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THE DURATION OF TRANSIENTS

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ABSTRACT. A transformation T defined on $X \subset \mathbf{R}^n$ for which $T(X) \supset X$ is considered. A *transient* in X is a trajectory $x, Tx, \ldots, T^m x \subset X$ so that $T^{m+1}x \notin X$. In this case, m is the duration of the transient. A method for estimating the average duration of transients is given, and an example of a transformation with exceedingly long transients is described.

Introduction. Imagine a particle moving in a box and making elastic collisions with the sides. Suppose there is a small hole in one side of the box. For many initial conditions the particle will bounce around for a long time and then leave the box. These trajectories are examples of transients. More generally, given a transformation T on $\mathbf{R}^n$ which leaves a set $X \subset \mathbf{R}^n$ partially invariant (i.e., $T(X) \supset X$) we call the trajectory through x a transient in X if $x, Tx, \ldots, T^n x$ all lie in X but $T^{n+1}x \notin X$. In this case n is the duration or lifetime of the transient. In this paper we investigate the average duration of transients for a certain class of transformations T.

Transient behavior has been observed in a number of physical systems like the one mentioned above, in the Lorenz model [**4**], and (recently) in a large class of dynamical systems in which (as a parameter changes) an attractor collides with the boundary of its basin of attraction [**2**]. Each repeller of a dynamical system has a neighborhood which is partially invariant. Thus we find transient behavior (for the inverse transformation) in the neighborhood of each attractor of a dynamical system.

In §1 we study conditionally invariant measures. These are tools for the study of transient behavior which were introduced in [**3**]. In §2 the average duration of transients is computed in terms of a quantity (the pressure of a function) which, at least in simple cases, can be determined or estimated. Finally, §3 contains a description of a one-parameter dynamical system for which the average duration of transients is exceedingly long. This example illustrates the effects of factors which determine the duration of transients. It also indicates that great care may be required to distinguish (by means of computer simulation) a system with a chaotic attractor from one with long transients.

1. Measures and transients. Assume that X is a bounded set in $\mathbf{R}^n$ and that $T\colon X \to \mathbf{R}^n$. We will always denote normalized Lebesgue measure on X by

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m. If $T(X) \supset X$, then $T^{-1}(X)$ is nonempty, as are all the sets $R_n = T^{-n}(X)$. The set $\Lambda = \bigcap_{n \geq 0} T^{-n}(X)$ is then nonempty and is, in fact, the maximal T-invariant set in X. For $x \in X$, let $\tau(x) = \sup\{j \geq 0\colon T^j(x) \in X\}$. Thus $\tau(x) = n$ if and only if $x \in R_n \backslash R_{n+1}$. Then $\tau(x)$ is the duration of the transient beginning at x. We are interested in the situations where the expected value of τ, that is $\overline{\tau} = \int_x \tau(x)\, dm$, is finite. This implies that m-almost every trajectory leaves X eventually.

Useful tools for the study of transients are conditionally invariant measures, introduced in [**3**].

DEFINITION. A probability measure μ on X is conditionally invariant for T if, for each measurable set A,

$$\mu(A) = \mu[T^{-1}(A)]/\mu[T^{-1}(X)].$$

This says that the μ-probability that a random point X is in A is the same as the probability that $T(y)$ is in A, *given that* y is a point for which $T(y)$ is in X. Note that any T-invariant measure is conditionally invariant.

The most important property of conditionally invariant measures is given by

LEMMA 1. *Let T be as above and let μ be a conditionally invariant measure which is not invariant. Then $\mu(R_n) = \alpha^n$, where $\alpha = \mu(T^{-1}X) < 1$.*

PROOF. Because $\mu(A) = \mu(T^{-1}A)/\mu(T^{-1}X)$ for any measurable set A, we obtain, by taking $A = T^{-n}X$, that

$$\mu(T^{-n}X) = \mu(T^{-(n+1)}X)/\mu(T^{-1}X).$$

Consequently for any n, $\mu(T^{-n}X)\alpha = \mu(T^{-(n+1)}X)$, and so $\mu(T^{-n}X) = \alpha^n$. $\square$

In the cases studied here, there are more conditionally invariant measures than invariant ones.

The utility of conditionally invariant measures in the study of transients comes from the fact that $\int \tau(x)\, d\mu$ is easy to evaluate. Since $\tau = n$ on $R_n \backslash R_{n+1}$, we obtain from Lemma 1

COROLLARY 1.

$$\int \tau\, d\mu = \frac{\alpha}{1-\alpha} \quad \textit{and} \quad \int \left(\tau - \int \tau\, d\mu\right)^2 d\mu = \frac{\alpha}{(1-\alpha)^2}.$$

PROOF. We compute

$$\int \tau\, d\mu = \sum_{n=0}^{\infty} n[\mu(R_n) - \mu(R_{n+1})] = \sum_{n=0}^{\infty} n(\alpha^n - \alpha^{n+1}) = (1-\alpha)\sum_{n=0}^{\infty} n\alpha^n.$$

But

$$\sum_{n=0}^{\infty} n\alpha^n = \alpha \sum_{n=0}^{\infty} n\alpha^{n-1} = \alpha\left[\frac{d}{d\alpha}\sum_{n=0}^{\infty}\alpha^n\right] = \alpha\left[\frac{d}{d\alpha}\frac{1}{1-\alpha}\right] = \frac{\alpha}{(1-\alpha)^2};$$

hence, $\int \tau\, d\mu = \alpha/(1-\alpha)$. The proof of the second assertion is a similar computation. $\square$

When the measure μ is equivalent to m (i.e., there is a constant $K \geq 1$ so that $m(A)/K \leq \mu(A) \leq Km(A)$ for each measurable set A), the corollary suffices to estimate $\overline{\tau}$. In [**3**] the authors give a condition sufficient to guarantee the existence of a conditionally invariant measure μ equivalent to m. This condition is that $T\colon X \to \mathbf{R}^n$ is expanding and piecewise C^2. Expanding means that at every point where the derivative $DT(x)$ exists, all its eigenvalues have absolute value greater than 1. Piecewise C^2 means that X can be written as a finite union of sets $X = X_1 \cup \cdots \cup X_k$ so that the boundary of each X_i is smooth, except for a finite number of corners, and so T extends to a C^2 function on closure (X_i), $i = 1, \dots, k$.

Conditionally invariant measures can be proved to exist under more general conditions than those just described. In particular, consider a diffeomorphism f which has an axiom A attractor Λ. Let U be a small neighborhood of Λ and consider f^{-1} which maps U onto itself: $f^{-1}(U) \supset U$. On each piece of stable manifold of Λ, i.e., $W^s(x) \cap U$, the map f^{-1} acts like an expanding map. A modification of the techniques of [**3**] can then be used to construct conditionally invariant measures on each leaf $W^s(x) \cap U$, and these can be integrated to prove the existence of a conditionally invariant measure for f^{-1}.

Unfortunately, the proof of the existence of μ is not constructive and so the number $\alpha = \mu(T^{-1}X)$ is not readily attainable. The goal of the next section is to provide another expression for α. We begin by recalling the definition of the pressure of a function.

2. Transients and pressure. If $T\colon \Lambda \to \Lambda$ is a continuous map of a compact metric space and $\phi\colon \Lambda \to \mathbf{R}$ is continuous, the pressure of ϕ is defined as follows:

Let $\mathcal{U} = \{U_1, \dots, U_k\}$ be a finite open cover of Λ. For each $n > 0$ denote by $\mathcal{U}^{(n)}$ the collection of all sets of the form

$$U_{i_1} \cap T^{-1}U_{i_2} \cap \cdots \cap T^{-(N-1)}U_{i_n} = U_{i_1,\dots,i_n}.$$

If the diameter of $\mathcal{U} = \max \operatorname{diameter}(U_j)$ is ε, each set $U_{i_1,\dots,i_n}$ in $\mathcal{U}^{(n)}$ represents a collection of points whose jth images under T lie in the same set of $\mathcal{U}$, and hence remain within ε of each other for $0 \le j \le n-1$. The points in $U_{i_1,\dots,i_n}$ might then be called (n, ε) indistinguishable.

When $\mathcal{A} \subset \mathcal{U}^{(n)}$ is a collection of sets which cover Λ we define

$$Q(\mathcal{A}, \phi, n) = \sum_{A \in \mathcal{A}} \sup_{x \in A} \exp \sum_{j=0}^{n-1} \phi(T^j x)$$

and

$$P(\mathcal{U}, \phi, n) = \inf_{\substack{\mathcal{A} \text{ a subcover} \\ \text{of } \mathcal{U}^{(n)}}} Q(\mathcal{A}, \phi, n).$$

It turns out that the limit

$$P(\phi, \mathcal{U}) = \lim_n \frac{1}{n} \log[P(\mathcal{U}, \phi, n)]$$

exists and is finite. The pressure of ϕ, denoted $P(\phi)$, is defined by

$$P(\phi) = \lim_{\text{diameter } \mathcal{U} \to 0} P(\phi, \mathcal{U}).$$

Think of $P(\phi, \mathcal{U}, n)$ as the sum of $f = \exp \sum_{j=0}^{n-1} \phi(T^j x)$ over a collection of (n, ε) indistinguishable orbits in Λ. $P(\phi, \mathcal{U})$ is the exponential growth rate of these sums.

If T has a conditionally invariant measure μ which is equivalent to m, then $\alpha = \mu(T^{-1}X)$ satisfies $\log \alpha = \lim_n \frac{1}{n} \log m(R_n)$. This is the idea behind Theorem 1.

THEOREM 1. *Suppose that* $T\colon X \to \mathbf{R}^n$ *is a piecewise* C^2 *expanding transformation, where* $X = X_1 \cup \cdots \cup X_k$ *is a decomposition of* X *into pieces on which* T *is* C^2. *Suppose* T *maps each* X_j *onto* X. *Set* $\phi(x) = -\log(|\det DT(x)|)$ *and let* $P(\phi)$ *be the pressure of* ϕ *on* $T\colon \Lambda \to \Lambda$, *where* Λ *is the maximal* T*-invariant set in* X. *Then* $\log \alpha = P(\phi)$.

PROOF. By hypothesis, $R_n = T^{-n}(X)$ consists of k^n pieces $\{C_j^n\}_{j=1}^{k^n}$, so that T^n maps each C_j^n onto X and $T^n|C_j^n$ is C^2. We ensure that $T\colon \Lambda \to \Lambda$ and $\phi\colon \Lambda \to \mathbf{R}$ are continuous by giving Λ the topology in which each of the sets $\Lambda_j^n = \Lambda \cap C_j^n$ are open.

There are constants $0 < \lambda < 1$ and $B > 0$ so that $\max_j \operatorname{diameter}(C_j^n) \leq B\lambda^n$. To see this, let $B = \operatorname{diameter}(T(X))$ and

$$\mu = \inf_{1 \leq j \leq k} \inf_{X_j} \{\lambda\colon \lambda \text{ is eigenvalue of } DT(x)\} > 1.$$

If $x, y \in C_j^n$, then $\operatorname{dist}(T^n x, T^n y) \geq \operatorname{dist}(xy)\mu^n$. But $T^n x$ and $T^n y \in T(X)$ so $\operatorname{dist}(T^n x T^n y) \leq B$. Hence

$$\operatorname{diameter}(C_j^n) = \sup_{x, y \in C_j^n} \operatorname{dist}(x, y) \leq B(1/\mu)^n.$$

Take $\lambda = \mu^{-1}$.

Because T^n maps C_j^n onto X, we have, for each n and j,

$$m(C_j^n) \cdot \sup_{C_j^n} |\det DT^n| \geq 1 \geq m(C_j^n) \inf_{C_j^n} |\det DT^n|,$$

so

$$\inf_{C_j^n} \frac{1}{|\det DT^n|} \leq m(C_j^n) \leq \sup_{C_j^n} \frac{1}{|\det DT^n|}.$$

Since $R_n = \bigcup_{j=1}^{k^n} C_j^n$ we sum over j and obtain

$$\sum_j \inf_{C_j^n} \frac{1}{|\det DT^n|} \leq m(R_n) \leq \sum_j \sup_{C_j^n} \frac{1}{|\det DT^n|}. \tag{1}$$

Since $1/|\det DT^n(x)| = \exp \sum_{k=0}^{n-1} (T^k x)$, we can write (1) as

$$\sum_j \inf_{C_j^n} \exp \sum_{k=0}^{n-1} \phi(T^k x) \leq m(R_n) \leq \sum_j \sup_{C_j^n} \exp \sum_{k=0}^{n-1} \phi(T^k x). \tag{2}$$

We now show that the exponential growth rates of the quantities in (2) are the same. To do this, note that if x, y are in the same set C_j^k, then $\operatorname{dist}(x, y) \leq B\lambda^k$ and hence there exists $K > 0$ so that

$$|\phi(x) - \phi(y)| \leq K \operatorname{dist}(x, y) \leq KB\lambda^k$$

because ϕ is C^1 on each C_j^k. Consequently

$$\sum_{k=0}^{n-1} \phi(T^k x) - \sum_{k=0}^{n-1} \phi(T^k y) \leq \sum_{k=0}^{n-1} |\phi(T^k x) - \phi(T^k y)| \leq \sum_{k=0}^{n-1} KB\lambda^{n-k}$$
$$< KB \frac{1}{1 - \lambda} = M,$$

independent of n. Then we have that

$$\lim_n \frac{1}{n} \log \frac{\sum_j \sup_{C_j^n} \exp \sum_{k=0}^{n-1} \phi(T^k x)}{\sum_j \inf_{C_j^n} \exp \sum_{k=0}^{n-1}(T^k x)}$$
$$\leq \lim_n \frac{1}{n} \log \sum_j \left(\frac{\inf_{C_j^n} \exp \sum_{k=0}^{n-1} \phi(T^k x)}{\sum_j \inf_{C_j^n} \exp \sum_{k=0}^{n-1} \phi(T^k x)} \right) e^M = \lim_n \frac{M}{n} = 0.$$

Since for any function f we have that

$$\sum_j \inf_{C_j^n} f \leq \sum_j \inf_{\Lambda_j^n} f \leq \sum_j \sup_{\Lambda_j^n} f \leq \sum_j \sup_{C_j^n} f,$$

we obtain from (2) that

$$\log \alpha = \lim_n \frac{1}{n} \log m(R_n) = \lim_n \log \sum_j \sup_{\Lambda_j^n} \exp \sum_{k=0}^{n-1} \phi(T^k x). \tag{3}$$

To complete the proof it only remains to show that the right-hand side of (3) is equal to $P(\phi)$. Note that it is equal to $P(\phi, \mathcal{U})$, where $\mathcal{U} = \{\Lambda \cap X_1, \ldots, \Lambda \cap X_k\}$. To find $P(\phi) = \lim_{\text{diam}(\mathcal{U}) \to 0} P(\phi, \mathcal{U})$ we use the open covers $\mathcal{U}^{(n)} = \{\Lambda_j^n\}$ which have diameter $(\mathcal{U}^{(n)}) \leq B\lambda^n$. We show that $P(\phi, \mathcal{U}^{(n)}) = P(\phi, \mathcal{U})$. This is because, for fixed m,

$$\begin{aligned} \sum_j^{k^n} k^m \inf_{\Lambda_j^n} \exp \sum_{k=0}^{n-1} \phi(T^k x) &\leq \sum_j^{k^{m+n}} \sup_{\Lambda_j^{m+n}} \exp \sum_{k=0}^{n-1} (T^k) \\ &\leq \sum_{j=\cdot}^{k^n} k^m \sup_{\Lambda_j^n} \exp \sum_{k=0}^{n-1} \phi(T^k x), \end{aligned} \tag{4}$$

which is obtained by replacing the sum over the k^m sets Λ_j^{m+n} which constitute a single Λ_j^n by k^m times the sup or inf over the Λ_j^n. But the right- and left-hand sides of (4) have the same exponential growth rate (in n) as the terms in (2). □

The usefulness of Theorem 1 in estimating $\bar{\tau}$ comes from the variational inequality which says that for every T-invariant measure μ on Λ,

$$h_\mu(T) + \int \phi \, d\mu \leq P(\phi), \tag{5}$$

where $h_\mu(T)$ is the μ-entropy of T restricted to Λ. (See reference [**1**] for details.) Rather than enter into details we illustrate the theorem by estimating $\int \tau \, d\mu$ for a particular example.

Take $X = \{(x,y)\colon 0 \leq x \leq 1,\ |y| < 1\}$. Define $T\colon X \to \mathbf{R}^2$ by $T(x,y) = (2x (\operatorname{mod} 1), f(x,y))$, where f is a C^2 function on $X - \{(x,y)\colon x = \frac{1}{2}\}$ with the properties that $f(x,0) = 0$ and $\partial f(x,y)/\partial y \geq \gamma > 1$. Then $\Lambda = \{(x,y)\colon y = 0\}$ and $|\det DT(x,y)| = 2|\partial f(x,y)/\partial y|$. In this setting, equality is attained in (5) for a unique measure μ^* on Λ and $h_{\mu^*}(T) = \log 2$. Therefore

$$\log \alpha = P(\phi) = h_{\mu^*}(T) - \int \log |\det DT| \, d\mu^* \geq \log 2 - \sup_\Lambda \log \left[2 \frac{\partial f}{\partial y}(x,y) \right]$$

so that $\alpha \geq \inf_\Lambda (\partial f/\partial y)^{-1}$. Denoting this inf by s, we obtain $\int \tau \, d\mu \geq s/(1-s)$. Similarly, $\int \tau \, d\mu \leq m/(1-m)$, where $m = \sup_\Lambda (\partial f/\partial y)^{-1}$.

3. An example. In this section we consider one-parameter families $\{T_\varepsilon\}$ of transformations from $[0,1]$ to $\mathbf{R}$ and study the dependence of $\overline{\tau}$ on ε. In particular, we will describe a family $\{T_\varepsilon\}$ for which $\lim_{\varepsilon\to 0}\overline{\tau}_\varepsilon\varepsilon^n = +\infty$ for every $n \geq 0$. This says that the expected duration of the transients is large for ε near 0, and remains large for a substantial range of parameter values. This phenomenon has been observed numerically by Grebogi, et al. [**2**].

We begin by defining a function T_0 on $[0,1]$ which has very thin cusps at $x = \frac{1}{2}$ and $x = 1$. To do this, select δ and $d_1 > 0$ and define

$$T_0(x) = \begin{cases} 1 - \sqrt{-1/\log\left(\frac{1}{2} - x\right)} & \text{for } \frac{1}{2} - d_1 < x < \frac{1}{2}, \\ (2-\delta)x & \text{for } 0 \leq x < \frac{1}{4}. \end{cases}$$

On $(\frac{1}{4}, \frac{1}{2} - d_1)$ define T_0 in such a way that T_0 is a C^2, concave up, monotone increasing function on $[0, \frac{1}{2}]$. In order to do this, d_1 must be quite small. For later convenience we record the fact that $d_1 < \frac{1}{32}$. Define T_0 on $[\frac{1}{2}, 1]$ by translation: $T_0(x) = T_0(x - \frac{1}{2})$. Note that the graph of $T_0(x)$ approaches the line $x = \frac{1}{2}$ in the same manner that $\exp(-1/x^2)$ approaches the x-axis at $x = 0$. Finally, define $T_\varepsilon(x) = (1+\varepsilon)T_0(x)$ for $\varepsilon \geq 0$.

Denoting $\{x \colon T_\varepsilon(x) \notin [0,1]\} = I \backslash T_\varepsilon^{-1}(I)$ by A_ε we observe that there is an $\varepsilon_0 > 0$ such that $A_\varepsilon \subset [\frac{1}{2} - d_1, \frac{1}{2}] \cup [1 - d_1, 1]$ for all $\varepsilon < \varepsilon_0$. Concerning T_ε we show

Observation 1. For each $\varepsilon > 0$, T_ε has an absolutely continuous conditionally invairant measure μ_ε, the density of which, $d\mu_\varepsilon/dx = \rho_\varepsilon$, is a decreasing function of $x \in [0,1]$.

PROOF. The proof is a modification of that given in [**3**]. Define an operator Q_ε on $\mathcal{L}^1([0,1], m)$ by

$$Q_\varepsilon(f)(x) = \frac{\sum_{T_\varepsilon(x)=x}(f(y)/|T'_\varepsilon(y)|)}{\int_0^1 \sum_{T_\varepsilon(y)=x}(f(y)/|T'_\varepsilon(y)|)\,dx}.$$

Nonnegative fixed points of Q_ε are densities of absolutely continuous conditionally invariant measures. In [**3**] it is proved that there is a compact, convex set C in $\mathcal{L}^1([0,1])$ which is mapped into itself by Q_ε. We observe that Q_ε also preserves the set D of decreasing functions on $[0,1]$, and so $Q_\varepsilon(C \cap D) \subset C \cap D$. Therefore Q_ε has a fixed point in $C \cap D$. This fixed point is ρ_ε. The next two remarks provide bounds on $\rho_\varepsilon(x)$ which are independent of ε.

Observation 2. There is a number $b > 1(2-\delta)$ such that for $\varepsilon < \varepsilon_0$,

$$1 \geq \mu_\varepsilon(T_\varepsilon^{-1}(I)) \geq b.$$

PROOF. Denote $\mu_\varepsilon(T_\varepsilon^{-1}(I))$ by α_ε. This means $\alpha_\varepsilon = 1 - \int_{A_\varepsilon}\rho_\varepsilon\,dx$. Since ρ_ε is decreasing and has integral 1, $1 \geq \int_0^{1/4}\rho_\varepsilon\,dx \geq \rho_\varepsilon(\frac{1}{4})\cdot\frac{1}{4}$, and so $\rho_\varepsilon(\frac{1}{4}) \leq 4$. Recall that $A_\varepsilon \subset [\frac{1}{2} - d_1, \frac{1}{2}] \cup [1 - d_1, 1]$ when $\varepsilon > \varepsilon_0$ and that $d_1 < \frac{1}{32}$. Then

$$\int_{A_\varepsilon}\rho_\varepsilon\,dx \leq \rho_\varepsilon\left(\tfrac{1}{4}\right)\cdot m(A_\varepsilon) \leq 4\cdot 2d_1 \leq \tfrac{1}{4}$$

and so $\alpha_\varepsilon \geq 1 - \frac{1}{4}$. For b select any number satisfying $\frac{3}{4} \geq b > 1/(2-\delta)$.

Observation 3. For $\varepsilon < \varepsilon_0$, the functions ρ_ε are uniformly bounded.

PROOF. Since each ρ_ε is a decreasing function of x, it is enough to find a bound on $\rho_\varepsilon(0)$ which is independent of ε. Because ρ_ε is a fixed point of Q, we have that

$$\rho_\varepsilon(0) = \left(\frac{\rho_\varepsilon(0)}{T'_\varepsilon(0)} + \frac{\rho_\varepsilon(1/2)}{T'_\varepsilon(1/2)}\right)\alpha_\varepsilon^{-1} = \left[\rho_\varepsilon(0) + \rho_2\left(\frac{1}{2}\right)\right]\frac{\alpha_\varepsilon^{-1}}{(2-\delta)}.$$

Therefore, $b(2-\delta)\rho_\varepsilon(0) \leq \rho_\varepsilon(0) + \rho_2(\frac{1}{2})$. As before, $\rho_\varepsilon(\frac{1}{2}) \leq 2$, and so $\rho_\varepsilon(0) \leq 2/(b(2-\delta)-1) = \Delta$ for each $\varepsilon < \varepsilon_0$. We now have the estimate that $\int \tau_\varepsilon \, d\mu_\varepsilon = \int \tau_\varepsilon \, \rho_\varepsilon \, dx \leq \Delta \int \tau_\varepsilon \, dx$, and therefore

$$\overline{\tau}_\varepsilon \geq \frac{1}{\Delta}\int \tau_\varepsilon \, d\mu_\tau = \frac{1}{\Delta}\frac{\alpha_\varepsilon}{1-\alpha_\varepsilon} \geq \frac{1}{\Delta}\mu_\varepsilon(A_\varepsilon)^{-1} \geq \frac{1}{\Delta^2}m(A_\varepsilon)^{-1}.$$

To complete the argument, note that $m(A_\varepsilon) = 2e^{-1/\varepsilon^2} \leq 2\varepsilon^n$ when ε is small enough.

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