INVARIANT DENSITIES FOR RANDOM MAPS
OF THE INTERVAL

BY

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ABSTRACT. A *random map* is a discrete time process in which one of a number of
functions is selected at random and applied. Here we study random maps of $[0, 1]$
which represent dynamical systems on the square $[0, 1] \times [0, 1]$. Sufficient conditions
for a random map to have an absolutely continuous *invariant measure* are given, and
the number of *ergodic components* of a random map is discussed.

1. Introduction. In this paper we investigate the following kind of question: consider two maps from $[0, 1]$ to itself—for example, let $T_1(x) = x/2$ and $T_2(x) = 2x(\bmod 1)$. Select an initial point $x_0 \in [0, 1]$, and choose one of the maps at random (T_1 with probability p_1 and T_2 with probability p_2). Define $x_1 = T_i(x_0)$ with probability p_i, $i = 1, 2$. Continue in this manner, selecting maps at random and setting $x_{n+1} = T_i(x_n)$ (prob. p_i). How is the "trajectory" $\{x_n\}$ distributed? We say that a Borel probability measure μ describes the distribution of $\{x_n\}$ if

$$\lim_{n \to \infty} \frac{1}{n} \operatorname{card}(\{x_0, \dots, x_{n-1}\} \cap A) = \mu(A)$$

for every open set A with $\mu(\partial A) = 0$. In the specific example mentioned above we prove that when $p_2 > 1/2$ there is a unique measure, which is absolutely continuous with respect to Lebesgue measure m and which describes the distribution of the sequence $\{x_n\}$ for m-almost every choice of initial condition x_0 and "almost every" selection of the maps T_i.

More generally, let $T_1(x), \dots, T_n(x)$ be maps of the unit interval and define a "random map" T by $T(x) = T_i(x)$ with probability p_i. A measure μ will be called T-invariant if $\mu(A) = \sum_{i=1}^{n} p_i \mu(T_i^{-1}A)$ for each measurable set A. We study a class of random maps for which it is possible to describe the distribution of almost every trajectory. This description is accomplished through the demonstration of three facts. The first fact is that every random map may be realized as a transformation of the square $[0, 1] \times [0, 1]$ to itself. This construction, the pseudo-skew product (see §2), makes the phrase "almost every selection of the maps T_i" precise.

The second component of the description is Proposition 6 (§5) which shows that sets which are invariant for a pseudo-skew product map are products. The third fact

is contained in Theorem 1, which gives sufficient conditions for a random map to have an invariant measure which is absolutely continuous with respect to Lebesgue measure. Such a measure has a finite number of ergodic components (§5); taken together these components describe the distribution of almost every trajectory of the random map T.

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2. Background.

(A) *Definitions.* A random map of $[0, 1]$ is a process $T(x)$ specified by a finite collection of measurable functions T_i: $[0, 1] \times [0, 1]$, $i = 1, \dots, n$, and a probability vector $P = (P_1, \dots, P_n)$. That is, $P_i \geqslant 0$ and $\sum_{i=1}^{n} P_i = 1$. We define $T(x) = T_i(x)$ with probability P_i, and assume that the selection of the functions is an independent identically distributed process so that $T^m(x) = T_{i_m} \circ T_{i_{m-1}} \circ \cdots \circ T_{i_1}(x)$ with probability $\prod_{j=1}^{m} P_{i_j}$. A measure μ on $[0, 1]$ is defined to be T-invariant if $\mu(A) = \sum_{i=1}^{n} P_i \mu(T_i^{-1}(A))$ for each measurable set A.

(B) *Pseudo-skew products.* Random maps of $[0, 1]$ may always be realized as transformations of a larger space. Suppose that $T = T_i$ (prob. P_i), $i = 1, \dots, n$, is a random map. Let $\Omega = \{w = (w_i)_{i=0}^{\infty}: w_i \in \{1, 2, \dots, n\}\}$ be the set of all infinite one-sided sequences of the symbols in $S = \{1, 2, \dots, n\}$. The left shift $\sigma: \Omega \to \Omega$ is defined by $(\sigma(w_i))_j = w_{j+1}, j = 0, 1, 2, \dots$. The topology on Ω is the product of the discrete topology on S, and the Borel measure μ_p on Ω is defined as the product of the distribution on S given by $\text{Prob}(j) = P_j$. Note that there is a 1-1 correspondence between the ways in which the maps $\dots T_{i_n}, T_{i_{n-1}}, \dots, T_{i_1}$ can be selected and points in Ω. The phrase "for almost every selection of the maps T_i" means for selections of the maps T_i which corresponds to a subset of Ω having μ_p measure one.

In the space $\Omega \times I$ define a map R by $R(w, x) = (\sigma w, T_{w_0} x)$. The measure ν is T invariant if and only if $\mu_p \times \nu$ is R invariant. We say the random map T is ergodic wrt ν when R is ergodic wrt $\mu_p \times \nu$.

Because there may be no measure on $[0, 1]$ which is preserved by each of the T_i, the map R is not a skew product in the traditional sense [**AR**]. We will refer to R as the pseudo-skew product (PSP) determined by the random map T.

To realize the random map T as a map S of $[0, 1] \times [0, 1]$ to itself we use the following measure theoretic isomorphism: $(\Omega, \sigma, \mu_p) \approx ([0, 1], m, f_p)$, where f is defined as follows. Let $I_1, \dots, I_n$ be a partition of $[0, 1]$ into intervals of length $P_1, \dots, P_n$, so that $I_1 = [0, P_1]$, and $I_j = [\sum^{j-1} P_i, \sum^{j} P_i]$. For $x \in \mathring{I}_j$ define (Figure 1)

$$f_p(x) = \frac{1}{P_j} x - \frac{\sum^{j-1} P_i}{P_j}.$$

The random map T is then to be represented by $S(x, y) = (T_i(x), f_p(y))$ for $y \in I_i$. Note that T has an absolutely continuous invariant measure $\nu \ll m$ if and only if S: $[0, 1] \times [0, 1] \circlearrowleft$ has an invariant measure $\nu \times m$ which is absolutely continuous wrt 2-dimensional Lebesgue measure.
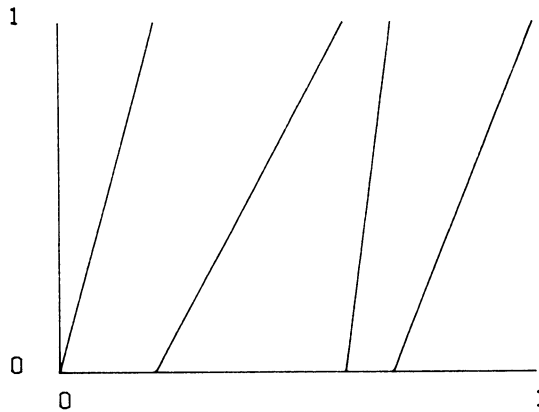
FIGURE 1. The graph of f_p

(C) *Lasota-Yorke maps and the Frobenius-Perron operator*. A transformation T: $[0, 1] \rightarrow [0, 1]$ is called nonsingular (wrt Lebesgue measure) if $m(T^{-1}A) = 0$ whenever $m(A) = 0$. If T is a nonsingular transformation, the action of T^* on measures $(T^*\mu(A) = \mu(T^{-1}A))$ preserves the class of measures absolutely continuous wrt Lebesgue measure. By restricting the domain of T^* one obtains an operator on $\mathcal{L}^1([0, 1], m)$ given by $T^*(f) = [dT^*\mu/dm]$, where $d\mu = fdx$ with $f \in \mathcal{L}^1$, or equivalently, $\int_A T^*(f)\,dx = \int_{T^{-1}A} f\,dx$ for $f \in \mathcal{L}^1$ and every measurable set A. This operator is called the Frobenius-Perron (F-P) operator of T and will be denoted by P_T.

We say $T: [0, 1] \rightarrow [0, 1]$ is a Lasota-Yorke type map (L-Y type map) if T is piecewise monotone and C^2, and T is nonsingular. Piecewise monotone and C^2 means that there is a partition of $[0, 1]$, $0 = a_0 < a_1 < \cdots < a_k = 1$, so that for each $i = 0, 1, \ldots, k - 1$, $T|_{(a_i, a_{i+1})}$ is monotone and extends to a C^2 map on $[a_i, a_{i+1}]$.

If $T = T_i$ (prob. p_i), $i = 1, \ldots, n$, is a random map and each T_i is nonsingular, we define the F-P operator of T to be $P_T(f) := \sum_{i=1}^n p_i P_{T_i}(f)$. Then P_T has the following properties:

(i) P_T is linear,

(ii) $\|P_T\| \leqslant 1$,

(iii) P_T is nonnegative ($f \geqslant 0 \Rightarrow P_T(f) \geqslant 0$),

(iv) $P_T(f) = f \Leftrightarrow d\mu = f\,dx$ is a T-invariant measure,

(v) $P_T^K(f) = P_{T^K}(f)$.

When each T_i is a L-Y map we obtain an explicit formula for P_T as follows. Let $I_1, \ldots, I_k$ be a partition of $[0, 1]$ into intervals of joint monotonicity of the T_i's. That is, each T_i is monotone on $\mathring{I}_j$ and extends to a C^2 function on $\overline{I}_j$. Set

$$H_i^j = T_i(I_j), \quad \psi_i^j = \left(T_i|_{I_j}\right)^{-1} \quad \text{and} \quad \sigma_i^j = \left|\frac{d}{dx}\psi_i^j\right|.$$

One then computes that

$$P_T(f)(x) = \sum_{i,j} p_i f(\psi_i^j x) \sigma_i^j(x) \chi_{H_i^j}(x),$$

where χ_I is the characteristic function of the set I.

By (iv), we can prove the existence of an absolutely continuous T-invariant measure by showing that P_T has an eigenvector for the eigenvalue 1. This will be accomplished by the use of the mean ergodic theorem [DS]. To show that P_T satisfies the hypotheses of the theorem we estimate $\bigvee_0^1 P_T(f)$ in terms of $\bigvee_0^1 f$, for f a function of bounded variation. The following lemmas are useful for this. The first two appear in [K].

3. The main theorem. In what follows BV denotes $BV([0,1])$, m is Lebesgue measure, $\mathcal{L}^1$ denotes $\mathcal{L}^1([0,1], m)$ and '$f \in BV$' means $\exists\, g \in BV$ so that $f = g$ a.e.(m).

LEMMA 1. *Let $f \in BV([0,1])$ and $\phi \in \mathcal{L}^1([0,1], m)$. Set $\Phi(x) = \int_0^x \phi(t)\,dt$. Then*

$$\left| \int_0^1 f(t)\phi(t)\,dt \right| \leq \left(\bigvee_0^1 f(t) + \sup_x |f(x)| \right) \|\Phi\|_\infty.$$

LEMMA 2. *Let $f \in BV([0,1])$. Then $\bigvee_0^1 f = \sup_\phi | \int_0^1 f(t)\phi(t)\,dt |$, where the sup is taken over those $\phi \in \mathcal{L}^1([0,1], m)$ with the properties that (i) $\int_0^1 \phi\,dt = 0$ and (ii) $\sup_x | \int_0^x \phi(t)\,dt | \leq 1$.*

LEMMA 3 (CHANGE OF VARIABLES). *Let $I = [a, b]$ and $f \in BV(I)$. Suppose $T: I \to J$ is a C^1 function with $T'(x) \neq 0$. Set $\psi = T^{-1}(x)$ and $\sigma = |d\psi(x)/dx|$. Then*

$$\bigvee_J f(\psi x)\sigma(x) \leq \left(\bigvee_I f + \operatorname*{Sup}_I (f) \right)\left(\bigvee_J \sigma + \operatorname*{Sup}_J \sigma \right).$$

PROOF. Let ϕ be as in Lemma 2,

$$\int_J (f \circ \psi) \cdot \sigma \cdot \phi\,dx = \int_I f \cdot z_\phi\,dt, \quad \text{where } z_\phi(\psi(x)) = \phi(x).$$

Hence, by Lemma 1,

$$\left| \int_J (f \circ \psi) \cdot \sigma \cdot \phi\,dx \right| = \left| \int_I f \cdot z_\phi\,dt \right| \leq \left(\bigvee_I f + \sup_I |f(x)| \right)\left(\sup_x \left| \int_a^x z_\phi(t)\,dt \right| \right).$$

But

$$\int_a^x z_\phi\,dt = \int_a^x \phi(Tx)\,dx = \int_a^x \phi(Tx)\sigma(Tx)|T'_x|\,dx$$

$$= \int_{T([a,x])} \phi(t)\sigma(t)\,dt \leq \left(\bigvee_J \sigma + \sup_J \sigma \right)\|\Phi\|_\infty.$$

Taking sup over ϕ's as in Lemma 2 completes the proof.   □

LEMMA 4. *Let T, ψ and σ be as in Lemma 3, but assume that T is C^2. Then there exist constants $\beta > 0$ and $K > 0$ so that, for each $f \in BV(I)$,*

$$\bigvee_J (f \circ \psi)\sigma \leq \beta \bigvee_I f + K\|f\|_1.$$

PROOF. First note that

$$\bigvee_I f + \sup_I |f| \le 2 \bigvee_I (f) + \|f\|_1.$$

Let $J_1, \ldots, J_m$ be a partition of J, and set $I_i = \psi(J_i)$. Put $\alpha_i = \sup_{J_i} \sigma$ and $\alpha = \sup_J \sigma$. Then applying Lemma 3 to each J_i yields

$$(1) \qquad \sum_{i=1}^m \bigvee_{J_i} f(\psi(x))\sigma(x) \le \sum_{i=1}^m \left(2 \bigvee_{I_i} f + \int_{I_i} |f| dx\right)\left(\alpha_i + \bigvee_{J_i} \sigma\right)$$

$$\le 2\alpha \bigvee_I f + 2\sum \bigvee_{I_i}(f) \bigvee_{J_i} \sigma + \alpha \sum \int_{I_i} f |dx|$$

$$+ \sum \bigvee_{J_i} \sigma \int_{I_i} |f| dx.$$

Since $\bigvee_{J_i} \sigma = \int_{J_i} |\sigma'| \, dx \le (\sup |\sigma'|/\inf \sigma) \int_{J_i} \sigma \, dx := K_1 \operatorname{length}(J_i)$,

$$(2) \qquad \sum \bigvee_{J_i} \sigma \int_{I_i} |f| dx + \alpha \sum \int_{I_i} |f| dx \le (K_1 + \alpha)\|f\|_1 := K\|f\|_1.$$

Since $\bigvee_{J_i} \sigma \le \sup_J |\sigma'| \operatorname{length}(J_i)$, we have

$$(3) \qquad 2\sum \bigvee_{I_i} f \bigvee_{J_i} \sigma \le 2 \sup_J |\sigma'|\left(\sum \bigvee_{I_i} f\right) \cdot \max\{\operatorname{length} J_i\},$$

and this term may be made arbitrarily small by refining the partition $J_1, \ldots, J_m$. Combining 1, 2 and 3 gives $\bigvee_J f(\psi(x))\sigma(x) \le 2\alpha \bigvee_I f + K\|f\|_1$. Note that the β of the lemma may be taken as $2 \cdot \sup_J \sigma$. $\square$

Lemma 4 may be used to provide another proof of Theorem 1 of [LY]. We now prove

THEOREM 1. *Let* $T(x) = T_i(x)$ *(prob. p_i), $i = 1, \ldots, m$, be a random map of* $[0,1]$, *where each T_i is a L-Y type map. If, for all $x \in [0,1]$*,

$$\sum_{i=1}^m \frac{p_i}{|T_i'(x)|} \le \gamma < 1,$$

then for all $f \in L^1([0,1], m)$:
  (1) *The limit*

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} P_T^j(f) = f^* \quad \text{exists in } \mathcal{L}^1.$$

  (2) $P(f^*) = f^*$.
  (3) $\bigvee_0^1 f^* \le C\|f\|_1$ *for some constant $C > 0$ which is independent of f.*

REMARKS. (i) $T_i'(x)$ may have two values at some points in $[0,1]$, which we denote by $T_i'(x+)$ and $T_i'(x-)$. The hypotheses of the theorem require only that

$$\sum_{i=1}^m \frac{p_i}{|T_i'(x+)|} \le \gamma < 1 \quad \text{and} \quad \sum_{i=1}^m \frac{p_i}{|T_i'(x-)|} \le \gamma < 1.$$

(ii) Keller [**K**] has proved a similar theorem under the stronger hypotheses that, for all i and x, $1/|T_i'(x)| < 1$.

PROOF. Our goal is to show that there are constants $0 < \alpha < 1$ and $K > 0$ so that $\bigvee_0^1 P^N(f) \leq \alpha \bigvee_0^1 f + K\|f\|_1$, for some $N \in \mathbf{N}$. This will suffice, as in [**LY**], to show that $P^* = \lim_n \frac{1}{n}\sum_{j=0}^{n-1} P^j$ exists and is a (nonzero) projection onto the eigenspace E_1 of eigenvectors of P_T with eigenvalue 1.

Select N so that $\gamma^N < 1/3$, and consider the map T^N. This is a random map defined by L-Y type functions: $T^N(x) = T_{j_N} \circ T_{j_{N-1}} \circ \cdots \circ T_{j_1}(x)$ with probability $\prod_{i=1}^N P_{j_i}$. The maps defining T^N may be indexed by $\{1, 2, \ldots, n\}^N$. Set $T_{j_N} \circ \cdots \circ T_{j_1}(x) := T_w(x)$ where $w = (j_1, \ldots, j_N)$ and $P_w = \prod_{i=1}^N P_{j_i}$. We then have that

$$\sum_{w \in \{1,2,\ldots,n\}^N} \frac{p_w}{|T_w'(x)|} = \sum_{\bar{w} \in \{1,\ldots,n\}^{N-1}} \sum_{i=1}^m \frac{p_i p_{\bar{w}}}{|T_i'(T_{\bar{w}}(x))| \, |T_w'(x)|}$$

$$= \sum_{\bar{w}} \frac{p_{\bar{w}}}{|T_{\bar{w}}'(x)|} \sum_{i=1}^m \frac{p_i}{|T_i'(T_{\bar{w}}x)|} \leq \gamma \sum_{\bar{w}} \frac{p_{\bar{w}}}{|T_{\bar{w}}'x|} \leq \cdots \leq \gamma^N < \frac{1}{3}.$$

Thus, to simplify notation we may assume that $\gamma < 1/3$ and show that $f \in BV \Rightarrow \bigvee P_T f \leq \alpha \bigvee f + K\|f\|_1$ for $\alpha \in (0,1)$ and $K > 0$.

Recall that $I_1, \ldots, I_l$ (where $I_i = (a_{i-1}, a_i)$) is the partition of $[0,1]$ into maximal intervals of joint monotonicity and C^2-ness of the maps $\{T_i\}_{i=1}^m$. We define

$$H_i^j = T_i(I_j), \qquad i = 1, \ldots, m, j = 1, \ldots, l,$$

$$\psi_i^j = \left(T_i|_{I_j}\right)^{-1} \quad \text{and} \quad \sigma_i^j = \left|\frac{d}{dx}\psi_i^j\right|.$$

Let $\mathcal{J}$ be the partition of $[0,1]$ by all the endpoints of the intervals H_i^j, and $\mathcal{K}$ be the partition of $[0,1]$ defined by all inverse images of endpoints of $\mathcal{J}$ under all possible T_i's. Each interval of $\mathcal{K}$ is mapped by each T_i into exactly one interval of $\mathcal{J}$. Finally, let $\mathcal{L} = \{L_k\}$ be a refinement of $\mathcal{K}$ into intervals the maximum length of which will be specified below.

To estimate $\bigvee_0^1 P_T(f)$ we compute as follows: suppose $f \in BV([0,1])$,

$$(4) \qquad \bigvee_0^1 \sum_{i=1}^m p_i \sum_{j=1}^l f(\psi_i^j)\sigma_i^j(x)\chi_{H_i^j}$$

$$\leq \sum_{i=1}^m \sum_{j=1}^l p_i\left(|f(a_{j-1})|\sigma_i^j(T_i(a_{j-1}))| + |f(a_j)|\sigma_i^j(T_ia_j)\right)$$

$$+ \sum_{i=1}^m \sum_{j=1}^l p_i \bigvee_{H_i^j} f(\psi_i^j x)\sigma_i^j(x).$$

The first sum on the right-hand side of (4) is

$$(5) \qquad \sum_{j=1}^l \left(\sum_{i=1}^m p_i\sigma_i^j(T_i(a_{j-1}))\right)|f(a_{j-1})| + \left(\sum_{i=1}^m p_i\sigma_i^j(T_i(a_j))\right)|f(a_j)|$$

$$\leq \gamma \sum_{j=1}^l |f(a_j)| + |f(a_{j-1})| \leq \gamma\left(\bigvee_I f + 2\|f\|_1\right)$$

because

$$\left| f(a_{j-1}) \right| + \left| f(a_j) \right| \leqslant \bigvee_{I_j} f + 2 \inf_{I_j} |f| \leqslant \bigvee_{I_j} f + 2 \int_{I_j} |f| dx.$$

The second sum in (4) we estimate as follows:

$$\sum_{i=1}^{m} \sum_{j=1}^{l} p_i \bigvee_{H_i^j} f(\psi_i^j x) \sigma_i^j(x) \leqslant \sum_{i=1}^{m} \sum_{j=1}^{l} p_i \sum_{\substack{k \text{ with} \\ T_i(L_K) \subset H_i^j}} \bigvee_{T_i(L_k)} f(\psi_i^j x) \sigma_i^j(x).$$

By Lemmas 3 and 4,

$$(6) \qquad \bigvee_{T_i(L_k)} f(\psi_i^j x) \sigma_i^j(x) \leqslant \left(\bigvee_{L_k} f + \sup_{L_k}(f) \right) \left(\bigvee_{T_i(L_k)} \sigma_i^j + \sup_{T_i(L_k)}(\sigma_i^j) \right)$$

$$\leqslant 2 \left(\sup_{T_i(L_k)} \sigma_i^j \right) \bigvee_{L_K} f + K_{ik} \int_{L_k} |f| dx$$

$$= 2 \frac{1}{\inf_{L_k} |T_i'(x)|} \bigvee_{L_k} f + K_{ik} \int_{L_k} |f| dx,$$

where $K_{ik} = \sup |\sigma_i^{j''}| / \inf \sigma_i^j + \sup \sigma_i^j$. Combining (5) and (6) gives

$$(7) \qquad \sum_{i=1}^{m} \sum_{j=1}^{n} p_i \bigvee_{H_i^j} f(\psi_i^j x) \sigma_i^j(x) \leqslant \sum_k \sum_{i=1}^{m} \frac{2 p_i}{\inf_{L_K} |T_i'x|} \bigvee_{L_k} f + K \|f\|_1$$

$$\leqslant 2 \gamma^1 \bigvee_0^1 f + K \|f\|_1,$$

where $\gamma^1 = \max_k \sum_{i=1}^{m} (p_i / \inf_{L_K} |T_i'(x)|)$ and $K = \Sigma_{i,k} K_{ik}$. Combining (4), (5) and (7) gives

$$(8) \quad \bigvee_0^1 P_T(f) \leqslant (\gamma + 2\gamma^1) \bigvee_0^1 (f) + (2\gamma + K) \|f\|_1 \leqslant 3\gamma^1 \bigvee_0^1 f + (2\gamma + K) \|f\|_1.$$

We now show that the partition $\mathcal{L}$ can be chosen so that $3\gamma^1 < 1$. Select δ so small that

$$\sum \frac{p_i}{|T_i'(x)| + \varepsilon_i} < \frac{1}{3} \quad \text{whenever } |\varepsilon_i| < \delta, i = 1, \dots, m.$$

Then, since $T_i|_{L_K}$ is C^2, select ε so that $|T_i'(x) - T_i'(y)| < \delta$ whenever x and y are in the same interval of $\mathcal{L}$ and $|x - y| < \varepsilon$. Finally, select $\mathcal{L} = \{L_K\}$ so that $\max_k \{\text{length}(L_K)\} < \varepsilon$. Then

$$\gamma^1 = \max_k \sum_{i=1}^{m} \frac{p_i}{\inf_{L_K} |T_i'x|} < \frac{1}{3}.$$

We have shown that, when $f \in BV[0,1]$, $\bigvee_0^1 P_T(f) \leqslant \alpha \bigvee_0^1 f + K \|f\|_1$ for some $0 < \alpha < 1$ and $K > 0$ independent of f. The assertions of the theorem now follow immediately, as in [**LY** or **T**]. $\quad \square$

4. An example. In this section we describe a family of random maps which demonstrates that the hypotheses of Theorem 1 are not necessary for the existence of an absolutely continuous invariant measure. The random maps discussed here have the form

$$T(x) = \begin{cases} f(x) & \text{prob. } p, \\ g(x) & \text{prob. } q, \end{cases}$$

where f is an expanding map ($|f'(x)| \geq k > 1$) and g contracts no faster than f expands ($1 > |g'(x)| \geq 1/k$). We will impose extra hypotheses and show that when the probability of applying the expanding map is larger than $1/2$, there is an absolutely continuous invariant measure. We assume

(1) g is C^2, $g(0) = 0$, $g''(x) \geq 0$ and $\frac{1}{2} \leq g' < 1$ and

(2) f is a L-Y type map with partition $I_0, I_1, \ldots, I_n$; that $f'(x) \geq 2$ whereever defined, $f(0) = 0$ and $f''(x) \geq 0$ for $x \in I_0$, while $f(x) \leq x$ for $x \in I_1, \ldots, I_n$.

The specific case where $f(x) = 2x \pmod 1$ and $g(x) = \beta x$ ($\frac{1}{2} \leq \beta < 1$) is an example of a pair of maps which possess properties (1) and (2).

THEOREM 2. *Let T be as above, where f and g satisfy properties* (1) *and* (2). *If $p > \frac{1}{2}$ then T has an absolutely continuous invariant measure.*

PROOF. Select $r \in (0, 1)$ so that $2^{1-r} \in (1, p/q)$. This can be done since $\lim_{r \to 1} 2^{1-r} = 1$ while $p/q > 1$. Changing variables by $y = y(x) = x^{1-r}$ gives a new random map

$$\tilde{T}(y) = \begin{cases} F(y) & \text{prob. } p, \\ G(y) & \text{prob. } q, \end{cases}$$

where $F(y) = y(f(x(y)))$ and $G(y) = y(g(x(y)))$. Note that F and G are still L-Y type maps. By the chain rule,

$$G'(y) = \frac{dy(g(x(y)))/dx}{dy(x(y))/dx} \cdot \frac{dg}{dx}(x(y)) \geq \left(\frac{g(x)}{x}\right)^{-r} \cdot g'(x)$$

$$\geq \frac{g(x)^{1-r}}{x^{1-r}} \geq \left(\frac{1}{2}\right)^{1-r}$$

since, by convexity, $g'(x) \geq g(x)/x$ and $g(x) \geq x/2$. On the interval $y(I_0)$ we compute, as with $G(y)$, that $F'(y) \geq 2^{1-r}$, while on $y(I_1), \ldots, y(I_n)$ we have

$$\left|\frac{dF}{dy}\right| = \left|\frac{dy(f(x(y)))/dx}{dy(x(y))/dx}\right| \cdot \left|\frac{df}{dx}\right| \geq 2 \geq 2^{1-r}$$

since y' is decreasing and $f(x) \leq x$ on $I_1, \ldots, I_n$. Hence

$$\frac{q}{|G'|} + \frac{p}{|F'|} \leq q2^{1-r} + \frac{p}{2^{1-r}} < 1$$

because $qx + p/x < 1$ for $x \in (1, p/q)$. Thus $\tilde{T}$ satisfies the hypotheses of Theorem 1, and so $\tilde{T}$ has an invariant density ϕ of bounded variation. From ϕ we obtain an invariant density for the random map T by setting $\psi(x) = \phi(y(x)) \, dy/dx$. Note that $\psi(x)$ is not of bounded variation if $\phi(0) > 0$, since $dy/dx = (1-r)x^{-r}$.
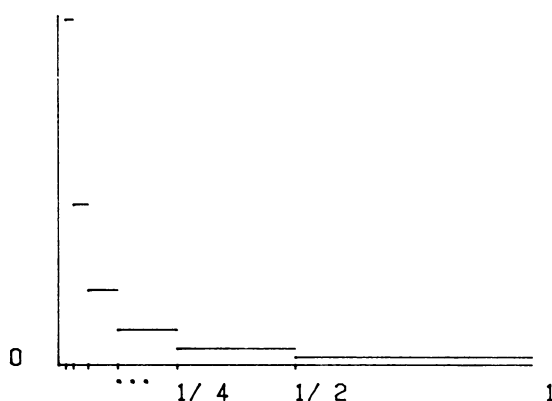
FIGURE 2. The invariant density of the random map

$$T(x) = \begin{cases} x/2, & q, \\ 2x \bmod 1, & p, \end{cases} \text{ with } 1/2 < p < 2/3.$$

In the specific case where $f(x) = 2x(\bmod 1)$ and $g(x) = x/2$ it is possible to construct the density of an absolutely continuous T invariant measure explicitly (Figure 2). In this case we can prove the

Claim. When $p > 1/2$ the random map T has a unique absolutely continuous invariant measure ν_p. The support of ν_p is all of $[0, 1]$. The density of the measure, $[d\nu_p/dm]$, is unbounded when $p < 2/3$. With respect to ν_p the random map T is ergodic in the sense that the associated PSP is ergodic.

PROOF. We begin by finding the F-P operator for the random map

$$T(x) = \begin{cases} T_1(x) = 2x \bmod 1, & \text{prob. } p, \\ T_2(x) = x/2, & \text{prob. } q. \end{cases}$$

Since T_2 is invertible,

$$P_{T_2}(f) = d\big(T_2^{-1}(x)\big)/dx \cdot f\big(T_2^{-1}(x)\big) \cdot \chi_{T_2[0,1]} = 2f(2x) \cdot \chi_{[0,1/2]}.$$

The map T_1 is monotone on 2 intervals on which it has the inverses $x \to x/2$ and $x \to (x + 1)/2$. Hence

$$P_{T_1}(f) = \frac{1}{2}f\left(\frac{x}{2}\right) + \frac{1}{2}f\left(\frac{x+1}{2}\right).$$

The F-P operator of T is then

$$(9) \qquad P_T(f) = pP_{T_1}(f) + qP_{T_2}(f)$$

$$= \frac{p}{2}\left(f\left(\frac{x}{2}\right) + f\left(\frac{x+1}{2}\right)\right) + 2qf(2x) \cdot \chi_{[0,1/2]}.$$

Assuming that $P_T(f) = f$, (9) can be rewritten as

$$(10) \qquad f\left(\frac{x}{2}\right) = \frac{2}{p}f(x) - \frac{4q}{p}\chi_{[0,1/2]}(x)f(2x) - f\left(\frac{x+1}{2}\right).$$

Then, assuming f is a positive constant on $(1/2, 1)$, (10) can be used to define f successively on the intervals $(1/2^{n+1}, 1/2^n)$, $n = 1, 2, 3, \ldots$. It is immediate that, defined in this way, f is constant on each of these intervals. We now show that $f \geqslant 0$ and that $f \in \mathcal{L}^1([0, 1], m)$.

Define $r_n = \int_{1/2^{n+1}}^{1/2^n} f(x)\, dx$, and $a_n = f|_{(1/2^{n+1}, 1/2^n)}$. Integrating (10) and changing variables we have

$$r_n = \frac{1}{p} r_{n-1} - \frac{q}{p} r_{n-2} - \frac{1}{2^{n+1}} r_0, \qquad n \geqslant 2,$$

$$a_n = \frac{2}{p} a_{n-1} - \frac{4q}{p} a_{n-2} - a_0, \qquad n \geqslant 2 \text{ and } 2^{n+1} r_n = a_n.$$

Then

$$\begin{pmatrix} r_{n+1} \\ r_n \end{pmatrix} = A \begin{pmatrix} r_n \\ r_{n-1} \end{pmatrix} - \frac{1}{2^{n+1}} \begin{pmatrix} r_0 \\ 0 \end{pmatrix}, \quad \text{where } A = \begin{pmatrix} 1/p, & -q/p \\ 1, & 0 \end{pmatrix},$$

so that

$$(11) \qquad \begin{pmatrix} r_{n+1} \\ r_n \end{pmatrix} = A^{n+1} \begin{pmatrix} r_0 \\ 0 \end{pmatrix} - \sum_{j=0}^{n} A^{n-j} \frac{1}{2^{j+1}} \begin{pmatrix} r_0 \\ 0 \end{pmatrix}$$

$$= \left(A^{n+1} - \frac{1}{2^{n+1}} \sum_{j=0}^{n} 2^j A^j \right) \begin{pmatrix} r_0 \\ 0 \end{pmatrix}.$$

The eigenvalues (eigenvectors) of A are $\lambda_1 = 1$, $e_1 = (1, 1)$ and $\lambda_2 = q/p$, $e_2 = (1, p/q)$. With respect to the basis e_1, e_2 the matrix in (11) may be rewritten as

$$A^{n+1} - \sum_{j=0}^{n} A^{n-j} \cdot \frac{1}{2^{j+1}} = \begin{pmatrix} \dfrac{1}{2}^{n+1} & 0 \\ 0 & \left(\dfrac{q}{p}\right)^{n+1} - \dfrac{(1^{n+1}/2) - (q/p)^{n+1}}{(p - 2q)/p} \end{pmatrix}.$$

Hence, $r_{n+1}/r_n = (ab - qc_n)/(ab - pc_n)$ where

$$a = p/(p - q), \quad b = 1/(p - q)(p - 2q)$$

and

$$c_n = 2^{n+1} \big(2(q/p)^n (1 - p^{n-1}) - p \big).$$

When $2q/p > 1$, so that $p < 2/3$, we have that $c_n \to \infty$ as $n \to \infty$ and $\lim_n (r_{n+1}/r_n) = q/p$. But $q/p < 1$ for $p > 1/2$, and so $\int_0^1 f\, dx = \sum_{i=0}^{\infty} r_i < \infty$ converges by the ratio test. For $1/2 < p < 2/3$ we also have that

$$\frac{a_{n+1}}{a_n} = \frac{2^{n+1} r_{n+1}}{2^n r_n} \sim \frac{2q}{p} > 1,$$

so that $\{a_n\}$ is increasing and unbounded. Thus, when normalized, f is the density of an invariant probability measure ν_p. The assertion concerning the uniqueness of ν_p and the ergodicity of T follows from the fact that ν_p is equivalent to Lebesgue measure and Corollary 7 of the next section. $\square$

5. Ergodic components. In this section we continue to assume that $T(x) = T_i(x)$ (prob. p_i), $i = 1, \ldots, n$, is a random map of $[0, 1]$ specified by L-Y type maps, and that T satisfies the hypotheses of Theorem 1. R denotes the PSP associated with T (see §2) and μ_f denotes the measure with $d\mu_f/dm = f$.

The proof of Theorem 1 shows that some power of P_T is quasicompact as an operator on BV. This implies (see [**K**]) that the eigenspace E_1 corresponding to the eigenvalue 1 is a finite-dimensional vector sublattice of $BV \subset \mathcal{L}^1$. In particular, E_1 has a basis of nonnegative functions $v_1(x), \ldots, v_k(x)$ with $\min(v_i, v_j) = 0$ when $i \neq j$. Set $A_i = \text{support}(v_i) = \{x: v_i(x) > 0\}$. Since $v_i \in BV$, each A_i differs from a union of intervals by a set of zero Lebesgue measure.

In this section it will be shown that the PSP R is ergodic with respect to each measure $\mu_p \times \mu_{v_i}$. The projection

$$P^*(\chi_{[0,1]}) = \lim_n \frac{1}{n} \sum_{j=0}^{n-1} P^j(\chi_{[0,1]}) := 1^*$$

is nonzero onto each basis vector $v_i(x)$. Hence, the number of ergodic components of R wrt $\mu_p \times \mu_{1*}$ is the maximum possible for any product measure the second factor of which is absolutely continuous wrt Lebesgue measure. The sets A_i are invariant under T in the sense of the following

LEMMA 5. $m(A_i \triangle \bigcup_{j=1}^n T_j(A_i)) = 0$.

PROOF. Set $S = A_i \setminus \bigcup_{j=1}^n T_i(A_i)$. If $m(S) > 0$, then $\mu_{v_i}(S) > 0$ since $S \subset A_i$. But $T^{-1}(S) = \bigcup_{j=1}^n T_i^{-1}(S) \subset A_i^c$ so $\mu_{v_i}(T^{-1}S) = 0$, which contradicts the fact μ_{v_i} is T-invariant. Similarly, suppose $S = \bigcup_{j=1}^n T_j(A_i) \setminus A_i$ has $m(S) > 0$. Then $\mu_{v_i}(S) = 0$ because $S \subset A_i^c$ while $\mu_{v_i}(T^{-1}S) > 0$ because each T_i is nonsingular with respect to m and hence $m(T^{-1}S \cap A_i) > 0$. Again, this contradicts the assumption that μ_{v_i} is T invariant. $\square$

The significance of the sets A_i is seen by the following characterization of the invariant sets of the PSP R.

PROPOSITION 6. *Suppose that* $\mu = \mu_p \times \nu$ *is R-invariant and that* $A \subset \Omega \times I$ *is an invariant set of positive* μ *measure* ($R(A) \subset A$ *and* $\mu(A) > 0$). *Then there is a set* $B \subset [0, 1]$ *so that* $\mu(A \triangle (\Omega \times B)) = 0$, *and* B *is T-invariant in the sense that* $\nu(B \triangle \bigcup_{j=1}^n T_j(B)) = 0$.

PROOF. First note that if a measurable set $U \subset \Omega$ has $\mu_p(U) > 0$, then $\lim_n \mu_p(\sigma^n(U)) = 1$. Suppose that A is not of the form $\Omega \times B$. Then $\exists S \subset [0, 1]$ with $\nu(S) > 0$ so that for ν almost every $x \in S$ we have $0 < \mu_p(A_x) < 1 - \varepsilon$, where $A_x = \{\omega \in \Omega: (\omega, x) \in A\}$ is the x-section and ε is some positive number. Define a sequence of measurable functions by $f_n(x) = \mu_p(\sigma^n(A_x))$. Then $f_n \to 1$ pointwise almost everywhere on S, and there is a set $S' \subset S$ with $\nu(S') > 0$ so that $f_n(x) \to 1$ uniformly on S'. Select N so that $n \geq N$ and $x \in S'$ implies $f_n(x) \geq 1 - \varepsilon/2$. Poincaré recurrence guarantees that almost every point of $A \cap (\Omega \times S')$ returns to it

under the iterates $R^{N+j}, j = 0, 1, 2, \ldots$, and consequently ν-almost every $x \in S'$ has $\mu_p(A_x) > 1 - \varepsilon/2$, contradicting the definition of the set S. We have shown that $A = \Omega \times B$ (except for a set of zero $\mu_p \times \nu$ measure). By the invariance of the set A, it is immediate that $\cup_{j=1}^n T_j(B) \subset B$. Since $R(A) \subset A$, we have $\mu(R^{-1}(A) \backslash A) = 0$. This implies that $\mu(A \backslash TA) = 0$, and hence $\nu(B \backslash \cup_{j=1}^n T_j B) = 0$. $\square$

The assumption that $v_1(x), \ldots, v_k(x)$ is a basis for E_1 implies that no A_i has a proper subset $(S \subset A,\ 0 < \mu_{v_i}(S) < \mu_{v_i}(A))$ which is invariant in the sense of Lemma 5. This fact, together with the previous proposition establishes a 1-1 correspondence between the sets A_i and the ergodic components of R wrt $\mu_p \times \mu_{1^*}$.

We now have enough information to describe "almost every" trajectory of the random map T. Set $A_j^{-\infty} = \{(w, x):\ R^k(w, x) \in \Omega \times A_j$ for some $k \geqslant 0\} = \cup_{k=0}^\infty R^{-k}(\Omega \times A_j)$. If $(w, x) \in A_j^{-\infty}$, the distribution of the sequence $\{x_n\}$ (where $x_n = T_{w_n} \circ \cdots \circ T_{w_0}(x)$) is given by μ_{v_j}. Since, $\cup_{j=1}^k A_j^{-\infty} = \Omega \times I$, up to sets of measure zero wrt $\mu_p \times m$, this is a "complete" description of the ways in which trajectories of the random map T can be distributed. We now assume that one of the maps specifying T, say T_1, is an expanding map $(|T_1'(x)| > 1$ wherever defined). For such a map Li and Yorke [**LiY**] have shown that (with respect to any absolutely continuous T_1-invariant measure) there are a finite number of ergodic components. They also show that each component must contain at least one point of discontinuity of the function $dT_1(x)/dx$. This result gives an easy method for determining an upper bound for the number of ergodic components of an expanding L-Y type map. Because of the fact that the ergodic components of the PSP R associated with a random map T are products, we may apply the Li-Yorke result as follows.

COROLLARY 7. *Suppose T_1 is an expanding map. Let $C_1, \ldots, C_k$ be the ergodic components of T_1 wrt the measure $\mu(d\mu = P_{T_1}^*(\chi_{[0,1]})\, dx)$. Define $C_i \sim C_j$ if $m(T_{i_l} \circ T_{i_{l-1}} \circ \cdots \circ T_{i_1}(C_i) \cap C_j) > 0$, for some $(i_l, \ldots, i_1)$. Then $\sim$ is an equivalence relation, and the number of ergodic components of R wrt $\mu_p \times \mu_{1^*}$ is the number of $\sim$ equivalence classes. In particular, the number of ergodic components of R is bounded by the number of discontinuities of dT_1/dx.*

PROOF. Under the iterates of T_1 almost every (m) point is mapped into $C = \cup C_i$. This implies, because A_i is invariant in the sense of Lemma 5, that each set $\Omega \times A_i$ contains at least one set of the form $\Omega \times C_j$. Thus each $C_j \subset \text{support}(1^*)$ and $\mu_{1^*}(C_j) > 0$. To show that $\sim$ is an equivalence relation: $C_i \sim C_i$ since $T_1(C_i) \subset C_i$. It follows from the definition that $\sim$ is transitive. The fact that $\sim$ is symmetric follows from the Poincaré recurrence theorem, since $\mu_{1^*}(C_i) > 0$. By Proposition 6, $C_i \sim C_j$ iff $\Omega \times C_i$ and $\Omega \times C_j$ are in the same ergodic component of R. $\square$

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