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INVERSE PRODUCING EXTENSION OF A BANACH ALGEBRA WHICH ELIMINATES THE RESIDUAL SPECTRUM OF ONE ELEMENT

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ABSTRACT. If A is a commutative unital Banach algebra and $G \subset A$ is a collection of nontopological zero divisors, the question arises whether we can find an extension A' of A in which every element of G has an inverse. Shilov [**1**] proved that this was the case if G consisted of a single element, and Arens [**2**] conjectures that it might be true for any set G. In [**3**], Bollobás proved that this is not the case, and gave an example of an uncountable set G for which no extension A' can contain inverses for more than countably many elements of G. Bollobás proved that it was possible to find inverses for any countable G, and gave best possible bounds for the norms of the inverses in [**4**].

In this paper, it is proved that inverses can always be found if the elements of G differ only by multiples of the unit; that is, we can eliminate the residual spectrum of one element of A. This answers the question posed by Bollobás in [**5**].

1. Preliminary definitions and statement of the main result.

Throughout this paper, a Banach algebra is assumed to be commutative and to possess a unit.

If A is a Banach algebra, $x \in A$, then the *essential spectrum* of x in A is the set

$$\sigma_e(x) = \{\lambda \in \mathbf{C}\colon\ \lambda \cdot 1_A - x \text{ is a topological zero divisor}\},$$

and the *residual spectrum* of x in A is the set

$$\sigma_r(x) = \{\lambda \in \mathbf{C}\colon\ \lambda \cdot 1_A - x \text{ is not invertible, but is not a topological zero divisor}\}.$$

Thus our main theorem may be stated as follows.

THEOREM 1. *Let A be a commutative Banach algebra, $x \in A$. Then there is an extension A' of A in which the spectrum of x is precisely the essential spectrum of x in A.*

Before proving Theorem 1, we prove the weaker result stated here as Theorem 2.

THEOREM 2. *Let A be a Banach algebra, $x \in A$, and let K be a compact set in the residual spectrum of x in A. Then there is an extension A' of A, such that the spectrum of x in A' does not intersect K.*

The method used to prove Theorem 2 is to take an open neighbourhood U of the essential spectrum of x in A, whose closure does not intersect K. (Such a

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neighbourhood can always be found since the essential spectrum and the set K are two nonintersecting compact sets in $\mathbf{C}$.) We then consider the algebra X of bounded analytic maps $U \to A$, and give it the supremum norm; this is a Banach algebra in which A is embedded isometrically as the constant functions.

We then let $J \subset X$ denote the closed ideal generated by the function $h \in X$, where

$$\begin{aligned} h\colon\ & U \to A \\ :\ & \lambda \to \lambda \cdot 1_A - x \end{aligned}$$

(we shall see that the ideal generated by this function is closed anyway).

We shall find that the algebra A is embedded (up to isomorphism) in the quotient space X/J. But then X/J is an extension of A, and the spectrum of x when embedded in X/J does not intersect K.

To see this, let $\mu \in K$, and let us find an inverse for $\mu \cdot 1_A - x$ when embedded in X/J. Let

$$\begin{aligned} R_\mu\colon\ & U \to A \\ :\ & \lambda \to (\mu - \lambda)^{-1} \cdot 1_A. \end{aligned}$$

This is a bounded analytic map $U \to A$ (since K and U are a finite distance apart). So $R_\mu \in X$. Also, the element $\mu \cdot 1_A - x$ is embedded in X as the constant function

$$\begin{aligned} c_\mu\colon\ & U \to A \\ :\ & \lambda \to \mu \cdot 1_A - x. \end{aligned}$$

Let $q\colon X \to X/J$ denote the quotient map. Then,

$$\begin{aligned} R_\mu \cdot c_\mu\colon\ & U \to A \\ :\ & \lambda \to ((\mu - \lambda)^{-1} \cdot 1_A) \cdot (\mu \cdot 1_A - x) \\ & = (\mu - \lambda)^{-1} \cdot 1_A \cdot ((\mu - \lambda) \cdot 1_A + (\lambda \cdot 1_A - x)) \\ & = 1_A + (\mu - \lambda)^{-1} \cdot (\lambda \cdot 1_A - x). \end{aligned}$$

Thus

$$q(R_\mu) \cdot q(c_\mu) = q(1_x) + q(h),$$

where $h\colon \lambda \to (\mu - \lambda)^{-1} \cdot (\lambda \cdot 1_A - x)$ and

$$\begin{aligned} 1_x\colon\ & U \to A \\ :\ & \lambda \to 1_A. \end{aligned}$$

But h is in the ideal J, so $q(h) = 0$. Thus $q(R_\mu) \cdot q(c_\mu) = q(1_x)$, so the element $\mu \cdot 1_A - x \in A$ has an inverse $q(R_\mu)$, when embedded as $q(c_\mu) \in X/J$.

Thus Theorem 2 will be proved.

An important tool in proving Theorem 2 and, later, Theorem 1, is the following

LEMMA 3. *Let B be a Banach algebra, $x \in B$, and let U and V be open sets in $\mathbf{C}$ such that*

(1) *U contains the essential spectrum of x in B,*

(2) *V contains U, and*

(3) *every component of V intersects U.*

Suppose we have $f(\lambda) = (\lambda - x)g(\lambda)$ (all $\lambda \in U$), where f and g are analytic functions $f\colon V \to B$ and $g\colon U \to B$.

Then there is an analytic extension g: $V \to B$ of g.

PROOF OF LEMMA 3. We can find an open set $W \subset \mathbf{C}$, such that $V = U \cup W$ and $\overline{W} \cap \sigma_e(x) = \varnothing$. Each component of W will intersect U, and our problem is to extend the analytic germs of g from $U \cap W$ to all of W.

Since

$$\overline{W} \cap \sigma_e(x) = \varnothing, \tag{$*$}$$

we claim that there is an $\varepsilon > 0$ such that $\lambda \in W$, $a \in A$ implies

$$\|(\lambda \cdot 1_A - x) \cdot a\| \geq \varepsilon \|a\|. \tag{3.1}$$

For if not, there are sequences $(\lambda_n)_{n=1}^{\infty} \subset W$, and $(a_n)_{n=1}^{\infty} \subset A$, with each $\|a_n\| = 1$, and

$$\|(\lambda_n \cdot 1_A - x) \cdot a_n\| \to 0 \quad \text{as } n \to \infty.$$

Then $\{|\lambda_n|: n \in \mathbf{N}\}$ must be bounded, so we may assume (taking a subsequence if necessary) that $\lambda_n \to \lambda \in \overline{W}$. Then

$$\begin{aligned} \|(\lambda \cdot 1_A - x) \cdot a_n\| &\leq \|(\lambda_n \cdot 1_A - x)a_n\| + \|(\lambda - \lambda_n)a_n\| \\ &= \|(\lambda_n \cdot 1_A - x)a_n\| + |\lambda - \lambda_n| \\ &\to 0 \quad \text{as } n \to \infty. \end{aligned}$$

Thus $\lambda \cdot 1_A - x$ is a topological zero divisor in A, so $\lambda \in \overline{W} \cap \sigma_e(x)$, contradicting our observation ($*$) that this set is empty. Let us choose an $\varepsilon > 0$ such that condition (3.1) is satisfied.

Now,

$$f(\lambda) = (\lambda \cdot 1_A - x) \cdot g(\lambda)$$

which implies that, for each $n = 1, 2, \ldots,$

$$f^{(n)}(\lambda) = (\lambda \cdot 1_A - x) \cdot g^{(n)}(\lambda) + n g^{(n-1)}(\lambda),$$

where $h^{(r)}$ denotes the rth derivative of a function h.

It follows that, if we have an analytic germ of g at some point $\lambda_0 \in W$, then

$$\|g^{(n)}(\lambda_0)\| \leq \sum_{r=0}^{n} \|f^{(r)}(\lambda_0)\| \cdot \frac{n!}{r!} \cdot \left(\frac{1}{\varepsilon}\right)^{n-r}$$

for all $n \in \mathbf{N}$.

So if the power series for f at λ_0 has radius of convergence $\delta > 0$, then the power series for g at λ_0 has radius of convergence greater than or equal to $\varepsilon \cdot \delta$.

Therefore this radius of convergence is bounded away from zero on any compact set in W; hence it must be possible to extend g throughout W, as required. Thus Lemma 3 is proved.

COROLLARY 4. *Let B be a Banach algebra, $c \in B$, and suppose that*

$$c = (\lambda \cdot 1_A - x) \cdot g(\lambda) \quad (\text{all } \lambda \in U),$$

where g: $U \to B$ is analytic, and U is a neighbourhood of the essential spectrum of c in $\mathbf{B}$. *Then $c = 0$.*

PROOF. The constant function c can be extended to all of $\mathbf{C}$, hence, by Lemma 3, the function g extends to all of $\mathbf{C}$, and the extension is a bounded entire function, which must be constant. Thus we have $c = (\lambda \cdot 1_A - x) \cdot g$ (all $\lambda \in \mathbf{C}$) for some $g \in A$; therefore $c = g = 0$.

We now prove Theorem 2.

PROOF OF THEOREM 2. We are given a Banach algebra A, an element $x \in A$, and a compact set K in the residual spectrum of X in A. We wish to exhibit an extension A' of A, such that the spectrum of x in A does not intersect K.

Let us choose a bounded open set $U \subset \mathbf{C}$ such that $U \supset \sigma_e(x)$ and $\overline{U} \cap K = \varnothing$. Let X be the Banach algebra of bounded analytic functions $U \to A$, with the supremum norm,

$$\|f\colon U \to A\|_X = \sup_{\lambda \in U} \|f(\lambda)\|_A.$$

(Note: we do not demand that such a function have a continuous extension to $\overline{U}$; this is important when we come to prove Theorem 1.) Let J be the closed ideal in X generated by the function $h \in X$, where $h(\lambda) = \lambda \cdot 1_A - x$ (all $\lambda \in U$).

Consider the isometric embedding $j\colon A \to X$ sending $c \in A$ to the constant function $j(c)\colon \lambda \to c$ (all $\lambda \in U$).

We wish to show that the morphism $\psi\colon A \to X/J$ obtained by composing j and the quotient map $q\colon X \to X/J$, is still an isomorphism.

Now it is evident that

$$\|\psi(a)\| \leq \|a\| \quad \text{for all } a \in A;$$

so we need to check that there is no sequence $(c_i)_{i=1}^{\infty} \subset A$, such that each $\|c_i\| = 1$, but

$$\|\psi(c_i)\|_{X/J} \to 0 \quad \text{as } i \to \infty.$$

Now J is the closure of the set of all functions $H \in X$ of form

$$\begin{aligned} H\colon\ & U \to A \\ :\ & \lambda \to f(\lambda) \cdot (\lambda \cdot 1_A - x), \end{aligned}$$

where $f \in X$. Thus if $\|\psi(c_i)\|_{X/J} \to 0$ as $i \to \infty$, there must be functions $(f_i)_{i=1}^{\infty} \subset X$ such that

$$\sup_{\lambda \in U} \|f_i(\lambda) \cdot (\lambda \cdot 1_A - x) - c_i\|_A \to 0 \quad \text{as } i \to \infty.$$

We have to show that such constants and functions cannot exist. We state this as a separate lemma:

LEMMA 2.1. *If A is a Banach algebra, $x \in A$, and $U \subset \mathbf{C}$ is an open set containing the essential spectrum of x in A, then there is an $\varepsilon > 0$ such that for all $c \in A$, and all analytic functions $f\colon U \to \mathbf{C}$, we have*

$$\sup_{\lambda \in U} \|f(\lambda) \cdot (\lambda \cdot 1_A - x) - c\|_A \geq \varepsilon \|c\|.$$

PROOF OF LEMMA 2.1. Let B be the Banach algebra of bounded sequences $(a_i)_{i=1}^{\infty} \subset A$, with pointwise addition and multiplication, and the norm

$$\|(a_i)_{i=1}^{\infty}\|_B = \sup_{i \in \mathbf{N}} \|a_i\|_A.$$

Let I be the closed ideal of B consisting of those sequences in B which norm converge to zero. Let π be the natural projection $\pi\colon B \to B/I$.

Now, for all $(a_i)_{i=1}^\infty \in B$,

$$\|\pi((a_i)_{i=1}^\infty)\|_{B/I} = \limsup_{i\to\infty} \|a_i\|_A;$$

and B/I is a commutative Banach algebra with unit $1_{B/I} = \pi[(1_A, 1_A, 1_A, \ldots)]$ (the equivalence class of the sequence in B consisting entirely of 1's).

A is embedded in B/I by the isometry

$$\begin{aligned} \phi\colon\ & A \to B/I \\ :\ & a \to \pi[(a, a, a, \ldots)]. \end{aligned}$$

In fact, our element x will have exactly the same spectrum and essential spectrum as $\phi(x) \in B/I$ as it did in A.

Suppose our lemma is false. Let $(c_i)_{i=1}^\infty$ be a sequence of norm 1 elements of A, and $(f_i)_{i=1}^\infty$ a sequence of analytic functions $U \to A$, such that

$$\sup_{\lambda\in U} \|f_i(\lambda)\cdot(\lambda\cdot 1_A - x) - c_i\| \to 0 \quad \text{as } i \to \infty.$$

Now if $W \subset U$ is any set which is bounded away from the essential spectrum of x, then there is an $\eta > 0$ such that

$$\|a\cdot(\lambda\cdot 1_A - x)\| \geq \eta\|a\| \quad \text{for all } a \in A,\ \lambda \in W.$$

Therefore for all $i \in \mathbf{N}$, $\lambda \in W$ we have

$$\|f_i(\lambda)\cdot(\lambda\cdot 1_A - x) - c_i\| \geq \eta\|f_i(\lambda)\| - \|c_i\| = \eta\|f_i(\lambda)\| - 1.$$

Thus there is a uniform bound on the values $\|f_i(\lambda)\|$ for all i, throughout W.

However, since the essential spectrum is a compact set within U, we may pick a suitable set W so that, by the Maximum Modulus principle, each f_i approaches its supremum norm $\sup_{\lambda\in U}\|f_i(\lambda)\|$ somewhere on W (we must say "approaches" rather than "achieves" since U is an open set so the supremum need not be achieved anywhere).

It follows that the collection $(f_i)_{i=1}^\infty$ is uniformly norm bounded. Similarly, the collection is uniformly differentiable at any point $u \in U$.

Now consider the map

$$\begin{aligned} F\colon\ & U \to B/I \\ :\ & \lambda \to \pi[(f_1(\lambda), f_2(\lambda), f_3(\lambda), \ldots)]. \end{aligned}$$

Since the f_i's are uniformly bounded, the sequence $(f_1(\lambda), f_2(\lambda), f_3(\lambda), \ldots)$ is always in B; and since they are uniformly differentiable, we find that F is a uniformly bounded analytic map. It is easy to see that for all $\lambda \in U$,

$$(\lambda\cdot 1_{B/I} - \phi(x))\cdot F(\lambda) = \pi[(c_1, c_2, c_3, \ldots)].$$

Therefore, since U contains the essential spectrum of $\phi(x) \in B/I$, we have by Corollary 4 that $\pi[(c_1, c_2, c_3, \ldots)] = 0$. Therefore $\limsup_{i\to\infty}\|c_i\|_A = 0$, but this is a contradiction since by hypothesis each $\|c_i\| = 1$. Thus Lemma 2.1 is proved.

COROLLARY 2.2. *The map ψ: $A \to X/J$ is an isomorphism.*

By the result of [**6**], we can put an equivalent norm on X/J so that A is now embedded isometrically by the map ψ. But this proves the theorem, for X/J contains an inverse to $\mu \cdot 1_A - x$ for every μ in K, namely the element $q(R_\mu)$, where

$$\begin{aligned} R_\mu &: U \to A \\ &: \lambda \to (\mu - \lambda)^{-1} \cdot 1_A. \end{aligned}$$

Having proved Theorem 2, we now make some definitions which lead towards a proof of Theorem 1.

Let Ω denote the collection of all countable ordinals. With each ordinal $\alpha \in \Omega$, we shall associate a Banach algebra $A(\alpha)$; the collection $\{A(\alpha)\colon \alpha \in \Omega\}$ will be directed upwards, in the sense that for all $\alpha, \beta \in \Omega$, $\alpha < \beta$, there will be an isometric embedding

$$\tau_{\alpha,\beta}\colon A(\alpha) \to A(\beta).$$

Furthermore, if $\alpha, \beta, \gamma \in \Omega$ $(\alpha < \beta < \gamma)$, we will have $\tau_{\beta,\gamma} \circ \tau_{\alpha,\beta} = \tau_{\alpha,\gamma}$.

We now proceed to define our sequence of Banach algebras, using transfinite induction.

(1) We define $A(1)$ to be our original Banach algebra A.

(2) Given $\alpha \in \Omega$ and the Banach algebra $A(\alpha)$, we define $A(\alpha + 1)$ as follows: Let $B(\alpha)$ be the Banach algebra of sequences $(a_i)_{i=1}^\infty$, $a_i \in A(\alpha)$, with pointwise addition and multiplication and the supremum norm

$$\|(a_i)_{i=1}^\infty\|_{B(\alpha)} = \sup_{i \in \mathbf{N}} \|a_i\|_{A(\alpha)},$$

let $I(\alpha)$ be the closed ideal consisting of all sequences in $B(\alpha)$ which norm converge to zero, and let π_α denote the natural projection

$$\pi_\alpha\colon B(\alpha) \to B(\alpha)/I(\alpha).$$

We define $A(\alpha+1) = B(\alpha)/I(\alpha)$, and the map $\tau_{\alpha,\alpha+1}$ is the isometric embedding

$$\begin{aligned} j_\alpha &: A(\alpha) \to A(\alpha + 1) \\ &: a \to \pi_\alpha[(a, a, a, a, \ldots)]. \end{aligned}$$

We must then define $\tau_{\beta,\alpha+1} = j_\alpha \circ \tau_{\beta,\alpha}$ for each ordinal $\beta < \alpha$.

(3) If $\alpha_i \in \Omega$, $i = 1, 2, \ldots$, then we require that $A(\bigcup_i \alpha_i)$ be the completion of the direct limit of the collection $\{A(\alpha_i),\ i = 1, 2, 3, \ldots\}$ of Banach algebras, which is directed by the maps τ_{α_i,α_j} $(\alpha_i < \alpha_j)$. For $\beta < \bigcup_i \alpha_i$, the map

$$\tau_{\beta,\bigcup_i \alpha_i}\colon A(\beta) \to A\left(\bigcup_i \alpha_i\right)$$

is the direct limit of the maps $\tau_{\beta,\alpha_i}\colon \beta < \alpha_i$, followed by the map which sends the direct limit of the algebras $A(\alpha_i)$ to its completion, which is $A(\bigcup_i \alpha_i)$.

Now these three conditions define the collection $\{A(\alpha),\ \alpha \in \Omega\}$ uniquely, together with the linking maps $\tau_{\beta,\alpha}\colon \beta < \alpha$.

LEMMA 5. *If* $\alpha, \beta \in \Omega$ $(\alpha \leq \beta)$ *and* $a \in A(\alpha)$, *then*

$$(*) \qquad \inf_{y \in A(\beta)} \left(\frac{\|\tau_{\alpha,\beta}(a) \cdot y\|}{\|y\|} \right) = \inf_{y_0 \in A(\alpha)} \left(\frac{\|ay_0\|}{\|y_0\|} \right).$$

PROOF. The left-hand side of this identity is an infimum similar to that on the right-hand side, but evaluated with the element $a \in A(\alpha)$ embedded in a larger space $A(\beta)$. It is therefore less than or equal to the right-hand side.

The opposite inequality is proved by transfinite induction on β; the result is trivially true if $\beta = \alpha$.

If the result is true for $\beta = \beta_0$, then given $y \in A(\beta_0 + 1)$ let us say

$$y = \pi_\alpha[(y_i)_{i=1}^\infty] \qquad (y_i \in A(\beta_0));$$

then

$$\begin{aligned}
\|y \cdot \tau_{\alpha,\beta_0+1}(a)\|_{A(\beta_0+1)} &= \limsup_{i \to \infty} (\|y_i \cdot \tau_{\alpha,\beta_0}(a)\|_{A(\beta_0)}) \\
&\geq \limsup_{i \to \infty} \|y_i\| \cdot \inf_{y' \in A(\beta_0)} \left(\frac{\|y' \cdot \tau_{\alpha,\beta_0}(a)\|}{\|y'\|} \right) \\
&\geq \limsup_{i \to \infty} \|y_i\| \cdot \inf_{y_0 \in A(\alpha)} \left(\frac{\|y_0 a\|}{\|y_0\|} \right) \\
&\qquad \text{(by induction hypothesis)} \\
&= \|y\|_{A(\beta_0+1)} \cdot \inf_{y_0 \in A(\alpha)} \left(\frac{\|y_0 a\|}{\|y_0\|} \right).
\end{aligned}$$

This is the result for $\beta_0 + 1$. But it is clear that an equality such as $(*)$ is preserved under direct limits and completions. So the result is true for all $\beta \in \Omega$.

We now prove Theorem 1.

PROOF OF THEOREM 1. Given a Banach algebra A and $x \in A$, let us choose a sequence $(U_i)_{i=1}^\infty$ of bounded open sets in $\mathbf{C}$, such that:

(1) For each i, $U_i \supset U_{i+1}$.

(2) For each i, every component of U_i intersects U_{i+1}.

(3) $\bigcap_{i=1}^\infty U_i = \sigma_e(x)$.

DEFINITION. A sequence $(\varepsilon_i)_{i=1}^n$ of strictly positive real numbers is said to be "admissible" for $\alpha \in \Omega$ if, whenever there are bounded analytic functions

$$g_i\colon U_i \to A(\alpha) \qquad (i = 1, 2, \ldots, n),$$

an analytic function $f\colon U_n \to A(\alpha)$ and a constant $c \in A(\alpha)$ such that, for all $\lambda \in U_n$,

$$c = \sum_{i=1}^n g_i(\lambda) + \tau_{1,\alpha}(\lambda \cdot 1_A - x) \cdot f(\lambda),$$

then

$$\|c\|_{A(\alpha)} \leq \sum_{i=1}^n \varepsilon_i^{-1} \cdot \sup_{\lambda \in U_i} \|g_i(\lambda)\|_{A(\alpha)}.$$

The most important step in proving Theorem 1 is the following

LEMMA 6. *There is a sequence* $(\varepsilon_i)_{i=1}^{\infty}$ *such that, for each* n, *the sequence* $(\varepsilon_i)_{i=1}^{n}$ *is admissible for every* $\alpha \in \Omega$.

PROOF OF LEMMA 6. It is enough to show the following:

(1) There is an ε_1, which is admissible, as a sequence of length 1, for every $\alpha \in \Omega$.

(2) If the sequence $(\varepsilon_i)_{i=1}^{n}$ is admissible for every $\alpha \in \Omega$ and $\delta > 0$, then there is an $\varepsilon_{n+1} > 0$ such that the sequence

$$(1+\delta)^{-1} \cdot \varepsilon_1, (1+\delta)^{-1} \cdot \varepsilon_2, \ldots, (1+\delta)^{-1} \cdot \varepsilon_n, \varepsilon_{n+1}$$

is also admissible for every $\alpha \in \Omega$.

Let us apply Lemma 2.1 with A replaced by $A(\alpha)$ for some ordinal α, x replaced by $\tau_{1,\alpha}(x)$ and U replaced by U_1. We find that there is an $\varepsilon > 0$ such that, for all $c \in A(\alpha)$ and all analytic functions f: $U_1 \to A(\alpha)$, we have

$$\sup_{\lambda \in U_1} \|f(\lambda) \cdot (\lambda \cdot 1_A - x) - c\|_{A(\alpha)} \geq \varepsilon \|c\|.$$

Writing $g_1(\lambda) = c - f(\lambda) \cdot (\lambda \cdot 1_A - x)$ $(\lambda \in U_1)$ we see the value ε, a "sequence" of length 1, is admissible for α.

Denote by $\varepsilon_1(\alpha)$ the supremum of all admissible values of ε for a given α. This decreases with increasing α, since the Banach algebras involved are always getting larger.

But $\varepsilon_1(\alpha)$ must be bounded away from zero. For if we could find a sequence $(\alpha_i)_{i=1}^{\infty}$ such that $\alpha_i \in \Omega$ and $\varepsilon_1(\alpha_i) \to 0$ as $i \to \infty$, then we must have $\varepsilon_1(\bigcup_{i=1}^{\infty} \alpha_i) = 0$, which is impossible.

Thus there is an ε_1 which is admissible for every $\alpha \in \Omega$. This proves assertion (1).

Now suppose that $(\varepsilon_i)_{i=1}^{n}$ is admissible for every $\alpha \in \Omega$. Choose a particular $\alpha \in \Omega$, and, given $\delta > 0$, suppose that we cannot find an ε_{n+1} such that

$$(1+\delta)^{-1}\varepsilon_1, (1+\delta)^{-1}\varepsilon_2, \ldots, (1+\delta)^{-1}\varepsilon_n, \varepsilon_{n+1}$$

is admissible for α.

Then we must be able to find constants $(c_j)_{j=1}^{\infty}$ in $A(\alpha)$, and analytic functions

$$g_i^{(j)}\colon U_i \to A(\alpha) \qquad (i = 1, 2, \ldots, n+1;\ j \in \mathbf{N})$$

and

$$f^{(j)}\colon U_{n+1} \to A(\alpha) \qquad (j \in \mathbf{N}),$$

such that, for each j, $\|c_j\| = 1$, and for all $\lambda \in U_{n+1}$,

$$c_j = \sum_{i=1}^{n+1} g_i^{(j)}(\lambda) + \tau_{1,\alpha}(\lambda \cdot 1_A - x) \cdot f^{(j)}(\lambda),$$

but

$$1 > (1+\delta) \sum_{i=1}^{n+1} \sup_{\lambda \in U_i} \|g_i^{(j)}(\lambda)\| + 2^j \cdot \sup_{\lambda \in U_{n+1}} \|g_{n+1}^{(j)}(\lambda)\|.$$

Extracting a subsequence from $(c_k)_{k=1}^{\infty}$, we may assume that

$$\sup_{\lambda \in U_i} \|g_i^{(j)}(\lambda)\|_{A(\alpha)} \to \eta_i \quad \text{as } j \to \infty,\ 1 \leq i \leq n,$$

where

$$(*) \qquad (1+\delta)\cdot\sum_{i=1}^{n}\varepsilon_i^{-1}\cdot\eta_i \le 1.$$

But now, for each $i \le n$ and $\lambda \in U_i$, define

$$G_i(\lambda) = \pi_\alpha[(g_i^{(1)}(\lambda), g_i^{(2)}(\lambda), \ldots, g_i^{(j)}(\lambda), \ldots)].$$

Define also, for $\lambda \in U_{n+1}$,

$$F(\lambda) = \pi_\alpha[(f^{(1)}(\lambda), f^{(2)}(\lambda), \ldots, f^{(j)}(\lambda), \ldots)] \in A(\alpha+1).$$

These functions are analytic by arguments similar to those used in the proof of Theorem 2.

Then, for every $\lambda \in U_{n+1}$,

$$\sum_{i=1}^{n} G_i(\lambda) + (\lambda\cdot 1_{A(\alpha+1)} - \tau_{1,\alpha+1}(x))\cdot F(\lambda) = \pi_\alpha[(c_1, c_2, c_3, \ldots)].$$

Therefore, by Lemma 3, we can extend F to U_n (this being the domain of definition of G_n), and so, since $(\varepsilon_i)_{i=1}^n$ is admissible for $\alpha+1$, we must have

$$1 = \|\pi_\alpha[(c_1, c_2, c_3, \ldots)]\| \le \sum_{i=1}^{n} \sup_{\lambda\in U_i} \|G_i(\lambda)\|\cdot\varepsilon_i^{-1}.$$

But $\sup_{\lambda\in U_i}\|G_i(\lambda)\| \le \eta_i$ so, by $(*)$, we have a contradiction.

Thus, for each $\alpha \in \Omega$, there is a suitable $\varepsilon_{n+1} > 0$ so that

$$(1+\delta)^{-1}\varepsilon_1, (1+\delta)^{-1}\varepsilon_2, \ldots, (1+\delta)^{-1}\varepsilon_n, \varepsilon_{n+1},$$

is an admissible sequence for α. Let $\varepsilon_{n+1}(\alpha)$ denote the supremum of possible values of ε_{n+1} for a given α.

By the transfinite induction argument of part (1) of this lemma, $\inf_{\alpha\in\Omega}\varepsilon_{n+1}(\alpha) > 0$.

Thus there is an ε_{n+1} such that

$$(1+\delta)^{-1}\varepsilon_1(1+\delta)^{-1}\varepsilon_2, (1+\delta)^{-1}\varepsilon_3, \ldots, (1+\delta)^{-1}\varepsilon_n, \varepsilon_{n+1}$$

is admissible for every $\alpha \in \Omega$. Thus Lemma 6 is proven.

Let $(\varepsilon_i)_{i=1}^\infty$ be a sequence such that $(\varepsilon_i)_{i=1}^n$ is admissible for each n and $\alpha \in \Omega$. Assume each $\varepsilon_i < 1$. Let Z be the algebra of all analytic functions taking values in A, which are defined on a neighbourhood of $\sigma_e(x)$.

For each $g \in Z$, define

$$\|g\|^{(1)} = \inf\left\{\|c\|_A + \sum_{i=1}^{N}\varepsilon_i^{-1}\cdot\sup_{\lambda\in U_i}\|g_i(\lambda)\|_A : (\lambda\cdot 1_A - x)f(\lambda) + g(\lambda) = c + \sum_{i=1}^{N} g_i(\lambda) \text{ (all } \lambda\in U_N)\right\}$$

(this is a seminorm on Z); and

$$\|g\|^{(2)} = \sup_{h\in Z, \|h\|^{(1)}\neq 0} (\|gh\|^{(1)}/\|h\|^{(1)}).$$

We shall see that $\|\cdot\|^{(2)}$ is also a seminorn on Z; it is always finite, and has the algebra norm property that $\|g_1 g_2\|^{(2)} \leq \|g_1\|^{(2)} \cdot \|g_2\|^{(2)}$ for all $g_1, g_2 \in Z$.

First, we claim that A is isometrically embedded in $(Z, \|\cdot\|^{(2)})$. For, by the definition of $\|\cdot\|^{(1)}$, we see that for all $c \in A$, $h \in Z$,

$$\|c \cdot h\|^{(1)} \leq \|c\|_A \cdot \|h\|^{(1)};$$

thus $\|c\|^{(2)} \leq \|c\|_A$. However,

$$\begin{aligned}\|c\|^{(1)} = \inf \Bigg\{ \|d\|_A + \sum_{i=1}^{N} \varepsilon_i^{-1} \cdot \sup_{\lambda \in U_i} \|g_i(\lambda)\|_A &: \\ c = (\lambda \cdot 1_A - x) f(\lambda) + d + \sum_{i=1}^{N} g_i(\lambda) \ (\text{all } \lambda \in U_N) \Bigg\} \\ \geq \inf\{\|d\|_A + \|c - d\|_A\} = \|c\|_A,\end{aligned}$$

since $(\varepsilon_i)_{i=1}^{N}$ is admissible for every N. So

$$\|c\|^{(2)} \geq \|c \cdot 1_A\|^{(1)} / \|1_A\|^{(1)} = \|c\|_A;$$

therefore A is indeed embedded isometrically. Next, we show that there is an inverse to $\mu \cdot 1_A - x$ in $(Z, \|\cdot\|^{(2)})$ for all $\mu \notin \sigma_e(x)$; and to do this we must show that the function $R_\mu \colon \lambda \to (\mu - \lambda)^{-1} \cdot 1_A$ has finite norm $\|R_\mu\|^{(2)}$. But this is true of any function $g \in Z$. For if g is defined on a neighbourhood of $\sigma_e(x)$, then for some n it will be defined and bounded on U_n. Then, if $h \in Z$, let us say $\delta > 0$ and

$$h(\lambda) = (\lambda - x) k(\lambda) + c + \sum_{i=1}^{M} \varepsilon_i^{-1} \sup_{\lambda \in U_i} \|g_i(\lambda)\|_A - \delta,$$

with

$$\|h\|^{(1)} \geq \|c\|_A + \sum_{i=1}^{M} \varepsilon_i^{-1} \sup_{\lambda \in U_i} \|g_i(\lambda)\| - \delta.$$

Then

$$gh(\lambda) = (\lambda - x) gk(\lambda) + \sum_{i=1}^{M} g f_i(\lambda) + c g(\lambda) \qquad (\lambda \in U_m \cap U_n)$$

and

$$\begin{aligned}\|gh\|^{(1)} &\leq \left(\sup_{\lambda \in U_n} \|g(\lambda)\| \right) \cdot \left[\left(\|c\| + \sum_{i=1}^{n} \sup_{\lambda \in U_i} \|g_i(\lambda)\| \right) \cdot \varepsilon_n^{-1} \right. \\ &\qquad \left. + \sum_{i=n+1}^{M} \varepsilon_i^{-1} \sup_{\lambda \in U_i} \|g_i(\lambda)\| \right] \\ &\leq \varepsilon_n^{-1} \sup_{\lambda \in U_n} \|g(\lambda)\| (\|h\|^{(1)} + \delta) \quad (\text{for each } \varepsilon_i \leq 1).\end{aligned}$$

Hence

$$\|g\|^{(2)} \leq \varepsilon_n^{-1} \cdot \sup_{\lambda \in U_n} \|g(\lambda)\|.$$

The third remark we make about $\|\cdot\|^{(2)}$ is that is has the algebra norm property that

$$\|g_1 g_2\|^{(2)} \leq \|g_1\|^{(2)} \cdot \|g_2\|^{(2)}$$

for all $g_1, g_2 \in Z$.

For

$$(*) \qquad \begin{aligned} \|g_1, g_2\|^{(2)} &= \sup_{\substack{h \in Z \\ \|h\|^{(1)} \neq 0}} \left(\frac{\|g_1 g_2 h\|^{(1)}}{\|h\|^{(1)}} \right) \\ &= \sup_{\substack{h \in Z \\ \|h\|^{(1)} \neq 0 \\ \|g_2 h\|^{(1)} \neq 0}} \left(\frac{\|g_1 g_2 h\|^{(1)}}{\|q_2 h\|^{(1)}} \cdot \frac{\|g_2 h\|^{(1)}}{\|h\|^{(1)}} \right). \end{aligned}$$

It is legitimate to restrict our attention to functions h such that $\|g_2 h\|^{(1)} \neq 0$ since if $\|g_2 h\|^{(1)} = 0$, then the expression $(*)$ is certainly zero for this function h; thus $\|g_1 g_2\|^{(2)} \leq \|g_1\|^{(2)} \cdot \|g_2\|^{(2)}$.

Now, let I be the ideal in Z consisting of all functions g whose seminorm $\|g\|^{(2)}$ is zero. Let A' denote the completion of the quotient space Z/I. Then A' is a Banach algebra in which A is embedded isometrically, just as it is embedded isometrically in $(Z, \|\cdot\|^{(2)})$. But for all $\mu \notin \sigma_e(x)$, there is an inverse for the element $\mu \cdot 1_A - x$ in Z, hence also in A'. Thus A' is an extension of A in which the spectrum of x is precisely the essential spectrum of x in A.

This concludes the proof of Theorem 1.

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