

FINITELY ADDITIVE F -PROCESSES¹

BY

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ABSTRACT. If one replaces random variables by finitely additive measures one obtains instead of an F -process a finitely additive F -process. Finitely additive F -processes on a decreasing collection of Boolean algebras form a dual base norm ordered Banach space. When the collection is linearly ordered they form a dual Kakutani L -space. This L -space may be represented as the L -space of all finitely additive bounded measures on the Boolean ring of predictable subsets of the extreme points of the positive face of the unit ball. Of independent interest is the fact that any bounded supermartingale is a decreasing process in contrast to the usual case where only the supermartingales of class DL are decreasing processes.

0. Introduction. Let $(\Omega, \mathfrak{F}, P)$ be a probability space and let $\{\mathfrak{F}_t: t \in T\}$ be a filtration so that $T \subset [0, \infty]$ and $\{\mathfrak{F}_t: t \in T\}$ is an increasing family of sub- σ -algebras of \mathfrak{F} . A stochastic process $\{f_t: t \in T\}$ adapted to $\{\mathfrak{F}_t: t \in T\}$ is a family of random variables on Ω so that, for $t \in T$, f_t is \mathfrak{F}_t measurable. When it is true for $t < s$ in T that $E(f_s | \mathfrak{F}_t) = f_t$, the stochastic process is called a martingale; when $E(f_s | \mathfrak{F}_t) \leq f_t$ holds the process is called a supermartingale. When $E(f_s | \mathfrak{F}_t) \geq f_t$ holds one has a submartingale. These types of processes were introduced by Doob [21], where supermartingales were called upper semimartingales and submartingales were called lower semimartingales.

Fisk, in [24], studied processes $\{f_t: t \in T\}$ which are of bounded variation in the sense that there is a least constant k so that if $t_1 < t_2 < \dots < t_n$ are chosen from T then $\sum_{i=1}^{n-1} |E(f_{t_{i+1}} | \mathfrak{F}_{t_i}) - f_{t_i}| \leq k$. Fisk introduced the term quasimartingales for such processes. Orey, in [42], further studied the properties of quasimartingales, which he now called F -processes. He noted in particular that the differences of two bounded nonnegative supermartingales is an F -process. Rao, in [46], established that any F -process is a difference of nonnegative supermartingales.

Functions of bounded variation on T into $(-\infty, \infty)$ may be represented as distribution functions of bounded measures on T which are countably additive provided the functions of bounded variation are right continuous. Otherwise, a function of bounded variation is unambiguously represented only as the distribution function of a finitely additive measure on the Jordan algebra of subsets of T generated by the order intervals of T .

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At least for right continuous F -processes adapted to right continuous filtrations, it is possible to obtain a representation as measures on a certain measurable space. This program has been carried out by a number of people. This has culminated, perhaps, in what Metivier and Pellaumail term, in [34], the Doleans-Föllmer measure. Specifically Doleans-Dade, in [20], constructs representing measures for non-negative supermartingales on the Boolean ring of predictable events. Pellaumail [44], Föllmer [26], and Airault and Föllmer [1], and Metivier and Pellaumail [34] have extended and refined this representation to its present state. Perhaps the best source about this is the book of Metivier and Pellaumail [35] where it is argued that semimartingales, an extension of the notion of an F -process, should provide the foundation of the theory of stochastic integration. The Doleans-Föllmer representation as a result nearly brings stochastic integration into the fold of classical measure theory (we should remark that another closely related approach is in Bichteler [15]).

Let us describe the ring of predictable events and the Doleans-Föllmer representation for the case $T = (0, \infty)$. The ring of predictable events, in this case, is the ring of subsets of $(0, \infty] \times \Omega$ generated by $\{(t, \infty] \times \mathfrak{F}_t; 0 \leq t < \infty\}$. A basis for this ring consists of predictable rectangles of the form $(t, s] \times F$ where $F \in \mathfrak{F}_t$ and $t < s \leq \infty$. The events $(t, \infty] \times F$ where $F \in \mathfrak{F}_t$ and $t < \infty$ are those predictable at time t . The measure μ_f on the predictable events corresponding to a process $f = \{f_t; 0 \leq t < \infty\}$ is defined by $\mu_f((t, \infty] \times F) = \int_F f_t dP$ for $F \in \mathfrak{F}_t$. Thus, $\mu_f((t, s] \times F) = \int_F f_t dP - \int_F f_s dP$ if $t < s \leq \infty$ and one sets $f_\infty = 0$. The measure μ_f is finitely additive on the ring of predictable events. μ_f is nonnegative iff f is a supermartingale. μ_f is of bounded variation iff f is an F -process. The measures μ on the predictable events which are of the form μ_f for some F -process f are those measures of bounded variation so that if one defines μ' on \mathfrak{F}_t for $t > 0$ by $\mu'(F) = \mu((t, \infty] \times F)$ for $F \in \mathfrak{F}_t$ then $\mu' \ll P$. Given such a μ there is an essentially unique F -process $f = \{f_t; t \in T\}$ so that $\mu = \mu_f$.

Besides its use in the theory of stochastic integration, one of the chief reasons for the importance of the Doleans-Föllmer measure is that very many of the important decompositions of F -processes are mirrored in corresponding decompositions of the associated Doleans-Föllmer measures. For instance, the Riesz-Doob decomposition of an F -process f into potential and martingale parts is mirrored into the decomposition of μ_f into a part which gives full measure to the increasing union of events $\{(0, t) \times \Omega; t < \infty\}$ and a part which annihilates all such events. Thus, $\mu_f^*(\{\infty\} \times \Omega) = 0$ iff f is a potential and $\mu_f^*(\{\infty\} \times \Omega) = \|f\|$ iff f is a martingale. The decomposition of an F -process f into a (minimal) difference of nonnegative supermartingales corresponds to the Jordan decomposition $\mu_f = \mu_f^+ - \mu_f^-$ of μ_f into its positive and negative parts. The Hewitt-Yosida decomposition of μ_f into countably additive and purely finitely additive parts corresponds to the decomposition of f into an F -process of class DL and a part which may be said to be singular to all F -processes of class DL [34]. This latter part is an F -process $\{g_t; t \in T\}$ so that $\lim_{t \rightarrow \infty} f_t = 0$, P a.s. yet $\{f_t; t \in T\}$ is a local martingale [34].

Unless one limits one's attention to F -processes of class DL one is forced to deal with f with μ_f only finitely additive. Astbury in [14] finds that (in the case where T is

an index set which is not linearly ordered) questions of order convergence of martingales necessarily involves the concepts of purely finitely additive measures.

We propose to replace the stochastic processes $\{f_t: t \in T\}$ with the corresponding family of measures $\{f_t \cdot P|_{\mathfrak{F}_t}: t \in T\} = \{X_t: t \in T\}$. Here each X_t is the measure on \mathfrak{F}_t absolutely continuous with respect to $P|_{\mathfrak{F}_t}$ with density f_t . Since two processes $\{f_t: t \in T\}$ and $\{\tilde{f}_t: t \in T\}$ giving rise to $\{X_t: t \in T\}$ are equivalent (up to modification) nothing has been lost except ambiguity inherent in sets of measure 0. Most properties (excepting maybe almost sure convergence) of processes are conveniently phrased in terms of $\{X_t: t \in T\}$. At this stage one realizes that very few properties of stochastic processes depend on P but rather only on $\{X_t: t \in T\}$ as a class of "mass distributions" varying as t varies in T . That is, P could be replaced by a mutually absolutely continuous Q for most purposes. Once one chooses to ignore P and concentrate on the process $\{X_t: t \in T\}$ adapted to $\{\mathfrak{F}_t: t \in T\}$, one might assume that one framework which captures most of the properties one is interested in is a Boolean algebra \mathfrak{F} , a possibly directed but usually linearly ordered set $\{\mathfrak{F}_t: t \in T\}$ of subalgebras, and a process $\{X_t: t \in T\}$ of finitely additive measures "adapted" to $\{\mathfrak{F}_t: t \in T\}$ in that each X_t is defined on \mathfrak{F}_t . This is the framework we shall adopt. This framework allows us to use the theory of convex compact sets to represent our F -processes as a Kakutani L -space with a Bauer simplex as the positive unit ball \diamond^+ . Full usage of Stonian analysis allows us to represent F -processes as the space of all finitely additive measures of bounded variation on a certain Boolean ring of predictable events. This ring of predictable events is just the ring of compact open subsets of $\xi(\diamond^+) \setminus \{0\}$ but may be described in terms close to the usual description. Finally, it is shown that the usual space of F -processes adapted to a filtration $\{\mathfrak{F}_t\}$ in $(\Omega, \mathfrak{F}, P)$ may be embedded in the L -space of finitely additive F -processes adapted to the same filtration. That is, in this case our framework essentially extends the allowable processes from those with representing measures μ on the ring of predictable events with $\mu^t \ll P|_{\mathfrak{F}_t}$ for all $t \in T$ to arbitrary finitely additive measures of bounded variation. Thus, in our setting, purely finitely additive representing measures are not pathological but rather are to be expected. We also should remark that nowhere do we make the usual assumption of right continuity of the filtration $\{\mathfrak{F}_t\}$.

It is useful to note that the question of when an F -process is of class DL or not does not arise in our context. We are able to show that every finitely additive supermartingale is a decreasing process with respect to *some* reference probability measure (but not necessarily with respect to one given a priori). Supermartingales of random variables fail to be of class DL iff they may not be represented as decreasing processes with respect to the given probability measure. However, there is always some probability measure, with a possibly nontrivial purely finitely additive part, so that an F -process of random variables may be represented as a decreasing process.

1. Additive functions on Boolean algebras and Stone spaces. \mathfrak{B} denotes a fixed Boolean algebra with supremum N and infimum \emptyset . The lattice operations on \mathfrak{B} will be denoted by \cup and \cap . We will think of \mathfrak{B} as an algebra of sets; hence we will

speak of supersets and subsets with \subset and \supset having the obvious meaning. \mathfrak{B}^* will denote the complete lattice of subalgebras of \mathfrak{B} when ordered by inclusion.

If μ is an additive function on \mathfrak{B} into $[-\infty, \infty]$ it has a positive variation μ^+ , a negative variation μ^- , and an absolute variation $|\mu| = \mu^+ + \mu^-$ defined in the usual fashion as additive functions on \mathfrak{B} into $[0, \infty]$. $FA(\mathfrak{B})$ denotes the real vector space of real valued additive functions on \mathfrak{B} . Denote by $BA^+(\mathfrak{B})$ the cone of nonnegative elements of $FA(\mathfrak{B})$. The total variation $\|\mu\|$ of an additive μ is equal to $|\mu|(N)$. The space $BA(\mathfrak{B})$ of additive functions of bounded variation is the set $\{\mu \text{ additive, } |\mu|(N) < \infty\}$ and is equal to $BA^-(\mathfrak{B}) - BA^+(\mathfrak{B})$. The function $\mu \rightarrow \|\mu\|$ is a norm on $BA(\mathfrak{B})$. We set $\mathfrak{P}(\mathfrak{B}) = \{\mu \in BA^+(\mathfrak{B}): \|\mu\| = 1\}$ and call this space the set of probability measures B on \mathfrak{B} . $BA^+(\mathfrak{B})$ is a lattice ordering cone under which order $BA(\mathfrak{B})$ becomes a Banach lattice with $\mu^+ = \sup(\mu, 0)$, $\mu^- = \sup(-\mu, 0)$ and $|\mu| = \sup(\mu, -\mu)$.

If \mathfrak{F} is a family of additive functions on \mathfrak{B} , its pointwise supremum, $\bigvee \mathfrak{F}$, is subadditive and its pointwise infimum, $\bigwedge \mathfrak{F}$, is superadditive on \mathfrak{B} . The least additive majorant of \mathfrak{F} will be denoted by $\sup \mathfrak{F}$ when it exists. When all $f \in \mathfrak{F}$ omit $-\infty$ as a value, $\sup \mathfrak{F}$ always exists. Similarly, $\inf \mathfrak{F}$ is defined as the greatest additive minorant of $\bigwedge \mathfrak{F}$ and this always exists if all $f \in \mathfrak{F}$ omit $+\infty$ as a value. Note that the cone of nonnegative $[0, \infty]$ -valued additive functions is a complete lattice with \inf and \sup as the infinite lattice operations. $FA(\mathfrak{B})$ and $BA(\mathfrak{B})$ are Dedekind complete lattices with \inf and \sup as infinite lattice operations. If $\mathfrak{F} \subset BA(\mathfrak{B})$ then $\sup \mathfrak{F}$ exists as an additive $(-\infty, \infty]$ -valued function and $\inf \mathfrak{F}$ exists as an additive $[-\infty, \infty)$ -valued function.

If $\mathcal{A}_1 \subset \mathcal{A}_2$ are in \mathfrak{B}^* and μ is an additive function on \mathcal{A}_2 , its restriction to \mathcal{A}_1 will be denoted by $E(\mu | \mathcal{A}_1)$ and is additive on \mathcal{A}_1 . This notation is entirely consistent with the usual notion of conditional expectation of random variables from one probability algebra to a smaller one. Note that $\mu \rightarrow E(\mu | \mathcal{A}_1)$ is norm decreasing for the variation norm and that, in fact, $E(|\mu| | \mathcal{A}_1) \geq |E(\mu | \mathcal{A}_1)|$. If ν is superadditive and has greatest additive minorant μ then $E(\mu | \mathcal{A}_1)$ is dominated by the greatest additive minorant of, $\nu | \mathcal{A}_1$, the restriction of ν to \mathcal{A}_1 . From this it follows that if \mathfrak{F} is a family of additive functions on \mathcal{A}_2 we have $E(\sup \mathfrak{F} | \mathcal{A}_1) \geq \sup\{E(f | \mathcal{A}_1): f \in \mathfrak{F}\}$ and $E(\inf \mathfrak{F} | \mathcal{A}_1) \leq \inf\{E(f | \mathcal{A}_1): f \in \mathfrak{F}\}$ when these exist.

The Stone space $X_{\mathfrak{B}}$ of a Boolean algebra \mathfrak{B} is the set of ultrafilters on \mathfrak{B} . For any $A \in \mathfrak{B}$, $[A]$ denotes the set $\{x \in X_{\mathfrak{B}}: x \in A\}$. The assignment $A \rightarrow [A]$ is a Boolean isomorphism of \mathfrak{B} into 2^X whose image $[\mathfrak{B}]$ may at times be identified with \mathfrak{B} . $[\mathfrak{B}]$ forms the base of a totally disconnected compact Hausdorff topology on $X_{\mathfrak{B}}$, the Stone topology. The clopen algebra for the Stone topology is precisely $[\mathfrak{B}]$. Any compact totally disconnected Hausdorff space may be considered to be $X_{\mathfrak{B}}$ where \mathfrak{B} is the clopen algebra for X . This is because, for algebras \mathcal{A} and \mathfrak{B} , $X_{\mathcal{A}}$ is homeomorphic to $X_{\mathfrak{B}}$ iff \mathcal{A} and \mathfrak{B} are isomorphic. If $j: \mathcal{A} \rightarrow \mathfrak{B}$ is a Boolean injection, there is a dual continuous surjection $j^*: X_{\mathfrak{B}} \rightarrow X_{\mathcal{A}}$ defined by the requirement that $(j^*)^{-1}[A] = [j(A)]$ for $A \in \mathcal{A}$. If \mathcal{A} is regarded as a subalgebra of \mathfrak{B} , j^* restricts an ultrafilter from \mathfrak{B} to \mathcal{A} . If $\pi: \mathfrak{B} \rightarrow \mathcal{A}$ is a homomorphism of \mathfrak{B} onto \mathcal{A} with kernel \mathfrak{I} , there is a dual homeomorphic embedding $\pi^*: X_{\mathcal{A}} \rightarrow X_{\mathfrak{B}}$. π^* assigns to an ultrafilter x on \mathcal{A} the

ultrafilter $\pi^*(x)$ on \mathfrak{B} given by $\pi^*(x) = \{A \in \mathfrak{B} : \pi(A) \in x\}$. Thus, to \mathcal{G} is associated the open set $\cup \{[A] : [A] \cap \pi^*(X_{\mathcal{G}}) = \emptyset\} = \theta_{\mathcal{G}}$ in $X_{\mathfrak{B}}$. We have $\theta_{\mathcal{G}} = \cup \{[A] : A \in I\}$. The map $\mathcal{G} \rightarrow \theta_{\mathcal{G}}$ is lattice isomorphic from the lattice of ideals of \mathfrak{B} onto the lattice of open sets of $X_{\mathfrak{B}}$. Dually there is a lattice isomorphism between filters of \mathfrak{B} and closed sets of $X_{\mathfrak{B}}$ which associates with a filter \mathcal{F} the closed set $\cap \{[A] : \mathcal{F}\} = K_{\mathcal{F}}$. If $K \subset X_{\mathfrak{B}}$ is closed it has as a base for its filter of neighborhoods $\{[A] : K \subset [A]\} = \{[A] : A \in \mathcal{F}\}$ if $K = K_{\mathcal{F}}$.

If \mathfrak{B} is a Boolean algebra with Stone space $X_{\mathfrak{B}}$, the Baire σ -algebra of $X_{\mathfrak{B}}$ is the monotone sequential closure of $[\mathfrak{B}]$. If μ is any element of $BA(\mathfrak{B})$ there is a corresponding countably additive Baire measure $\tilde{\mu}$ on $X_{\mathfrak{B}}$ satisfying $\tilde{\mu}([A]) = \mu(A)$ for $A \in \mathfrak{B}$. $\tilde{\mu}$ may be inductively defined on the Baire algebra by the requirement of monotone sequential continuity. The measure $\tilde{\mu}$ automatically extends to a regular Borel measure on $X_{\mathfrak{B}}$. For $\mu \geq 0$ and θ open in $X_{\mathfrak{B}}$, $\tilde{\mu}(\theta) = \sup\{\mu(A) : [A] \subset \theta\}$. For $K \subset X_{\mathfrak{B}}$ closed $\tilde{\mu}(K) = \inf\{\mu(A) : [A] \supset K\}$. The correspondence, $\mu \rightarrow \tilde{\mu}$, called the Stone correspondence, is a Banach lattice isomorphism from $BA(\mathfrak{B})$ to the Radon measures, $\mathfrak{M}(X_{\mathfrak{B}})$, on $X_{\mathfrak{B}}$. Thus, $\|\mu\| = \|\tilde{\mu}\|$ and $\mu \geq 0$ iff $\tilde{\mu} \geq 0$.

The concept of absolute continuity, when lifted from $\mathfrak{M}(X_{\mathfrak{B}})$ to $BA(\mathfrak{B})$ via the Stone correspondence, is that $\mu \ll \nu$ iff $|\mu| \ll |\nu|$ iff for all $\epsilon > 0$ there is a $\delta > 0$ so that when $|\nu|(A) < \delta$ then $|\mu|(A) < \epsilon$. The corresponding concept of singularity is that $\mu \perp \nu$ iff $|\mu| \perp |\nu|$ iff for all $\epsilon > 0$ there is an $A \in \mathfrak{B}$ so that $|\mu|(A) < \epsilon$ and $|\nu|(A^c) < \epsilon$.

If $\nu \in BA(\mathfrak{B})$ then $\mathfrak{N}_{\nu} = \{A : |\nu|(A) = 0\}$ as an ideal of \mathfrak{B} . The set $\text{supp}(\tilde{\nu}) \subset X_{\mathfrak{B}}$ is the complement of $\cup \{[A] : A \in \mathfrak{N}_{\nu}\}$. As a result, if one defines $\mu \ll_w \nu$ to mean that $|\mu|(A) = 0$ when $|\nu|(A) = 0$ for $\{\mu, \nu\} \subset BA(\mathfrak{B})$, then $\mu \ll_w \nu$ iff $\text{supp}(\tilde{\mu}) \subset \text{supp}(\tilde{\nu})$. We will call \ll_w *weak absolute continuity*. The corresponding concept of singularity, called *strong singularity*, is that μ and ν are strongly singular iff $|\mu|(\text{supp}(\tilde{\nu})) = 0 = |\tilde{\nu}|(\text{supp}(\tilde{\mu}))$. If μ and ν are strongly singular then for all $\epsilon > 0$ there are $A_1 \in \mathfrak{N}_{\nu}$ and $A_2 \in \mathfrak{N}_{\mu}$ with $|\mu|(A_1^c) < \epsilon$ and $|\nu|(A_2^c) < \epsilon$. We write $\mu \perp_s \nu$ iff μ and ν are strongly singular. If μ and ν have the property that there is an $A \in \mathfrak{B}$ with $|\mu|(A) = |\nu|(A^c) = 0$ we say that μ and ν are *disjoint* and write $\mu \perp_d \nu$. We have $\mu \perp_d \nu$ iff $\text{supp}(\tilde{\nu}) \cap \text{supp}(\tilde{\mu}) = \emptyset$. We have $\mu \perp_d \nu \Rightarrow \mu \perp_s \nu \Rightarrow \mu \perp \nu$ with these implications proper in general, see [10] for a further discussion.

If \mathcal{G} is an ideal of the algebra \mathfrak{B} and ν is an element of $BA(\mathfrak{B})$ with $\mathcal{G} \subset \mathfrak{N}_{\nu}$, then one may define the measure ν on \mathfrak{B}/\mathcal{G} by requiring that $\nu(A\Delta\mathcal{G}) = \nu(A)$ for $A \in \mathfrak{B}$. Here Δ is symmetric difference and $A\Delta\mathcal{G} = \{A\Delta I : I \in \mathcal{G}\}$. This defines a surjection from $\{\nu \in BA(\mathfrak{B}) : \mathcal{G} \subset \mathfrak{N}_{\nu}\}$ to $BA(\mathfrak{B}/\mathcal{G})$. The most common \mathcal{G} is \mathfrak{N}_{μ} for some $\mu \in BA(\mathfrak{B})$. One has an identification of $BA(\mathfrak{B}_{\mu})$ with the elements $\nu \in BA(\mathfrak{B})$ with $\nu \ll_w \mu$. Here \mathfrak{B}_{μ} is the *measure algebra*, $\mathfrak{B}/\mathfrak{N}_{\mu}$, of μ . In the Stone space the corresponding situation is the identification of $\mathfrak{M}(K)$ as a subset of $\mathfrak{M}(X_{\mathfrak{B}})$ where K is a closed set whose complement corresponds to \mathcal{G} .

If \mathfrak{R} is a Boolean ring without a unit it may be embedded as a maximal ideal in a (unique up to isomorphism) Boolean algebra \mathfrak{B} . \mathfrak{B} is obtained by formally adjoining the ultrafilter $\infty = \{A^c : A \in \mathfrak{R}\}$ of "complements" of elements of \mathfrak{R} . The ultrafilters on \mathfrak{B} other than ∞ are denoted by $X_{\mathfrak{R}}$ and are precisely extensions of

ultrafilters on \mathfrak{R} to \mathfrak{B} . $X_{\mathfrak{R}}$ is an open subset of $X_{\mathfrak{B}}$; hence is a locally compact totally disconnected Hausdorff space. The ring of compact open subsets $X_{\mathfrak{R}}$ is $[\mathfrak{R}] = \{[A]: A \in \mathfrak{R}\}$. $X_{\mathfrak{R}}$ is called the *Stone space* of \mathfrak{R} . Any locally compact totally disconnected Hausdorff space “is” the Stone space of its ring of compact open sets. The bounded Radon measures on $X_{\mathfrak{R}}$ are identified with the elements of $\mathfrak{M}(X_{\mathfrak{R}})$ annihilating ∞ , hence with $\mu \in BA(\mathfrak{B})$ with $\lim_{A \in \infty} \mu(A) = 0$. Such μ however are precisely the extensions of additive functions of bounded variation on \mathfrak{R} . That is, there is a Stone correspondence between $BA(\mathfrak{R})$ and $\mathfrak{M}_b(X_{\mathfrak{R}})$ defined by assigning to $\mu \in BA(\mathfrak{R})$ that $\tilde{\mu} \in \mathfrak{M}_b(X_{\mathfrak{R}})$ with $\tilde{\mu}([A]) = \mu(A)$ for $A \in \mathfrak{R}$. The Stone correspondence is a Banach lattice isomorphism of $BA(\mathfrak{R})$ and $\mathfrak{M}_b(X_{\mathfrak{R}})$ where both spaces are equipped with their variation norms. The *vague topology* $\sigma(\mathfrak{M}_b(X_{\mathfrak{R}}), \mathcal{C}_c(X_{\mathfrak{R}}))$ on $\mathfrak{M}_b(X_{\mathfrak{R}})$ (where $\mathcal{C}_c(X_{\mathfrak{R}})$ denotes the continuous functions of compact support on $X_{\mathfrak{R}}$) induces the weakest topology on $BA(\mathfrak{R})$ making the maps $\mu \rightarrow \mu(A)$ continuous for all $A \in \mathfrak{R}$. When \mathfrak{R} is an algebra, so $\mathfrak{R} = \mathfrak{B}$, the unit ball of $BA(\mathfrak{R})$ for the variation norm is compact for this vague topology (and conversely). The positive face $\mathfrak{P}(\mathfrak{R})$ of the unit ball of $BA(\mathfrak{R})$ is the set of (finitely additive) probability measures on \mathfrak{R} and is identified with $\mathfrak{M}_1^+(X_{\mathfrak{R}})$, the probability Radon measures on $X_{\mathfrak{R}}$. Thus, $\mathfrak{P}(\mathfrak{R})$ is a simplex (more precisely a K -simplex [7]) as the positive face of the unit ball of a Kakutani L -space. When $\mathfrak{R} = \mathfrak{B}$ is an algebra, $\mathfrak{P}(\mathfrak{R})$ is vaguely compact; hence is a Choquet simplex. In this case the extreme points $\xi(\mathfrak{P}(\mathfrak{B}))$ of $\mathfrak{P}(\mathfrak{B})$, which are the $\{0, 1\}$ -valued elements, are vaguely closed so $\mathfrak{P}(\mathfrak{B})$ is a Bauer simplex isomorphic to $\mathfrak{M}_1^+(\xi(\mathfrak{P}(\mathfrak{B})))$. As a result, $X_{\mathfrak{B}}$ is homeomorphic to $\xi(\mathfrak{P}(\mathfrak{B}))$. When \mathfrak{R} is a ring without unit $\xi(\mathfrak{P}(\mathfrak{B}))$ remains homeomorphic with $X_{\mathfrak{R}}$ and $\mathfrak{P}(\mathfrak{R})$ is affinely equivalent to $\mathfrak{M}_1^+(\xi(\mathfrak{P}(\mathfrak{R})))$. See [10] for similar results.

When \mathfrak{R} is a ring without unit $X_{\mathfrak{R}}$ is locally compact. The dual of $\mathcal{C}_c(X_{\mathfrak{R}})$ is $\mathfrak{M}(X_{\mathfrak{R}})$, the space of (possibly) unbounded Radon measures on $X_{\mathfrak{R}}$. $\mathfrak{M}(X_{\mathfrak{R}})$ is the inductive limit of Banach spaces $\{\mathfrak{M}([A]): A \in \mathfrak{R}\}$ and may be given either the inductive limit topology generated by the seminorms $\|\mu\|_A: \tilde{\mu} \rightarrow \|\tilde{\mu}|_{[A]}\|$ with $A \in \mathfrak{R}$ or the vague topology $\sigma(\mathfrak{M}(X_{\mathfrak{R}}), \mathcal{C}_c(X_{\mathfrak{R}}))$. We have $\mathfrak{M}(X_{\mathfrak{R}}) = \mathfrak{M}^+(X_{\mathfrak{R}}) - \mathfrak{M}^+(X_{\mathfrak{R}})$. Corresponding, à la Stone, to $\mathfrak{M}^+(X_{\mathfrak{R}})$ is the cone $FA^+(\mathfrak{R})$ of additive $\mu: \mathfrak{R} \rightarrow [0, \infty)$. Call the space $FA^+(\mathfrak{R}) - FA^+(\mathfrak{R})$ the space of additive real-valued functions on \mathfrak{R} *locally of bounded variation* and denote it by $BA_{\text{loc}}(\mathfrak{R})$. There is a linear isomorphism $\mu \rightarrow \tilde{\mu}$ between $BA_{\text{loc}}(\mathfrak{R})$ and $\mathfrak{M}(X_{\mathfrak{R}})$, again called the *Stone correspondence*. For $A \in \mathfrak{R}$ we have $\mu(A) = \tilde{\mu}([A])$ and this requirement defines $\tilde{\mu}$ given μ . The vague topology on $\mathfrak{M}(X_{\mathfrak{R}})$ determines the vague topology on $BA_{\text{loc}}(\mathfrak{R})$ which is the weakest rendering the map $\mu \rightarrow \mu(A)$ continuous for $A \in \mathfrak{R}$. The inductive limit definition of $\mathfrak{M}(X_{\mathfrak{R}})$ shows that $BA_{\text{loc}}(\mathfrak{R})$ is the inductive limit of the Banach spaces $\{BA(\mathfrak{R}_A): A \in \mathfrak{R}\}$ where $\mathfrak{R}_A = \{A' \in \mathfrak{R}: A' \subset A\}$. The inductive limit topology on $BA_{\text{loc}}(\mathfrak{R})$ is generated by the seminorms $\|\mu\|_A = \mu \rightarrow \|\chi_A \mu\| = \|\mu|_{\mathfrak{R}_A}\|$ where $\mu|_{\mathfrak{R}_A}$ is the restriction of μ to \mathfrak{R}_A and $\chi_A \mu$ is the measure equal to μ on A and to 0 on A^c . For further details see [10]. We remark that if one considers a $\mu: \mathfrak{B} \rightarrow [-\infty, \infty]$ which is additive on the algebra \mathfrak{B} one’s attention is drawn to $\mathfrak{R}(\mu) = \{A: \mu|_{\mathfrak{B}_A} \in BA(\mathfrak{B}_A)\}$, the largest ideal in \mathfrak{B} on which μ is locally

of bounded variation, then μ may be identified with an element of $BA_{loc}(\mathfrak{R}(\mu))$. If $\mu \geq 0$ then $\mathfrak{R}(\mu) = \{A: \mu(A) < \infty\}$.

Let $\mathfrak{A}_1 \subset \mathfrak{A}_2$ be algebras. Suppose $\nu \ll P$ for some $P \in \mathfrak{P}(\mathfrak{A}_2)$. Passing to the Stone space of \mathfrak{A}_2 we have $\tilde{\nu} \ll \tilde{P}$. Let \tilde{f} be a Baire measurable version of $d\tilde{\nu}/d\tilde{P}$ on $X_{\mathfrak{A}_2}$. \mathfrak{A}_2 may be regarded as the clopen algebra of $X_{\mathfrak{A}_2}$ and \mathfrak{A}_1 the clopen algebra of $X_{\mathfrak{A}_1}$. Let $j_{\mathfrak{A}_2, \mathfrak{A}_1}$ denote the inclusion mapping $\mathfrak{A}_1 \subset \mathfrak{A}_2$ and $p_{\mathfrak{A}_1, \mathfrak{A}_2}$ the dual projection of $X_{\mathfrak{A}_2}$ onto $X_{\mathfrak{A}_1}$. The map $p_{\mathfrak{A}_1, \mathfrak{A}_2}$ restricts an ultrafilter on \mathfrak{A}_2 to \mathfrak{A}_1 . $\{p_{\mathfrak{A}_1, \mathfrak{A}_2}^{-1}(A): A \in \mathfrak{A}_1\} = p_{\mathfrak{A}_1, \mathfrak{A}_2}^{-1}(\mathfrak{A}_1)$ is the embedding of the clopen algebra of $X_{\mathfrak{A}_1}$ into that of $X_{\mathfrak{A}_2}$. The Baire algebra of $X_{\mathfrak{A}_1}$ has as inverse image under $p_{\mathfrak{A}_1, \mathfrak{A}_2}$ the σ -algebra $\sigma(p_{\mathfrak{A}_1, \mathfrak{A}_2}^{-1}(\mathfrak{A}_1))$ which is a subalgebra of the Baire algebra of $X_{\mathfrak{A}_2}$. The inverse image of the Borel algebra of $X_{\mathfrak{A}_1}$ may be similarly described as a subalgebra of the Borel algebra of $X_{\mathfrak{A}_2}$. Corresponding under the Stone correspondence to $E(\nu | \mathfrak{A}_1)$ is the Radon measure $\tilde{\omega}$ on $X_{\mathfrak{A}_1}$. $\tilde{\omega}$ induces a measure $\bar{\omega}$ on $\sigma(p_{\mathfrak{A}_1, \mathfrak{A}_2}^{-1}(\mathfrak{A}_2))$ defined by $\bar{\omega}(p_{\mathfrak{A}_1, \mathfrak{A}_2}^{-1}(A)) = \tilde{\omega}(A)$ for A Baire in $X_{\mathfrak{A}_1}$. For $A \in \mathfrak{A}_1$, $\bar{\omega}(p_{\mathfrak{A}_1, \mathfrak{A}_2}^{-1}(A)) = E(\nu | \mathfrak{A}_1)(A) = \nu(A)$. As a result, $\bar{\omega} = E(\tilde{\nu} | \sigma(p_{\mathfrak{A}_1, \mathfrak{A}_2}^{-1}(\mathfrak{A}_1)))$. Consequently, $\bar{\omega} = E(\tilde{f} | \sigma(p_{\mathfrak{A}_1, \mathfrak{A}_2}^{-1}(\mathfrak{A}_1))) \cdot P$ (where we have here the usual conditional expectation of \tilde{f}). That is, $E(\nu | \mathfrak{A}_1)$ corresponds to $E(\tilde{f} | \sigma(p_{\mathfrak{A}_1, \mathfrak{A}_2}^{-1}(\mathfrak{A}_1)))$ under the Stone correspondence.

LEMMA 1-1. Let $\mathfrak{A}_1 \subset \mathfrak{A}_2$ be Boolean algebras. Let $\mu_1 \in BA^+(\mathfrak{A}_1)$ and $\mu_2 \in BA^+(\mathfrak{A}_2)$ with $\mu_1 \ll E(\mu_2 | \mathfrak{A}_1)$. There is an extension $\bar{\mu}_1$ of μ_1 to \mathfrak{A}_2 with $\bar{\mu}_1 \ll \mu_2$. If $\lambda \in [0, \infty)$ with $\mu_1 \leq \lambda E(\mu_2 | \mathfrak{A}_1)$ then $\bar{\mu}_1 \leq \lambda \mu_2$.

PROOF. Let \tilde{f} be the Radon-Nikodym derivative of $\bar{\mu}_1$ with respect to $\mu_2^\#$, the Radon measure on $X_{\mathfrak{A}_1}$ corresponding to $E(\mu_2 | \mathfrak{A}_1)$. Let $\tilde{\omega} = (\tilde{f} \circ P_{\mathfrak{A}_1, \mathfrak{A}_2}) \cdot \tilde{\mu}_2 \in \mathfrak{N}^+(X_{\mathfrak{A}_2})$. Let μ_1 be the element of $BA^+(\mathfrak{A}_2)$ corresponding to $\tilde{\omega}$ under the Stone correspondence. \square

The following lemma was initially established for conditional expectations of random variables by Jeulin [31]. We use here the fact that

$$E(Y | \mathfrak{A}_1)^+ \leq E(Y^+ | \mathfrak{A}_1)$$

for any $Y \in BA(\mathfrak{A}_2)$ with $\mathfrak{A}_1 \subset \mathfrak{A}_2$.

LEMMA 1-2. Let $\mathfrak{A}_1 \subset \mathfrak{A}_2$ be Boolean algebras. Let $Y_j \in BA(\mathfrak{A}_j)$ for $j = 1, 2$.

- (i) $[Y_1^+ - E(Y_2^+ | \mathfrak{A}_1)]^+ \leq [Y_1 - E(Y_2 | \mathfrak{A}_1)]^+$,
- (ii) $[Y_1^+ - E(Y_2^+ | \mathfrak{A}_1)] + |Y_1^+ - E(Y_2^+ | \mathfrak{A}_1)| \leq 2 |Y_1 - E(Y_2 | \mathfrak{A}_1)|$.

PROOF. (i) We have $Y_1^+ - E(Y_2^+ | \mathfrak{A}_1) \leq Y_1^+ - E(Y_2 | \mathfrak{A}_1)^+$. Since

$$Y_1^+ \leq E(Y_2 | \mathfrak{A}_1)^+ + [Y_1 - E(Y_2 | \mathfrak{A}_1)]^+$$

it follows that $Y_1^+ - E(Y_2^+ | \mathfrak{A}_1) \leq [Y_1 - E(Y_2 | \mathfrak{A}_1)]^+$. Statement (i) is now immediate since $[Y_1 - E(Y_2 | \mathfrak{A}_1)]^+ \geq 0$.

(ii) When \mathfrak{A}_1 and \mathfrak{A}_2 are random variables and $\mathfrak{A}_1 \subset \mathfrak{A}_2$ are σ -algebras, this has been established by Jeulin [31]. Otherwise, extend Y_1 to Y_1' on \mathfrak{A}_2 . Regard \mathfrak{A}_1 and \mathfrak{A}_2 as algebras of clopen sets in $X_{\mathfrak{A}_2}$. Set $P = (|Y_1| + |Y_2|) \cdot (\|Y_1'\| + \|Y_2\|)^{-1}$. Set f_2 equal to the Radon-Nikodym derivative \tilde{Y}_2 with respect to \tilde{P} , f_1' the derivative of \tilde{Y}_1' ,

and $f_1 = E(f_1' | \sigma(\mathcal{A}_1))$ which corresponds to Y_1 . By Jeulin,

$$|f_1^+ - E(f_2^+ | \sigma(\mathcal{A}_1))| + |f_1^- - E(f_2^- | \sigma(\mathcal{A}_1))| \leq 2 |f_1 - E(f_2 | \sigma(\mathcal{A}_1))|.$$

Upon appeal to the Stone correspondence we have (ii). \square

2. Finitely additive processes and supermartingales. If \mathfrak{B} is a Boolean algebra and T is a directed index set, then a *filtration* on \mathfrak{B} is a family $\{\mathcal{A}_t : t \in T\}$ of subalgebras of \mathfrak{B} so that $\mathcal{A}_t \subset \mathcal{A}_s$ if $t < s$. Here subalgebras of \mathfrak{B} are ordered by inclusion and, in most instances, $t \rightarrow \mathcal{A}_t$ is an order isomorphic injection. For this reason it is convenient to consider T to be a directed family of subalgebras of \mathfrak{B} . We will be considering only families Γ of subalgebras of \mathfrak{B} ordered by inclusion.

If Γ is a family of algebras in \mathfrak{B} by a *finitely additive stochastic process* on Γ , or *process* for short, we understand an assignment Y to each $\mathcal{A} \in \Gamma$ an element of $BA(\mathcal{A})$ denoted by $Y_{\mathcal{A}}$. That is, processes are elements of $\Pi\{BA(\mathcal{A}) : \mathcal{A} \in \Gamma\}$. This space is a vector lattice with the pointwise operations which is Dedekind complete in that if $\{Y^\alpha : \alpha \in \Lambda\}$ is a family of processes bounded above by a process Z then a supremum $Y = \bigvee\{Y^\alpha : \alpha \in \Lambda\}$ exists and is given by $Y_{\mathcal{A}} = \sup\{Y_{\mathcal{A}}^\alpha : \alpha \in \Lambda\} \leq Z_{\mathcal{A}}$ for all $\mathcal{A} \in \Gamma$. We note that the space of processes may be identified with the space of Radon measures on the free disjoint union $X = \bigcup\{X_{\mathcal{A}} : \mathcal{A} \in \Gamma\}$ of the Stone spaces of algebras in Γ for $\mathfrak{M}(X) \approx \Pi\{\mathfrak{M}(X_{\mathcal{A}}) : \mathcal{A} \in \Gamma\}$. Thus, the vague topology $\sigma(\mathfrak{M}(X), \mathcal{C}_c(X))$ carries over to a vague topology on processes. This is the coarsest topology rendering each map $Y \rightarrow Y_{\mathcal{A}}(A)$ continuous for $A \in \mathcal{A} \in \Gamma$.

A family \mathfrak{F} of processes on Γ is vaguely relatively compact iff for each $\mathcal{A} \in \Gamma$ the ensemble $\{Y_{\mathcal{A}} : Y \in \mathfrak{F}\}$ is bounded in $BA(\mathcal{A})$. This is a consequence of the Alaoglu theorem. For the most part we will be dealing with bounded processes Y on Γ for which $\|Y\| = \{\|Y_{\mathcal{A}}\| : \mathcal{A} \in \Gamma\} < \infty$. The norm $\|\cdot\|$ on processes is called the *uniform norm*. The family of bounded processes forms a vector space $[\Sigma\{BA(\mathcal{A}) : \mathcal{A} \in \Gamma\}]_{\infty}$ which is the l^{∞} -direct sum of the Banach lattices $\{BA(\mathcal{A}) : \mathcal{A} \in \Gamma\}$ [52]. The unit ball of this space is vaguely compact; hence this space is a dual Banach lattice. Since $[\Sigma\{BA(\mathcal{A}) : \mathcal{A} \in \Gamma\}]_{\infty}$ may be identified with $[\Sigma\mathfrak{M}(X_{\mathcal{A}}) : \mathcal{A} \in \Gamma]_{\infty}$, the predual may be identified with the l^1 -direct sum $[\Sigma\mathcal{C}(X_{\mathcal{A}}) : \mathcal{A} \in \Gamma]_1$.

A *martingale* on Γ is a process Y on Γ so that if $\mathcal{A}_1 \subset \mathcal{A}_2$ then $E(Y_{\mathcal{A}_2} | \mathcal{A}_1) = Y_{\mathcal{A}_1}$. A *supermartingale* on Γ is a process Y on Γ so that if $\mathcal{A}_1 \subset \mathcal{A}_2$ are in Γ then $E(Y_{\mathcal{A}_2} | \mathcal{A}_1) \leq Y_{\mathcal{A}_1}$. Notice that martingales form a vaguely closed vector space of processes and that supermartingales form a vaguely closed cone of processes on Γ . *Submartingales* are negatives of supermartingales and the martingales are the intersection of the cones of supermartingales and submartingales. We will denote by $\text{MART}(\Gamma)$ the vector space of bounded martingales of Γ and by $\text{SMART}(\Gamma)$ the cone of bounded supermartingales on Γ . $\text{MART}^+(\Gamma)$ and $\text{SMART}^+(\Gamma)$ denote the vaguely closed cones of nonnegative elements of $\text{MART}(\Gamma)$ and $\text{SMART}(\Gamma)$ respectively. Note that $E = \{Y \in \text{SMART}(\Gamma) : \|Y\| \leq 1\}$ is vaguely compact as are $E \cap \text{MART}(\Gamma)$, $E \cap \text{MART}^+(\Gamma)$, $E \cap \text{SMART}^+(\Gamma) = \diamond^+$, $\{M \in \text{MART}^+(\Gamma) : \|M\| = 1\}$.

Since $\{M \in \text{MART}(\Gamma): \|M\| \leq 1\}$ is vaguely compact $\text{MART}(\Gamma)$ is a dual Banach space under $\|\cdot\|$ with predual the vaguely continuous affine functions on the unit ball which vanish at 0. When Γ is increasing (= upper directed by inclusion, = filtering increasing) we can say somewhat more. We denote, in this case, by $\mathcal{Q}_\infty = \mathcal{Q}_\infty(\Gamma)$ the supremum of the algebras in Γ . If Y is any process on Γ and $\mathcal{Q}_\infty \notin \Gamma$ we denote by $Y_{\mathcal{Q}_\infty^-}$ the vague limit of the net $\{Y_{\mathcal{Q}}: \mathcal{Q} \in \Gamma\}$. That is, for $A \in \mathcal{Q}_\infty$, $Y_{\mathcal{Q}_\infty^-}(A)$ is defined to be $\lim\{Y_{\mathcal{Q}}(A): \mathcal{Q} \in \Gamma\}$ provided this limit exists. Notice that $Y_{\mathcal{Q}_\infty^-}$ is always additive on \mathcal{Q}_∞ when it exists. If $\mathcal{Q}_\infty \subset \mathcal{Q}_0$ and μ is an additive function on \mathcal{Q}_0 we may define a martingale $M(\mu) = M(\mu, \Gamma)$ on Γ associated with μ by setting $M(\mu)_{\mathcal{Q}} = E(\mu | \mathcal{Q})$ for $\mathcal{Q} \in \Gamma$. Notice that if $\mathcal{Q}_\infty \in \Gamma$ then a martingale Y is precisely of the form $M(Y_{\mathcal{Q}_\infty})$. Otherwise it is of the form $M(Y_{\mathcal{Q}_\infty^-})$. We have, as a result, the following proposition.

PROPOSITION 2-1. *Let Γ be an increasing family in \mathfrak{B} . $\text{MART}(\Gamma)$ is isomorphic, as an ordered Banach space, with $BA(\mathcal{Q}_\infty)$. The isomorphism is the map $\mu \rightarrow M(\mu)$ from $BA(\mathcal{Q}_\infty)$ to $\text{MART}(\Gamma)$.*

Thus, as an ordered Banach space, $\text{MART}(\Gamma)$ is a dual L -space. The positive face of the unit ball, $\text{MART}_1^+(\Gamma) = \{\mu \in \text{MART}^+(\Gamma): \|\mu\| = \mu(N) = 1\}$, is a Bauer simplex with extreme points equal to the $\{0, 1\}$ -valued martingales which are homeomorphic to the Stone space of \mathcal{Q}_∞ .

Suppose that Y is a process on the increasing family Γ . If $\mathcal{Q}_\infty \in \Gamma$ we set $\mathfrak{M}Y = M(Y_{\mathcal{Q}_\infty})$. Otherwise we set $\mathfrak{M}Y = M(Y_{\mathcal{Q}_\infty^-})$ provided that $Y_{\mathcal{Q}_\infty^-}$ exists. We set $\mathfrak{P}Y = Y - \mathfrak{M}Y$. We call $\mathfrak{M}Y$ the *martingale part* of Y and $\mathfrak{P}Y$ the *potential part*. Notice that, where defined, $\mathfrak{M}: Y \rightarrow \mathfrak{M}Y$ and $\mathfrak{P}: Y \rightarrow \mathfrak{P}Y$ are linear functions of processes Y . On the cone $\text{SMART}(\Gamma)$, \mathfrak{M} and \mathfrak{P} are very well behaved. This is because $\mathfrak{M}Y_1 \geq \mathfrak{M}Y_2$ whenever Y_1 and $-Y_2$ are supermartingales with $Y_1 \geq Y_2$. We will call any supermartingales P with $\mathfrak{M}P = 0$ a *potential*. We will call a process Y *lower bounded* iff $Y \geq B$ for some bounded process B iff $\{Y_{\mathcal{Q}}^-: \mathcal{Q} \in \Gamma\}$ is a bounded process. The proof of this theorem is immediate and well known.

THEOREM 2-2 (RIESZ-DOOB DECOMPOSITION). *Let Γ be an increasing family of subalgebras of \mathfrak{B} .*

- (1) *If Y is a lower bound supermartingale on Γ then $\mathfrak{M}Y$ is the largest submartingale minorant of Y and $\mathfrak{P}Y$ is a potential.*
- (2) *If Y is a supermartingale which is of the form $M + P$ with M a bounded martingale and P a potential, then Y is a lower bounded supermartingale with $\mathfrak{M}Y = M$ and $\mathfrak{P}Y = P$.*

REMARK. As a result the cone $\text{SMART}(\Gamma)$ is the direct sum of the cone of potentials and the space $\text{MART}(\Gamma)$. Since the potentials lie in $\text{SMART}^+(\Gamma)$ we will usually work with only $\text{SMART}^+(\Gamma)$.

PROPOSITION 2-3. *Let Γ be an increasing family of subalgebras of \mathfrak{B} with supremum \mathcal{Q}_∞ .*

- (a) *If $\mathcal{Q}_\infty \in \Gamma$ the potentials form a vaguely closed subcone of $\text{SMART}^+(\Gamma)$.*
- (b) *If $\mathcal{Q}_\infty \notin \Gamma$ the potentials are vaguely dense in $\text{SMART}^+(\Gamma)$.*

PROOF. (a) If $\mathcal{Q}_\infty \in \Gamma$ the potentials are those elements Y of $\text{SMART}^+(\Gamma)$ with $Y_{\mathcal{Q}_\infty} = 0$, hence are vaguely closed.

(b) If $\mathcal{Q}_\infty \notin \Gamma$ and $Y \in \text{SMART}^+(\Gamma)$ let, for $\mathcal{Q} \in \Gamma$, $Y^\mathcal{Q}$ be the potential defined by $Y_{\mathcal{Q}'}^\mathcal{Q} = Y_{\mathcal{Q}'}$, if $\mathcal{Q}' \subset \mathcal{Q}$ and by $Y_{\mathcal{Q}'}^\mathcal{Q} = 0$ otherwise. The net $\{Y^\mathcal{Q}: \mathcal{Q} \in \Gamma\}$ of potentials is vaguely convergent to Y . \square

At times when $\mathcal{Q}_\infty \notin \Gamma$ it will be convenient to extend processes to $\Gamma \cup \{\mathcal{Q}_\infty\}$. This will be done by setting $Y_{\mathcal{Q}_\infty} = Y_{\mathcal{Q}_\infty^-}$ if $Y_{\mathcal{Q}_\infty^-}$ exists. This extends martingales on Γ to martingales on $\Gamma \cup \{\mathcal{Q}_\infty\}$ and supermartingales on Γ to supermartingales on $\Gamma \cup \{\mathcal{Q}_\infty\}$. If \bar{Y} is a process on Γ which is extended to \bar{Y} on $\Gamma \cup \{\mathcal{Q}_\infty\}$ then $\overline{\mathfrak{P}}\bar{Y} = \mathfrak{P}\bar{Y}$ and $\overline{\mathfrak{N}}\bar{Y} = \mathfrak{N}\bar{Y}$.

3. Strategies: an example of martingales from gambling theory. Strategies in gambling theory yield an example of martingales in which finite additivity is an inescapable feature. As we shall see, although initially defined in terms of a linearly ordered family of algebras, the principal utility of strategies in gambling theory is as martingales on a nonlinearly ordered family of stopping algebras corresponding to stop rules or stopping times. There are many other instances where martingales on nonlinearly ordered filtrations play an important role. Examples are multiple parameter martingales and spatially indexed martingales.

In Dubins and Savage [22], a *fortune space* F is given as a discrete set. A *partial history of length n* is a sequence $(f_1, \dots, f_n) \in F^n$. A *history h* is an element of $H = F^N = F^\infty$. If $\{m, n\} \subset N \cup \{\infty\}$ with $m > n$ the restriction operation $p_{m,n}: F^m \rightarrow F^n$ assigns to a partial history or history $h = (f_1, \dots, f_m)$ its initial segment (f_1, \dots, f_n) . $\{(F^n), \{p_{m,n}\}, m, n < \infty\}$ is a projective system with projective limit H with associated projections $\{p_{n,\infty}: n \in N\}$. For $n < \infty$, F^n is discrete with the product topology and H , as the projective limit, is fitted with the product topology. Let \mathcal{Q}_n be the algebra of subsets of H which depend only on the first n coordinates. Thus, $\mathcal{Q}_n = \sigma(p_{n,\infty})$ for $n \in N$. The sequence $\{\mathcal{Q}_n: n \in N\}$ forms an increasing family of algebras of subsets of H whose union \mathcal{Q}_∞ consists of the subsets of H depending on only finitely many coordinates. \mathcal{Q}_∞ consists entirely of clopen subsets of H and is equal to the clopen algebra \mathfrak{B} of H iff F is finite. An element of \mathfrak{B} is called a *finitary set*. Continuous functions on H are called *finitary functions* and are uniformly approximable by finite linear combinations of characteristic functions of finitary sets.

A *strategy* $\sigma = (\sigma_0, \sigma_1, \dots, \sigma_n, \dots)$ is a sequence of Markov kernels σ_n from F^n to $\mathfrak{P}(2^F)$. Thus, σ_0 is an element of $\mathfrak{P}(2^F)$ (i.e. a gamble), and, for each n and $(f_1, \dots, f_n) \in F^n$, $\sigma_n(f_1, \dots, f_n) \in \mathfrak{P}(2^F)$. Each strategy induces a sequence $(\sigma^0, \sigma^1, \sigma^2, \dots)$ of gambles with $\sigma^0 = \sigma_0$ and σ^n an element of $\mathfrak{P}(2^{F^{n+1}})$ defined inductively via the integration formula

$$\begin{aligned} & \int_{F^{n+1}} s(f_1, \dots, f_n, f_{n+1}) d\sigma^n \\ &= \int \left[\int_F s(f_1, \dots, f_n, f_{n+1}) \sigma_n(f_1, \dots, f_n, df_{n+1}) \right] \sigma^{n-1}(df_1, \dots, df_n) \end{aligned}$$

valid for bounded functions s on F^{n+1} . Since 2^N is isomorphic to \mathcal{Q}_n under $P_{n,\infty}$ there is a σ^{n-1} on \mathcal{Q}_n with $\sigma^{n-1} \circ p_{n,\infty}^{-1} = \sigma^{n-1}$ called a *strategic measure* on \mathcal{Q}_n . The sequence $\{\sigma^{n-1}; n \in N\}$ is a martingale on $\{\mathcal{Q}_n; n \in N\}$. Let σ^∞ be the element of $\mathfrak{P}(\mathcal{Q}_\infty)$ corresponding to the martingale $\{\sigma^{n-1}; n \in N\}$ so that $\{\sigma^{n-1}; n \in N \cup \{\infty\}\}$ is a martingale on $\{\mathcal{Q}_n; n \in N \cup \{\infty\}\}$.

For many purposes the algebra \mathcal{Q}_∞ is too small. In Dubins and Savage [22] it is shown how to extend σ_∞ from \mathcal{Q}_∞ to \mathfrak{B} yielding a measure σ called the *strategic measure* in $\mathfrak{P}(\mathfrak{B})$ corresponding to the strategy σ . This is done simultaneously for all strategies σ . The extension is unique subject to the condition that for all strategies $\sigma = (\sigma_0, \sigma_1, \dots, \sigma_n, \dots)$ and finitary functions s , one has

$$\int_H s \, d\sigma = \int_F \left[\int_H s f_1 \, d\sigma[f_1] \right] \sigma_0(df_1)$$

where $s f_1(f_2, \dots, f_n, \dots) = s(f_1, f_2, \dots, f_n, \dots)$ and $\sigma[f_1] = (\sigma_1[f_1], \sigma_2[f_1], \dots)$ where $\sigma_n[f_1](f_2, \dots, f_n, \dots) = \sigma_n(f_1, f_2, \dots, f_n, \dots)$ for all n .

A *stop rule* τ is a map on H to N so that if $\tau(f_1, \dots, f_n, \dots) = n$ and $h = (f_1, \dots, f_n, f'_{n+1}, \dots, f'_m, \dots)$ then $\tau(h) = n$. All stop rules are finitary functions. For any stop rule τ we let $\mathcal{Q}_\tau \subset \mathfrak{B}$ consist of those sets A with $A \cap \{\tau \leq n\} \in \mathcal{Q}_n$, or, equivalently, $A \cap \{\tau = n\} \in \mathcal{Q}_n$, for $n \in N$. \mathcal{Q}_τ is called the *stopping algebra* for τ . \mathcal{Q}_τ is a complete algebra of subsets of H since each \mathcal{Q}_n is a complete algebra of subsets of H . If one chooses an arbitrary $A_n \subset \{\tau = n\}$ with $A_n \in \mathcal{Q}_n$ and sets $A = \bigcup \{A_n; n \in N\}$ one obtains an element of \mathcal{Q}_τ with

$$H \setminus A = \bigcup \{ \{\tau = n\} \setminus A_n; n \in N \}.$$

One notes that the A_n are open in H as are $\{\tau = n\} \setminus A_n$ for $n \in N$. Thus, A is clopen in H for $A \in \mathcal{Q}_\tau$. As a result $\mathcal{Q}_\tau \subset \mathfrak{B}$. Denote by σ^τ the element $E(\sigma | \mathcal{Q}_\tau)$ of $\mathfrak{P}(\mathcal{Q}_\tau)$. The ensemble $\{\sigma^\tau; \tau \text{ stop rule}\}$ is a martingale on $\{A_\tau; \tau \text{ stop rule}\}$ (note that for stop rules τ_1 and τ_2 , $\mathcal{Q}_{\tau_1} \subset \mathcal{Q}_{\tau_2}$ iff $\tau_1 \leq \tau_2$). If $n \in N$ denotes the constant stop rule always equal to n , then we have an extension of the original martingale $\{\sigma_n; n \in N\}$ to an enlarged family of subalgebras of \mathfrak{B} .

One important property of $\{\mathcal{Q}_\tau; \tau \text{ stop rule}\}$ is that this family is increasing with supremum \mathfrak{B} . To see that a clopen set A is in some \mathcal{Q}_τ one sets, for any n , A_n equal to the largest element of \mathcal{Q}_n contained in A . Define $(A^c)_n$ similarly. Then $\bigcup_n A_n = A$ and $\bigcup_n (A^c)_n = A^c$. Define the stop rule τ on H inductively by the requirement that $\{\tau = n\} = [A_n \cup (A^c)_n] \setminus \{\tau < n\}$. This τ is the smallest stop rule so that $A \in \mathcal{Q}_\tau$ and we have $A_n = A \cap \{\tau \leq n\}$ for all n .

The stopping algebras \mathcal{Q}_τ , for τ a stop rule, describe events which may be observed in the finitary length of time τ . The algebras \mathcal{Q}_τ describe events observable with sequential sampling techniques whereas \mathfrak{B} is too large for all events to be observed by a single sequential sampling technique.

4. The base norm ordered Banach space FPROC(Γ). We denote by FPROC(Γ) the vector space SMART⁺(Γ) – SMART⁺(Γ) for Γ a family of subalgebras of \mathfrak{B} . Note that FPROC(Γ) consists of bounded processes. Elements of FPROC(Γ) will be

called F -processes on Γ . We let $\diamond^+ = \diamond^+(\Gamma)$ denote the vaguely compact convex set $\{Y \in \text{SMART}^+(\Gamma) : \|Y\| \leq 1\}$ and we let $\diamond = \diamond(\Gamma)$ denote the vaguely compact set $\text{conv}(\diamond^+, -\diamond^+)$. \diamond is a compact convex symmetric absorbent subset of $\text{FPROC}(\Gamma)$. Thus, \diamond is the unit ball for a norm $\|\cdot\|_{\text{Dev}}$ on $\text{FPROC}(\Gamma)$ under which $\text{FPROC}(\Gamma)$ is a dual Banach space. Note that $\|\cdot\| = \|\cdot\|_D$ on $\text{SMART}^+(\Gamma)$. The predual of $\text{FPROC}(\Gamma)$ may be identified with the space $A_0(\diamond^+)$ consisting of vaguely continuous affine functions on \diamond^+ which vanish at $0 \in \xi(\diamond^+)$. $A_0(\diamond^+)$ is equipped with the pointwise order and uniform norm (as a subspace of $\mathcal{C}(\diamond^+)$). $\text{FPROC}(\Gamma)$ is the order dual of the ordered Banach space $A_0(\diamond^+)$. The cone $\text{SMART}^+(\Gamma)$ is the nonnegative cone of the ordered Banach space $\text{FPROC}(\Gamma)$.

Recall from [2, 13 or 3] that an ordered Banach space $(E, \|\cdot\|)$ with closed nonnegative cone E^+ is base normed iff $\|\cdot\|$ is additive on E^+ in that $\|x + y\| = \|x\| + \|y\|$ for all $\{x, y\} \subset E^+$ and if $z \in E$ there is a sequence $\{x_n, y_n\} \subset E^+$ with $z = x_n - y_n$ and $\|x_n\| + \|y_n\| \rightarrow \|z\|$. In this case $D = \{x \in E^+ : \|x\| = 1\}$ is a base for C and the unit ball of $\|\cdot\|$ is the norm closure of $\text{conv}(D, -D)$. If there are, for any $z \in E$, $\{x, y\} \subset E^+$ with $z = x - y$ and $\|z\| = \|x\| + \|y\|$ then E is said to be 1-generated [13]. E is 1-generated iff $\text{conv}(D, -D)$ is the unit ball for $\|\cdot\|$. The dual of a base norm ordered space E with base D may be identified with $\text{ba}(D)$, the bounded affine functions on D . The space $\text{ba}(D)$ is, as a subspace of the bounded functions on D , an ordered Banach space with the uniform norm and pointwise order. $\text{ba}(D)$ has unit ball $\{f : -1 \leq f \leq 1\} = [-1, 1]$, hence is an order unit space (here 1 is the constant function 1). As a result the (order) dual E' is order isomorphic to the order unit space $\text{ba}(D)$.

For a family Γ of algebras in \mathfrak{B} denote by \mathcal{A}_0 the infimum of Γ and by $\Delta = \Delta(\Gamma)$ the subset of \diamond^+ consisting of nonnegative supermartingales of norm 1.

PROPOSITION 4-1. *Let Γ be a family of subalgebras of \mathfrak{B} .*

(a) *FPROC(Γ) is a base norm ordered Banach space under $\|\cdot\|_D$ iff Γ is a decreasing family.*

(b) *When Γ is decreasing FPROC(Γ) is a 1-generated base norm ordered Banach space with base Δ .*

(c) *When Γ is decreasing with infimum \mathcal{A}_0 , Δ is vaguely compact if $\mathcal{A}_0 \in \Gamma$.*

PROOF. (a) If Γ is decreasing then it forms a net ordered by reverse inclusion. If $Y \in \text{SMART}^+(\Gamma)$ then $\|Y\|_D = \|Y\| = \lim\{\|Y\|_{\mathcal{A}} : \mathcal{A} \in \Gamma\}$. From this the additivity of $\|\cdot\|_D$ on $\text{SMART}^+(\Gamma)$ is easily seen. Conversely, suppose that Γ is not decreasing so that there are $\{\mathcal{A}_1, \mathcal{A}_2\} \subset \Gamma$ which are not minorized in Γ . Pick $\mu_1 \in \mathfrak{P}(\mathcal{A}_1)$ and $\mu_2 \in \mathfrak{P}(\mathcal{A}_2)$. Define $Y^j \in \text{SMART}^+(\Gamma)$ by $Y^j_{\mathcal{A}} = E(\mu_j | \mathcal{A})$ if $\mathcal{A} \subset \mathcal{A}_j$ and by $Y^j_{\mathcal{A}} = 0$ otherwise. We have $\|Y_1\| = \|Y_2\| = 1 = \|Y_1 + Y_2\|$. Thus, $\|\cdot\|_D = \|\cdot\|$ is not additive on $\text{SMART}^+(\Gamma)$. As a result $\|\cdot\|_D$ is additive on $\text{SMART}^+(\Gamma)$ iff Γ is decreasing.

(b) If Γ is decreasing then Δ is easily seen to be a base for $\text{SMART}^+(\Gamma)$. Since $\text{conv}(0, \Delta) = \diamond^+$ is vaguely compact, $\diamond = \text{conv}(\diamond^+, -\diamond^+) = \text{conv}(\Delta, -\Delta)$ is the unit ball for $\|\cdot\|_D$. As a result $\text{FPROC}(\Gamma)$ is a 1-generated base norm ordered Banach space with base Δ .

If $\mathfrak{A}_0 \in \Gamma$ then $\Delta = \{Y \in \diamond^+ : \|Y_{\mathfrak{A}_0}\| = 1\}$ is vaguely compact. Otherwise, as before, for $\mathfrak{A} \in \Gamma$ let $\mu_{\mathfrak{A}}$ be chosen in $\mathfrak{P}(\mathfrak{A})$ for all $\mathfrak{A} \in \Gamma$. Set $Y_{\mathfrak{A}'}^{\mathfrak{A}} = E(\mu_{\mathfrak{A}} | \mathfrak{A}')$ if $\mathfrak{A}' \subset \mathfrak{A}$ and set $Y_{\mathfrak{A}'}^{\mathfrak{A}} = 0$ otherwise. One thus obtains the net $\{Y^{\mathfrak{A}} : \mathfrak{A} \in \Gamma\} \subset \Delta$ which is vaguely convergent to $0 \notin \Delta$. This establishes (c). \square

When $\mathfrak{A}_0 \in \Gamma$, a predual to $FPROC(\Gamma)$ is $A(\Delta)$, the vaguely continuous affine functions fitted with the uniform norm and pointwise order.

When $\mathfrak{A}_0 \notin \Gamma$ it is often convenient to extend processes Y on the decreasing Γ to $\{\mathfrak{A}_0\} \cup \Gamma$ by setting $Y_{\mathfrak{A}_0} = \lim\{E(Y_{\mathfrak{A}} | \mathfrak{A}_0) : \mathfrak{A} \in \Gamma\}$ when this limit exists along Γ . When $\Gamma \setminus \{\mathfrak{A}_0\}$ is decreasing we set $Y_{\mathfrak{A}_0} = \lim\{E(\mathfrak{A} | \mathfrak{A}_0) : \mathfrak{A} \in \Gamma \setminus \{\mathfrak{A}_0\}\}$ which may or may not coincide with $Y_{\mathfrak{A}_0}$. When Γ is not decreasing it is often convenient to extend a supermartingale Y to $\Gamma \cup \{\mathfrak{A}_0\}$ by setting $Y_{\mathfrak{A}_0} = \sup\{E(\mathfrak{A} | \mathfrak{A}_0) : \mathfrak{A} \in \Gamma\}$ thus obtaining a supermartingale on $\Gamma \cup \{\mathfrak{A}_0\}$.

Since, when Γ is decreasing, $FPROC(\Gamma)$ is a 1-generated base norm ordered Banach space, each Z with $\|Z\|_D = 1$ may be expressed, as in the Jordan decomposition, as $Y_1 - Y_2$ with $\|Y_1\| + \|Y_2\| = \|Z\|$. One might ask if there is, as in the Jordan decomposition for measures, a canonical decomposition of this form (uniqueness is not to be expected unless Δ is a simplex [23]). The answer is yes. Note that $\{Y_1, Y_1 - Z\} \in SMART^+(\Gamma)$ and $\{Y_2, Y_2 + Z\} \in SMART^+(\Gamma)$. It is easily verified that if $\mathfrak{F} \subset SMART^+(\Gamma)$ then the infimum of \mathfrak{F} in the vector lattice $\Pi\{BA(\mathfrak{A}) : \mathfrak{A} \in \Gamma\}$ is in $SMART^+(\Gamma)$. As a result, the infimum D^+Z of those $Y \in SMART^+(\Gamma)$ with $Y - Z \in SMART^+(\Gamma)$, the infimum D^-Z of those $Y \in SMART^+(\Gamma)$ with $Y + Z \in SMART^+(\Gamma)$ and the infimum DZ of those $Y \in SMART^+(\Gamma)$ with $\{Y + Z, Y - Z\} \in SMART^+(\Gamma)$ all exist. When $Z = Y_1 - Y_2$ with $\{Y_1, Y_2\} \subset SMART^+(\Gamma)$ then $Y_1 \geq D^+Z$, $Y_2 \geq D^-Z$ and $Y_1 + Y_2 \geq DZ$.

PROPOSITION 4-2. *Let Γ be a decreasing family of subalgebras of \mathfrak{B} and let Z be in $FPROC(\Gamma)$.*

- (a) $Z = D^+Z - D^-Z$,
- (b) $DZ = D^+Z + D^-Z$,
- (c) $\|Z\|_D = \|DZ\|_D = \|D^+Z\|_D + \|D^-Z\|_D$.

PROOF. If Y is a process on Γ let $Y' = Y - Z$ and $Y'' = 2Y - Z$. Note that $\{Y, Y - Z\} \subset SMART^+(\Gamma)$ iff $\{Y', Y' + Z\} \subset SMART^+(\Gamma)$ iff $\{Y'' - Z, Y'' + Z\} \subset SMART^+(\Gamma)$. Consequently, as Y decreases to D^+Z , Y' decreases to D^-Z and Y'' decreases to DZ . From this all statements of the proposition follow easily. \square

5. $FPROC(\Gamma)$ is an L -space if Γ is a chain. We will establish in Theorem 5-2 that when Γ is a chain of subalgebras of \mathfrak{B} (so that it is linearly ordered) then $SMART^+(\Gamma)$ is lattice ordering. Thus, \diamond^+ is a Choquet simplex. $FPROC(\Gamma)$ under $\|\cdot\|_D$ is, as a result, an L -space with Δ a K -simplex (as the positive face of the unit ball of an L -space). The case where Γ is finite is handled first in Proposition 5-1. An alternative definition of $\|Z\|_D$ is given in terms of the coordinates of the process. When Z arises from an F -process of random variables this definition of $\|Z\|_D$ is the one usually given. As an application the result of Youerp that $Z \in FPROC(\Gamma)$ iff $|Z| \in FPROC(\Gamma)$ is given in the finitely additive setting.

PROPOSITION 5-1. Let $\Gamma = \{\mathcal{Q}_0 \subset \mathcal{Q}_1 \subset \dots \subset \mathcal{Q}_n\}$ be a chain of subalgebras of \mathfrak{B} .

(a) $\text{FPROC}(\Gamma)$ is lattice ordered by $\text{SMART}^+(\Gamma)$.

(a') If $Z \in \text{FPROC}(\Gamma)$ then $D^+ Z$, $D^- Z$ and DZ are respectively the positive part, the negative part and the absolute value of Z in the vector lattice $\text{FPROC}(\Gamma)$.

(b) For $Z \in \text{FPROC}(\Gamma)$, DZ has as its \mathcal{Q}_k -coordinate

$$D_{\mathcal{Q}_k} Z = \sum_{j=k}^{n-1} E(|Z_{\mathcal{Q}_j} - E(Z_{\mathcal{Q}_{j+1}} | \mathcal{Q}_j)| | \mathcal{Q}_k) + E(|Z_{\mathcal{Q}_n}| | \mathcal{Q}_k).$$

(c) For $Z \in \text{FPROC}(\Gamma)$, $\|Z\|_D$ is given by

$$\|Z_{\mathcal{Q}_n}\| + \sum_{j=0}^{n-1} \|Z_{\mathcal{Q}_j} - E(Z_{\mathcal{Q}_{j+1}} | \mathcal{Q}_j)\|.$$

PROOF. To establish (a) it is necessary to show that $\text{FPROC}(\Gamma)$ is lattice ordered by $\text{SMART}^+(\Gamma)$. We will do this by showing that if $Z \in \text{FPROC}(\Gamma)$ then $D^+(Z)$ is the supremum for the ordering induced by $\text{SMART}^+(\Gamma)$. We must show that if $\{Y, Y - Z\} \subset \text{SMART}^+(\Gamma)$ then $Y - D^+ Z \in \text{SMART}^+(\Gamma)$. This will be established for all $Z \in \text{FPROC}(\Gamma)$ based on an induction on the length $n + 1$ of the chain Γ . We shall, in the process, establish the analogue of (b) and (c) for $D_{\mathcal{Q}_k}^+ Z$ which is that

$$D_{\mathcal{Q}_k}^+ Z = \sum_{j=0}^{k-1} E\left(\left(Z_{\mathcal{Q}_j} - E(Z_{\mathcal{Q}_{j+1}} | \mathcal{Q}_j)\right)^+ | \mathcal{Q}_k\right) + E(Z_{\mathcal{Q}_n}^+ | \mathcal{Q}_k) \quad \text{for } k \leq n.$$

From this (b) and (c) are nearly immediate.

If $n = 0$ then $Z_{\mathcal{Q}_0}^+$ is $D_{\mathcal{Q}_0}^+ Z$. Furthermore in this case $D^+ Z$ is immediately the supremum of Z and 0 in $\text{FPROC}(\Gamma) \cong BA(\mathcal{Q}_0)$. Assume by induction that the assertion is valid for $\Gamma' = \{\mathcal{Q}_1, \dots, \mathcal{Q}_n\}$. That is, in $\text{FPROC}(\Gamma')$, $D^+ Z$ is given by the indicated formula and is the supremum in $\text{FPROC}(\Gamma')$ of Z and 0 if Z is regarded as an element of $\text{FPROC}(\Gamma')$ upon restriction to Γ' . Here the formula for $D_{\mathcal{Q}_0}^+ Z$ is $\sum_{j=0}^{n-1} E((Z_{\mathcal{Q}_j} - E(Z_{\mathcal{Q}_{j+1}} | \mathcal{Q}_j))^+ | \mathcal{Q}_0) + E(Z_{\mathcal{Q}_n}^+ | \mathcal{Q}_0)$. That is,

$$D_{\mathcal{Q}_0}^+ Z = (Z_{\mathcal{Q}_0} - E(Z_{\mathcal{Q}_1} | \mathcal{Q}_0))^+ + E(D_{\mathcal{Q}_1}^+ Z | \mathcal{Q}_0).$$

It is immediate that

$$D_{\mathcal{Q}_0}^+ Z \geq E(D_{\mathcal{Q}_1}^+ Z | \mathcal{Q}_0) \quad \text{and} \quad D_{\mathcal{Q}_0}^+ Z - Z_{\mathcal{Q}_0} \geq E(D_{\mathcal{Q}_1}^+ Z - Z_{\mathcal{Q}_1} | \mathcal{Q}_0).$$

Since $D^+ Z$ and $D^+ Z - Z$ are in $\text{SMART}^+(\Gamma')$ when restricted to Γ' it follows that $D^+ Z$ and $D^+ Z - Z$ are in $\text{SMART}^+(\Gamma)$. It is also clear that this definition of $D_{\mathcal{Q}_0}^+ Z \in BA(\mathcal{Q}_0)$ is the smallest possible for $D^+ Z$ and $D^+ Z - Z$ to be in $\text{SMART}^+(\Gamma)$. That is, $D^+ Z$ is correctly defined on Γ .

If $Y \in \text{SMART}^+(\Gamma)$ with $Y - Z \in \text{SMART}^+(\Gamma)$ then, when restricted to Γ' , $\{Y, Y - Z\} \subset \text{SMART}^+(\Gamma')$. Thus, by induction, $Y - D^+ Z \in \text{SMART}^+(\Gamma')$. We have $Y_{\mathcal{Q}_0} - Z_{\mathcal{Q}_0} \geq E(Y_{\mathcal{Q}_1} | \mathcal{Q}_0) - E(Z_{\mathcal{Q}_1} | \mathcal{Q}_0)$. Thus,

$$Y_{\mathcal{Q}_0} - E(Y_{\mathcal{Q}_1} | \mathcal{Q}_0) \geq [Z_{\mathcal{Q}_0} - E(Z_{\mathcal{Q}_1} | \mathcal{Q}_0)]^+.$$

As a result,

$$\begin{aligned} Y_{\mathcal{Q}_0} &\geq E(Y_{\mathcal{Q}_1} | \mathcal{Q}_0) + [Z_{\mathcal{Q}_0} - E(Z_{\mathcal{Q}_1} | \mathcal{Q}_0)]^+ \\ &\geq E(D_{\mathcal{Q}_1}^+ Z | \mathcal{Q}_0) + [Z_{\mathcal{Q}_0} - E(Z_{\mathcal{Q}_1} | \mathcal{Q}_0)]^+ = D_{\mathcal{Q}_0}^+ Z. \end{aligned}$$

That is, $Y - D^+ Z \in \text{SMART}^+(\Gamma)$. This shows that $D^+ Z$ is the supremum, for the order on $\text{FPROC}(\Gamma)$, of Z and 0 .

Replacing Z by $-Z$, $D^- Z$ is seen to be defined by

$$D_{\mathcal{Q}_k}^- Z = \sum_{j=k}^{n-1} E\left(\left(Z_{\mathcal{Q}_j} - E(Z_{\mathcal{Q}_{j+1}} | \mathcal{Q}_j)\right)^- | \mathcal{Q}_k\right) + E(Z_{\mathcal{Q}_n}^- | \mathcal{Q}_k)$$

and is the supremum of $-Z$ on 0 in $\text{FPROC}(\Gamma)$. Adding $D^- Z$ to $D^+ Z$ we obtain DZ . The formula in (b) for DZ is easily seen to be valid. The formula of (c) is an immediate consequence of (b) with $k = 0$ and the fact that $\|Z\|_D = \|D_{\mathcal{Q}_0} Z\|$. \square

To obtain this result for $\text{FPROC}(\Gamma)$ with Γ an arbitrary chain of algebras in \mathfrak{B} , we are going to heavily utilize the vague compactness of \diamond^+ . We have already encountered the process of restricting an element of $\text{SMART}^+(\Gamma)$ or of $\text{FPROC}(\Gamma)$ to $\Gamma' \subset \Gamma$ to obtain an element of $\text{SMART}^+(\Gamma')$ or of $\text{FPROC}(\Gamma')$. Restriction is vaguely continuous and a norm contraction for $\|\cdot\|_D$. There is a corresponding extension operation inverse to restriction. Since $\Gamma' \subset \Gamma$, $\Gamma \setminus \Gamma'$ is a union of intervals of Γ . We extend a $Y \in \text{SMART}^+(\Gamma')$ to a $Y \in \text{SMART}(\Gamma)$ by setting $Y_{\mathcal{Q}} = Y_{\mathcal{Q}}$ for $\mathcal{Q} \in \Gamma'$ and by requiring that Y be a martingale on each interval of $\Gamma \setminus \Gamma'$. Thus, if $\mathcal{Q} \in \Gamma \setminus \Gamma'$ then $Y_{\mathcal{Q}}$ is defined to be $\sup\{E(Y_{\mathcal{Q}'} | \mathcal{Q}) : \mathcal{Q}' \subset \mathcal{Q}' \in \Gamma'\}$, and to be 0 if no $\mathcal{Q}' \in \Gamma'$ includes \mathcal{Q} . This extension is a minimal extension of Y from Γ' to Γ in that the extension is the pointwise infimum in $\text{SMART}^+(\Gamma)$ of the Z with $Z \geq Y$ on Γ' . This extension process may be carried out for $Y \in \text{FPROC}(\Gamma)$ by linearity. Notice that the extension process is vaguely continuous and is a norm isometry. $\text{FPROC}(\Gamma')$ is, as a result, identified with those $Z \in \text{FPROC}(\Gamma)$ which are martingales on intervals of $\Gamma \setminus \Gamma'$ and have $Z_{\mathcal{Q}} = 0$ if \mathcal{Q} is not included in an $\mathcal{Q}' \in \Gamma'$.

THEOREM 5-2. *Let Γ be a chain of algebras of \mathfrak{B} .*

- (a) $\text{FPROC}(\Gamma)$ is an L -space with norm $\|\cdot\|_D$ and nonnegative cone $\text{SMART}^+(\Gamma)$.
- (a') For $Z \in \text{FPROC}(\Gamma)$, $D^+ Z$, $D^- Z$ and DZ are the positive part, negative part and absolute value of Z .
- (b) \diamond^+ is a Choquet simplex for the vague topology. Δ is a K -simplex for the vague topology. Δ is a Choquet simplex iff $\mathcal{Q}_0 \in \Gamma$.
- (c) If $Z \in \text{FPROC}(\Gamma)$ then $D_{\mathcal{Q}} Z$ is given by

$$D_{\mathcal{Q}} Z = \sup \left\{ \sum_{j=0}^{n-1} E\left(\left| Z_{\mathcal{Q}_j} - E(Z_{\mathcal{Q}_{j+1}} | \mathcal{Q}_j) \right| | \mathcal{Q}\right) + E(Z_{\mathcal{Q}_n} | \mathcal{Q}) : \right. \\ \left. \mathcal{Q} = \mathcal{Q}_0 \subset \mathcal{Q}_1 \subset \dots \subset \mathcal{Q}_n \text{ are in } \Gamma \right\}.$$

(d) If $Z \in \text{FPROC}(\Gamma)$ then $\|Z\|_D$ is

$$\sup \left\{ \sum_{j=0}^{n-1} \|Z_{\mathcal{A}_j} - E(Z_{\mathcal{A}_{j+1}} | \mathcal{A}_j)\| + \|Z_{\mathcal{A}_n}\| : \mathcal{A}_0 \subset \mathcal{A}_1 \subset \dots \subset \mathcal{A}_n \text{ are in } \Gamma \right\}.$$

PROOF. To establish (a) and (b) it suffices to show that DZ is the supremum of Z and $-Z$ for the ordering of $\text{FPROC}(\Gamma)$. To do this we will show that DZ is the limit of $\{D^{\Gamma'}Z : \Gamma' \text{ finite } \subset \Gamma\}$ in the vague topology as Γ' increases to Γ . Here $D^{\Gamma'}Z$ is obtained by restricting Z to Γ' , computing DZ in $\text{FPROC}(\Gamma')$ and extending DZ to $D^{\Gamma'}Z$ in $\text{FPROC}(\Gamma)$ in the usual fashion.

Notice that the net $\{D^{\Gamma'}Z : \Gamma' \text{ finite } \subset \Gamma\}$ is monotone hence vaguely convergent to a process W in $\|Z\|_D \cdot \diamond^+$. Notice that if $\Gamma_1 \subset \Gamma_2$ are finite in Γ then $D^{\Gamma_2}Z - Z$ and $D^{\Gamma_2}Z + Z$ are in $\text{SMART}^-(\Gamma_1)$ when restricted to Γ_1 . Passing to the limit as Γ_2 increases to Γ , we find that $W - Z$ and $W + Z$ are in $\text{SMART}^+(\Gamma_1)$ when restricted to Γ_1 . Thus, $W + Z$ and $W - Z$ are in $\text{SMART}^+(\Gamma)$ since Γ_1 is arbitrary. Thus, $W \geqslant DZ$. On the other hand, $DZ \geqslant D^{\Gamma'}Z$ when restricted to Γ' finite in Γ . Passage to the limit as Γ' increases to Γ shows that $DZ \geqslant W$. Consequently, $DZ = \lim\{D^{\Gamma'}Z : \Gamma' \text{ finite } \subset \Gamma\}$.

Similar reasoning shows that if $\{Y + Z, Y - Z\} \subset \text{SMART}^+(\Gamma)$ then $\{Y + DZ, Y - DZ\} \subset \text{SMART}^+(\Gamma)$. This establishes that DZ is the supremum of Z and $-Z$ in the order of $\text{FPROC}(\Gamma)$. Similarly, D^+Z and D^-Z are the positive and negative parts of Z .

(b) is a consequence of (a) for Δ ; hence \diamond^+ is a simplex as Δ is the base of a lattice ordering cone. \diamond^+ is vaguely compact, hence a Choquet simplex. Δ is vaguely compact iff $\mathcal{A}_0 \in \Gamma$ by Proposition 4-1(c).

(c) and (d) follow from (b) and (c) of Proposition 5-1. \square

As a corollary one finds that if $Z \in \text{FPROC}(\Gamma)$ then $\|\mathfrak{N}Z\|_D = \|Z_{\mathcal{A}_\infty}\|$ if $\mathcal{A}_\infty \in \Gamma$ or $\|\mathfrak{N}Z\|_D = \|Z_{\mathcal{A}_\infty^-}\|$ if $\mathcal{A}_\infty \notin \Gamma$.

$$\|\mathfrak{P}Z\|_D + \sup \left\{ \sum_{j=0}^{n-1} \|Z_{\mathcal{A}_j} - E(Z_{\mathcal{A}_{j+1}} | \mathcal{A}_j)\| : \mathcal{A}_0 \subset \mathcal{A}_1 \subset \dots \subset \mathcal{A}_n \text{ in } \Gamma \right\}.$$

Thus, $\|Z\|_D = \|\mathfrak{N}Z\|_D + \|\mathfrak{P}Z\|_D$.

The seminorm $Z \rightarrow \|\mathfrak{P}Z\|_D$ is the one usually used for F -processes of random variables [42].

COROLLARY 5-2-1. If Z is a process then $Z \in \text{FPROC}(\Gamma)$ iff

$$K = \sup \left\{ \sum_{j=0}^{n-1} \|Z_{\mathcal{A}_j} - E(Z_{\mathcal{A}_{j+1}} | \mathcal{A}_j)\| + \|Z_{\mathcal{A}_n}\| : \mathcal{A}_0 \subset \mathcal{A}_1 \subset \dots \subset \mathcal{A}_n \text{ in } \Gamma \right\}$$

is finite.

PROOF. If $K < \infty$ then Z is a vague limit of elements of $K\diamond$, hence is in $K\diamond \subset \text{FPROC}(\Gamma)$. \square

Even with ordinary processes of random variables it is surprising that when a process Z is an F -process so are $|Z|$, Z^+ and Z^- where $|Z| = \{|Z_{\mathcal{A}}| : \mathcal{A} \in \Gamma\}$,

$Z^+ = \{Z_{\mathcal{Q}}^+ : \mathcal{Q} \in \Gamma\}$ and $Z^- = \{Z_{\mathcal{Q}}^- : \mathcal{Q} \in \Gamma\}$. This however is a result of Youerp which deserves wider currency. As an application of Theorem 5-2 we give a proof along the lines of Jeulin [31].

PROPOSITION 5-3. *Let Z be a process on the chain Γ of subalgebras of \mathfrak{B} . $Z \in \text{FPROC}(\Gamma)$ iff $\{Z^+, Z^-\} \subset \text{FPROC}(\Gamma)$.*

PROOF. One need only show that $\|Z\|_D < \infty$ implies that $\|Z^+\|_D < \infty$. We may assume $\mathfrak{N}Z = \mathfrak{N}Z^+ = 0$ without loss of generality. Lemma 1-2 implies that if $\mathcal{Q}_1 \subset \mathcal{Q}_2$ are in Γ then

$$2|Z_{\mathcal{Q}_1} - E(Z_{\mathcal{Q}_2} | \mathcal{Q}_1)| \geq [Z_{\mathcal{Q}_1}^+ - E(Z_{\mathcal{Q}_2}^+ | \mathcal{Q}_1)] + |Z_{\mathcal{Q}_1}^+ - E(Z_{\mathcal{Q}_2}^+ | \mathcal{Q}_1)|.$$

If $\Gamma' = \{\mathcal{Q}_0 \subset \mathcal{Q}_1 \subset \dots \subset \mathcal{Q}_n\}$ one obtains, upon adding up such inequalities, that $2D_{\mathcal{Q}_0}Z \geq Z_{\mathcal{Q}_0}^+ - E(Z_{\mathcal{Q}_n}^+ | \mathcal{Q}_0) + D_{\mathcal{Q}_0}Z^+$ in $\text{FPROC}(\Gamma')$. Thus, in $\text{FPROC}(\Gamma')$, $2\|Z\|_D + \|Z_{\mathcal{Q}_0}^+\| + \|Z_{\mathcal{Q}_n}^+\| \geq \|Z^+\|_D$. Since Z is a bounded process, $\|Z_{\mathcal{Q}}^+\| \leq K$ for some $K < \infty$ and all \mathcal{Q} . Letting $\Gamma' \uparrow \Gamma$ we find $2\|Z\|_D + 2K \geq \|Z^+\|_D$. This establishes the proposition. \square

6. Decreasing and increasing processes. Let Γ be a collection of subalgebras of \mathfrak{B} with supremum \mathcal{Q}_∞ . The power $[BA(\mathcal{Q}_\infty)]^\Gamma$ is equipped with the product topology where each factor $BA(\mathcal{Q}_\infty)$ has the vague topology. An element $\mu = \{\mu_{\mathcal{Q}} : \mathcal{Q} \in \Gamma\}$ of $[BA(\mathcal{Q}_\infty)]^\Gamma$ is said to be decreasing (increasing) if it is an increasing function on the ordered set Γ to the lattice $BA(\mathcal{Q}_2)$. The decreasing (increasing) elements of $[BA(\mathcal{Q}_2)]^\Gamma$ are closed for the product topology. The map $\{\mu_{\mathcal{Q}} : \mathcal{Q} \in \Gamma\} \rightarrow \{E(\mu_{\mathcal{Q}} | \mathcal{Q}) : \mathcal{Q} \in \Gamma\}$ is continuous for the vague topology on processes. If $\{\mu_{\mathcal{Q}} : \mathcal{Q} \in \Gamma\}$ is decreasing (increasing) then $\{E(\mu_{\mathcal{Q}} | \mathcal{Q}) : \mathcal{Q} \in \Gamma\}$ is called a decreasing (increasing) process. Any decreasing (increasing) process is a supermartingale (submartingale). For a $K \in [0, \infty)$ the decreasing (increasing) processes of uniform norm at most K form a vaguely compact set of processes.

Suppose that $P \in \mathfrak{P}(\mathcal{Q}_\infty)$. If $\{\mu_{\mathcal{Q}} : \mathcal{Q} \in \Gamma\}$ is a decreasing (increasing) element of $[BA(\mathcal{Q}_\infty)]^\Gamma$ with $\mu_{\mathcal{Q}} \ll P$ for all $\mathcal{Q} \in \Gamma$, then the process $\{E(\mu_{\mathcal{Q}} | \mathcal{Q}) : \mathcal{Q} \in \Gamma\}$ is said to be decreasing (increasing) with respect to P . Even when Y is a supermartingale with $Y_{\mathcal{Q}} \ll E(P | \mathcal{Q})$ for all $\mathcal{Q} \in \Gamma$, it need not be the case that Y is a decreasing process with respect to P . When Y is a supermartingale of random variables it is a decreasing process with respect to P iff it is of type DL . Here, for Y to be of type DL it is necessary that $\{Y_\tau : \tau \text{ stopping time } \leq t\}$ be a uniformly integrable family of random variables for all $t \in [0, \infty)$ [34]. In contrast to this, with the absence of the reference measure P , we find that, for a chain Γ , all supermartingales are decreasing processes (Proposition 6-1). An intermediate concept is that of a process Y which is weakly decreasing (increasing) with respect to P . Here one does not require for $Y = \{E(\mu_{\mathcal{Q}} | \mathcal{Q}) : \mathcal{Q} \in \Gamma\}$ that $\mu_{\mathcal{Q}} \ll P$ but rather that each $\mu_{\mathcal{Q}}$ be weakly absolutely continuous, $\mu_{\mathcal{Q}} \ll_w P$, with respect to P . Here we mean that $P(A) = 0$ implies that $\mu_{\mathcal{Q}}(A) = 0$. We find, in Corollary 6-1-1, that if Y is a supermartingale (submartingale) with $Y_{\mathcal{Q}} \ll_w P$ for all \mathcal{Q} then Y is a process weakly decreasing (increasing) with respect to P .

PROPOSITION 6-1. *Let Γ be a chain of subalgebras of \mathfrak{B} with supremum \mathcal{Q}_∞ .*

(a) *Any bounded supermartingale (submartingale) on Γ is a decreasing (increasing) process.*

(b) *If Y is a bounded nonnegative supermartingale (submartingale) there is a decreasing (increasing) $\{\mu_\mathcal{Q}: \mathcal{Q} \in \Gamma\} \in [BA^+(\mathcal{Q}_\infty)]^\Gamma$ with $E(\mu_\mathcal{Q}|\mathcal{Q}) = Y_\mathcal{Q}$ for all $\mathcal{Q} \in \Gamma$.*

PROOF. We will establish the proposition only for supermartingales, the case of submartingales being easier.

For $Y \in \text{SMART}(\Gamma)$ and Γ' a finite subset of Γ , let $\mathfrak{D}(Y, \Gamma')$ denote all decreasing processes on Γ with uniform norm at most $\|Y\|_D$ and which agree with Y on Γ' . $\mathfrak{D}(Y, \Gamma')$ is convex and vaguely compact for all Γ' . Furthermore, if $\Gamma'' \subset \Gamma'$ then $\mathfrak{D}(Y, \Gamma') \subset \mathfrak{D}(Y, \Gamma'')$. If it is shown that $\mathfrak{D}(Y, \Gamma') \neq \emptyset$ for all Γ' it follows from $\{Y\} = \bigcap \{\mathfrak{D}(Y, \Gamma'): \Gamma' \text{ finite } \subset \Gamma\}$, that Y is a decreasing process. This will establish (a). In the course of establishing the nonemptiness of $\mathfrak{D}(Y, \Gamma')$ it will be clear that if $Y \in \text{SMART}^+(\Gamma)$ then the compact convex set of $\{\mu_\mathcal{Q}: \mathcal{Q} \in \Gamma\} \in [BA(\mathcal{Q}_\infty)]^\Gamma$, so that $\{E(\mu_\mathcal{Q}|\mathcal{Q}): \mathcal{Q} \in \Gamma\} \in \mathfrak{D}(Y, \Gamma')$ is nonempty for each finite set $\Gamma' \subset \Gamma$. From this (b) follows.

The nonemptiness of $\mathfrak{D}(Y, \Gamma')$ will be established via induction on $|\Gamma'|$. If $|\Gamma'| = 1$ this is immediate. Assume that $\mathfrak{D}(Y, \Gamma') \neq \emptyset$ for all Y and for Γ' with $|\Gamma'| \leq n$. Let, for the induction step, $\Gamma' = \{\mathcal{Q}_1 \subset \mathcal{Q}_2 \subset \dots \subset \mathcal{Q}_n \subset \mathcal{Q}_{n+1}\}$. Let $\{\nu_{\mathcal{Q}_j}; j = 1, \dots, n\}$ be decreasing in $BA(\mathcal{Q}_\infty)$ with $E(\nu_{\mathcal{Q}_j}|\mathcal{Q}_j) = Y_{\mathcal{Q}_j}$. We have

$$Y_{\mathcal{Q}_j} - E(Y_{\mathcal{Q}_n}|\mathcal{Q}_j) = E(\nu_{\mathcal{Q}_j} - \nu_{\mathcal{Q}_n}|\mathcal{Q}_j) \quad \text{for all } j.$$

Let $\mu_{\mathcal{Q}_{n+1}} \in BA(\mathcal{Q}_\infty)$ be an extension of $Y_{\mathcal{Q}_{n+1}}$ to \mathcal{Q}_∞ which is nonnegative if $Y_{\mathcal{Q}_{n+1}}$ is nonnegative. Let $\omega \in BA^+(\mathcal{Q}_\infty)$ be an extension of $Y_{\mathcal{Q}_n} - E(Y_{\mathcal{Q}_{n+1}}|\mathcal{Q}_n)$. Set $\mu_{\mathcal{Q}_n} = \mu_{\mathcal{Q}_{n+1}} + \omega$ and set $\mu_{\mathcal{Q}_j} = (\nu_{\mathcal{Q}_j} - \nu_{\mathcal{Q}_n}) + \mu_{\mathcal{Q}_n}$ if $j = 1, \dots, n-1$. Set $\mu_{\mathcal{Q}_j} = \mu_{\mathcal{Q}_j}$ if $\mathcal{Q}_{j-1} \subset \mathcal{Q} \subset \mathcal{Q}_j$ when $j = 2, \dots, n$, set $\mu_{\mathcal{Q}} = \mu_{\mathcal{Q}_1}$ if $\mathcal{Q} \subset \mathcal{Q}_1$ and set $\mu_{\mathcal{Q}} = \mu_{\mathcal{Q}_{n+1}}$ if $\mathcal{Q}_{n+1} \subset \mathcal{Q}$. It is immediate that $\{E(\mu_\mathcal{Q}|\mathcal{Q}): \mathcal{Q} \in \Gamma\} \in \mathfrak{D}(Y, \Gamma')$. This suffices to establish the proposition. \square

COROLLARY 6-1-1. (a) *Any $Y \in \text{SMART}(\Gamma)$ may be written as $M - I$ where M is a bounded martingale and I is an increasing process with $I_{\mathcal{Q}_0}$ (or $I_{\mathcal{Q}_0^+}$) equal to 0.*

(b) *FPROC(Γ) is the vector space of differences of bounded nonnegative submartingales.*

PROOF. (a) Y is $\{E(\mu_\mathcal{Q}|\mathcal{Q}): \mathcal{Q} \in \Gamma\}$ where $\{\mu_\mathcal{Q}: \mathcal{Q} \in \Gamma\}$ is a decreasing family in $BA(\mathcal{Q}_\infty)$. Let $\mu_{\mathcal{Q}_0} = \sup\{\mu_\mathcal{Q}: \mathcal{Q} \in \Gamma\}$. Set $M = M(\mu_{\mathcal{Q}_0})$ and set $I = M - Y$.

(b) Any bounded martingale M is a difference of two nonnegative submartingales. The reason for this is that if $M = M(\mu)$ for some $\mu \in BA(\mathcal{Q}_\infty)$ then $M(\mu^+)$ and $M(\mu^-)$ are nonnegative martingales with $M = M(\mu^+) - M(\mu^-)$. Thus, because of (a), any $Y \in \text{SMART}^+(\Gamma)$ is the difference of two nonnegative bounded submartingales. Thus, $\text{FPROC}(\Gamma)$ consists of differences of nonnegative submartingales. Conversely, if Y is a nonnegative bounded submartingale then $-Y = \mathfrak{N}(-Y) + \mathfrak{P}(-Y)$ so $Y = -\mathfrak{N}(-Y) - \mathfrak{P}(-Y)$ is in $\text{SMART}^+(\Gamma) - \text{SMART}^+(\Gamma) = \text{FPROC}(\Gamma)$. As a result any difference of bounded nonnegative submartingales is in $\text{FPROC}(\Gamma)$. This establishes (b). \square

COROLLARY 6-1-2. (a) *For any nonnegative bounded supermartingale (submartingale) Y there is a $P \in \mathfrak{P}(\mathfrak{A}_\infty)$ so that Y is a decreasing (increasing) process with respect to P .*

(b) *If $P \in \mathfrak{P}(\mathfrak{A}_\infty)$ is such that the supermartingale (submartingale) Y satisfies $Y_{\mathfrak{A}} \ll_w E(P|\mathfrak{A})$ for $\mathfrak{A} \in \Gamma$ there exists a decreasing (increasing) family $\{\mu_{\mathfrak{A}}: \mathfrak{A} \in \Gamma\} \subset BA(\mathfrak{A}_\infty)$ with $E(\mu_{\mathfrak{A}}|\mathfrak{A}) = Y_{\mathfrak{A}}$ for $\mathfrak{A} \in \Gamma$ so that $\mu_{\mathfrak{A}} \ll P$ for $\mathfrak{A} \in \Gamma$.*

PROOF. (a) If $\{\mu_{\mathfrak{A}}|\mathfrak{A} \in \Gamma\}$ is a decreasing (increasing) family in $BA(\mathfrak{A}_\infty)$ with $E(\mu_{\mathfrak{A}}|\mathfrak{A}) = Y_{\mathfrak{A}}$ for all $\mathfrak{A} \in \Gamma$ choose $P = \sup\{\mu_{\mathfrak{A}}: \mathfrak{A} \in \Gamma\}$.

(b) One may suppose under these conditions that P is a strictly positive measure on \mathfrak{A}_∞ (upon passage to the quotient algebra of \mathfrak{A}_∞ modulo the ideal of P -negligible elements of \mathfrak{A}_∞) in which case the assertion is immediate. \square

REMARK. Results similar to (a) and (b) of Corollary 6-1-2 hold for arbitrary $Y \in FPROC(\Gamma)$.

Even when all $Y_{\mathfrak{A}}$ are absolutely continuous with respect to $E(P|\mathfrak{A})$ the conclusion of (b) of Corollary 6-1-2 is best possible. This is true even in the very simple case where $\mathfrak{B} = \mathfrak{A}_\infty$ is the Cantor algebra of clopen subsets of $\{0, 1\}^N$ and $\Gamma = \{\mathfrak{A}_n: n \in N\}$ where \mathfrak{A}_n is the algebra of clopen sets dependent only on the first n coordinates. Let P in this case be ordinary fair coin toss measure on \mathfrak{A}_∞ so P is strictly positive. Notice that each Y of the form $M(\mu)$ with $\mu \in P(\mathfrak{A}_\infty)$ satisfies $Y_{\mathfrak{A}_n} \ll E(P|\mathfrak{A}_n)$. If $\{\mu_{\mathfrak{A}_n}: n \in N\}$ is an increasing sequence in $\mathfrak{P}(\mathfrak{A}_\infty)$ with $E(\mu_{\mathfrak{A}_n}|\mathfrak{A}_n) = Y_{\mathfrak{A}_n}$ for $n \in N$ then $\sup\{\mu_{\mathfrak{A}_n}: n \in N\} \geq \mu$. Similarly, if $\{\mu_{\mathfrak{A}_n}: n \in N\}$ is a decreasing sequence in $\mathfrak{P}(\mathfrak{A}_\infty)$ with $E(\mu_{\mathfrak{A}_n}|\mathfrak{A}_n) = Y_{\mathfrak{A}_n}$ then $\mu_{\mathfrak{A}_0} \geq \mu$. Thus, if $\mu \perp P$ and $\{\mu_{\mathfrak{A}_n}: n \in N\}$ is a decreasing process with $E(\mu_{\mathfrak{A}_n}|\mathfrak{A}_n) = E(\mu|\mathfrak{A}_n)$ for all n , then for no n is it true that $\mu_{\mathfrak{A}_n} \ll P$. However it is always possible to find an increasing sequence $\{\mu_{\mathfrak{A}_n}: n \in N\}$ of elements of $\mathfrak{P}(\mathfrak{A}_\infty)$ with $\mu_{\mathfrak{A}_n} \ll P$ and $E(\mu_{\mathfrak{A}_n}|\mathfrak{A}_n) = E(\mu|\mathfrak{A}_n)$ for all n .

PROPOSITION 6-2. *Let $\Gamma = \{\mathfrak{A}_n: n \in N\}$ be an increasing sequence of subalgebras of \mathfrak{B} with supremum \mathfrak{A}_∞ . If Y is a submartingale on Γ with $P \in \mathfrak{P}(\mathfrak{A}_\infty)$ and $Y_{\mathfrak{A}_n} \ll P$ for all n , then Y is an increasing process with respect to P .*

PROOF. Use Lemma 1-1 to extend $Y_{\mathfrak{A}_1}$ to $\mu_1 \ll P$ on \mathfrak{A}_∞ and to extend each $Y_{\mathfrak{A}_j} - E(Y_{\mathfrak{A}_{j+1}}|\mathfrak{A}_j)$ to ω_{j+1} with $0 \leq \omega_{j+1}$ and $\omega_{j+1} \ll P$ for all $j \geq 1$. Set $\mu_j = \mu_1 + \sum_{n=2}^j \omega_n$ to obtain an increasing sequence in $BA(\mathfrak{A}_\infty)$ with $\mu_j \ll P$ for all j and $E(\mu_j|\mathfrak{A}_j) = Y_{\mathfrak{A}_j}$. \square

7. Extreme points of \diamond^+ are $\{0, 1\}$ -valued. In Proposition 5-2 it was established that \diamond^+ and Δ are simplexes when Γ is a chain of subalgebras of \mathfrak{B} . One task of this section is to examine the extreme points $\xi(\Delta)$ of Δ and $\xi(\diamond^+)$ and characterize them (as one might expect) as the $\{0, 1\}$ -valued elements. From this it will follow that \diamond^+ is a Bauer simplex with $\xi(\diamond^+)$ closed. \diamond^+ is affinely isomorphic (with the vague topology) to the simplex $\mathfrak{N}_1^+(\xi(\diamond^+))$ of probability Radon measures on $\xi(\diamond^+)$ and Δ is affinely homeomorphic to the simplex of probability measures $\bar{\mu}$ on $\xi(\diamond^+)$ with $\bar{\mu}(\{0\}) = 0$. When $\mathfrak{A}_0 \in \Gamma$ then Δ is a Bauer simplex isomorphic to $\mathfrak{N}_1^+(\xi(\Delta))$ where $\xi(\Delta)$ is the compact set $\xi(\diamond^+) \setminus \{0\}$. In any case, $FPROC(\Gamma)$ is isomorphic as an ordered Banach space to $\mathfrak{N}_b(\xi(\Delta))$, the bounded Radon measures on $\xi(\Delta)$.

The vague topology on $\text{FPROC}(\Gamma)$ corresponds to the vague topology $\sigma(\mathfrak{N}_b(\xi(\Delta)), \mathcal{C}_c(\xi(\Delta)))$ in the usual sense.

PROPOSITION 7-1. *Let Γ be a chain of subalgebras of \mathfrak{B} . $Y \in \diamond^+$ is extreme iff it is $\{0, 1\}$ -valued.*

PROOF. If Y is extreme it is either 0 or is in Δ . Thus, it may be assumed that $Y \in \Delta$. We have, by the Riesz-Doob Decomposition, $Y = \mathfrak{N}Y + \mathfrak{P}Y$. If Y is extreme it is either a potential or a martingale. If it is a martingale then it is $M(\mu)$ for some $\mu \in BA_1^+(\mathcal{Q}_\infty)$. This μ must in turn be extreme hence $\{0, 1\}$ -valued. In this case $Y = \{E(\mu|\mathcal{Q}) : \mathcal{Q} \in \Gamma\}$ is $\{0, 1\}$ -valued. Thus, it may be assumed that Y is a potential. That is, the limit of $\|Y_\mathcal{Q}\|$ is 0 as \mathcal{Q} increases in Γ to \mathcal{Q}_∞ . The set $t^-(Y) = \{\mathcal{Q} \in \Gamma : \|Y_\mathcal{Q}\| = 1\}$ is an initial segment of Γ on which Y is a martingale. The set $t^+(Y) = \{\mathcal{Q} \in \Gamma : \|Y_\mathcal{Q}\| = 0\}$ is a final segment of Γ . We now show that $t^-(Y) \cup t^+(Y) = \Gamma$. Otherwise, there is some $\mathcal{Q} \in \Gamma$ with $0 < \|Y_\mathcal{Q}\| < 1$. Set $Y_\mathcal{Q}' = Y_\mathcal{Q} - E(Y_\mathcal{Q}|\mathcal{Q}')$ if $\mathcal{Q}' \subset \mathcal{Q}$ and $Y_\mathcal{Q}' = 0$ if $\mathcal{Q}' \not\subset \mathcal{Q}$. Set $Y^2 = Y - Y^1$ so $Y_\mathcal{Q}'^2 = E(Y_\mathcal{Q}'|\mathcal{Q}')$ if $\mathcal{Q}' \subset \mathcal{Q}$ and $Y_\mathcal{Q}'^2 = Y_\mathcal{Q}'$ otherwise. We have $\{Y^1, Y^2\} \subset \text{SMART}^+(\Gamma) \setminus \{0\}$ and $Y^1 + Y^2 = Y$. As a result, Y is not extreme. Thus, $t^-(Y) \cup t^+(Y) = \Gamma$. If the restriction of Y to $t^-(Y)$ is not an extreme martingale on $t^-(Y)$ then Y is not extreme. Thus, Y is $\{0, 1\}$ -valued on $t^-(Y)$ as well as on $t^+(Y)$. This establishes the proposition. \square

COROLLARY 7-1-1. \diamond^+ is a Bauer simplex and Δ is a Bauer simplex iff $\mathcal{Q}_\infty \in \Gamma$ (under the vague topology).

PROOF. The $\{0, 1\}$ -valued elements of \diamond^+ are vaguely closed. That is, $\xi(\diamond^+)$ is closed. When $\mathcal{Q}_\infty \in \Gamma$, Δ is compact as in $\Delta \cap \xi(\diamond^+) = \xi(\Delta)$. \square

COROLLARY 7-1-2. $\text{FPROC}(\Gamma)$ is isometric under $\|\cdot\|_D$ and order isomorphic to the Banach lattice $\mathfrak{N}_b(\xi(\Delta))$ of bounded Radon measures on the locally compact space $\xi(\Delta)$. The vague topology on $\text{FPROC}(\Gamma)$ is that induced by the vague topology $\sigma(\mathfrak{N}_b(\xi(\Delta)), \mathcal{C}_c(\xi(\Delta)))$ on $\mathfrak{N}_b(\xi(\Delta))$.

We shall denote by $\tilde{\mu}_Y$ the "representing" measure for $Y \in \text{FPROC}(\Gamma)$ on $\xi(\Delta)$ given by the isomorphism between $\text{FPROC}(\Gamma)$ and $\mathfrak{N}_b(\xi(\Delta))$. This is the unique measure on $\xi(\Delta)$ so that if $A \in \mathcal{Q} \in \Gamma$ has corresponding to it the linear functional $i_{\mathcal{Q}, A} : Z \rightarrow Z_{\mathcal{Q}}(A)$ on $\text{FPROC}(\Gamma)$ then $Y_{\mathcal{Q}}(A) = \int i_{\mathcal{Q}, A}(x) \tilde{\mu}_Y(dx)$ for all such choices of A and \mathcal{Q} .

In the proof of Proposition 7-1 we associated to an extreme Y the pair (t^-, t^+) of subsets of Γ and essentially showed that (t^-, t^+) is a Dedekind cut of Γ as a linearly ordered set. This suggests the following procedure for constructing elements of $\xi(\diamond^+)$. Pick δ a $\{0, 1\}$ -valued element of $BA_1^+(\mathcal{Q}_\infty)$ and pick a Dedekind cut $t = (t^-, t^+)$ of Γ . Define $Y(\delta, t)$ to be the process with $Y_{\mathcal{Q}}(\delta, t) = 0$ if $\mathcal{Q} \in t^+$ and to be $E(\delta|\mathcal{Q})$ if $\mathcal{Q} \in t^-$. If we call the ensemble $T = T(\Gamma)$ of Dedekind cuts of Γ the ideal time set of Γ then the map $t \rightarrow Y(\delta, t)$ is an injection of T into $\xi(\diamond^+)$ for all $\delta \in \xi(BA^+(\mathcal{Q}_\infty))$. We shall denote by $-\infty$ the cut (ϕ, Γ) and by ∞ the cut (Γ, ϕ) of T . Note that $Y(\delta, -\infty) = 0$ and $Y(\delta, \infty) = M(\delta)$.

REMARKS. Föllmer in [25] obtained Proposition 7-1 in the case of supermartingales of random variables. Pavlov in [43] has shown that supermartingales of random variables cannot be represented as barycenters of measures on the extremal supermartingales in general.

8. The ring of predictable sets. We shall examine, for a chain Γ , the structure of $\xi(\Delta)$ in more detail in this section. We first show that $\xi(\Delta)$ is totally disconnected, hence is the Stone space of its ring of compact-open sets. These sets will be called the predictable subsets of $\xi(\Delta)$. A very concrete description of this ring of predictable sets is given in terms of the time set T and the Stone space $X_{\mathcal{Q}_\infty}$ of $\mathcal{Q}_\infty = \sup \Gamma$. Finally, an isomorphic ring of predictable subsets of $T \times X_{\mathcal{Q}_\infty}$ is given which is the analogue of the ring of predictable events used in representing F -processes of random variables. Because of the Stone correspondence and the fact that $\text{FPROC}(\Gamma)$ is isomorphic to $\mathfrak{N}_b(\xi(\Delta))$ it follows that $\text{FPROC}(\Gamma)$ is isomorphic to the Banach lattice of bounded finitely additive measures on the ring of predictable sets in $\xi(\Delta)$ or in $T \times X_{\mathcal{Q}_\infty}$.

PROPOSITION 8-1. $\xi(\diamond^+)$ is totally disconnected as is $\xi(\Delta)$.

PROOF. The topology on $\xi(\diamond^+)$ is the coarsest which makes each $\{0, 1\}$ -valued functional $i_{\mathcal{Q}, A}: Y \rightarrow Y_{\mathcal{Q}}(A)$ continuous for $A \in \mathcal{Q} \in \Gamma$. \square

The ring of compact open sets in $\xi(\Delta)$ will be denoted by $\mathfrak{R} = \mathfrak{R}(\Gamma)$ and is called the ring of *predictable subsets* of $\xi(\Delta)$.

PROPOSITION 8-2. $\text{FPROC}(\Gamma)$ is Banach lattice isomorphic to $BA(\mathfrak{R})$. The vague topologies of these spaces coincide under this isomorphism. The isomorphism is effected by the isomorphism between $\text{FPROC}(\Gamma)$ and $\mathfrak{N}_b(\xi(\Delta))$ and by the Stone correspondence for finitely additive measures on Boolean rings.

PROOF. Immediate. \square

The structure of T , the ideal time set, must be examined more closely for a concrete description of \mathfrak{R} to be given. Recall that T is the set of Dedekind cuts of Γ . An element t of T is a pair (t^-, t^+) . T is naturally ordered by the relation $t_1 \leq t_2$ iff $t_1^- \subset t_2^-$. Γ is naturally embedded in T in an order isomorphic fashion. This is done by assigning to $\mathcal{Q} \in \Gamma$ the time $t_{\mathcal{Q}} \in \Gamma$ which has $(t_{\mathcal{Q}})^- = \{\mathcal{Q}' \in \Gamma: \mathcal{Q}' \subset \mathcal{Q}\}$. T is then the order completion of Γ . T equipped with the order topology is compact. The order topology for T has as a subbase all intervals of the form $[t_{\mathcal{Q}}, \infty)$ or $(-\infty, t_{\mathcal{Q}}^-]$ where $t_{\mathcal{Q}}^-$ is the time with $(t_{\mathcal{Q}}^-)^- = \{\mathcal{Q}' \in \Gamma: \mathcal{Q}' \subset \mathcal{Q}, \mathcal{Q}' \neq \mathcal{Q}\}$. One may identify a time t with the characteristic function $\chi_{t^-} \in \{0, 1\}^\Gamma$ of t^- . One has, for $\mathcal{Q} \in \Gamma$, $\chi_{t^-}(\mathcal{Q}) = 1$ iff $\mathcal{Q} \in t^-$ so $(-\infty, t_{\mathcal{Q}}^-] = \{t: \chi_{t^-}(\mathcal{Q}) = 0\}$ and $[t_{\mathcal{Q}}, \infty) = \{t: \chi_{t^-}(\mathcal{Q}) = 1\}$. Thus, the order topology on T is precisely that induced from the product topology on $\{0, 1\}^\Gamma$ via the map $t \rightarrow \chi_{t^-}$.

Recall from §7 that if $t \in T$ and $\delta \in \xi(\mathcal{P}(\mathcal{Q}_\infty))$ then $Y(\delta, t) \in \xi(\diamond^+)$ is defined by setting $Y_{\mathcal{Q}}(\delta, t) = E(\delta | \mathcal{Q})$ and $\mathcal{Q} \in t^-$ and by $Y_{\mathcal{Q}}(\delta, t) = 0$ otherwise.

PROPOSITION 8-3. (a) For any $\delta \in \xi(\mathcal{P}(\mathcal{Q}_\infty))$ the map $t \rightarrow Y(\delta, t)$ is a homeomorphism of T into $\xi(\diamond^+)$ when $\xi(\diamond^+)$ has the vague topology and T the order topology.

(b) The map $(\delta, t) \rightarrow Y(\delta, t)$ is continuous from $\xi(\mathcal{P}(\mathcal{Q}_\infty)) \times T$ onto $\xi(\diamond^+)$ when $\xi(\mathcal{P}(\mathcal{Q}_\infty)) \times T$ is equipped with the product of the vague and order topologies.

PROOF. Only (b) will be established.

Let $\mathcal{A}_1 \subset \mathcal{A}_2 \subset \dots \subset \mathcal{A}_n$ be in Γ , let $A_j \in \mathcal{A}_j$ for $j = 1, \dots, n$ and let $\epsilon_j \in \{0, 1\}$ for $j = 1, \dots, n$.

For each $j = 1, \dots, n$ we have $\{(\delta, t): Y_{\mathcal{A}_j}(\delta, t)(A_j) = \epsilon_j\} = \{(\delta, t): \mathcal{A}_j \in t^- \text{ and } \delta(A_j) = \epsilon_j\}$. This set is clopen in $\xi(\mathcal{P}(\mathcal{A}_\infty)) \times T$. As a result

$$\bigcap_j \{(\delta, t): Y_{\mathcal{A}_j}(\delta, t)(A_j) = \epsilon_j\}$$

is clopen in $\xi(\mathcal{P}(\mathcal{A}_\infty)) \times T$. Since such sets are basic for the vague topology on $\xi(\diamond^+)$ the continuity of $(\delta, t) \rightarrow Y(\delta, t)$ is established. \square

Since $\xi(\mathcal{P}(\mathcal{A}_\infty))$ is homeomorphic to $X_{\mathcal{A}_\infty}$ we have a continuous map $\tilde{Y}: (x, t) \rightarrow \tilde{Y}(x, t) \in \xi(\diamond^+)$ obtained by associating to the ultrafilter x its characteristic function $\chi_x \in \xi(\mathcal{P}(\mathcal{A}_\infty))$ and setting $\tilde{Y}(x, t) = Y(\chi_x, t)$. The inverse image of $\xi(\Delta)$ under Y is the set $X_{\mathcal{A}_\infty} \times T'$ where T' is the open set $T \setminus \{-\infty\} = (-\infty, \infty]$ in T . We call $\tilde{\mathcal{R}} = \tilde{Y}^{-1}(\mathcal{R})$ the *ring of predictable subsets* of $X_{\mathcal{A}_\infty} \times T$. $\tilde{\mathcal{R}}$ and \mathcal{R} are isomorphic under \tilde{Y} .

Note that if $A \in \mathcal{A} \in \Gamma$ then $[A] \times [t_{\mathcal{A}}, \infty] = \{(x, t): \tilde{Y}_{\mathcal{A}}(x, t)(A) = 1\}$ is in $\tilde{\mathcal{R}}$. In fact, such sets form a subbasis for $\tilde{\mathcal{R}}$ in that each element of \mathcal{R} is a finite union of finite intersections of such sets. Actually, the ensemble consisting of sets of the form $[A] \times [t_{\mathcal{A}}, \infty]$ or of the form

$$[A] \times [t_{\mathcal{A}_1}, t_{\mathcal{A}_2}] = ([A] \times [t_{\mathcal{A}_1}, \infty]) \setminus ([A] \times [t_{\mathcal{A}_2}, \infty])$$

form a basis for $\tilde{\mathcal{R}}$ when it is required that $A \in \mathcal{A}_1 \subset \mathcal{A}_2$ for $\{\mathcal{A}_1, \mathcal{A}_2\} \subset \Gamma$. Here A is regarded as an element of \mathcal{A}_∞ with $[A]$ the corresponding clopen set. We denote by $A \times_{\circ} [t_{\mathcal{A}}, \infty]$ or $A \times_{\circ} [t_{\mathcal{A}_1}, t_{\mathcal{A}_2}]$ the image of $[A] \times [t_{\mathcal{A}}, \infty]$ or $[A] \times [t_{\mathcal{A}_1}, t_{\mathcal{A}_2}]$ in $\xi(\Delta)$ under \tilde{Y} .

Each $Y \in \text{FPROC}(\Gamma)$ has corresponding to it, by §7, $\tilde{\mu}_Y \in \mathfrak{M}_b(\xi(\Delta))$. Corresponding to $\tilde{\mu}_Y$ under the Stone correspondence is a $\mu_Y \in \text{BA}(\tilde{\mathcal{R}}) \simeq \text{BA}(\mathcal{R})$. The correspondence $Y \rightarrow \tilde{\mu}_Y \rightarrow \mu_Y$ are all vaguely continuous. If one wishes to describe μ_Y on \mathcal{R} it is only necessary to define $\mu_Y([A] \times [t_{\mathcal{A}}, \infty])$ for $A \in \mathcal{A} \in \Gamma$.

PROPOSITION 8-4. For $Y \in \text{FPROC}(\Gamma)$ and $A \in \mathcal{A} \in \Gamma$, $\mu_Y([A] \times [t_{\mathcal{A}}, \infty]) = Y_{\mathcal{A}}(A)$.

PROOF. The formula is easily verified if Y is $\{0, 1\}$ -valued. By linearity the formula holds for all Y which are linear combinations of $\{0, 1\}$ -valued elements of Δ . Because such linear combinations are vaguely dense in Δ the formula holds for all $Y \in \Delta$, hence for all Y . \square

REMARK. Y is a martingale iff $|\mu_Y|(N \times (-\infty, \infty)) = 0$ and Y is a potential iff $|\mu_Y|(N \times \{\infty\}) = 0$ where the notation is the obvious.

When $\mathcal{A}_0 = \inf(\Gamma) \notin \Gamma$ then $\xi(\Delta)$ is noncompact. In this case, when Γ does not have uncountable cofinality when directed by reverse inclusion, $\mathfrak{M}_b(\xi(\Delta)) \neq \mathfrak{M}(\xi(\Delta))$. There is an easy interpretation of $\mathfrak{M}(\xi(\Delta))$ in terms of processes on Γ . Elements of $\mathfrak{M}(\xi(\Delta))$ correspond to unbounded F -processes on Γ whose restriction to final segments of Γ of the form $\{\mathcal{A}' \in \Gamma: \mathcal{A} \subset \mathcal{A}'\} = \Gamma^{\mathcal{A}}$ belong to $\text{FPROC}(\Gamma^{\mathcal{A}})$

for all $\mathcal{Q} \in \Gamma$. Call such processes *locally bounded F -processes* and denote them by $\text{FPROC}_{\text{loc}}(\Gamma)$. $\text{FPROC}_{\text{loc}}(\Gamma)$ may be equipped with the seminorms $\| \cdot \|_D^{\mathcal{Q}}$ for $\mathcal{Q} \in \Gamma$ which are defined by $\|Z\|_D^{\mathcal{Q}} = \|Z|_{\Gamma^{\mathcal{Q}}}\|_D$ for $Z \in \text{FPROC}_{\text{loc}}(\Gamma)$. In this case $\text{FPROC}_{\text{loc}}(\Gamma)$ is realized as the inductive limit of the Banach lattices $\{\text{FPROC}(\Gamma^{\mathcal{Q}}): \mathcal{Q} \in \Gamma\}$ under the natural maps. This proposition is immediate upon noticing that $\{N \times [t_{\mathcal{Q}}, \infty]: \mathcal{Q} \in \Gamma\}$ is cofinal in the net of compacts in $\xi(\Delta)$ where Γ is now directed by reverse inclusion (N is the supremum of \mathfrak{B} and $N \times [t_{\mathcal{Q}}, \infty]$ is defined as $\{Y \in \xi(\Delta): Y_{\mathcal{Q}}(N) = 1\}$ for all $\mathcal{Q} \in \Gamma$).

PROPOSITION 8-5. $\text{FPROC}_{\text{loc}}(\Gamma)$, $\mathfrak{M}(\xi(\Delta))$, and $BA_{\text{loc}}(\mathfrak{R})$ are isomorphic as inductive limits of Banach lattices. To any Z in $\text{FPROC}_{\text{loc}}(\Gamma)$ corresponds a unique $\mu_Z \in BA_{\text{loc}}(\mathfrak{R})$ defined by the formula $\mu_Z(\mathcal{Q} \times [t_{\mathcal{Q}}, \infty]) = Z_{\mathcal{Q}}(A)$ for $A \in \mathcal{Q} \in \Gamma$.

The vague topologies on $\text{FPROC}_{\text{loc}}(\Gamma)$, $\mathfrak{M}(\xi(\Delta))$ and $BA_{\text{loc}}(\mathfrak{R})$ all coincide under these isomorphisms.

COROLLARY 8-5-1. $\text{FPROC}_{\text{loc}}(\Gamma)$ consists of differences of nonnegative supermartingales on Γ .

PROOF. $\mathfrak{M}(\xi(\Delta)) = \mathfrak{M}^+(\xi(\Delta)) - \mathfrak{M}^+(\xi(\Delta))$. \square

We conclude with some remarks on the connections between the representation of elements of $\text{FPROC}(\Gamma)$ as finitely additive bounded measures on $\tilde{\mathfrak{R}}$ or \mathfrak{R} and the Doleans-Föllmer representation of F -processes of random variables adapted to a filtration on a probability measure space.

Let Γ be a chain of subalgebras of \mathfrak{B} with supremum \mathcal{Q}_{∞} . Let $X_{\mathcal{Q}_{\infty}}$ be the Stone space of \mathcal{Q}_{∞} and let T be the ideal time set of Γ . For $A \in \mathcal{Q}_{\infty}$, $[A]$ denotes, as usual, the clopen set in $X_{\mathcal{Q}_{\infty}}$ corresponding to A . For $\mathcal{Q} \in \Gamma$ let $[\mathcal{Q}] = \{[A]: A \in \mathcal{Q}\}$. If $\pi_{\mathcal{Q}}: X_{\mathcal{Q}_{\infty}} \rightarrow X_{\mathcal{Q}}$ is the canonical surjection then $[\mathcal{Q}]$ is the inverse image of the clopen algebra of $X_{\mathcal{Q}}$. The σ -algebra $\sigma[\mathcal{Q}]$ generated by $[\mathcal{Q}]$ is the inverse image of the Baire algebra of $X_{\mathcal{Q}}$ under $\pi_{\mathcal{Q}}$. One may consider the ring $\mathfrak{R}^{\#}$ in $X_{\mathcal{Q}_{\infty}} \times T$ generated by $\{\sigma[\mathcal{Q}] \times [t_{\mathcal{Q}}, \infty]: \mathcal{Q} \in \Gamma\}$. $\mathfrak{R}^{\#}$ is a subalgebra of the Baire algebra of $X_{\mathcal{Q}_{\infty}} \times T$. Each μ_Y , for $Y \in \text{FPROC}(\Gamma)$ has a unique extension $\mu_Y^{\#}$ from $\tilde{\mathfrak{R}}$ to $\mathfrak{R}^{\#}$ such that $\mu_Y^{\#}$ is countably additive on $\sigma[\mathcal{Q}] \times [t_{\mathcal{Q}}, \infty]$ for all $\mathcal{Q} \in \Gamma$. In fact, $Y \rightarrow \mu_Y^{\#}$ is a bijection from $\text{FPROC}(\Gamma)$ to the elements μ of $BA(\mathfrak{R}^{\#})$ so that μ has countably additive restrictions to each σ -algebra $\sigma[\mathcal{Q}] \times [t_{\mathcal{Q}}, \infty]$. Let $P \in \mathcal{P}(\mathcal{Q}_{\infty})$ be such that $Y_{\mathcal{Q}} \ll E(P|\mathcal{Q})$ for $\mathcal{Q} \in \Gamma$, as guaranteed by Proposition 6-2. Let $\tilde{\mathfrak{P}}$ be the corresponding Radon probability measure on $X_{\mathcal{Q}_{\infty}}$. Let $Y_{\mathcal{Q}}^{\#}$ be the countably additive measure on $\sigma[\mathcal{Q}]$ corresponding to $Y_{\mathcal{Q}}$, and let $f_{\mathcal{Q}}^{\#}$ be the Radon-Nikodym derivative on $Y_{\mathcal{Q}}^{\#}$ with respect to $E(\tilde{P}|\sigma[\mathcal{Q}])$. It is then true that $\{f_{\mathcal{Q}}^{\#}: \mathcal{Q} \in \Gamma\}$ is an F -process of random variables adapted to the filtration $\{\sigma[\mathcal{Q}]: \mathcal{Q} \in \Gamma\}$. All F -processes of random variables adapted to $\{\sigma[\mathcal{Q}]: \mathcal{Q} \in \Gamma\}$ arise in this fashion. Set $P_{\mathcal{Q}}^{\#}$ equal to the measure on $\sigma[\mathcal{Q}] \times [t_{\mathcal{Q}}, \infty]$ defined by $P_{\mathcal{Q}}^{\#}(E \times [t_{\mathcal{Q}}, \infty]) = \tilde{P}(E)$ for $E \in \sigma[\mathcal{Q}]$. The bounded measures μ on $\mathfrak{R}^{\#}$ with, for $\mathcal{Q} \in \Gamma$, $E(\mu|\sigma[\mathcal{Q}] \times [t_{\mathcal{Q}}, \infty]) \ll P_{\mathcal{Q}}^{\#}$ are in bijective correspondence with the F -processes of random variables adapted to the filtration $\{\sigma[\mathcal{Q}]: \mathcal{Q} \in \Gamma\}$.

We remark that there are some differences between the filtration $\{\sigma[\mathcal{Q}]: \mathcal{Q} \in \Gamma\}$ and those usually encountered when representing F -processes of random variables. First, this filtration is left continuous but rarely is it right continuous. Second, the σ -algebra $\sigma[\mathcal{Q}]$, although isomorphic to the Baire algebra of a compact Hausdorff space, is rarely a standard Borel algebra isomorphic to the Borel algebra of $[0, 1]$.

One may consider an F -process $f = \{f_t: t \in T_0\}$ of random variables adapted to a filtration $\{\mathcal{F}_t: t \in T_0\}$ in a probability measure space (Ω, \mathcal{F}, P) and carry out the representation of f as a finitely additive measure μ_F on the ring $\tilde{\mathcal{R}}$ of predictable sets in $X_{\mathcal{F}_\infty} \times T$ where T is the ideal time set. Since each \mathcal{F}_t is σ -complete, $\sigma[\mathcal{F}_t]$ has the property that each element of it differs from an element of $[\mathcal{F}_t]$ by a \tilde{P} -null set. A related result is that the F -process $\{f_t: t \in T_0\}$ of random variables adapted to the filtration $\{\sigma[\mathcal{F}_t]: t \in T_0\}$ may be taken so that each \tilde{f}_t is a continuous function from $X_{\mathcal{F}_\infty}$ to $[-\infty, \infty]$. For an $\omega \in \Omega$ we let $j(\omega)$ denote the ultrafilter in \mathcal{F}_∞ of supersets of the singleton $\{\omega\}$. The map $j: \omega \rightarrow j(\omega)$ embeds Ω into $X_{\mathcal{F}_\infty}$. For any $t \in T_0$, $\tilde{\mathcal{F}}_t = j^{-1}(\sigma[\mathcal{F}_t])$. Let $i: T_0 \rightarrow T$ be the usual inclusion. We have that $(j, i)^{-1}(\tilde{\mathcal{R}})$ is the usual ring of predictable events in $\Omega \times T_0$, and the Doleans-Föllmer representation is that induced by $(j, i)^{-1}$.

REFERENCES

1. H. Airault and H. Föllmer, *Relative densities of semi-martingales*, Invent. Math. **27** (1974), 299–327.
2. E. M. Alfsen, *Compact convex sets and boundary integrals*, Springer-Verlag, New York, 1971.
3. T. Armstrong, *Split faces in base norm ordered Banach spaces* (unpublished).
4. _____, *L-spaces do not have pre-duals of all orders*, Proc. Amer. Math. Soc. **74** (1979), 285–290.
5. _____, *Polyhedrality of infinite dimensional compact cubes*, Pacific J. Math. **70** (1977), 297–307.
6. _____, *Simplicial subdivisions of infinite dimensional compact cubes*, Pacific J. Math **81** (1979), 1–17.
7. _____, *Barycentric simplicial subdivisions of infinite dimensional simplexes and octahedra*, Pacific J. Math. **98** (1982), 251–270.
8. _____, *Full houses and cones of excessive functions*, Indiana Univ. J. Math. **29** (1980), 737–746.
9. _____, *The excessive cone of a stop-or-go gambling house* (unpublished).
10. _____, *Strong singularity, disjointness and strong finite additivity of finitely additive measures* (unpublished).
11. T. Armstrong and K. Prikry, *Singularity and absolute continuity with respect to strategic measures*, Illinois J. Math. (to appear).
12. T. Armstrong and W. Sudderth, *On nearly strategic measures*, Pacific J. Math. **94** (1981), 251–257.
13. L. Asimov and A. J. Ellis, *Convexity theory and its applications in functional analysis*, Academic Press, London, 1980.
14. K. A. Astbury, *Order convergence of martingales in terms of countably additive and purely finitely additive martingales*, Ann. Probability **9** (1981), 266–275.
15. K. Bichteler, *Stochastic integration and the L^p -theory of semi-martingales*, Ann. Probability **9** (1981), 49–89.
16. E. Briem and N. G. Loc, *Conditional Banach spaces, conditional projections and generalized martingales*, Math. Scand. **27** (1970), 227–236.
17. H. D. Brunk, *Stone spaces in probability*, Tech. Rep. 52, Department of Statistics, University of Oregon, 1976.
18. C. Dellacherie, *Une représentation intégrale des surmartingales a temps discret*, Publ. Inst. Statist. Univ. Paris **17** (1968), no. 2, 1–18.
19. J. Dixmier, *Sur un théorème de Banach*, Duke Math. J. **15** (1948), 1057–1071.
20. C. Doleans-Dade, *Existence du processus croissant naturel associé d'un potentiel de classe (D)*, Z. Wahrsch. Verw. Gebiete **9** (1968), 309–314.
21. J. Doob, *Stochastic processes*, Wiley, New York, 1953.

22. L. E. Dubins and L. J. Savage, *Inequalities for stochastic processes; how to gamble if you must*, Wiley, New York, 1969.
23. A. J. Ellis, *Minimal decompositions in partially ordered normed vector spaces*, Proc. Cambridge Phil. Soc. **64** (1968), 989–1000.
24. D. L. Fisk, *Quasi-martingales*, Trans. Amer. Math. Soc. **120** (1965), 359–389.
25. H. Föllmer, *The exit measure of a supermartingale*, Z. Warhsch. Verw. Gebiete **21** (1972), 154–166.
26. _____, *On the representation of semi-martingales*, Ann. Probability **1** (1973), 580–589.
27. D. Gilat, *Every nonnegative submartingale is the absolute value of a martingale*, Ann. Probability **5** (1977), 475–481.
28. P. Halmos, *Lectures on Boolean algebras*, Van Nostrand, Princeton, N. J., 1963.
29. C. A. Hayes and C. Y. Pauc, *Derivation and martingales*, Springer-Verlag, New York, 1970.
30. E. Hewitt and K. Yosida, *Finitely additive measures*, Trans. Amer. Math. Soc. **72** (1952), 46–66.
31. T. Jeulin, *Partie positive d'une quasimartingale*, C. R. Acad. Sci. Paris Sér. A–B **287** (1978), 351–352.
32. Z. Lipecki, *Extensions of additive set functions with values in a topological group*, Bull. Acad. Polon. Sci. Sér. Sci. Math. **22** (1974), 19–27.
33. J. Memin, *Espaces de semi-martingales et changement de probabilité*, Z. Warhsch. Verw. Gebiete **52** (1980), 9–39.
34. M. Metivier and J. Pellaumail, *On Doleans-Föllmer measure for quasimartingales*, Illinois J. Math. **19** (1975), 491–504.
35. _____, *Stochastic integration*, Academic Press, London, 1980.
36. P. A. Meyer, *Probability and potentials*, Blaisdell, Waltham, 1966.
37. _____, *Intégrales stochastiques*, Séminaire de Probabilités I (Univ. Strasbourg, 1967), Springer-Verlag, New York, 1967, pp. 72–167.
38. _____, *A decomposition theorem for supermartingales*, Illinois J. Math. **6** (1962), 193–205.
39. _____, *Decomposition for supermartingales: the uniqueness theorem*, Illinois J. Math. **7** (1963), 1–17.
40. J. S. Misura, *The Doob-Meyer decomposition for two parameter semimartingales*, Dokl. Akad. Nauk Ukrain. SSR Ser. A **92** (1980) 19–21.
41. K. F. Ng and Y. C. Wong, *Partially ordered topological vector spaces*, Clarendon Press, Oxford, 1973.
42. S. Orey, *F-processes*, Fifth Berkeley Sympos. Math. Statist. and Probability, 2, Univ. of California Press, Berkeley, 1967, pp. 301–313.
43. I. V. Pavlov, *On extreme rays and the integral representation in a cone of supermartingales*, Theory Probab. Appl. **25** (1980), 602–605.
44. J. Pellaumail, *Sur l'intégrale stochastique et la décomposition de Doob-Meyer*, Asterisque **9** (1973).
45. R. R. Phelps, *Lectures on Choquet's theorem*, Van Nostrand, Princeton, N. J., 1966.
46. M. M. Rao, *On decomposition theorems of Meyer*, Math. Scand. **24** (1969), 66–78.
47. _____, *Quasi-martingales*, Math. Scand. **24** (1969), 79–92.
48. _____, *Foundations of stochastic analysis*, Academic Press, London, 1981.
49. H. H. Schaefer, *Topological vector spaces*, Macmillan, New York, 1966.
50. K. Schmidt, *Sur la valeur d'un processus de fonctions d'ensemble*, C. R. Acad. Sci. Paris Sér. A-B **288** (1979), 431–434.
51. _____, *Espaces vectoriels réticules, décompositions de Riesz, et caractérisations de certains processus de fonctions d'ensembles*, C. R. Acad. Sci. Paris Sér. A-B **289** (1979), 75–78.
52. Z. Semadeni, *Banach spaces of continuous functions*, PWN, Warsaw, 1971.
53. R. Sikorsky, *Boolean algebras*, Springer-Verlag, New York, 1964.
54. C. Stricker, *Quasimartingales, martingales locales, semi-martingales et filtration naturelle*, Z. Warhsch. Verw. Gebiete **39** (1977), 55–63.
55. T. Watanabe, *The general Riesz decomposition and the specific order of excessive functions*, Math. Scand. **28** (1971), 33–44.

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