TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 281, Number 1, January 1984

# ON SOME SUBALGEBRAS OF A VON NEUMANN ALGEBRA CROSSED PRODUCT

BY

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ABSTRACT. We study conditions for a nonselfadjoint subalgebra of a von Neumann crossed product $\mathfrak{L}$ to be an algebra of analytic operators with respect to a flow on $\mathfrak{L}$. We restrict ourselves to the case where $\mathfrak{L}$ is constructed from a finite von Neumann algebra $M$ with a trace preserving *-automorphism $\alpha$ that acts ergodically on the center of $M$.

**1. Introduction.** In [**9**] we studied subalgebras of a von Neumann algebra $\mathfrak{L}$, constructed as a crossed product of a finite von Neumann algebra $M$ by a trace preserving *-automorphism $\alpha$, that contain the nonselfadjoint crossed product $\mathfrak{L}_+$. With the results of [**9**, §4] we can prove (see Corollary 3.5) that each such subalgebra is an algebra of analytic operators with respect to some flow $\beta$; i.e. it has the form $\mathfrak{L}^\beta[0,\infty)$ (see [**2**]).

In this paper we study conditions on subalgebras $\mathfrak{B}$ of $\mathfrak{L}$ (weaker than the condition $\mathfrak{B} \supseteq \mathfrak{L}_+$) that are sufficient to ensure $\mathfrak{B}$ has the form $\mathfrak{L}^\beta[0,\infty)$ for some flow $\beta$ on $\mathfrak{L}$. We do this with the assumption that $\alpha$ acts ergodically on the center $Z$ of $M$.

We deal separately with the cases $Z$ nonatomic and $Z$ atomic (in the latter case we add the assumption that $\alpha^k$ is outer for each $k \in \mathbf{Z}$). The assumptions allow us to use the results of [**6**].

In both cases we find (Theorem 3.4 and Corollary 3.5) sufficient conditions for the algebra to be of the form $\mathfrak{L}^\beta[0,\infty)$. Moreover, we show how the flow $\beta$ is derived from the algebra in question.

For the case where $Z$ is atomic we also find that any subalgebra of $\mathfrak{L}$ containing a subalgebra of the form $\mathfrak{L}^\beta[0,\infty)$ (for a flow $\beta$) has the form $\mathfrak{L}^\gamma[0,\infty)$ (for some other flow $\gamma$) (Theorem 4.2).

**2. Preliminaries and the definition of $\phi$.** Let $M$ be a finite von Neumann algebra with a faithful and normal finite trace $\varphi$. We assume $M$ is in standard form and identify it with the von Neumann algebra of left multiplications on $L^2(M,\varphi)$ (see [**8**]). The algebra $M'$ is its commutant on $L^2(M,\varphi)$. Since $M$ has a generating and

Received by the editors January 24, 1983.

1980 *Mathematics Subject Classification*. Primary 46L10; Secondary 46L40, 47L25.

*Key words and phrases*. Crossed product, nonselfadjoint subalgebra, analyticity, flow, spectral subspaces for groups of automorphisms.

[1]Supported by the Fund for Basic Research, administered by the Israeli Academy of Sciences and Humanities.

separating vector, $M'$ is also finite. We write $Z$ for $M \cap M'$ and identify it with $L^\infty(X, \nu)$ for some locally compact Hausdorff space $X$ with a probability measure $\nu$ such that

$$\int_X f\,d\nu = \varphi(f), \qquad f \in L^\infty(X, \nu).$$

We fix once and for all a normal, *-automorphism $\alpha$ of $M$ which preserves $\varphi$; i.e., $\varphi \circ \alpha = \varphi$. The following proposition appears in [**4**].

PROPOSITION 2.1. *Let* $L_0^2 = \{f\colon \mathbf{Z} \to M;\ f(n) = 0$ *for all but finitely many* $n\}$. *Then with respect to pointwise addition, scalar multiplication and the operations defined by equations* (1)–(3), $L_0^2$ *is a Hilbert algebra with identity* $\psi$ *defined by* $\psi(0) = I_M$ *and* $\psi(n) = 0, n \neq 0$.

$$(1) \qquad (f * g)(n) = \sum_{k \in Z} f(k)\alpha^k(g(n-k)),$$

$$(2) \qquad (f^*)(n) = [\alpha^n(f(-n))]^*,$$

$$(3) \qquad \langle f, g \rangle = \sum_{k \in Z} (f(k), g(k))_{L^2(M,\varphi)}.$$

Note that the Hilbert space completion $L^2$ of $L_0^2$ is

$$\left\{ f\colon \mathbf{Z} \to L^2(M, \varphi);\ \sum_{n \in \mathbf{Z}} \|f(n)\|^2_{L^2(M,\varphi)} < \infty \right\}.$$

For $f$ in $L_0^2$ we define operators $L_f$ and $R_f$ on $L^2$ by $L_f g = f * g$ and $R_f g = g * f$, $g \in L^2$. Both $L_f$ and $R_f$ are well-defined, bounded operators, and we set $\mathfrak{L} = \{L_f\colon f \in L_0^2\}''$, $\mathfrak{R} = \{R_f\colon f \in L_0^2\}''$. Also, we define $L^\infty$ to be the achieved Hilbert algebra of all bounded elements in $L^2$. For such an $f$, we write $L_f$ and $R_f$ for the operators it determines. It is known that the map $f \to L_f$ [resp. $f \to R_f$] is a *-isomorphism [resp. *-anti-isomorphism] from $L^\infty$ onto $\mathfrak{L}$ [resp. $\mathfrak{R}$]. Moreover, $\mathfrak{L}$ and $\mathfrak{R}$ are finite von Neumann algebras with $\mathfrak{R}' = \mathfrak{L}$. We call $L^\infty$ the *selfadjoint* or *von Neumann algebra crossed product* determined by $M$, $\varphi$ and $\alpha$, and refer to $\mathfrak{L}$ and $\mathfrak{R}$ as the left and right regular representations of it.

The original algebra $M$ is identified with the subalgebra $\{x\psi\colon x \in M\}$ of $L^\infty$, and we write $L_x$ (and $R_x$) for $L_{x\psi}$ (and $R_{x\psi}$). We have, for $f \in L^2$, $(L_x f)(n) = xf(n)$ and $(R_x f)(n) = f(n)\alpha^n(x)$. We write $\mathfrak{L}(M) = \{L_x\colon x \in M\}$ and $\mathfrak{R}(M) = \{R_x\colon x \in M\}$.

If we let $\delta$ be defined by $\delta(n) = 0$ if $n \neq 1$, $\delta(1) = I_M$, then it is easy to check that $\mathfrak{L}$ is the von Neumann algebra generated by $\mathfrak{L}(M)$ and $L_\delta$ and, similarly, $\mathfrak{R}$ is generated by $\mathfrak{R}(M)$ and $R_\delta$.

The automorphism group $\{\hat{\alpha}_t\}_{t \in \mathbf{R}}$ of $\mathfrak{L}$ dual to $\alpha$ in the sense of Takesaki [**10**] is implemented by the unitary representation of $\mathbf{R}$, $\{W_t\}_{t \in \mathbf{R}}$, defined by

$$(W_t f)(n) = e^{2\pi i n t} f(n), \qquad f \in L^2;$$

that is, $\hat{\alpha}_t(L_f) = W_t L_f W_t^*$. Similarly, $\hat{\alpha}_t(R_f) = W_t R_f W_t^*$. It is easy to see that $\hat{\alpha}_t(L_f) = L_{W_t f}$ for $f$ in $L^\infty$, and similarly for $R_f$. One can check that the spectral resolution of $\{W_t\}_{t\in\mathbf{R}}$ is given by

$$W_t = \sum_{n=-\infty}^{\infty} e^{2\pi int}E_n,$$

where $E_n$ is the projection on $L^2$ defined by

$$(E_n f)(k) = \begin{cases} f(n), & k = n, \\ 0, & k \neq n. \end{cases}$$

We denote the restriction of $E_n$ to $L^\infty$ by $\varepsilon_n$ and write $\varepsilon_n(L_f) = L_{\varepsilon_n(f)}$. We have

$$\varepsilon_n = \int_0^1 e^{-2\pi int}\hat{\alpha}_t\, dt,$$

where the integral converges in the $\sigma$-weak operator topology.

REMARK 2.1. If $f$ lies in $L^\infty$, then $\varepsilon_k(L_f) = L_{f(k)}L_\delta^k$ (see [**9**, Remark preceding Theorem 4.5]) for each $k \in \mathbf{Z}$. Hence,

$$\begin{aligned}\varepsilon_k(L_f^*)^* &= \varepsilon_k(L_{f^*})^* = \left(L_{f^*(k)}L_\delta^k\right)^* = \left(L_{\alpha^k(f(-k))^*}L_\delta^k\right)^* \\ &= \left(L_\delta^k L_{f(-k)^*}\right)^* = L_{f(-k)}L_\delta^{-k} = \varepsilon_{-k}(L_f).\end{aligned}$$

Thus $\varepsilon_k(L_f^*) = (\varepsilon_{-k}(L_f))^*$.

We let $H^2$ be $\{f \in L^2: f(n) = 0, n < 0\}$, and $H^\infty$ be $\{f \in L^\infty: f(n) = 0, n < 0\}$.

More details concerning these algebras can be found in [**4** and **5**].

We will define an $\mathcal{L}(Z)$-trace following [**1**, Chapter III, §4]. First we let $\mathfrak{X}$ be the set of all nonnegative measurable functions, finite or not, on $X$, and $\mathfrak{X}'$ the set of all real valued (finite) measurable functions on $X$ (we identify two functions in $\mathfrak{X}$, or $\mathfrak{X}'$, if they are different only on a set of measure zero). We identify the bounded functions in $\mathfrak{X}'$ with $\mathcal{L}(Z)$. Hence $\mathcal{L}(Z) \subseteq \mathfrak{X}'$ and $\mathcal{L}(Z)_+ \subseteq \mathfrak{X}$.

DEFINITION. An $\mathcal{L}(Z)$-trace on $\mathcal{L}(M)'_+$ (the positive cone of the commutant of $\mathcal{L}(M)$) is a map $\phi$ defined on $\mathcal{L}(M)'_+$, with values in $\mathfrak{X}$ and satisfying:

(1) If $T, S \in \mathcal{L}(M)'_+$, then $\phi(S + T) = \phi(S) + \phi(T)$;

(2) If $S \in \mathcal{L}(M)'_+$, $T \in \mathcal{L}(Z)_+$, then $\phi(TS) = T\phi(S)$; and

(3) If $S \in \mathcal{L}(M)'_+$, and $U \in \mathcal{L}(M)'$ is a unitary operator, then $\phi(USU^*) = \phi(S)$.

$\phi$ is *semifinite* if for every $S \neq 0$ in $\mathcal{L}(M)'_+$, there is an operator $T \in \mathcal{L}(M)'_+$, $T \neq 0$, such that $T \leqslant S$ and $\phi(T) \in \mathcal{L}(Z)_+$. $\phi$ is *faithful* if $\phi(T) = 0$ only when $T = 0$, and $\phi$ is *normal* if, for each increasing net $\{S_\alpha\} \subseteq \mathcal{L}(M)'_+$, $\operatorname{Sup}_\alpha \phi(S_\alpha) = \phi(\operatorname{Sup}_\alpha S_\alpha)$.

It is shown in [**9**] that there is a unique, faithful, normal semifinite $\mathcal{L}(Z)$-trace that maps $E_0$ into $I$. Henceforth, we let $\phi$ be this $\mathcal{L}(Z)$-trace.

REMARK 2.2. Consider $\alpha$ as acting on $Z$. By a theorem of Mackey [**3**] there is a measurable transformation $\tau$ on $X$ implementing $\alpha$; i.e. $\alpha(f)(x) = f(\tau(x))$. Using this we can extend the action of $\alpha$ to both $\mathfrak{X}$ and $\mathfrak{X}'$. We claim that $\phi(L_\delta T L_\delta^*) = \alpha(\phi(T))$ for $T \in \mathcal{L}(M)'$ with $\phi(T)$ in $\mathfrak{X}$. Indeed, let $\phi_1$ be defined on $\mathcal{L}(M)'_+$ by $\phi_1(T) = \alpha^{-1}(\phi(R_\delta^* L_\delta T L^* R_\delta))$; then $\phi_1$ is a faithful, normal, semifinite $\mathcal{L}(Z)$-trace that maps $E_0$ into $I$ (because $\phi$ has these properties and $R_\delta^* L_\delta E_0 L_\delta^* R_\delta = E_0$). Since $\phi$ is unique with these properties, the claim follows.

Finally, we recall that if $\{\beta_t\}_{t\in\mathbf{R}}$ is a group of *-automorphisms on $\mathfrak{L}$ such that $t \to \beta_t(T)$ is $\sigma$-weakly continuous for each $T \in \mathfrak{L}$, then $\mathfrak{L}^{\beta}[0,\infty)$ is the spectral subspace associated with $[0,\infty) \subseteq \mathbf{R}$. Such a group $\beta$ will be called a *flow* and $\mathfrak{L}^{\beta}[0,\infty)$ is a $\sigma$-weakly closed subalgebra of $\mathfrak{L}$ (see **[2]** for more details). We also refer to $\mathfrak{L}^{\beta}[0,\infty)$ as the algebra of analytic operators with respect to the flow $\beta$.

**3. Subalgebras of $\mathfrak{L}$ when $\mathfrak{L}(Z)$ is nonatomic.** In the section and the next we frequently refer to the following four conditions, where $\mathfrak{B}$ is a $\sigma$-weakly closed subalgebra of $\mathfrak{L}$.

(i) $\mathfrak{B} + \mathfrak{B}^*$ is $\sigma$-weakly dense in $\mathfrak{L}$.

(ii) $\mathfrak{L}(M) \subseteq \mathfrak{B}$.

(iii) For $f \in L^{\infty}$, $f$ lies in $[\mathfrak{B}]_2$ ($=$ the closure of the subspace $\{T\psi\colon T \in \mathfrak{B}\}$) if and only if $L_f \in \mathfrak{B}$.

(iv) $\phi(P - PL_{\delta}PL_{\delta}^*)$ and $\phi(L_{\delta}PL_{\delta}^* - PL_{\delta}PL_{\delta}^*)$ are finite a.e., where $P$ is the projection on $[\mathfrak{B}]_2$. (We shall refer to this condition only when it is known that $P$ and $L_{\delta}PL_{\delta}^*$ commute.)

We write $P(\mathfrak{B})$ for the projection onto $[\mathfrak{B}]_2$.

Throughout this section we assume $\mathfrak{L}(Z)$ is nonatomic and $\alpha$ acts ergodically on $\mathfrak{L}(Z)$.

LEMMA 3.1. *If $\mathfrak{B}$ satisfies conditions* (i)–(iii) *there is a sequence $\{e_k\}_{k=-\infty}^{\infty}$ of projections in $\mathfrak{L}(Z)$ such that $P(\mathfrak{B}) = \sum_{k=-\infty}^{\infty} e_k E_k$ and*

$$\mathfrak{B} = \left\{T \in \mathfrak{L} \colon \varepsilon_k(T) \in e_k\mathfrak{L}(M)L_{\delta}^k \text{ for each } k \in \mathbf{Z}\right\}.$$

*Moreover, for each $k$, $n \in \mathbf{Z}$, we have*

$$e_n\alpha^n(e_k) \leq e_{n+k}, \tag{1}$$

$$1 - \alpha^{-n}(e_n) \leq e_{-n}, \tag{2}$$

$$e_n\big(1 - \alpha^k(e_{n-k})\big) \leq e_k. \tag{3}$$

PROOF. Let $P$ be $P(\mathfrak{B})$, then since $\mathfrak{L}(M) \subseteq \mathfrak{B}$, $P \in \{\mathfrak{L}(M), \mathfrak{R}(M)\}'$. Using **[6**, Proposition 3.1**]** there is a sequence $\{e_k\}_{k=-\infty}^{\infty}$ of projections in $\mathfrak{L}(Z)$ such that $P = \sum_{k=-\infty}^{\infty} e_k E_k$. (iii) now implies $\mathfrak{B} = \{T \in \mathfrak{L}\colon \varepsilon_k(T) \in e_k\mathfrak{L}(M)L_{\delta}^k \text{ for each } k \in \mathbf{Z}\}$ since $\varepsilon_k(T) = L_{E_k(f)}$ where $T = L_f$, $f \in L^{\infty}$.

For each $k$, $n \in \mathbf{Z}$, $e_nL_{\delta}^n$ and $e_kL_{\delta}^k$ lie in $\mathfrak{B}$; hence, $e_n\alpha^n(e_k)L_{\delta}^{n+k} = e_nL_{\delta}^n e_k L_{\delta}^k \in \mathfrak{B}$ and (1) follows. For (2) note that, since $\varepsilon_k(L^*) = (\varepsilon_{-k}(L_f))^*$ (see Remark 2.1), $\varepsilon_k(T^*) \in e_k\mathfrak{L}(M)L_{\delta}^k$ for $T \in \mathfrak{L}$ if and only if

$$\varepsilon_{-k}(T) \in \big(e_k\mathfrak{L}(M)L_{\delta}^k\big)^* = L_{\delta}^{-k}e_k\mathfrak{L}(M) = L_{\delta}^{-k}e_kL_{\delta}^{-k}\mathfrak{L}(M)L_{\delta}^k = \alpha^{-k}(e_k)\mathfrak{L}(M)L_{\delta}^k.$$

Hence,

$$\begin{aligned}\mathfrak{B}^* = \{T\colon T^* \in \mathfrak{B}\} &= \left\{T \in \mathfrak{L}\colon \varepsilon_k(T^*) \in e_k\mathfrak{L}(M)L_{\delta}^k \text{ for each } k \in \mathbf{Z}\right\}.\\ &= \left\{T \in \mathfrak{L}\colon \varepsilon_k(T) \in \alpha^k(e_{-k})\mathfrak{L}(M)L_{\delta}^{-k} \text{ for each } k \in \mathbf{Z}\right\},\end{aligned}$$

and

$$\mathfrak{B} + \mathfrak{B}^* \subseteq \left\{T \in \mathfrak{L}\colon \varepsilon_k(T) \in \big(e_k \vee \alpha^k(e_{-k})\big)\mathfrak{L}(M)L_{\delta}^k \text{ for each } k \in \mathbf{Z}\right\}.$$

Since $\mathfrak{B} + \mathfrak{B}^*$ is $\sigma$-weakly dense in $\mathfrak{L}$ and the set on the right-hand side is $\sigma$-weakly closed (as $\varepsilon_k$ are $\sigma$-weakly continuous), $e_k \vee \alpha^k(e_{-k}) = 1$ and (2) follows.

As a consequence we have, for $n, k \in \mathbf{Z}$,

$$e_n(1 - \alpha^k(e_{n-k})) = e_n\alpha^n(1 - \alpha^{k-n}(e_{n-k})) \leq e_n\alpha^n(e_{n-k}) \leq e_k;$$

hence, (3) holds. □

Let $f$ lie in $\mathfrak{Z}'$ and let the sequence $\{f_n\}_{n=-\infty}^{\infty}$ of elements of $\mathfrak{Z}'$ be defined by

$$f_n = \begin{cases} \sum_{k=0}^{n-1} \alpha^k(f), & n > 0, \\ 0, & n = 0, \\ -\alpha^n(f_{-n}), & n < 0. \end{cases}$$

Let $U_t^{(n)}$ be the unitary operator in $\mathfrak{L}(Z)$ defined by

$$U_t^{(n)} = \exp(itf_n), \qquad n \in \mathbf{Z}, t \in \mathbf{R}.$$

Since $E_n E_m = 0$ for $n \neq m$ and $U_t^{(n)}E_n(L^2) \subseteq E_n(L^2)$, $U_t = \sum_{n=-\infty}^{\infty} U_t^{(n)}E_n$ is a unitary operator on $L^2$. Moreover, $t \to U_t$ is a strongly continuous representation of $\mathbf{R}$ (since $t \to U_t^{(n)}$ is strongly continuous for each $n \in \mathbf{Z}$). We let $\{\beta_t\}_{t\in\mathbf{R}}$ be the group of *-automorphisms on $\mathfrak{L}$ defined by

$$\beta_t(T) = U_t T U_t^*, \qquad T \in \mathfrak{L}.$$

We call $\{U_t\}_{t\in\mathbf{R}}$ the group of unitary operators and $\{\beta_t\}_{t\in\mathbf{R}}$ the group of *-automorphisms given rise to by $f$ (in $\mathfrak{Z}'$).

A calculation similar to [**4**, p. 390] reveals

$$\beta_t(L_g) = L_{U_t g}, \qquad t \in \mathbf{R}, L_g \in \mathfrak{L}.$$

There is an obvious correspondence between projections in $\mathfrak{L}(Z)$ and measurable subsets of $X$ (where the projection $e$ that corresponds to $\hat{e} \subseteq X$ is of the form $L_h$, and $h$ is the characteristic function of $\hat{e}$ viewed as an element of $Z \simeq L^\infty(X, \nu)$).

LEMMA 3.2. *Let $f$ (in $\mathfrak{Z}'$) give rise to a group of *-automorphisms $\{\beta_t\}_{t\in\mathbf{R}}$. Then $t \to \beta_t(T)$ is $\sigma$-weakly continuous for each $T \in \mathfrak{L}$ and the algebra $\mathfrak{L}^\beta[0, \infty)$ of analytic operators is the set $\{T \in \mathfrak{L}: \varepsilon_k(T) \in c_k\mathfrak{L}(M)L_\delta^k$ for each $k \in \mathbf{Z}\}$, where $c_k$ is the projection in $\mathfrak{L}(Z)$ corresponding to the set $\hat{c}_k = \{x \in X: f_k(x) \geq 0\}$. Therefore $\mathfrak{L}^\beta[0, \infty)$ satisfies conditions* (i)–(iii).

PROOF. For $T \in \mathfrak{L}$, $t \to \beta_t(T) = U_t T U_t^*$ is $\sigma$-weakly continuous since $t \to U_t$ is strongly continuous.

Let $P$ ($P_n$) be the spectral measure associated with the group $\{U_t\}_{t\in\mathbf{R}}$ ($\{U_t^{(n)}\}_{t\in\mathbf{R}}$, $n \in \mathbf{Z}$) by Stone's Theorem.

Since $U_t = \sum_{n=-\infty}^{\infty} U_t^{(n)}E_n$ and $U_t^{(n)}E_n(L^2) \subseteq E_n(L^2)$ for all $n \in \mathbf{Z}$, $t \in \mathbf{R}$, we have

$$P[s, \infty) = \sum_{n=-\infty}^{\infty} P_n[s, \infty)E_n \quad \text{for each } s \in \mathbf{R}.$$

Since $U_t^{(n)}$ lies in $\mathfrak{L}(Z)$, $P_n[s,\infty) \in \mathfrak{L}(Z)$ and, in fact, $P_n[s,\infty)$ is the projection, in $\mathfrak{L}(Z)$, corresponding to the subset $\hat{c}_n^{(s)} = \{x \in X\colon f_n(x) \geqslant s\}$ of $X$ (since $U_t^{(n)} = \exp(itf_n)$). Hence, $P[0,\infty) = \Sigma c_n E_n$, where $c_n$ corresponds to $\hat{c}_n = \{x \in X\colon f_n(x) \geqslant 0\}$.

We now wish to show that $g$, in $L^\infty$, lies in $P[0,\infty)(L^2)$ if and only if $L_g$ lies in $\mathfrak{L}^\beta[0,\infty)$.

Let $L_g$ be in $\mathfrak{L}^\beta[0,\infty)$. Then by [**2**, Theorem 2.9], $L_g P[0,\infty)(L^2) \subseteq P[0,\infty)(L^2)$. But $\psi \in P[0,\infty)(L^2)$ (since $P_0[0,\infty) = I$ and $\psi(0) = I$, $\psi(n) = 0$, $n \neq 0$) and $L_g\psi = g$; hence, $g \in P[0,\infty)(L^2)$.

For the converse note that, since $L^1(T)$ has an approximate identity consisting of trigonometric polynomials, say $\{k_n\}_{n=1}^\infty$, each $T \in \mathfrak{L}$ is the $\sigma$-weak limit of (finite) linear combinations of $\{\varepsilon_k(T)\}_{k=-\infty}^\infty$ (namely $\int_\pi \hat{\alpha}_t(T)k_n(t)\,dt$). Hence, it suffices to prove, for $g$ in $L^\infty \cap P[0,\infty)(L^2)$, that $\varepsilon_k(L_g)$ lies in $\mathfrak{L}^\beta[0,\infty)$ for each $k \in \mathbf{Z}$.

As previously noted (Remark 2.1), $\varepsilon_k(L_g) = L_{g(k)}L_\delta^k$, hence, we now fix $k$ and prove that

$$L_{g(k)}L_\delta^k P[s,\infty)(L^2) \subseteq P[s,\infty)(L^2) \quad \text{for all } s \in \mathbf{R}.$$

This will imply that $\varepsilon_k(L_g) \in \mathfrak{L}^\beta[0,\infty)$ (by Theorem 2.9 of [**2**]). Since $g \in P[0,\infty)(L^2)$, $E_k(g) \in P_k[0,\infty)(L^2)$, and if we let $p_k$ be the projection in $Z$ with $L_{p_k} = P_k[0,\infty)$, then $g(k) = p_k g(k)$ and $L_{g(k)} = P_k[0,\infty)L_{g(k)}$. Fix $s \in \mathbf{R}$ and $h$ in $P[s,\infty)(L^2)$. Then for $n \in \mathbf{Z}$,

$$\begin{aligned} E_n\big(\varepsilon_k(L_g)h\big) &= E_n L_{g(k)}L_\delta^k h = L_{g(k)}E_n L_\delta^k h = L_{g(k)}L^k E_{n-k}h \\ &\in L_{g(k)}L_\delta^k P_{n-k}[s,\infty)E_{n-k}(L^2) = L_{g(k)}\alpha^k\big(P_{n-k}[s,\infty)\big)E_n(L^2) \\ &\subseteq P_k[0,\infty)\alpha^k\big(P_{n-k}[s,\infty)\big)E_n(L^2). \end{aligned}$$

But, from the definition of the functions $\{f_k\}$ in $\mathfrak{Z}'$, if $f_k(x) \geqslant 0$ and $\alpha^n(f_{n-k})(x) \geqslant s$, then $f_n(x) \geqslant s$. Hence, $\hat{c}_k \cap \alpha^n(\hat{c}_{n-k}^{(s)}) \subseteq \hat{c}_n^{(s)}$ and $P_k[0,\infty)\alpha^k(P_{n-k}[s,\infty)) \subseteq P_n[s,\infty)$. It follows that, for each $n \in \mathbf{Z}$,

$$E_n\big(\varepsilon_k(L_g)h\big) \in P_n[0,\infty)E_n(L^2) \subseteq P[s,\infty)(L^2).$$

Therefore, $\varepsilon_k(L_g)h \in P[s,\infty)(L^2)$ and this completes the proof that $L_g \in \mathfrak{L}^\beta[0,\infty)$. We conclude that

$$\begin{aligned} \mathfrak{L}^\beta[0,\infty) &= \big\{L_g \in \mathfrak{L}\colon g \in P[0,\infty)(L^2)\big\} \\ &= \big\{L_g \in \mathfrak{L}\colon E_n(g) \in c_n E_n(L^2) \text{ for each } n \in \mathbf{Z}\big\} \\ &= \big\{L_g \in \mathfrak{L}\colon \varepsilon_n(L_g) \in c_n\mathfrak{L}(M)L_\delta^n \text{ for each } n \in \mathbf{Z}\big\}. \end{aligned}$$

Now $\mathfrak{L}(M) \subseteq \mathfrak{L}^\beta[0,\infty)$ since $c_0 = I$. Condition (i) is satisfied because of Theorem 3.15 in [**2**]. Condition (iii) follows from the fact that $\mathfrak{L}^\beta[0,\infty)$ is $\{L_g \in \mathfrak{L}\colon g \in P[0,\infty)(L^2)\}$, because $[\mathfrak{L}^\beta[0,\infty)]_2 \subseteq P[0,\infty)(L^2)$. □

LEMMA 3.3. *Let $\mathfrak{B}$ be a subalgebra of $\mathfrak{L}$ satisfying* (i)–(iv) *and let $P$ be $P(\mathfrak{B})$. For $k \in \mathbf{Z}$ let $g_1$ be $\phi(P - PL_\delta^k PL_\delta^{-k})$ and $g_2$ be $\phi(L_\delta^k PL_\delta^{-k} - PL_\delta^k PL_\delta^{-k})$. Then*:

(1) $g_1 - g_2 = f_k$, *where* $f = \phi(P - PL_\delta PL_\delta^*) - \phi(L_\delta PL_\delta^* - PL_\delta^*) \in \mathfrak{Z}'$;

(2) $g_1 g_2 = 0$.

PROOF. First note that $P$ commutes with $L_\delta^k P L_\delta^{-k}$ ($k \in \mathbf{Z}$) because $P = \sum_{n=-\infty}^{\infty} e_n E_n$, $L_\delta^k P L_\delta^{-k} = \sum_{n=-\infty}^{\infty} \alpha^k(e_n) E_{n+k}$ and $e_n \in \mathfrak{L}(Z)$ (Lemma 3.1).

We can extend the definition of $\phi$ to all operators $T \in \mathfrak{L}(M)'$ that can be written as $T = T_1 - T_2$, where $T_1, T_2 \in \mathfrak{L}(M)'_+$ and $\phi(T_i), \phi(T_2) \in \mathfrak{Z}'$, simply by $\bar{\phi}(T) = \phi(T_1) - \phi(T_2)$. Since (iv) is satisfied,

$$f = \phi(P - PL_\delta PL_\delta^*) - \phi(L_\delta PL_\delta^* - PL_\delta PL_\delta^*) = \bar{\phi}(P - L_\delta PL_\delta^*).$$

Remark 2.2 can be seen to hold for $\bar{\phi}$ in place of $\phi$, thus

$$\alpha^n(f) = \bar{\phi}\left(L_\delta^n PL_\delta^{-n} - L_\delta^{n+1} PL_\delta^{-n-1}\right),$$

and, for $k > 0$,

$$f_k = \sum_{k=0}^{n-1} \alpha^n(f) = \bar{\phi}\left(P - L_\delta^k PL_\delta^{-k}\right),$$

while for $k < 0$,

$$f_k = -\alpha^k(f_{-k}) = -\alpha^k\left(\phi\left(P - L_\delta^{-k} PL_\delta^k\right)\right) = \bar{\phi}\left(P - L_\delta^k PL_\delta^{-k}\right).$$

Thus

$$f_k = \bar{\phi}\left(P - L_\delta^k PL_\delta^{-k}\right) = \phi\left(P - PL_\delta^k PL_\delta^{-k}\right) - \phi\left(L_\delta^k PL_\delta^{-k} - PL_\delta^k PL_\delta^{-k}\right) = g_1 - g_2.$$

For (2), note that $P - L_\delta^k PL_\delta^{-k} = \Sigma d_n E_n$, where $d_n = e_n(1 - \alpha^k(e_{n-k})) \leqslant e_k$ (see Lemma 3.1). Hence,

$$g_1 = \phi(\Sigma d_n E_n) = \Sigma \phi(d_n E_n) = \Sigma d_n \quad \text{and} \quad g_1 e_k = g_1.$$

On the other hand, $L_\delta^k PL_\delta^{-k} - PL_\delta^k PL_\delta^{-k} = \Sigma c_n E_n$, where

$$c_n = \alpha^k(e_{n-k})(1 - e_n) = \alpha^k\left(e_{n-k}\left(1 - \alpha^{-k}(e_n)\right)\right) \leqslant \alpha^k(e_{-k}) \leqslant 1 - e_k;$$

thus $g_2 = g_2(1 - e_k)$ and $g_1 g_2 = 0$. □

THEOREM 3.4. *Let $\mathfrak{B}$ be a σ-weakly closed subalgebra of $\mathfrak{L}$ satisfying* (i)–(iv). *Then $\mathfrak{B} = \mathfrak{L}^\beta[0, \infty)$ for some flow $\beta$ on $\mathfrak{L}$. In fact $\beta$ is the group introduced in the discussion preceding Lemma* 3.2 *for*

$$f = \phi(P - PL_\delta PL_\delta^*) - \phi(L_\delta PL_\delta^* - PL_\delta PL_\delta^*) \in \mathfrak{Z}'$$

(*where* $P = P(\mathfrak{B})$).

PROOF. Keeping the notation of Lemmas 3.1 and 3.2, it suffices to show that $c_k = e_k$ for each $k \in \mathbf{Z}$.

$e_k \leqslant c_k$. Let $e$ be the projection $e_k - e_k c_k \in \mathfrak{L}(Z)$. Since $e \leqslant 1 - c_k$, $f_k < 0$ a.e. on $\hat{e}$ (the subset of $X$ that corresponds to $e$). Using the notation of Lemma 3.3, $f_k = -g_2$ on $\hat{e}$ (since $g_1 g_2 = 0, f_k = g_1 - g_2$). If $e \neq 0$, we have $g_2 e \neq 0$, but we saw, in the proof of Lemma 3.3, that $g_2 = g_2(1 - e_k)$, and from the definition of $e$, $e \leqslant e_k$. This is a contradiction and it proves that $e_k \leqslant c_k$ for all $k \in \mathbf{Z}$.

$c_k \leqslant e_k$. Let $c$ be the projection $c_k(1 - e_k)$. Since $c_k \leqslant c_k$, $f_k \geqslant 0$ a.e. on $\hat{c}$ and $g_2 c = 0$ (in the notations of Lemma 3.3). But $g_2 = g_2(1 - e_k)$ and $e \leqslant 1 - e_k$; hence $c = 0$ and $c_k \leqslant e_k$. □

COROLLARY 3.5. *Let* $\mathfrak{B}$ *be a* $\sigma$*-weakly closed proper subalgebra of* $\mathfrak{L}$ *satisfying* (i)–(iii). *If in addition,* $\mathfrak{B}$ *contains* $L_\delta^n\mathfrak{L}_+$ *for some* $n \in \mathbf{Z}$, *it satisfies* (iv) *and, consequently, has the form* $\mathfrak{L}^\beta[0,\infty)$ *for some flow* $\beta$ *on* $\mathfrak{L}$.

PROOF. Let $P$ be $P(\mathfrak{B})$.

To prove (iv), let $\{e_k\}_{k=-\infty}^{\infty}$ be the sequence of projections introduced in Lemma 3.1. If $\phi(L_\delta PL_\delta^* - L_\delta PL_\delta^*P)$ is not finite, then it is infinite everywhere on a subset $\hat{e}$ of $X$ with a corresponding projection $e \in \mathfrak{L}(Z)$. Since $\phi(L_\delta PL_\delta^* - L_\delta PL_\delta^*P) = \Sigma\alpha(e_{k-1})(1-e_k)$, we have

$$\Sigma\alpha(e_{k-1})(1-e_k)e = \infty \cdot e$$

and

$$\Sigma e_{k-1}(1-\alpha^{-1}(e_k))\alpha^{-1}(e) = \infty \cdot \alpha^{-1}(e).$$

We first show that, for each $k \in \mathbf{Z}$, $\alpha^{-1}(e) \leqslant e_k$. Suppose $\alpha^{-1}(e) \nleqslant e_{k_0}$ for some $k_0 \in \mathbf{Z}$. Then $c = \alpha^{-1}(e)(1-e_{k_0}) \neq 0$. Since $L_\delta^n\mathfrak{L}_+ \subseteq \mathfrak{B}$, $e_m = 1$ for each $m \geqslant n$, and also

$$e_{k_0-m} = e_{k_0-m}\alpha^{k_0-m}(e_m) \leqslant e_{k_0}, \qquad m \geqslant n.$$

Hence, $c \leqslant 1 - e_{k_0} \leqslant 1 - e_{k_0-m}$, $m \geqslant n$. Thus

$$\infty \cdot c = \sum_{k=-\infty}^{\infty} e_{k-1}(1-\alpha^{-1}(e_k))c = \sum_{k=k_0-n+1}^{\infty} e_{k-1}(1-\alpha^{-1}(e_k))c.$$

But, since $1-\alpha^{-1}(e_k) = 0$ for $k \geqslant n$,

$$\infty \cdot c = \sum_{k=k_0-n+1}^{n-1} e_{k-1}(1-\alpha^{-1}(e_k))c < \infty.$$

This contradiction shows that $\alpha^{-1}(e) \leqslant e_k$ for each $k \in \mathbf{Z}$. Since $\mathfrak{B} \neq \mathfrak{L}$, there is some $m \in \mathbf{Z}$ with $1 - e_m \neq 0$. By ergodicity, $\alpha^{-1}(e)\alpha^{-k}(1-e_m) \neq 0$ for some $k > n$. Thus $\alpha^{-k}(1-e_m)e_{m-k} \neq 0$ (since $\alpha^{-1}(e) \leqslant e_{m-k}$), and

$$0 \neq (1-e_m)\alpha^k(e_{m-k}) = (1-e_m)e_k\alpha^k(e_{m-k}) \leqslant (1-e_m)e_m = 0.$$

Therefore, $\phi(L_\delta PL_\delta^* - PL_\delta PL_\delta^*)$ is finite a.e. on $X$.

We now prove that $\phi(P - L_\delta PL_\delta^*)$ is finite a.e. on $X$. Assume the converse; i.e. $\phi(P - L_\delta PL_\delta^*)e = \infty \cdot e$ for some projection $e \in \mathfrak{L}(Z)$. As shown in the proof for $\phi(L_\delta PL_\delta^* - PL_\delta^*)$, it will suffice to show that $e \leqslant e_k$ for each $k \in \mathbf{Z}$. We assume $e(1-e_{k_0}) \neq 0$ for some $k_0 \in \mathbf{Z}$ and

$$\infty \cdot e = \phi(P - L_\delta PL_\delta^*)e = \sum_{k=-\infty}^{\infty} e_k(1-\alpha(e_{k-1}))e.$$

Hence,

$$\infty \cdot e(1-e_{k_0}) = \sum_{k=-\infty}^{\infty} e_k(1-\alpha(e_{k-1}))e(1-e_{k_0})$$
$$= \sum_{k=-n}^{n} e_k(1-\alpha(e_{k-1}))e(1-e_{k_0}) < \infty.$$

The contradiction proves that $e \leqslant e_k$ for each $k \in \mathbf{Z}$ and completes the proof. $\square$

**4. $\mathfrak{L}(Z)$ is atomic.** Suppose $\alpha$ acts ergodically on $\mathfrak{L}(Z)$ and $\mathfrak{L}(Z)$ is atomic. We also assume, in this case, that $\alpha^k$ is outer for each $k \in \mathbf{Z}$. Since $M$ is finite there is a family $\{p_n\}_{n=1}^n$ of mutually orthogonal minimal projections in $\mathfrak{L}(Z)$ with $\Sigma_{n=1}^n p_n = I$; $\alpha(p_n) = p_{n+1}$, $n < N$; and $\alpha(p_N) = p_1$.

Let $\mathfrak{B}$ be a $\sigma$-weakly closed subalgebra of $\mathfrak{L}$ containing $\mathfrak{L}(M)$. Then $P(\mathfrak{B})$ lies in $\{\mathfrak{L}(M), \mathfrak{R}(M)\}'$. But $\{\mathfrak{L}(M), \mathfrak{R}(M)\}' = \{\mathfrak{L}(Z), \{E_n\}_{n=-\infty}^{\infty}\}''$ (by Proposition 4.2 of **[6]**), hence there is a sequence $\{e_k\}_{k=-\infty}^{\infty}$ of projections in $\mathfrak{L}(Z)$ such that $P(\mathfrak{B}) = \Sigma e_k E_k$.

Under these assumptions the results of the preceding section hold here. In this case, however, we can say more about the algebras in question.

LEMMA 4.1. *Let $f$ be in $\mathfrak{Z}'$ with $\int_X f\,d\nu > 0$. Then the algebra $\mathfrak{B} = \mathfrak{L}^{\beta}[0, \infty)$ of Lemma* 3.2 *contains $L_\delta^{n_0}\mathfrak{L}_+$ for some integer $n_0 \geq 0$. Consequently, $\mathfrak{B}$ satisfies conditions* (i)–(iv).

PROOF. Let $c_n \in \mathfrak{L}(Z)$ be the projections introduced in Lemma 3.2. We can write $f = \Sigma_{n=1}^N \lambda_n p_n$ where $\lambda_n \in \mathbf{R}$ and $\Sigma_{n=1}^N \lambda_n > 0$; hence $\alpha^k(f) = \Sigma_{n=1}^N \lambda_n \alpha^k(p_n)$ for each $k \in \mathbf{Z}$, and $\Sigma_{k=0}^{N-1} \alpha^k(f) = (\Sigma_{n=1}^N \lambda_n) I > 0$. Let $r$ be $\Sigma_{n=1}^N \lambda_n$. Then, using the notation introduced in the discussion preceding Lemma 3.2, $f_N = rI > 0$; thus $c_N = 1$. Let $\lambda_0$ be $\min\{\lambda_n\colon 1 \leq n \leq N\}$. Then for some $m_0 \in \mathbf{Z}_+$, $N\lambda_0 + rm_0 > 0$. Let $n_0$ be $m_0 N$. Then for $n \geq n_0$, $n = m_1 N + m_2$ for some $m_1 \geq m_0$ and $0 \leq m_2 < N$; we get

$$\begin{aligned} f_n &= f_{m_1 N} + \alpha^{m_1 N}(f_{m_2}) = m_1 r + \alpha^{m_1 N}(f_{m_2}) \\ &\geq m_0 r + \alpha^{m_1 N}(f_{m_2}) \geq m_0 r + N\lambda_0 > 0. \end{aligned}$$

Hence, for $n \geq n_0$, $c_n = I$. This implies $L_\delta^{n_0}\mathfrak{L}_+ \subseteq \mathfrak{B}$. Since $f_{n_0} > 0$, $f_{-n_0} < 0$, and it follows that $c_{-n_0} = 0$, hence $\mathfrak{B} \neq \mathfrak{L}$. Thus we can use Corollary 3.5 to complete the proof. □

THEOREM 4.2. *Let $f$ lie in $\mathfrak{Z}'$ and let $\mathfrak{B}$ be a $\sigma$-weakly closed subalgebra of $\mathfrak{L}$ containing $\mathfrak{L}^{\beta}[0, \infty)$ (where $\beta$ arises from $f$ as in Lemma* 3.2). *Then $\mathfrak{B} = \mathfrak{L}^{\gamma}[0, \infty)$ for some flow $\gamma$ on $\mathfrak{L}$. Moreover, if $\int_X f\,d\nu = 0$, then $\mathfrak{B}$ is a nest subalgebra.*

PROOF. Since $\mathfrak{B}_0 = \mathfrak{L}^{\beta}[0, \infty)$ contains $\mathfrak{L}(M)$, and $\mathfrak{B}_0 + \mathfrak{B}_0^*$ is $\sigma$-weakly dense in $\mathfrak{L}$, $\mathfrak{B}$ satisfies (i) and (ii).

We now distinguish between two cases.

*Case* 1. $\int_X f\,d\nu \neq 0$. We can assume $\int_X f\,d\nu > 0$ (the other possibility can be handled similarly) and apply Lemma 4.1 to find that, for some $n_0 \in \mathbf{Z}$, $L_\delta^{n_0}\mathfrak{L}_+ \subseteq \mathfrak{B}_0 \subseteq \mathfrak{B}$. From this, using Corollary 3.5, it follows that $\tilde{f} = \phi(P - L_\delta P L_\delta^* P) - \phi(L_\delta P L_\delta^* - P L_\delta P L_\delta^*)$ lies in $\mathfrak{Z}'$ where $P = P(\mathfrak{B}_0)$. By Lemma 3.2, $\tilde{f}$ gives rise to a flow $\tilde{\beta}$ and an algebra $\tilde{\mathfrak{B}} = \mathfrak{L}^{\tilde{\beta}}[0, \infty) = \{T \in \mathfrak{L}\colon \varepsilon_k(T) \in \tilde{c}_k \mathfrak{L}(M) L_\delta^k$ for each $k \in \mathbf{Z}\}$, where $\hat{\tilde{c}}_k = \{x \in X\colon \tilde{f}_k(x) \geq 0\}$. Let $\hat{c}_k$ be $\{x \in X\colon f_k(x) \geq 0\}$ and $c_k \in \mathfrak{L}(Z)$ the corresponding projections in $\mathfrak{L}(Z)$ for $k \in \mathbf{Z}$. We now show that $\tilde{c}_k \leq c_k$ and this implies $\tilde{\mathfrak{B}} \subseteq \mathfrak{B}_0$, since $\mathfrak{B}_0 = \{T \in \mathfrak{L}\colon \varepsilon_k(T) \in c_k \mathfrak{L}(M) L_\delta^k$ for each $k \in \mathbf{Z}\}$. In fact, let $c$ be the projection $\tilde{c}_k(1 - c_k)$ and assume $c \neq 0$. Since $c \leq \tilde{c}_k$, $\tilde{f}_k \geq 0$ a.e. on $c$. Lemma 3.3, applied to the algebra $\mathfrak{B}_0$, shows that for almost every $x \in X$, if

$\tilde{f}_k(x) \geq 0$ then $\phi(L_\delta P L_\delta^* - P L_\delta P L_\delta^*)(x) = 0$; hence,

$$(*) \qquad \phi(L_\delta P L_\delta^* - P L_\delta P L_\delta^*)c = 0.$$

From the proof of Lemma 3.3, applied to $\mathfrak{B}_0$, we see that

$$\phi(L_\delta P L_\delta^* - P L_\delta P L_\delta^*)(1 - c_k) = \phi(L_\delta P L_\delta^* - P L_\delta P L_\delta^*),$$

but since $0 \neq c \leq 1 - c_k$, this contradicts $(*)$. This proves $\tilde{c}_k \leq c_k$ and, hence, $\tilde{\mathfrak{B}} \subseteq \mathfrak{B}_0 \subseteq \mathfrak{B}$. But $\tilde{\mathfrak{B}}$ is a maximal subdiagonal algebra in $\mathfrak{L}$. Indeed,

$$\tilde{f} = \phi(P - L_\delta P L_\delta^* P) - \phi(L_\delta P L_\delta^* - P L_\delta P L_\delta^*)$$
$$= \sum_{k=-\infty}^{\infty} e_k(1 - \alpha(e_{k-1})) - \sum_{k=-\infty}^{\infty} \alpha(e_{k-1})(1 - e_k)$$

and

$$\tilde{f} = \sum_{i=1}^{N} m_i p_i$$

where

$$m_i = \#\{k \in \mathbf{Z}: e_k(1 - \alpha(e_{k-1}))p_i \neq 0\}$$
$$- \#\{k \in \mathbf{Z}: \alpha(e_{k-1})(1 - e_k)p_i \neq 0\} \in \mathbf{Z}.$$

Hence, $\tilde{\beta}_t = \tilde{\beta}_{t+2\pi k}$ for each $k \in \mathbf{Z}$, and the map $\tilde{\varepsilon} = \int_0^{2\pi} \tilde{\beta}_t \, dt$ (where the integral converges in the $\sigma$-weak operator topology) defines a normal faithful expectation form $\mathfrak{L}$ onto $\mathfrak{L}^{\tilde{\beta}}(\{0\}) = \tilde{\mathfrak{B}} \cap \tilde{\mathfrak{B}}^*$ satisfying $\tilde{\varepsilon} \cdot \tilde{\beta}_t = \tilde{\varepsilon}$ and making $\tilde{\mathfrak{B}}$ a maximal subdiagonal algebra. (See [**2**, Theorem 3.15].) Now we can use [**7**, Theorem 1] to conclude that $\mathfrak{B}$ satisfies (iii) (with the notation of that theorem $\mathfrak{L}$ is $L^\infty$, $\tilde{\mathfrak{B}}$ is $H^\infty$, and $\mathfrak{B}$ is a $\sigma$-weakly closed subspace of $L^\infty$ satisfying $H^\infty \mathfrak{B} \subseteq \mathfrak{B}$). Thus we can apply Corollary 3.5 to $\mathfrak{B}$ to complete the proof in this case.

*Case* 2. $\int_X f \, d\nu = 0$. We have $f = \sum_{n=1}^N \lambda_n p_n$, where $\lambda_n \in \mathbf{R}$ and $\sum_{n=1}^N \lambda_n = 0$. Let $d_n$ be $\sum_{k=1}^n \lambda_k$ for $1 \leq n \leq N$; then $d_n - d_{n-1} = \lambda_n$ for $n > 1$ and $d_1 - d_N = \lambda_1$. Let $d$, in $\mathfrak{L}(Z)$, be $\sum_{n=1}^N d_n p_n$. Then $d - \alpha(d) = \sum d_n p_n - \sum d_n \alpha(p_n) = \sum \lambda_n p_n = f$ and, similarly, $d - \alpha^k(d) = f_k$ for each $k \in \mathbf{Z}$. Consequently,

$$\beta_t(L_x L_\delta^k) = (\exp itf_k) L_x L_\delta^k = \exp it(d - \alpha^k(d)) L_x L_\delta^k, \qquad x \in M, k \in \mathbf{Z}.$$

But

$$\exp(-it\alpha^k(d)) L_\delta^k = L_\delta^k(\exp(-itd));$$

hence,

$$\beta_t(L_x L_\delta^k) = \exp(itd) L_x L_\delta^k \exp(-itd);$$

i.e. $\beta_t$ is inner for each $t \in \mathbf{R}$.

By [**2**, Theorem 4.2.3], $\mathfrak{B}_0$ is a nest subalgebra of $\mathfrak{L}$. We shall show that $\mathfrak{B}$ is also.

As was seen in Lemma 3.2, $\mathfrak{B}_0$ is determined by the projections $\{c_k\}$ that correspond to the sets $\hat{c}_k = \{x \in X: f_k(x) \geq 0\}$. Here,

$$f_k = \sum_{n=1}^{N} d_n p_n - \sum_{n=1}^{N} d_n \alpha^k(p_n).$$

Hence $c_k = \Sigma_{n \in F_k} p_n$, where $F_k = \{n\colon d_n \geq d_m \text{ where } \alpha^k(p_m) = p_n\}$. Now, let $b_n$ be the number of $k$'s such that $d_k \leq d_n$; then $F_k = \{n\colon b_n \geq b_n \text{ where } \alpha^k(p_m) = p_n\}$; hence, we can replace $d$ by $b = \Sigma b_n p_n$ and still get the same algebra $\mathfrak{B}_0$. We denote by $\tilde{\beta}$ the new flow and we have $\tilde{\beta}_{t+2\pi n} = \tilde{\beta}_t$ for each $n \in \mathbf{Z}$ and $0 \leq t \leq 1$ (since $b_n$ are integers). Thus the map $\tilde{\varepsilon}$ defined by $\tilde{\varepsilon} = \int_0^{2\pi} \tilde{\beta}_t\, dt$ (where the integral converges in the $\sigma$-weak operator topology) is a faithful normal expectation onto $\mathfrak{L}^{\beta}(\{0\})$ satisfying $\tilde{\varepsilon} \cdot \tilde{\beta}_t = \tilde{\varepsilon}$ for all $t \in \mathbf{R}$. It makes $\mathfrak{B}_0$ a maximal subdiagonal algebra in $\mathfrak{L}$ (see [**2**, Theorem 3.15]). Since $\mathfrak{L} \supseteq \mathfrak{B} \supseteq \mathfrak{B}_0$, and $\mathfrak{B}_0$ is a maximal subdiagonal algebra in $\mathfrak{L}$, we can use [**7**, Theorem 1] to conclude that $\mathfrak{B}$ satisfies (iii). Thus, there is a sequence $\{e_k\}_{k=-\infty}^{\infty}$ of projections in $\mathfrak{L}(Z)$ such that $P(\mathfrak{B}) = \Sigma e_k E_k$ (see Lemma 3.1).

Recall that $\mathfrak{B}_0$ is determined by $\{c_k\}$ where $\hat{c}_k = \{x \in X\colon f_k(x) \geq 0\}$. Since $f_N \equiv 0$, $c_{kN} = 1$ for each $k$ in $\mathbf{Z}$. Since $\mathfrak{B} \supseteq \mathfrak{B}_0$, $e_{kN} = 1$ for each $k \in \mathbf{Z}$. Also, for $m, k \in \mathbf{Z}$.

$$e_{m+kN} = e_{-kN}\alpha^{-kN}(e_{m+kN}) \leq e_m = e_{kN}\alpha^{kN}(e_m) \leq e_{m+kN};$$

hence, $e_m = e_{m+kN}$.

For each $1 \leq m \leq N$, let $q_m$ be the projection $\Sigma_{k=0}^{N-1}\alpha^k(p_m)e_k$. We shall show that

$$(*) \qquad \mathfrak{B} = \mathfrak{L} \cap \operatorname{alg}\{q_m\colon 1 \leq m \leq N\};$$

hence, $\mathfrak{B}$ is a nest subalgebra. Denote the right-hand side of $(*)$ by $\tilde{\mathfrak{B}}$.

$\tilde{\mathfrak{B}} \subseteq \mathfrak{B}$. Take $T$ in $\tilde{\mathfrak{B}}$, then, for each $k \in \mathbf{Z}$, $1 \leq m \leq N$, $T$ maps $\alpha^k(p_m)e_k E_k(L^2)$ into $q_m E_k(L^2)$. Since $T \in \mathfrak{L}$, and $\Sigma_{n=-\infty}^{\infty} \alpha^n(p_m)E_n$ is the projection $R_{p_m}$ in $\mathfrak{R}\,(= \mathfrak{L}')$, $T$ maps $\alpha^k(p_m)e_k E_k(L^2)$ into $\Sigma_{n=-\infty}^{\infty} \alpha^n(p_m)E_n(L^2)$. But

$$\begin{aligned} q_m \sum_{n=-\infty}^{\infty} \alpha^n(p_m)E_n &= \sum_{j=0}^{N-1} e_j\alpha^j(p_m) \sum_{n=-\infty}^{\infty} \alpha^n(p_m)E_n \\ &= \sum_{n=-\infty}^{\infty} \alpha^n(p_m)e_n E_n \leq \sum_{n=-\infty}^{\infty} e_n E_n = P(\mathfrak{B}). \end{aligned}$$

Since

$$\sum_{k=-\infty}^{\infty} \sum_{m=1}^{N} \alpha^k(p_m)e_k E_k = \sum_{n=-\infty}^{\infty} e_n E_n = P(\mathfrak{B}),$$

$T$ maps $P(\mathfrak{B})(L^2) = [\mathfrak{B}]_2$ into itself. In particular, $T\psi \in [\mathfrak{B}]_2$ and, using condition (iii), $T$ lies in $\mathfrak{B}$.

$\mathfrak{B} \subseteq \tilde{\mathfrak{B}}$. Fix $T \in \mathfrak{B}$ and $0 \leq k \leq N-1$. Since $T$ maps $e_k E_k(L^2)$ into $\Sigma_{j=-\infty}^{\infty} e_j E_j(L^2)$, it maps $\alpha^k(p_m)e_k E_k(L^2)$ into $\Sigma_{j=-\infty}^{\infty} e_j E_j(L^2)$ for each $1 \leq m \leq N$. It also maps it into $\Sigma_{n=-\infty}^{\infty} \alpha^n(p_m)E_n(L^2) = R_{p_m}(L^2)$ since $T \in \mathfrak{L}$. But

$$(\Sigma e_j E_j)(\Sigma \alpha^n(p_m)E_n) = \Sigma e_n \alpha^n(p_m)E_n \leq q_m;$$

hence, $T$ maps $\alpha^k(p_m)e_k E_k(L^2)$ into $q_m(L^2)$.

For $n \in \mathbf{Z}$, $\alpha^k(p_m)e_k E_n = R_\delta^{n-k}\alpha^k(p_m)e_k E_k$ and

$$TR_\delta^{n-k}\alpha^k(p_m)e_k E_k(L^2) = R_\delta^{n-k}T\alpha^k(p_m)e_k E_k(L^2) \subseteq R_\delta^{n-k}q_m(L^2) \subseteq q_m(L^2).$$

Hence, $T$ maps $\alpha^k(p_m)e_k E_n(L^2)$ into $q_m(L^2)$ for each $m, n \in \mathbf{Z}$, $1 \leq m \leq N$. Since

$$\sum_{n=-\infty}^{\infty} \alpha^k(p_m)e_k E_n = \alpha^k(p_m)e_k \quad \text{and} \quad q_m = \sum_{k=0}^{N-1} \alpha^k(p_m)e_k,$$

$T$ maps $q_n(L^2)$ into $q_n(L^2)$. Hence $T \in \mathfrak{B}$.

Hence, $\mathfrak{B} = \tilde{\mathfrak{B}}$, and $\mathfrak{B}$ is a nest subalgebra. The fact that $\mathfrak{B}$ is $\mathfrak{L}^\gamma[0, \infty)$ for some flow $\gamma$ on $\mathfrak{L}$ follows from [**2**, Theorem 4.23].

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