TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 279, Number 1, September 1983

NAKAYAMA ALGEBRAS AND GRADED TREES

BY

B. ROHNES AND S. O. SMALØ

ABSTRACT. Let k be an algebraically closed field. We show that if T is a finite tree, then there is a grading g on T such that (T, g) is a representation finite graded tree, and such that the corresponding simply connected k-algebra is a Nakayama algebra (i.e. generalized uniserial algebra).

Introduction. Let k be an algebraically closed field. A simply connected algebra Λ over k is an algebra which is representation-finite, connected, basic, finite-dimensional and has a simply connected Auslander-Reiten quiver Γ_Λ. In order to study the simply connected algebras, K. Bongartz and P. Gabriel introduced the notion of graded trees [**2**]. If T is a finite tree, let T_0 denote the set of vertices of T. A grading of the tree T is a function $g\colon T_0 \to N$ (N is the nonnegative integers), satisfying the following conditions:

(a) $g(x) - g(y) \in 1 + 2Z$, whenever x and y are neighbours in T (Z the integers).

(b) $g^{-1}(0) \neq \varnothing$.

A graded tree is a pair (T, g) formed by a tree T and a grading g of T.

K. Bongartz and P. Gabriel show that there is a bijection between the isomorphism classes of representation-finite graded trees and the isomorphism classes of simply connected algebras. For the benefit of the reader we give a summary of their results in §1. They also show in [**2**] that every tree T admits only a finite number of representation-finite gradings. In this paper we show that for every tree T it is possible to find a grading g such that (T, g) is representation-finite. This answers a question raised by P. Gabriel. In fact, what we show is that given a tree T it is possible to find a grading g such that the associated simply connected algebra is a Nakayama algebra. Conversely, given a noncyclic Kupisch series for a Nakayama k-algebra Λ, one may associate a graded tree (T, g) such that the simply connected k-algebra obtained from (T, g) is Λ.

1. Simply connected algebras and graded trees. Let (T, g) be a graded tree. To this graded tree we associate a translation quiver Q_T in the following way. The vertices of Q_T are the points $(n, t) \in N \times T_0$ such that $n - g(t) \in 2N$, two such vertices (m, s) and (n, t) are joined by an arrow $(m, s) \to (n, t)$ if s, t are neighbours in T and $n = m + 1$. The projective vertices are the points $(g(t), t)$, the translate of a nonprojective vertex is defined by $\tau(n, t) = (n - 2, t)$.

Received by the editors September 13, 1982.

1980 *Mathematics Subject Classification*. Primary 16A64, 16A46.

Key words and phrases. Simply connected algebra, module, graded tree, Kupisch series.

For each graded tree $T = (T, g)$ there is a unique map $d\colon (Q_T)_0 \to N^{T_0}$ satisfying the following conditions:

(a) $d(g(t), t) = \delta_t + \Sigma_s d(g(t) - 1, s)$, where s ranges over the neighbours s of t such that $g(s) < g(t)$ and $d(g(t) - 1, s) > 0$ (where a function is > 0 if all its values are $\geqslant 0$ and at least one of them if > 0), and the Kronecker function δ_t takes the value 1 at t and 0 otherwise.

(b) $d(n, t) = \Sigma_s d(n - 1, s) - d(n - 2, t)$, whenever (n, t) is a nonprojective vertex of Q_T for which the functions $d(n - 2, t)$ and $\Sigma_s d(n - 1, s) - d(n - 2, t)$ are both > 0, when s ranges over the neighbours of t in T such that $g(s) < n$.

(c) For any other vertex (n, t) of Q_T we have $d(n, t) = 0$.

Using these conditions, $d(n, t)$ can be computed by induction on n, starting with $n = g(t)$. d is called the dimension map of Q_T. We denote by R_T the full subtranslation-quiver of Q_T formed by the vertices (n, t) such that $d(n, t) > 0$. The grading g is called admissible if R_T is a connected subquiver of Q_T, and T is then called an admissible graded tree. The grading is called representation-finite if it is admissible and R_T is finite. T is then called a representation-finite graded tree.

REMARK. We are using a definition of d different from the one given in [**2**, p. 356], since it was through our definition we saw the main result of this paper. Also with our definition the projective vertices in R_T coincide with those in Q_T regardless of the grading g. It is easy to see that the two definitions are the same when R_T is connected.

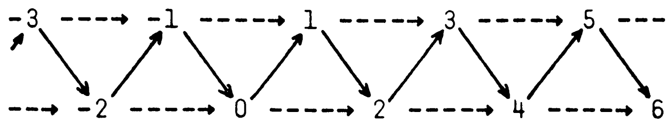
Let T be an admissible graded tree. Let A^T be the finite-dimensional algebra $A^T = \coprod_{p,q} k(R_T)(q, p)$, where $k(R_T)$ is the mesh category of R_T, and p, q range over all projective vertices of R_T. Then each vertex x of R_T is associated with an A^T-module $M(x) = \coprod_p k(R_T)(p, x)$, where p ranges over all projective vertices of R_T, and it is shown in [**2**] that for every vertex (n, t) of R_T, the A^T-module $M(n, t)$ is indecomposable and its dimension vector is $d(n, t)$, especially, $M(g(t), t)$ are the indecomposable projective modules, and if $M(n, t)$ is not projective, $DTr(M(n, t)) = M(n - 2, t) = M(\tau(n, t))$. In fact, if (T, g) is representation-finite, then there is a translation-quiver isomorphism of the Auslander-Reiten quiver Γ_{A^T} onto R_T.

If Γ is a locally finite translation-quiver, and x is a vertex of Γ, the set of all $n \in Z$ such that $\tau^n x$ is defined, is an inverval $\mathfrak{D}$ of Z. Then the set $x^\tau = \{\tau^n x, n \in \mathfrak{D}\}$ is called the τ-orbit of x. The vertex x is stable if $\mathfrak{D} = Z$, it is periodic if it is stable and has a finite τ-orbit. The τ-orbits of a connected component E of the stable part Γ_s of Γ are either all finite or all infinite. In the first case we call E a periodic component of Γ.

If $x \xrightarrow{\alpha} y$ is an arrow of Γ, where y is not projective, there is a unique arrow $\tau y \to x$, which we denote $\sigma\alpha$. The τ-orbit of α, denoted α^σ, is the set of all arrows of Γ of the form $\sigma^m\alpha$.

The graph G_Γ associated with Γ has as vertices the nonperiodic τ-orbits and the periodic components of Γ. To each periodic component, considered as a vertex of G_Γ, we associate a loop of G_Γ. Let α^σ be a σ-orbit connecting x^τ and y^τ. If both x and y are nonperiodic, we associate with α^σ an edge connecting the vertices x^τ and y^τ. If y is not periodic and x belongs to a periodic component E we associate with α^σ an edge of G_Γ connecting E and y^τ.

Now, if A is a simply connected algebra, and Γ_A is the Auslander-Reiten quiver of A, then the graph G_A associated with Γ_A is a tree [**2**, Theorem 4.2]. Since Γ_A is simply connected and finite, there is a unique quiver morphism K_A: $\Gamma_A \to ZA_2$ such that $0 = \operatorname{Min} K(x)$, the minimum taken over all vertices x of Γ_A. Here ZA_2 is the following translation quiver where $--\rightarrow$ indicates the translation. Since G_A is a tree, each τ-orbit t of Γ_A contains exactly one projective vertex p_t. We set $g_A(t) = K_A(p_t) \in N$. The function g_A is then a grading of G_A, and (G_A, g_A) is a graded tree. The maps $(T, g) \to A^T$ and $A \to (G_A, g_A)$ are inverse maps and therefore there is a bijection between the isomorphism classes of representation-finite graded trees and the isomorphism classes of simply connected algebras [**2**, 6.5].



ZA_2

2. The relation between Kupisch series and trees. In this section we examine the relation between the Nakayama algebras with noncyclic Kupisch series and trees. We show that given a tree T, it is possible to associate a Nakayama algebra Λ to this tree such that the graph of Λ is isomorphic to T. From this follows the main result of this paper: To every tree T it is possible to find a grading g such that (T, g) is representation-finite. The Nakayama algebra Λ is not uniquely given by the construction we use.

But first we show how to construct a tree T_Λ from a Nakayama algebra Λ with noncyclic Kupisch series. The construction determines T_Λ uniquely up to isomorphism, and later we will see that T_Λ is in fact the graph G_Λ associated with Γ_Λ. Therefore this gives us an easy way to construct G_Λ if Λ is a Nakayama algebra.

We recall that the Kupisch series for an indecomposable Nakayama algebra Λ is an ordered complete set of representatives $P_1, \ldots, P_n$ of the isomorphism classes of indecomposable projective Λ-modules, satisfying the following conditions:

(i) $P_i/\underline{r}P_i \cong \underline{r}P_{i+1}/\underline{r}^2 P_{i+1}$, or equivalently:

$$P_{i+1}/\underline{r}P_{i+1} = TrD(P_i/\underline{r}P_i).$$

(ii) $L(P_i) \geqslant 2$ for all i such that $2 \leqslant i \leqslant n$.

(iii) $L(P_{i+1}) \leqslant L(P_i) + 1$ for $i = 1, \ldots, n$, and $L(P_1) \leqslant L(P_n) + 1$.

($L(M) =$ the length of the Λ-module M.)

Any finite sequence of integers $c_1, \ldots, c_n$ satisfying (ii) and (iii) above when we put $c_i = L(P_i)$, is called an admissible sequence. Given an arbitrary admissible sequence, an algebra can be constructed such that its Kupisch series corresponds to this sequence. The Kupisch series is noncyclic if $L(P_1) = 1$. For details, see [**4**].

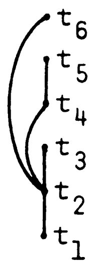
Let Λ be an indecomposable Nakayama algebra with a noncyclic Kupisch series. Let T_Λ be the following tree: The vertices of T_Λ are the representatives of the isomorphism classes of indecomposable projective Λ-modules. For each i, let t_i be the vertex corresponding to the projective P_i. If $i, j \in \{1, \ldots, n\}$, with $i \leqslant j$, there is an edge connecting t_i and t_j if i is the greatest integer less than j such that $L(P_j) = L(P_i) + 1$. T_Λ is connected, since for every $j \in \{2, \ldots, n\}$ it follows from

(iii) above that there always exists such an i, and it is not difficult to see that T_Λ really is a tree when constructed as above.

We define a walk in a tree T to be a sequence of vertices $S_1 \cdots S_n$, connected by edges $\alpha_1 \cdots \alpha_{n-1}$ in such a way that for each $i \in \{1, \ldots, n-1\}$, S_i and S_{i+1} are connected by the edge α_i of T.

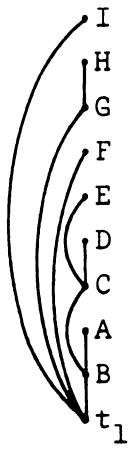
If u is a walk: $S_1 \overset{\alpha_1}{—} S_2 — \cdots \overset{\alpha_{k-1}}{—} S_k$ in a tree T, we define the length of u, $l(u) = k$. If S_i and S_j are two vertices of T, the shortest walk from S_i to S_j is the walk that does not pass through any vertex twice. It follows from the construction above that for any vertex t_i in T_Λ, $L(P_i)$ is equal to the length of the shortest walk in T_Λ from t_i to t_1.

Example. Given the admissible sequence $\{1, 2, 3, 3, 4, 3\}$ the corresponding tree T_Λ is:



Conversely, starting with a tree T, to this tree we can associate a noncyclic Kupisch-series for an indecomposable Nakayama algebra: Fix a point t_1 in the tree T and a walk V around the tree from t_1 to t_1 which passes through every edge in the tree exactly twice.

Example. If T is the tree, then V: t_1–B–A–B–C–D–C–E–C–B–t_1–F–t_1–G–H–G–t_1–I–t_1 is such a walk.



The order in which V passes through each vertex for the first time defines an ordering of the vertices of T, such that t_i is the ith new vertex which occurs in V. In the example above $t_2 = B$, $t_3 = A$, $t_4 = C$, $t_5 = D$, $t_6 = E$, $t_7 = F$, $t_8 = G$, $t_9 = H$, $t_{10} = I$.

Suppose $T_0 = \{t_1, \ldots, t_n\}$. Then for each $i \in \{1, \ldots, n\}$, let $C_i = l(U_i)$, where U_i is the shortest walk in T from t_1 to t_i. It is clear that $C_1 = 1$, and that $C_i \geqslant 2$ for $i \geqslant 2$. Further, if t_{i+1} is a neighbour of t_i, then $C_{i+1} = C_i + 1$, because it is clear that there is only one neighbour t_k of t_i with $l(U_k) < l(U_i)$, and it is the only neighbour with $k < i$. If t_{i+1} is not a neighbour of t_i, then t_{i+1} is a neighbour of a vertex t_j with $l(U_j) < l(U_i)$. So in that case $C_{i+1} < C_i + 1$. Therefore we have that $\{C_1, \ldots, C_n\}$ is an admissible sequence which corresponds to the noncyclic Kupisch series of an indecomposable Nakayama algebra.

We now claim that every indecomposable Nakayama algebra Λ with a noncyclic Kupisch series is simply connected. The ordinary quiver Q_Λ of an indecomposable Nakayama algebra Λ with a noncyclic Kupisch series is a tree of form $\rightarrow \cdots \rightarrow$, therefore the fundamental group $\pi(Q_\Lambda, x) = \{1\}$, and from [**3**, 2.2] we know that there is a surjective group homomorphism ϕ_x: $\pi(Q_\Lambda, x) \rightarrow \pi(\Gamma_\Lambda, x)$. Therefore $\pi(\Gamma_\Lambda, x)$ is trivial, and Λ is simply connected. See also [**2**, 6.1].

Therefore, to every tree T one may associate a simply connected algebra Λ, namely, the indecomposable Nakayama algebra constructed above. Remark that the Kupisch series of Λ depends on the choice of the basis point t_1 and the walk V, therefore given a tree T, there is usually more than one choice of a corresponding Nakayama algebra Λ. For our purposes, it is enough to look at one of these. Since Λ is simply connected, we know that the graph G_Λ is a tree [**2**, Theorem 4.2]. Because of the connection between simply connected algebras and graded trees, to show that the tree T has a representation-finite grading, it is enough to show that G_Λ is isomorphic to the tree T. (Remark that we consider a tree to be completely determined by the vertices and the edges connecting them, such that for instance, are considered to be isomorphic.)

$T_1 =$  $T_2 =$ 

The number of τ-orbits is equal to the number of projective Λ-modules, so the number of vertices of G_Λ is equal to the number of vertices of T. Now we define a map θ: $T_0 \rightarrow (G_\Lambda)_0$ such that $\theta(T_i)$ is the vertex representing the τ-orbit of the projective Λ-module P_i with $L(P_i) = C_i$, where C_i is as defined above. Then θ is a bijection. Denote $\theta(t_i)$ by S_i. Since G_Λ and T are trees with the same number of vertices, they also have the same number of edges, and to prove that G_Λ is isomorphic to T, it is enough to show that if there is an edge connecting the vertices t_i and t_j in T, there is an edge connecting the vertices S_i and S_j in G_Λ.

Let us recall some useful facts about Nakayama algebras. If Λ is a Nakayama algebra, then every indecomposable Λ-module is of the form $P_i/\underline{r}^k P_i$, where $k \geqslant 0$ and P_i is an indecomposable projective Λ-module. If $P_i/\underline{r}^k P_i$ is an indecomposable nonprojective Λ-module, then it is shown in [1] that the almost split sequence with $P_i/\underline{r}^k P_i$ as right-hand term has the form

$$0 \to P_{i-1}/\underline{r}^k P_{i-1} \to P_{i-1}/\underline{r}^{k-1} P_{i-1} \amalg P_i/\underline{r}^{k+1} P_i \to P_i/\underline{r}^k P_i \to 0.$$

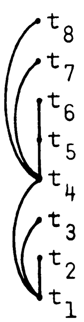
It follows from this that τ-orbits preserve the length of modules, and all simples belong to the same τ-orbit. We also recall that given the Kupisch series for a Nakayama algebra Λ, we always have an epimorphism $P_i \to \underline{r}P_{i+1}$. If $L(P_{i+1}) = L(P_i) + 1$, this epimorphism is also an isomorphism.

Now, suppose that t_i or t_j is t_1, say $t_i = t_1$. $L(P_1) = 1$, so P_1 is the unique simple projective. Since t_j is a neighbour of t_1, we see from the construction above, that $L(P_j) = 2$. But that means $\underline{r}P_j$ is simple, and then either $\underline{r}P_j \cong P_1$, or $\underline{r}P_j$ is in the τ-orbit determined by P_1, so S_j is a neighbour of S_1 in G_Λ. Suppose that neither t_i nor t_j is t_1, but that there is an edge $t_i - t_j$. Let $i < j$. Then $L(P_j) = L(P_i) + 1$ by the construction above. Therefore $L(\underline{r}P_j) = L(P_i)$. Since Λ is Nakayama, $\underline{r}P_j$ belongs to the τ-orbit of a projective module with the same length as P_i. We remember that the ordering of the projectives was defined by help of the walk V in T, and since T is a tree, and every edge in T appears in V exactly twice, we have $L(P_k) > L(P_i)$ for every edge k such that $i < k < j$. If P_m is an indecomposable Λ-module, the length of the τ-orbit determined by P_m, $l(P_m^\tau)$, is the number of nonisomorphic objects in the τ-orbit. For a Nakayama algebra Λ, the following formula is easily obtained, using the form of almost split sequences indicated above: $l(P_m^\tau) = h - m + 1$, where h is maximal with the property that $L(P_p) - L(P_m) > 0$ for all p such that $m < p \leqslant h$. Further if $L(P_m) = q$, then the modules in this τ-orbit are the modules of the form $P_p/\underline{r}^q P_p$, where $m \leqslant p \leqslant h$. In our case, if we let $L(P_i) = q$, it follows that $P_{j-1}/\underline{r}^q P_{j-1}$ is in the τ-orbit of P_i. But since we have an epimorphism $P_{j-1} \to \underline{r}P_j$, and $L(\underline{r}P_j) = L(P_i) = q$, we have $\underline{r}P_j \cong P_{j-1}/\underline{r}^q P_{j-1}$. Therefore $\underline{r}P_j$ is in the τ-orbit of P_i, and we have an edge $S_i - S_j$ in G_Λ.

We have now proved the main result of this paper:

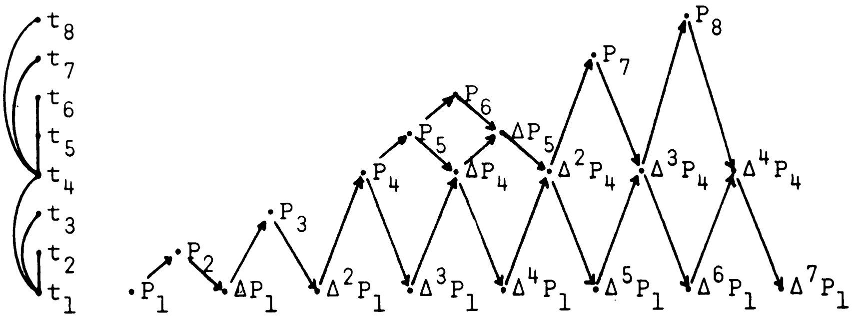
Theorem. *If T is a finite tree, then there is a grading g such that (T, g) is representation-finite, and such that the corresponding simply connected algebra Λ is a Nakayama algebra.*

Example. Let T be the tree:



Let V be the walk: $t_1–t_2–t_1–t_3–t_1–t_4–t_5–t_6–t_5–t_5–t_7–t_4–t_8–t_4–t_1$, which is a walk around the tree, passing through every edge in the tree exactly twice. To the tree T and the walk V we may associate the Kupisch series $\{P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8\}$ corresponding to the admissible sequence $\{1, 2, 2, 2, 3, 4, 3, 3\}$.

The AR-quiver Γ_Λ of the Nakayama algebra Λ is the following:



We see that P_2, P_3, P_6, P_7 and P_8 all are projective injectives. All arrows pointing upward correspond to irreducible monomorphisms, all arrows pointing downward correspond to irreducible epimorphisms. If there is an irreducible monomorphism $X \to Y$, $LY = LX + 1$, and if there is an irreducible epimorphism $X \to Y$, $LX = LY + 1$. If $X \in \operatorname{ind} \Lambda$, Soc X is the unique simple module S such that there is a chain of irreducible monomorphisms $S \to \cdots \to X$, $X/\underline{r}X$ is the simple module T such that there is a chain of irreducible epimorphisms $X \to \cdots \to T$. $L(X)$, the length of X, is equal to the shortest walk in Γ_Λ from Soc X to X.

If we start with a tree T, choose a point t, and a walk V around the tree, and construct the corresponding Nakayama algebra Λ in the way described above, it is possible to find the number of nonisomorphic indecomposable projective injective Λ-modules just by looking at the tree T.

PROPOSITION. *The number of projective injective Λ-modules is equal to the number of vertices in T, different from t_1, which have only one neighbour.*

PROOF. P_i is projective injective if and only if $L(P_{i+1}) < L(P_i) + 1$. If $t_i \neq t_1$ is a point in T having only one neighbour t_j, then every walk in T from t_1 to t_i must pass through t_j, therefore $j < i$, and t_{i+1} is not a neighbour of t_i. But then t_{i+1} is a neighbour of a point t_k which does not lie farther away from t_1 than t_j, and $L(P_{i+1}) \leqslant L(P_j) + 1 = L(P_i) < L(P_i) + 1$, which means that P_i is a projective injective module. On the other hand, if P_i is a projective injective module, then $L(P_{i+1}) < L(P_i) + 1$, and t_{i+1} is not a neighbour of t_i. But then t_i can have only one neighbour (recall that the walk V that defines the ordering passes through every edge exactly twice). The relation between V and Γ_Λ can be described in the following manner.

PROPOSITION. *Let θ be a chain of irreducible maps in $k(\Gamma_\Lambda)$ given by*

$$\theta\colon P_1 = M(0, t_1) \to P_2 \to \cdots \to P_i \to \cdots \to \underline{r}P_{i+1} \to \cdots$$
$$\to P_n \to \cdots \to M(2(n-1), t_1)$$

which passes through all the projectives in the order given by the Kupisch series, and which satisfies the condition that if P_n *is the last projective in the ordering, then* $P_n \to \cdots \to M(2(n-1), t_1)$ *is the unique path from the projective injective module* P_n *to the simple injective* Λ*-module* $M(2(n-1), t_1)$*. Then* V *is the walk in* T *constructed by taking for each module in* θ *the corresponding point in* T*, and passing through the points in the order defined by* θ*.*

PROOF. This can be proven in the same way as the main theorem above.

REFERENCES

1. M. Auslander and I. Reiten, *Representation theory of artin algebras*. IV; *Invariants given by almost split sequences*, Comm. Algebra **5** (1977), 443–518.

2. K. Bongartz and P. Gabriel, *Covering spaces in the representation theory*, Invent. Math. **65** (1982), 331–378.

3. P. Gabriel, *The universal cover of a representation-finite algebra*, Proc. Third Internat. Conf. Rep. Algebra, Puebla, 1980.

4. H. Kupisch, *Beiträge zur Theorie nichthalbeinfacher Ringe mit Minimalbedingung*, J. Reine Angew. Math. **201** (1959), 100–112.

DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF TRONDHEIM, NLHT, 7055 DRAGVOLL, NORWAY (Current address of S. O. Smalø)

Current address (B. Rohnes): Department of Mathematics, Brandeis University, Waltham, Massachusetts 02154