

A TWO WEIGHT WEAK TYPE INEQUALITY FOR FRACTIONAL INTEGRALS

BY

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ABSTRACT. For $1 < p \leq q < \infty$, $0 < \alpha < n$ and $w(x)$, $v(x)$ nonnegative weight functions on R^n we show that the weak type inequality

$$\int_{(T_\alpha f > \lambda)} w(x) dx \leq A \lambda^{-q} \left(\int |f(x)|^p v(x) dx \right)^{q/p}$$

holds for all $f \geq 0$ if and only if

$$\int_Q [T_\alpha(\chi_Q w)(x)]^{p'} v(x)^{1-p'} dx \leq B \left(\int_Q w \right)^{p'/q'} < \infty$$

for all cubes Q in R^n . Here T_α denotes the fractional integral of order α , $T_\alpha f(x) = \int |x-y|^{\alpha-n} f(y) dy$. More generally we can replace T_α by any suitable convolution operator with radial kernel decreasing in $|x|$.

1. Introduction. Weighted norm inequalities for fractional integrals have been treated by several authors. For example, B. Muckenhoupt and R. L. Wheeden have shown [8] that the one weight strong type inequality

$$(1) \quad \left(\int_{R^n} |T_\alpha f(x) w(x)|^q dx \right)^{1/q} \leq C \left(\int_{R^n} |f(x) w(x)|^p dx \right)^{1/p} \quad \text{for all } f \geq 0$$

where $1/q = 1/p - \alpha/n$ holds if and only if $w(x)^q$ satisfies the A_r condition with $r = 1 + q/p'$. Here $T_\alpha f(x) = \int_{R^n} |x-y|^{\alpha-n} f(y) dy$ is the fractional integral or Riesz potential of order α (see [10] for the basic properties of T_α) and the A_r condition on a function $v(x)$ is

$$(A_r) \quad \left(\int_Q v(x) dx \right)^{1/r} \left(\int_Q v(x)^{-r'/r} dx \right)^{1/r'} \leq C \int_Q dx \quad \text{for all cubes } Q$$

where the second factor on the left side is interpreted as $\|\chi_Q v^{-1}\|_\infty$ in the case $r = 1$. In a different direction, B. Dahlberg [3] has used a capacity strong type inequality to show that a positive measure ω satisfies the "trace" inequality

$$(2) \quad \int_{R^n} |T_\alpha f(x)|^p d\omega(x) \leq C \int_{R^n} |f(x)|^p dx \quad \text{for all } f \geq 0$$

Received by the editors March 22, 1983.

1980 *Mathematics Subject Classification*. Primary 42B25.

¹ Research supported in part by NSERC grant #A5149.

if and only if

$$(3) \quad \omega(E) \leq A \operatorname{Cap}(E) = A \inf \left\{ \int |f(x)|^p dx; T_\alpha f \geq 1 \text{ on } E \right\}$$

for all compact subsets E of R^n . See also D. Adams [1] and V. Maz'ya [7] for the case α integral. More recently, R. Kerman and the author [6] (see also [9]) have shown that (2) is equivalent to the simpler condition

$$(4) \quad \int_Q |T_\alpha(\chi_Q \omega)(x)|^{p'} dx \leq C \int_Q d\omega < \infty \quad \text{for all cubes } Q.$$

However, the characterization of the general two weight strong type inequality for fractional integrals remains open. In this note we address the simpler two weight weak type inequality and give a characterization of it in terms of a condition analogous to (4). As in [5 and 6] we will treat operators more general than fractional integrals, namely convolution operators of the form $Tf = K * f$ where $K(x)$ is a positive radial function decreasing in $|x|$. K. Hansson has recently obtained a capacity strong type inequality for such operators [5] and hence the equivalence of (2) and (3) for T in place of T_α (the corresponding equivalence of (2) and (4) is in [6]). If μ is a positive measure on R^n we use the notation $|E|_\mu = \int_E d\mu$ and $T(f\mu)(x) = K * (f\mu)(x) = \int K(x-y)f(y) d\mu(y)$.

THEOREM. *Suppose $1 < p \leq q < \infty$, ω and μ are positive borel measures on R^n and $Tf = K * f$ where $K(x)$ is a positive lower semicontinuous radial function decreasing in $|x|$. If $n \geq 2$ suppose, in addition, that $K(x)$ satisfies (A_1) . Then the weak type inequality*

$$(5) \quad |\{T(f\mu) > \lambda\}|_\omega \leq A\lambda^{-q} \left(\int |f|^p d\mu \right)^{q/p} \quad \text{for all } f \geq 0, \lambda > 0,$$

holds if and only if

$$(6) \quad \int_Q |T(\chi_Q \omega)|^{p'} d\mu \leq B|Q|_\omega^{p'/q'} < \infty \quad \text{for all cubes } Q.$$

Furthermore if A and B are the least such constants, then the ratio $A^{1/q}/B^{1/p'}$ is bounded between two positive constants independent of ω and μ .

REMARKS. I. The theorem is also valid for $p = 1$ if (6) is replaced by $\|T(\chi_Q \omega)\|_{L^\infty(\mu)} \leq C|Q|_\omega^{1/q'} < \infty$ for all cubes Q .

II. The result stated in the abstract follows from the Theorem with $K(x) = |x|^{\alpha-n}$, $d\omega(x) = w(x) dx$, $d\mu(x) = v(x)^{1-p'} dx$ and f replaced by $fv^{p'-1}$. Note that $|x|^{\alpha-n}$ satisfies (A_1) for $0 < \alpha < n$.

III. If μ is an A_∞ weight then condition (6) is sufficient for the strong type analogue of (5) (see [6]) but, in general, condition (6) is not sufficient. See D. Adams [1, Remark 2(iii)] for a counterexample in the case $p = q$ (note that (a'), p. 134 in [1] is equivalent to (6) with $d\omega = dx$, $T = T_m$ and p', q' replaced by q, p , respectively) and §3 below for the case $p \leq q$.

2. Proof of the Theorem. Assume (5) holds. Provided μ is nontrivial ($0 < |E|_\mu < \infty$ for some set E) the positivity of K together with (5) easily shows that $|Q|_\omega < \infty$ for all cubes Q . The remaining inequality in (6) is an easy consequence of duality (of Lorentz spaces). In fact,

$$\begin{aligned} \left(\int |T(\chi_Q \omega)|^{p'} d\mu \right)^{1/p'} &= \sup_{\|f\|_{L^p(\mu)} \leq 1} \int T(\chi_Q \omega) f d\mu = \sup_{\|f\|_{L^p(\mu)} \leq 1} \int_Q T(f\mu) d\omega \\ &= \sup_{\|f\|_{L^p(\mu)} \leq 1} \int_0^\infty |Q \cap \{T(f\mu) > \lambda\}|_\omega d\lambda \\ &\leq \int_0^\infty \min\{A\lambda^{-q}, |Q|_\omega\} d\lambda \quad \text{by (5)} \\ &= q' A^{1/q} |Q|_\omega^{1/q'} \end{aligned}$$

and so (6) holds with $B \leq (q')^{p'} A^{p'/q}$.

Conversely, suppose (6) holds and, without loss of generality, that f is nonnegative with compact support and satisfies $\int |f|^p d\mu < \infty$. The main idea of the proof is to establish a “good λ inequality” (in much the same manner as is done in R. Coifman [2]) for Tf relative to the maximal operator

$$Mf(x) = \sup_{x \in Q} \frac{1}{|Q|_\omega} \int_Q T(\chi_Q f) d\omega.$$

We begin with the case $n = 1$. Fix $0 < \beta < 1$ and $\lambda > 0$. Since $Tf\mu$ is lower semicontinuous we can write $\{Tf\mu > \lambda\} = \dot{\cup}_k I_k$ where the intervals $I_k = (a_k, b_k)$ are disjoint. Moreover, (6) implies that $|I_k|_\omega < \infty$ for all k (if I_k is infinite, then $\lim_{x \rightarrow \infty} K(x) > 0$ and it is easy to see that (6) implies $\int_{-\infty}^\infty d\omega < \infty$). We now discard those I_k with $|I_k|_\omega = 0$ and denote by F the set of indices k such that

$$(7) \quad \frac{1}{|I_k|_\omega} \int_{I_k} T(\chi_{I_k} f\mu) d\omega > \beta\lambda$$

and by G the set of k for which (7) fails. For k in F we have

$$\begin{aligned} (8) \quad \lambda^q |I_k|_\omega &< \beta^{-q} |I_k|_\omega^{1-q} \left(\int_{I_k} T(\chi_{I_k} f\mu) d\omega \right)^q \\ &= \beta^{-q} |I_k|_\omega^{1-q} \left(\int_{I_k} T(\chi_{I_k} \omega) f d\mu \right)^q \\ &\leq \beta^{-q} |I_k|_\omega^{1-q} \left(\int_{I_k} T(\chi_{I_k} \omega)^{p'} d\mu \right)^{q/p'} \left(\int_{I_k} |f|^p d\mu \right)^{q/p} \\ &\leq \beta^{-q} B^{q/p'} \left(\int_{I_k} |f|^p d\mu \right)^{q/p} \quad \text{by (6)}. \end{aligned}$$

Now observe that if \tilde{I}_k denotes the complement of I_k then $T(\chi_{\tilde{I}_k} f\mu) \leq 2\lambda$ on I_k by the maximum principle. Indeed, if x is in I_k then

$$\begin{aligned} T(\chi_{\tilde{I}_k} f\mu)(x) &= \left(\int_{-\infty}^{a_k} + \int_{b_k}^{\infty} \right) K(x-y)f(y) d\mu(y) \\ &\leq \int_{-\infty}^{a_k} K(a_k-y)f(y) d\mu(y) + \int_{b_k}^{\infty} K(b_k-y)f(y) d\mu(y) \\ &\leq Tf(a_k) + Tf(b_k) \leq 2\lambda. \end{aligned}$$

Thus for k in G

$$\begin{aligned} (9) \quad |I_k \cap \{T(f\mu) > 3\lambda\}|_\omega &\leq |I_k \cap \{T(\chi_{I_k} f\mu) > \lambda\}|_\omega \\ &\leq \frac{1}{\lambda} \int_{I_k} T(\chi_{I_k} f\mu) d\omega \leq \beta |I_k|_\omega \end{aligned}$$

since (7) fails. Combining (8) and (9) we obtain the “good λ inequality”

(10)

$$\begin{aligned} (3\lambda)^q |\{T(f\mu) > 3\lambda\}|_\omega &= \sum_k (3\lambda)^q |I_k \cap \{T(f\mu) > 3\lambda\}|_\omega \\ &\leq 3^q \sum_{k \in F} \lambda^q |I_k|_\omega + 3^q \lambda^q \sum_{k \in G} |I_k \cap \{T(f\mu) > 3\lambda\}|_\omega \\ &\leq \left(\frac{3}{\beta}\right)^q B^{q/p'} \sum_{k \in F} \left(\int_{I_k} |f|^p d\mu\right)^{q/p} + 3^q \beta \lambda^q \sum_{k \in G} |I_k|_\omega \\ &\leq \left(\frac{3}{\beta}\right)^q B^{q/p'} \left(\int |f|^p d\mu\right)^{q/p} + 3^q \beta \lambda^q |\{T(f\mu) > \lambda\}|_\omega \end{aligned}$$

since $q/p \geq 1$. Choose $\beta = \frac{1}{2}(\frac{1}{3})^q$ and take the supremum in (10) over $0 < \lambda \leq t/3$ to obtain

(11)

$$\sup_{0 < \lambda \leq t} \lambda^q |\{T(f\mu) > \lambda\}|_\omega \leq \frac{3^{q^2+q}}{2^q} B^{q/p'} \left(\int |f|^p d\mu\right)^{q/p} + \frac{1}{2} \sup_{0 < \lambda \leq t} \lambda^q |\{T(f\mu) > \lambda\}|_\omega$$

for all $t > 0$. If we can show that the left side of (11) is finite for all $t > 0$, we can subtract the second term on the right side of (11) from both sides to obtain (5) with $A \leq 3^{q^2+q} B^{q/p'} / 2^{q-1}$. To see that the left side of (11) is finite suppose that f is supported in an interval $I = (a, b)$ and that $r > 2b - a$. Then from (6) we have

$$B \left(\int_a^r d\omega\right)^{p'/q'} \geq \int_a^b |T(\chi_{(a,r)} \omega)(x)|^{p'} d\mu(x) \geq \int_I |K(r-x) \int_a^r d\omega|^{p'} d\mu(x)$$

and so $(\int_I K(r-x)^{p'} d\mu(x))(\int_a^r d\omega)^{p'/q} \leq B$. Thus for $r > 2b - a$ and $\lambda = T(f\mu)(r)$ we have

$$\begin{aligned} \lambda^q |\{T(f\mu) > \lambda\} \cap (a, \infty)|_\omega &\leq \left(\int_I K(r-x)f(x) d\mu(x)\right)^q \int_a^r d\omega \\ &\leq \left(\int_I f^p d\mu\right)^{q/p} \left(\int_I K(r-x)^{p'} d\mu(x)\right)^{q/p'} \int_a^r d\omega \\ &\leq \left(\int_I f^p d\mu\right)^{q/p} B^{q/p'} < \infty. \end{aligned}$$

Similarly one can show that for λ sufficiently small

$$\lambda^q |\{T(f\mu) > \lambda\} \cap (-\infty, b)|_\omega \leq \left(\int_I f^p d\mu\right)^{q/p} B^{q/p'} < \infty$$

and this shows that the left side of (11) is finite for $t > 0$ and completes the proof of the Theorem in the case $n = 1$.

We now turn to the case $n \geq 2$ and assume that K satisfies the (A_1) condition. Recall that f is nonnegative with compact support and satisfies $\int |f|^p d\mu < \infty$. Again fix $0 < \beta < 1$ and $\lambda > 0$. Let Mf denote the maximal function of f , i.e.

$$Mf(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f|$$

and for $r > 0$ denote by rQ the cube concentric with Q and having r times the side length of Q . Using a variant of the Whitney covering lemma in C. Fefferman [4, p. 16] we can write $\Omega_\lambda = \{M(T(f\mu)) > \lambda\} = \cup_k Q_k$ where the cubes Q_k satisfy both a Whitney condition

$$(12) \quad rQ_k \cap \tilde{\Omega}_\lambda \neq \emptyset \quad \text{for all } k$$

and a finite overlap condition

$$(13) \quad \sum_k \chi_{2Q_k} \leq D\chi_{\Omega_\lambda}$$

where r and D are positive constants depending only on the dimension n .

Since K satisfies (A_1) and is decreasing as a function of $|x|$ it is easy to show that

$$(14) \quad K(x) \leq CK(y) \quad \text{for } |y| \leq 2|x|.$$

Fix k for the moment and let $f_1 = f\chi_{2Q_k}$ and $f_2 = f - f_1$. For x in Q_k we have

$$\begin{aligned} T(f_2\mu)(x) &= \int_{y \notin 2Q_k} K(x-y)f(y) d\mu(y) \\ &\leq C' \int_{y \notin 2Q_k} \left(\frac{1}{|Q_k|} \int_{Q_k} K(z-y) dz\right) f(y) d\mu(y) \quad \text{by (14)} \\ &\leq \frac{C'}{|Q_k|} \int_{Q_k} T(f\mu)(z) dz \\ &\leq C'r^n \frac{1}{|rQ_k|} \int_{rQ_k} T(f\mu)(z) dz \leq C'r^n \lambda \quad \text{by (12)}. \end{aligned}$$

Thus for $\gamma > 2C'r^n$ we have

$$(15) \quad \{T(f\mu) > \gamma\lambda\} \cap Q_k \subset \{T(\chi_{2Q_k} f\mu) > \frac{1}{2}\gamma\lambda\} \cap Q_k.$$

Denote by F the set of indices k such that

$$(16) \quad \frac{1}{|2Q_k|_\omega} \int_{2Q_k} T(\chi_{2Q_k} f\mu) d\omega > \beta\lambda$$

and by G the set of k for which (16) fails. Using (15) and arguing as in the case $n = 1$ (see (8) and (9)) we obtain

$$|\{T(f\mu) > \gamma\lambda\} \cap Q_k|_\omega \leq \max \left\{ \beta|2Q_k|_\omega, (\beta\lambda)^{-q} B^{q/p'} \left(\int_{2Q_k} f^p d\mu \right)^{q/p} \right\}$$

for all k and then summing over k and using (13) we get

$$(17) \quad |\{T(f\mu) > \gamma\lambda\}|_\omega \leq \beta D |\Omega_\lambda|_\omega + (\beta\lambda)^{-q} B^{q/p'} \left(D \int f^p d\mu \right)^{q/p}$$

for $\gamma > 2C'r^n$ and $0 < \beta < 1$. However $T(f\mu)$ satisfies (A_1) with the same constant C that works for K and so $M(T(f\mu)) \leq CT(f\mu)$. Thus $|\Omega_\lambda|_\omega \leq |\{T(f\mu) > \lambda/C\}|_\omega$ and if we use this inequality on the first term on the right side of (17) we obtain an analogue of the good λ inequality (10) and the proof can now be completed as in the case $n = 1$.

3. An example. Fix $p = 2 \leq q < \infty$, $n = 1$ and $1/2 \leq \alpha + 1/q < 1$ with $\alpha > 0$. We construct a pair of weights w, v on the real line satisfying the condition in the abstract ($p = 2$)

$$(18) \quad \int_Q [T_\alpha(\chi_Q w)]^2 v^{-1} \leq B \left(\int_Q w \right)^{2/q'}$$
 for all intervals Q

but not the corresponding strong type inequality

$$(19) \quad \left(\int |T_\alpha f|^q w \right)^{1/q} \leq C \left(\int |f|^2 v \right)^{1/2}$$
 for all $f \geq 0$.

Let

$$f(x) = x^{-1} |\log x|^{-q'} \chi_{(0,1/2)}(x)$$

and set $v(x) = f(x)^{-1}$ and $w(x) = x^{q-q\alpha-1} |\log x|^{q'-1} \chi_{(0,1/2)}(x)$. Since $T_\alpha f(x) \approx x^{\alpha-1} |\log x|^{1-q'}$ we have that the left side of (19) is infinite while the right side is finite. On the other hand, using the estimates

$$\|T_\alpha(\chi_{(a,r)} w)\|_\infty \leq (r-a)^\alpha r^{q-q\alpha-1} |\log r|^{q'-1}$$

and

$$\int_a^r x^{-1} |\log x|^{-q'} dx \leq \begin{cases} |\log r|^{1-q'}, & 0 \leq a < r/2, \\ |\log r|^{-q'} \left(\frac{r-a}{r} \right), & r/2 \leq a < r, \end{cases}$$

which are valid for $1/2 \leq \alpha + 1/q < 1$, $\alpha > 0$ and $0 \leq a < r \leq 1/2$ we obtain

$$\int_a^r [T_\alpha(\chi_{(a,r)} w)]^2 v^{-1} \lesssim (r-a)^{2/q'} r^{2q+2\alpha-2-2/q'} |\log r|^{2-2/q'} \lesssim \left(\int_a^r w \right)^{2/q'}$$

for $0 \leq a < r \leq 1/2$ which is (18).

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