldots,n\}$ and numbers $a_j < b_j$, $j \in A$, such that if we set $L_j = \{f^\varepsilon\colon f_j^\varepsilon \leqslant a_j\}$ and $U_j = \{f^\varepsilon\colon f_j^\varepsilon \geqslant b_j\}$, then $(L_j, U_j)_{j\in A}$ is Boolean independent, in the language of the proof of Rosenthal's l_1 theorem [**10**]. In this sense our result may be viewed as a finite version of Rosenthal's theorem. Here, a_j and b_j are endpoints of some interval $L_{k,m}$. We may find a large subset of A such that (a_j, b_j) is the same for each j in the subset.

To prove the second part of the theorem, we use a much simpler argument which, however, does not work for small δ.

Let $\delta' = 2\delta - 1$. Then $(\delta-\delta')/(1-\delta') = 1/2$, so $|\mathfrak{S}| = 2^n/2 = 2^{n-1}$. Let $\beta = 1 - \sqrt{1-\delta'}$. For each $\varepsilon \in \mathfrak{S}$, there is a $A^\varepsilon \subseteq \{1,\ldots,n\}$ such that

$$|A^\varepsilon| \geqslant \beta n \quad\text{and}\quad \varepsilon_i f_i^\varepsilon \geqslant \beta \quad \text{for each } i \in A^\varepsilon.$$

For suppose $|\{i\colon \varepsilon_i f_i^\varepsilon < \beta\}| \geqslant (1-\beta)n$. Then

$$\sum_{i=1}^n \varepsilon_i f_i^\varepsilon < \beta(1-\beta)n + \beta n = \left[1-(1-\beta)^2\right]n = \delta' n,$$

a contradiction. Clearly there is no harm in assuming βn is an integer. Apparently there is a subset $\mathfrak{S}' \subseteq \mathfrak{S}$ and an $A' \subseteq \{1,\ldots,n\}$ such that $A^\varepsilon \supseteq A'$ for all $\varepsilon \in \mathfrak{S}'$, $|A'| \geqslant \beta n$, and $|\mathfrak{S}'| \geqslant |\mathfrak{S}|/\binom{n}{\beta n}$. Furthermore, there is a subset $\mathfrak{S}'' \subseteq \mathfrak{S}'$ such that $|\mathfrak{S}''| \geqslant |\mathfrak{S}'|/2^{(1-\beta)n}$, and if ε and ε' are in $\mathfrak{S}''$ with $\varepsilon \neq \varepsilon'$, then $(\varepsilon_i)_{i\in A'} \neq (\varepsilon_i')_{i\in A'}$. This is because there can be at most $2^{(1-\beta)n}$ sequences of signs ε which agree on the βn coordinates A'.

An application of Lemma 3 yields a subset $A \subseteq A'$ such that for any $\varepsilon \in \{-1,1\}^A$, there is $\varepsilon' \in \mathfrak{S}''$ such that $\varepsilon_i = \varepsilon_i'$ for all $i \in A$; we can have $|A| = m$ provided m satisfies

$$|\mathfrak{S}''| > \sum_{i=0}^{m-1}\binom{\beta n}{i}.$$

We need

$$2^{n-1}\Big/\left[\binom{n}{\beta n}2^{(1-\beta)n}\right] > \sum_{i=0}^{m-1}\binom{\beta n}{i}.$$

Let $m = \alpha\beta n$, where $\alpha < \frac{1}{2}$. Recall that

$$\sum_{i=0}^{\alpha\beta n-1}\binom{\beta n}{i} \leqslant \left[\alpha^\alpha(1-\alpha)^{(1-\alpha)}\right]^{-\beta n},$$

and also from Stirling's formula

$$\binom{n}{\beta n} \leqslant \frac{\left[\beta^{\beta}(1-\beta)^{(1-\beta)}\right]^{-n}}{2}, \quad \text{provided } 0 < \beta n < n.$$

So it suffices to have

$$2^{\beta}\beta^{\beta}(1-\beta)^{1-\beta}\left[\alpha^{\alpha}(1-\alpha)^{1-\alpha}\right]^{\beta} > 1.$$

This is useless if $\beta \leqslant \frac{1}{2}$, so our simple argument is of no help for small δ. Let $\alpha = \theta/2$, where $\theta < 1$. The requirement is

$$\beta^{\beta}(1-\beta)^{(1-\beta)}\left[\theta^{\theta}(2-\theta)^{2-\theta}\right]^{\beta/2} > 1.$$

We consider the function $f(\theta) = \theta^{\theta}(2-\theta)^{2-\theta}$. Now $f'(\theta) < 0$ for $\theta < 1$, so $f(\theta) > f(1) = 1$ when $\theta < 1$. So for any $\theta < 1$, the requirement is satisfied if β is sufficiently near 1, since $\beta^{\beta}(1-\beta)^{1-\beta} \to 1$ as $\beta \uparrow 1$. Now if $c < \frac{1}{2}$, let $\theta = c + \frac{1}{2}$, so $c < \alpha$. If β is sufficiently near 1, the above requirement is satisfied and also $\alpha\beta > c$. Since clearly $(e_i)_{i\in A} \overset{\beta^{-1}}{\sim}$ unit vector basis of $l_1^{|A|}$, and since $\beta \uparrow 1$ as $\delta \uparrow 1$, we see that we may choose $\beta \uparrow 1$ and $c \uparrow \frac{1}{2}$ as $\delta \uparrow 1$, as was claimed.

To prove the last part of the theorem, we observe that since there are only $2^{\alpha\beta n}$ sequences of signs on A (of the previous paragraph), there is some $\varepsilon^0 \in \{-1,1\}^A$ and $\mathcal{S}''' \subseteq \mathcal{S}''$ such that if $\varepsilon \in \mathcal{S}'''$, then $\varepsilon_i = \varepsilon_i^0$ for all $i \in A$, where $|\mathcal{S}'''| \geqslant |\mathcal{S}''|/2^{\alpha\beta n}$. Lemma 3, applied to $A' - A$, yields $A_1 \subseteq A' - A$ such that for any $\varepsilon \in \{-1,1\}^{A_1}$, there is $\varepsilon' \in \mathcal{S}'''$ such that $\varepsilon_i = \varepsilon_i'$ for all $i \in A_1$; $|A_1|$ may be taken to be as large as possible satisfying

$$|\mathcal{S}'''| > \sum_{i=0}^{|A_1|-1} \binom{(1-\alpha)\beta n}{i}.$$

Let $|A_1| = \gamma(1-\alpha)\beta n$. Estimating as before, we need

$$2^{(1-\alpha)\beta}\beta^{\beta}(1-\beta)^{1-\beta}\left[\gamma^{\gamma}(1-\gamma)^{1-\gamma}\right]^{(1-\alpha)\beta n} > 1.$$

Just as before, if $\gamma < \frac{1}{2}$, this is satisfied for β sufficiently near 1. We claim that

$$(e_i)_{i\in A\cup A_1} \overset{3/(2\beta-1)}{\sim} \text{unit vector basis of } l_1^{|A\cup A_1|}.$$

To see this, suppose $\varepsilon \in \{-1,1\}^{A\cup A_1}$ is given. Find ε' and ε'' in $\mathcal{S}'''$ such that $\varepsilon_i' = \varepsilon_i$ and $\varepsilon_i'' = -\varepsilon_i$ for all $i \in A_1$, and find ε''' in $\mathcal{S}''$ such that $\varepsilon_i''' = \varepsilon_i$ for all $i \in A$. If $i \in A$, $\varepsilon_i' = \varepsilon_i'' = \varepsilon_i^0$, and since $\varepsilon_i' f_i^{\varepsilon'} \geqslant \beta$ and $\varepsilon_i'' f_i^{\varepsilon''} \geqslant \beta$, we have $|f_i^{\varepsilon'} - f_i^{\varepsilon''}| \leqslant 1-\beta$ for all $i \in A$. If $i \in A_1$, we have

$$\varepsilon_i\left(f_i^{\varepsilon'} - f_i^{\varepsilon''}\right) = \varepsilon_i' f_i^{\varepsilon'} + \varepsilon_i'' f_i^{\varepsilon''} \geqslant 2\beta.$$

If $i \in A$, $\varepsilon_i f_i^{\varepsilon'''} \geqslant \beta$. So we let $g = (f^{\varepsilon'} - f^{\varepsilon''} + f^{\varepsilon'''})/3$. Then

$$\varepsilon_i g_i \geqslant [-(1-\beta)+\beta]/3 = (2\beta-1)/3 \quad \text{for } i \in A,$$

and

$$\varepsilon_i g_i \geqslant (2\beta-1)/3 \quad \text{for } i \in A_1$$

also. Since $\|g\| \leqslant 1$, it is clear that the claim is proved. Now $|A \cup A_1|$ is nearly $\frac{3}{4}n$ since $|A|$ is nearly $\frac{1}{2}n$. We could repeat this argument, if β is sufficiently near 1, to get a subset of (e_i) of cardinality nearly $\frac{7}{8}n$, with even larger equivalence constant, and so on. We omit the details.

This completes the proof of Theorem 1.

EXAMPLE. The following example due to S. Szarek, shows that it is not possible to have both $\beta \uparrow 1$ and $c \uparrow 1$ in the above theorem.

We let $\mathcal{S}$ be those length n sequences of signs ε for which

$$\left(\tfrac{1}{2} - \theta\right)n \leqslant |\{i: \varepsilon_i = +1\}| \leqslant \left(\tfrac{1}{2} + \theta\right)n,$$

where $\theta > 0$ is arbitrary. For a sequence of scalars $(a_i)_{i=1}^n$ define

$$\|(a_i)\| = \sup_{\varepsilon \in \mathcal{S}} \left| \sum_{i=1}^n \varepsilon_i a_i \right|.$$

Let $(e_i)_{i=1}^n$ be the usual unit vectors. Then $\|\sum_{i=1}^n \varepsilon_i e_i\| = n$ if $\varepsilon \in \mathcal{S}$. If we let $\delta = |\mathcal{S}|/2^n$, then $\text{Average}_{\varepsilon_i = \pm 1} \|\sum_{i=1}^n \varepsilon_i e_i\| \geqslant \delta n$, and clearly $\delta \uparrow 1$ as $n \to \infty$ (from the central limit theorem).

Let $A \subseteq \{1, \ldots, n\}$ with $|A| = cn \geqslant (\frac{1}{2} + \theta)n$. Then

$$\left\| \sum_{i \in A} \varepsilon_i \right\| = \left(\frac{1}{2} + \theta\right)n - \left[cn - \left(\frac{1}{2} + \theta\right)n\right] = (1 + 2\theta - c)n,$$

so

$$\left\| \sum_{i \in A} \varepsilon_i \right\| \Big/ |A| = \frac{1 + 2\theta - c}{c}.$$

Now $\theta > 0$ is arbitrary, so if β and c are as in the statement of Theorem 1, then $\beta \leqslant (1 - c)/c = 1/c - 1$, no matter how near δ is to 1. For example, if $c = \frac{3}{4}$, then $\beta \leqslant \frac{1}{3}$. This shows, in fact, that our proof of the last part of Theorem 1 even gives the right size for β.

PROOF OF THEOREM 2. As in the proof of Theorem 1, we first find $\mathcal{S} \subseteq \{-1, 1\}^n$ such that

$$\left\| \sum_{i=1}^n \varepsilon_i e_i \right\| > \frac{M}{2} \quad \text{for all } \varepsilon \in \mathcal{S}, \quad \text{where } |\mathcal{S}| \geqslant 2^{n-1}\frac{M}{n}.$$

For each $\varepsilon \in \mathcal{S}$, find $t^\varepsilon \in T$ such that $|\sum_{i=1}^n \varepsilon_i e_i(t^\varepsilon)| \geqslant M/2$. Let $f_i^\varepsilon = e_i(t^\varepsilon)$. So $f_i^\varepsilon = \pm 1$ for all i, all $\varepsilon \in \mathcal{S}$. Now if $f \in \{-1, 1\}^n$, then from Lemma 1, we have

$$\left| \left\{ \varepsilon : \left| \sum_{i=1}^n \varepsilon_i f_i \right| \geqslant \frac{M}{2} \right\} \right| \leqslant 2^n \cdot 2 \exp\left(-\frac{M^2}{16n}\right).$$

So there exists $\mathcal{S}' \subseteq \mathcal{S}$ such that

$$|\mathcal{S}'| \geqslant (M/n)2^{-2} \exp(M^2/16n),$$

and if ε and ε' are in $\mathcal{S}'$, then $f^\varepsilon \neq f^{\varepsilon'}$. Lemma 3 therefore gives us a subset $A \subseteq \{1, \ldots, n\}$ such that given any $f \in \{-1, 1\}^A$, there exists $\varepsilon \in \mathcal{S}'$ such that $f_i^\varepsilon = f_i$ for all i, where $|A| = m$ is as large as possible satisfying

$$\frac{M}{n} 2^{-2} \exp\left(\frac{M^2}{16n}\right) > \sum_{i=0}^{m-1} \binom{n}{i}.$$

Since, as before

$$\sum_{i=0}^{m-1}\binom{n}{i} \leqslant \frac{n^n}{m^m(n-m)^{n-m}} \leqslant \left(\frac{en}{m}\right)^m,$$

we require

$$\exp(M^2/16n) > 4(en/m)^m n/M, \quad \text{or} \quad M^2/16n \geqslant 2m\log(4n/m)$$

will do, if we assume $m \leqslant M$. It is easy to see that we may take $m = M^2/400n\log(2n/M)$, which proves the theorem.

PROOF OF COROLLARY (SZAREK). Let $e_i \in L_p(X)$, $i = 1,\ldots,n$, with $\|e_i\|_{L_p(X)} \leqslant 1$, such that

$$\left\|\sum_{i=1}^{n} \varepsilon_i e_i\right\|_{L_p(X)} \geqslant \frac{n}{K}$$

for all choices of signs $(\varepsilon_i)_{i=1}^n$. Thus

$$\int_0^1\int_\Omega\left\|\sum r_i(s)e_i(\omega)\right\|^p d\omega\, ds \geqslant \left(\frac{n}{K}\right)^p,$$

where (r_i) is the sequence of Rademacher functions. We consider the control function

$$g(\omega) = \left(n^{-1}\sum_{i=1}^{n}\|e_i(\omega)\|^p\right)^{1/p},$$

observing that $\int_\Omega g(\omega)^p\, d\omega \leqslant 1$. We will show that there exists $\varepsilon = \varepsilon(p, K) > 0$ so that

$$\int_0^1\int_\Omega\left\|\sum_{i\in A_\omega} r_i(s)e_i(\omega)\right\|^p d\omega\, ds \geqslant \left(\frac{n}{2K}\right)^p,$$

where $A_\omega = \{i\colon \varepsilon g(\omega) \leqslant \|e_i(\omega)\| \leqslant \varepsilon^{-1}g(\omega)\}$. Assuming this, we have

$$\int_\Omega\int_0^1\left\|\sum_{i\in A_\omega} r_i(s)e_i(\omega)\right\|^p ds\, d\omega \geqslant \left(\frac{n}{2K}\right)^p\int_\Omega g(\omega)^p\, d\omega,$$

so for some ω_0,

$$\int_0^1\left\|\sum_{i\in A_{\omega_0}} r_i(s)e_i(\omega_0)\right\|^p ds \geqslant \left(\frac{n}{2K}\right)^p g(\omega_0)^p.$$

Thus

$$\int_0^1\left\|\sum_{i\in A_{\omega_0}} r_i(s)e_i(\omega_0)\right\| ds \geqslant \frac{ng(\omega_0)}{2KC_p},$$

where C_p is the constant from Kahane's inequality [**6**, p. 74]. We let $v_i = e_i(\omega_0)\varepsilon/g(\omega_0)$ for $i \in A_{\omega_0}$, so $\|v_i\| \leqslant 1$, we get

$$\int_0^1\left\|\sum_{i\in A_{\omega_0}} r_i(s)v_i\right\| ds \geqslant \frac{n\varepsilon}{2KC_p}.$$

Now an application of Theorem 1 yields the desired result.

So to finish the proof, it is enough to show that for sufficiently small ε,

$$\int_0^1 \int_\Omega \Big\| \sum_{i \notin A_\omega} r_i(s) e_i(\omega) \Big\|^p d\omega\, ds \leq \left(\frac{n}{2K} \right)^p.$$

But

$$\int_\Omega \int_0^1 \Big\| \sum_{i \notin A_\omega} r_i(s) e_i(\omega) \Big\|^p ds\, d\omega \leq \int_\Omega \Big(\sum_{i \notin A_\omega} \|e_i(\omega)\| \Big)^p d\omega$$

$$\leq 2^{p-1} \left[\int_\Omega \Big(\sum_{i:\, \|e_i(\omega)\| < \varepsilon g(\omega)} \|e_i(\omega)\| \Big)^p d\omega + \int_\Omega \Big(\sum_{i:\, \|e_i(\omega)\| > \varepsilon^{-1} g(\omega)} \|e_i(\omega)\| \Big)^p d\omega \right].$$

To estimate the second integral, we observe that

$$\sum_{i:\, \|e_i(\omega)\| > \varepsilon^{-1} g(\omega)} \|e_i(\omega)\| = (\varepsilon^{-1} g(\omega))^{1-p} \sum_{i:\, \|e_i(\omega)\| > \varepsilon^{-1} g(\omega)} \|e_i(\omega)\| (\varepsilon^{-1} g(\omega))^{p-1}$$

$$\leq (\varepsilon^{-1} g(\omega))^{1-p} n^{-1} \sum_{i=1}^{n} \|e_i(\omega)\|^p n = \varepsilon^{p-1} g(\omega) n,$$

so

$$\int_\Omega \Big(\sum_{i:\, \|e_i(\omega)\| > \varepsilon^{-1} g(\omega)} \|e_i(\omega)\| \Big)^p d\omega \leq (\varepsilon^{p-1} n)^p \int_\Omega g(\omega)^p\, d\omega \leq (\varepsilon^{p-1} n)^p.$$

To estimate the first integral,

$$\int_\Omega \Big(\sum_{i:\, \|e_i(\omega)\| < \varepsilon g(\omega)} \|e_i(\omega)\| \Big)^p d\omega \leq \int_\Omega (n \varepsilon g(\omega))^p\, d\omega \leq (\varepsilon n)^p.$$

So it is enough to have

$$2^{p-1} \left[(\varepsilon n)^p + (\varepsilon^{p-1} n)^p \right] \leq (n/2K)^p, \quad \text{or} \quad 2^{p-1} [\varepsilon^p + \varepsilon^{p-1}] \leq (2K)^{-p},$$

which is true for small enough ε, provided $p > 1$.

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ADDED IN PROOF. Alain Pajor has recently shown that Theorem 1 holds for complex Banach spaces as well.

REFERENCES

1. J. Bourgain and H. P. Rosenthal, *Some applications of the theory of semi-embeddings to Banach space theory* (to appear).

2. H. Chernoff, *A measure of asymptotic efficiency for tests of a hypothesis based on the sum of observations*, Ann. Math. Statist. **23** (1952), 493–509.

3. W. B. Johnson and G. Schechtman, *Embedding l_p^m into l_1^n* (to appear).

4. M. G. Karpovsky and V. D. Milman, *Coordinate density of sets of vectors*, Discrete Math. **24** (1978), 177–184.

5. J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces*. I: *Sequence spaces*, Ergebnisse 92, Springer-Verlag, Berlin and New York, 1977.

6. ______, *Classical Banach spaces*. II: *Function spaces*, Ergebnisse 97, Springer-Verlag, Berlin and New York, 1979.

7. M. Loève, *Probability theory*. I, 4th ed., Springer-Verlag, Berlin and New York, 1977.

8. V. D. Milman, *Some remarks about embeddings of l_1^k in finite dimensional spaces* (to appear).

9. G. Pisier, *De nouvelles caracterisations des ensembles de Sidon*, Adv. in Math. (supplementary studies), 1981.

10. H. P. Rosenthal, *A characterization of Banach spaces containing l^1*, Proc. Nat. Acad. Sci. U.S.A. **71** (1974), 2411–2413.

11. ______, *Sign-embeddings of L^1* (to appear).

12. N. Sauer, *On the density of families of sets*, J. Combinatorial Theory Ser. A **13** (1972), 145–147.

13. G. Schechtman, *Random embeddings of Euclidean spaces in sequence spaces*, Israel J. Math. **40** (1981), 187–192.

14. S. Shelah, *A combinatorial problem; stability and order for models and theories in infinitary languages*, Pacific J. Math. **41** (1972).

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